Dynamic Games and Rational-Expectations Models of Macroeconomic Policies

Engelbert J. Dockner
Reinhard Neck

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Introduction

- Strategic interactions between the government and the aggregate private sector
- Asymmetry $\Rightarrow$ Stackelberg game
- Here: rather general linear-quadratic differential game with two decision makers: government and aggregate of private agents
- **Open-loop Stackelberg equilibrium** solution: government's strategies are equivalent to the optimal policies of a government for the linear rational-expectations model

A Differential Game

Linear-quadratic differential game with two decision makers:

- objective functions are quadratic
- system is linear
- model is deterministic
- no inequality restrictions
- coefficients of the system and the objective functions are time-invariant
- no exogenous non-controllable variables
- infinite time horizon
- discounting

A Differential Game

Dynamic economic system

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), \tag{1}$$

where

- $t \in [0, \infty)$,
- $x(t) \in \mathbb{R}^n$ state variables,
- $u_i(t) \in \mathbb{R}^{m_i}, i = 1, 2$, control (instrument) variables of the i-th decision maker (player)
- initial state $x(0) = x_0$ is known

A Differential Game

Two output equations:

$$y_i(t) = D_i x(t) + E_i u_i(t) + F_i u_j(t), \quad i, j = 1, 2, i \neq j, \tag{2}$$

$y_i(t) \in \mathbb{R}^{k_i}$ objective variables of the i-th decision maker

Quadratic objective functions to be minimized:

$$J_i = \int_0^\infty \exp(-rt)[(1/2)y_i'(t)W_i y_i(t) + w_i' y_i(t) + w_i]dt, \tag{3}$$

$r \geq 0$ common rate of discount

A Differential Game

Interpretation:

- Stackelberg equilibrium solution: player 1 is the leader, player 2 - the follower
- Open-loop information pattern for both players

The Open-Loop Stackelberg Equilibrium

Optimum control problem for the follower (player 2)

Hamiltonian:

$$\begin{aligned} H^2 &= (1/2)y_2'(t)W_2y_2(t) + w_2'y_2(t) + w_2 + \\ &+ \lambda_2'(t)[Ax(t) + B_1u_1(t) + B_2u_2(t)], \end{aligned} \tag{4}$$

$$\dot{\lambda}_2(t) = r\lambda_2(t) - \partial H^2/\partial x(t). \tag{5}$$

Transversality condition:

$$\lim_{t\to\infty} x'(t)\lambda_2(t)\exp(-rt) = 0. \tag{6}$$

The Open-Loop Stackelberg Equilibrium

Optimization problem of the leader (player 1)

Necessary conditions for optimality of the leader's strategy:

$$\dot{x}(t) = \partial H^1/\partial\lambda_{11}(t), \tag{7}$$

$$\dot{\lambda}_2(t) = \partial H^1/\partial\lambda_{12}(t), \tag{8}$$

$$\dot{\lambda}_{11}(t) = r\lambda_{11}(t) - \partial H^1/\partial x(t), \tag{9}$$

$$\dot{\lambda}_{12}(t) = r\lambda_{12}(t) - \partial H^1/\partial\lambda_2(t), \tag{10}$$

- Leader's current-value costate variables $\lambda_{11}(t)$ and $\lambda_{12}(t)$

The Open-Loop Stackelberg Equilibrium

Optimization problem of the leader (player 1)

Transversality conditions for $\lambda_2(t)$ and for $\lambda_{11}(t)$:

$$\lim_{t\to\infty} x'(t)\lambda_{11}(t)\exp(-rt) = 0, \tag{11}$$

- Initial conditions for $x(t)$: $x(0) = x_0$
- Initial conditions for $\lambda_{12}(t)$: $\lambda_{12}(t) = 0$

The Open-Loop Stackelberg Equilibrium

Notation:

$$G \equiv E_1 - F_1(E_2' W_2 E_2)^{-1} E_2' W_2 F_2. \tag{12}$$

Optimal (equilibrium) control of the leader:

$$\begin{aligned} u_1^S(t) &= -(G' W_1 G)^{-1} G' W_1 [D_1 - F_1(E_2' W_2 E_2)^{-1} E_2' W_2 D_2] x(t) + \\ &+ (G' W_1 G)^{-1} G' W_1 F_1 (E_2' W_2 E_2)^{-1} B_2' \lambda_2(t) - \\ &- (G' W_1 G)^{-1} [B_1' - F_2' W_2 E_2 (E_2' W_2 E_2)^{-1} B_2'] \lambda_{11}(t) + \\ &+ (G' W_1 G)^{-1} F_2' [I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2' D_2 \lambda_{12}(t) - \\ &- (G' W_1 G)^{-1} G' [w_1 - W_1 F_1 (E_2' W_2 E_2)^{-1} E_2' w_2]. \end{aligned} \tag{13}$$

The Open-Loop Stackelberg Equilibrium

Equilibrium control of the follower:

$$
\begin{aligned}
u_2(t) &= (E_2' W_2 E_2)^{-1} E_2' W_2 \{ F_2 (G' W_1 G)^{-1} G' W_1 [D_1 - \\
&- F_1 (E_2' W_2 E_2)^{-1} E_2' W_2 D_2] - D_2 \} x(t) - \\
&- (E_2' W_2 E_2)^{-1} [I + E_2' W_2 F_2 (G' W_1 G)^{-1} G' W_1 F_1 \cdot \\
&\cdot (E_2' W_2 E_2)^{-1}] B_2' \lambda_2(t) + \\
&+ (E_2' W_2 E_2)^{-1} E_2' W_2 F_2 (G' W_1 G)^{-1} \cdot \\
&\cdot [B_1' - F_2' W_2 E_2 (E_2' W_2 E_2)^{-1} B_2'] \lambda_{11}(t) - \\
&- (E_2' W_2 E_2)^{-1} E_2' W_2 F_2 (G' W_1 G)^{-1} F_2' \cdot \\
&\cdot [I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 D_2 \lambda_{12}(t) + \\
&+ (E_2' W_2 E_2)^{-1} E_2' W_2 F_2 (G' W_1 G)^{-1} G' w_1 - \\
&- (E_2' W_2 E_2)^{-1} [I + E_2' W_2 F_2 (G' W_1 G)^{-1} \cdot \\
&\cdot G' W_1 F_1 (E_2' W_2 E_2)^{-1}] E_2' w_2. \qquad (14)
\end{aligned}
$$

The Open-Loop Stackelberg Equilibrium

$$\begin{aligned} H_{11} &= A - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 D_2 - [B_1 - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 F_2] \cdot \\ &\cdot (G' W_1 G)^{-1} G' W_1 [D_1 - F_1 (E_2' W_2 E_2)^{-1} E_2' W_2 D_2], \end{aligned} \tag{15}$$

$$\begin{aligned} H_{12} &= [B_1 - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 F_2](G' W_1 G)^{-1} F_2' \cdot \\ &\cdot [I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 D_2, \end{aligned} \tag{16}$$

$$\begin{aligned} H_{13} &= -[B_1 - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 F_2](G' W_1 G)^{-1} \cdot \\ &\cdot [B_1' - F_2' W_2 E_2 (E_2' W_2 E_2)^{-1} B_2'], \end{aligned} \tag{17}$$

$$\begin{aligned} H_{14} &= -\{B_2 - [B_1 - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 F_2] \cdot \\ &\cdot (G' W_1 G)^{-1} G' W_1 F_1\} (E_2' W_2 E_2)^{-1} B_2', \end{aligned} \tag{18}$$

The Open-Loop Stackelberg Equilibrium

$$H_{15} = -[B_1 - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 F_2](G' W_1 G)^{-1} G', \tag{19}$$

$$\begin{aligned} H_{16} &= -\{B_2 - [B_1 - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 F_2] \cdot \\ &\cdot (G' W_1 G)^{-1} G' W_1 F_1\}(E_2' W_2 E_2)^{-1} E_2', \end{aligned} \tag{20}$$

$$\begin{aligned} H_{21} &= B_2(E_2' W_2 E_2)^{-1} F_1'[I - W_1 G(G' W_1 G)^{-1} G'] \cdot \\ &\cdot W_1[D_1 - F_1(E_2' W_2 E_2)^{-1} E_2' W_2 D_2], \end{aligned} \tag{21}$$

$$\begin{aligned} H_{22} &= A - B_2(E_2' W_2 E_2)^{-1}\{I - F_1' W_1 G(G' W_1 G)^{-1} F_2' \cdot \\ &\cdot [I - W_2 E_2(E_2' W_2 E_2)^{-1} E_2']\} W_2 D_2, \end{aligned} \tag{22}$$

The Open-Loop Stackelberg Equilibrium

$$\begin{aligned} H_{23} &= B_2(E_2'W_2E_2)^{-1}\{I - F_1'W_1G(G'W_1G)^{-1} \cdot \\ &\cdot \quad [B_1' - F_2'W_2E_2(E_2'W_2E_2)^{-1}\}B_2', \end{aligned} \tag{23}$$

$$\begin{aligned} H_{24} &= -B_2(E_2'W_2E_2)^{-1}F_1'[I - W_1G(G'W_1G)^{-1}G'] \cdot \\ &\cdot \quad W_1F_1(E_2'W_2E_2)^{-1}B_2', \end{aligned} \tag{24}$$

$$H_{25} = B_2(E_2'W_2E_2)^{-1}F_1'[I - W_1G(G'W_1G)^{-1}G'], \tag{25}$$

$$\begin{aligned} H_{26} &= -B_2(E_2'W_2E_2)^{-1}F_1'[I - W_1G(G'W_1G)^{-1}G'] \cdot \\ &\cdot \quad W_1F_1(E_2'W_2E_2)^{-1}E_2', \end{aligned} \tag{26}$$

The Open-Loop Stackelberg Equilibrium

$$\begin{aligned} H_{31} &= -[D_1' - D_2' W_2 E_2 (E_2' W_2 E_2)^{-1} F_1'][I - W_1 G (G' W_1 G)^{-1} G'] \cdot \\ &\cdot \quad W_1 [D_1 - F_1 (E_2' W_2 E_2)^{-1} E_2' W_2 D_2], \end{aligned} \tag{27}$$

$$\begin{aligned} H_{32} &= \{D_2' - [D_1' - D_2' W_2 E_2 (E_2' W_2 E_2)^{-1} F_1'] W_1 G \cdot \\ &\cdot \quad (G' W_1 G)^{-1} F_2'\}[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 D_2, \end{aligned} \tag{28}$$

$$\begin{aligned} H_{33} &= rI - A' + D_2' W_2 E_2 (E_2' W_2 E_2)^{-1} B_2' + \\ &+ \quad [D_1' - D_2' W_2 E_2 (E_2' W_2 E_2)^{-1} F_1'] W_1 G (G' W_1 G)^{-1} \cdot \\ &\cdot \quad [B_1' - F_2' W_2 E_2 (E_2' W_2 E_2)^{-1} B_2'], \end{aligned} \tag{29}$$

$$\begin{aligned} H_{34} &= [D_1' - D_2' W_2 E_2 (E_2' W_2 E_2)^{-1} F_1'] \cdot \\ &\cdot \quad [I - W_1 G (G' W_1 G)^{-1} G'] W_1 F_1 (E_2' W_2 E_2)^{-1} B_2', \end{aligned} \tag{30}$$

The Open-Loop Stackelberg Equilibrium

$$
\begin{aligned}
H_{35} &= -[D_1' - D_2' W_2 E_2 (E_2' W_2 E_2)^{-1} F_1'] \cdot \\
&\cdot \quad [I - W_1 G (G' W_1 G)^{-1} G'],
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
H_{36} &= [D_1' - D_2' W_2 E_2 (E_2' W_2 E_2)^{-1} F_1'] \cdot \\
&\cdot \quad [I - W_1 G (G' W_1 G)^{-1} G'] W_1 F_1 (E_2' W_2 E_2)^{-1} E_2',
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
H_{41} &= -D_2'[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 \cdot \\
&\cdot \quad \{D_2 - F_2 (G' W_1 G)^{-1} G' W_1 [D_1 - F_1 (E_2' W_2 E_2)^{-1} E_2' W_2 D_2]\},
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
H_{42} &= -D_2'[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 F_2 (G' W_1 G)^{-1} \cdot \\
&\cdot \quad F_2'[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 D_2,
\end{aligned} \tag{34}
$$

The Open-Loop Stackelberg Equilibrium

$$\begin{aligned} H_{43} &= D_2'[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 F_2 (G' W_1 G)^{-1} \cdot \\ &\cdot \quad [B_1' - F_2' W_2 E_2 (E_2' W_2 E_2)^{-1} B_2'], \end{aligned} \tag{35}$$

$$\begin{aligned} H_{44} &= rI - A' + D_2'\{W_2 E_2 - [I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] \cdot \\ &\cdot \quad W_2 F_2 (G' W_1 G)^{-1} G' W_1 F_1\} (E_2' W_2 E_2)^{-1} B_2', \end{aligned} \tag{36}$$

$$H_{45} = D_2'[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 F_2 (G' W_1 G)^{-1} G', \tag{37}$$

$$\begin{aligned} H_{46} &= -D_2'[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'][I + W_2 F_2 \cdot \\ &\cdot \quad (G' W_1 G)^{-1} G' W_1 F_1 (E_2' W_2 E_2)^{-1} E_2']. \end{aligned} \tag{38}$$

The Open-Loop Stackelberg Equilibrium

Notation:

$$H_1 \equiv \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}, H_2 \equiv \begin{bmatrix} H_{15} & H_{16} \\ H_{25} & H_{26} \\ H_{35} & H_{36} \\ H_{45} & H_{46} \end{bmatrix}, \qquad (39)$$

$$k(t) \equiv \begin{bmatrix} x(t) \\ \lambda_{12}(t) \\ \lambda_{11}(t) \\ \lambda_2(t) \end{bmatrix}, w \equiv \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \qquad (40)$$

The Open-Loop Stackelberg Equilibrium

Steady-state values of $k(t)$:

$$k^* = -H_1^{-1} H_2 w \tag{41}$$

System:

$$\dot{k}(t) = H_1 k(t) + H_2 w = H_1 [k(t) - k^*]. \tag{42}$$

- M - diagonal matrix of the eigenvalues of H_1
- μ^1 and μ^2 are the vectors of stable and unstable eigenvalues of H_1
- V - matrix of column eigenvectors of H_1: $H_1 V = VM$
- Canonical variables $z(t)$ defined by $k(t) - k^* \equiv Vz(t)$

The Open-Loop Stackelberg Equilibrium

Solution:

$$z(t) = Sz(0), \tag{43}$$

$$S \equiv \text{diag}[\exp(\mu_1 t), ..., \exp(\mu_{4n} t)]. \tag{44}$$

Initial conditions:

- $x(0) = x_0$,
- $\lambda_{12}(0) = 0$, $\lambda_{11}(0)$ and $\lambda_2(0)$ are chosen such that the system starts within its $2n$-dimensional stable manifold

Solution of the canonical system:

$$k(t) = Vz(t) + k^* = VSV^{-1}k(0) + [I - VSV^{-1}]k^* \tag{45}$$

A Dynamic Rational Expectations Model

- Economic rational-expectations models
- Predetermined and non-predetermined variables
- Linear dynamic deterministic continuous-time rational-expectations model Buiter (1984)
- Predetermined state variables $x(t) \in \mathbb{R}^n$, with n initial conditions given by $x(0) = x_0$
- Vector of non-predetermined state variables $v(t) \in \mathbb{R}^{n_1}$
- Transversality conditions
- Exogenous or forcing variables $b(t) \in \mathbb{R}^i$

A Dynamic Rational Expectations Model

Linear deterministic first-order differential equations rational-expectations model with constant coefficients:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}^e(t) \end{bmatrix} = K \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + Lb(t) + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{46}$$

superscript e denotes the value of the respective variable expected by the private sector, given the information available at time t.

A Dynamic Rational Expectations Model

Assumptions:

(A) Information set $I(t) = \{x(s), v(s), b(s), s \leq t; K, L\}$; perfect hindsight for $s < t$, weak consistency for $s = t$

(B) $I(t) \supseteq I(s)$ for $t > s$

(C) $b^e(s)$ bounded and continuous almost everywhere: exogenous variables do not explode too fast

(D) K is diagonalizable by $K = U^{-1}\Lambda U$;

U matrix whose rows are the linearly independent left-eigenvectors of K

$\Lambda = \text{diag}(\lambda_1, ..., \lambda_{n+n_1})$, where the $\lambda_i, i = 1, ..., n + n_1$ are the eigenvalues of K

(E) K has n stable eigenvalues and n_1 unstable eigenvalues

A Dynamic Rational Expectations Model

Buiter (1984): dynamic rational-expectations model $\Rightarrow$ solved analytically

1. K, L, U, U^{-1}, and Λ are partitioned conformably with $x(t)$ and $v(t)$:

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \tag{47}$$

2. Canonical variables $p(t) \in \mathbb{R}^n$, $q(t) \in \mathbb{R}^{n_1}$ are defined by

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} \equiv U \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}. \tag{48}$$

3. $\dot{q}^e(t)$ is expressed as a linear function of $q^e(t)$ and $b^e(t)$, and $\dot{q}^e(s)$ as a linear function of $q^e(s)$ and $b^e(s)$ for $s > t$.

A Dynamic Rational Expectations Model

4. Forward-looking solution for $q^e(s)$ for $s \geq t$ determined by integrating the linear differential equations obtained in step 3;

   n_1 boundary conditions are required for the convergence of the system. Justified as characterizing an optimal intertemporal plan in a model with an infinitely-lived private sector

5. Weak consistency implies $q(t) = q^e(t)$; the solution for $v(t)$ can be obtained from that of $q(t)$
6. Backward-looking solution can be obtained for the predetermined variables $x(t)$ with initial conditions $x(0) = x_0$.
7. Cases where assumption **(E)** above is not satisfied

A Dynamic Rational Expectations Model

Problem of a government designing optimal stabilization policies over an infinite time horizon, faced with a dynamic rational-expectations economic system of the form (46):

$$\dot{x}(t) = K_{11}x(t) + K_{12}v(t) + L_1 b(t) + c_1, \tag{49}$$

$$\dot{v}^e(t) = K_{21}x(t) + K_{22}v(t) + L_2 b(t) + c_2, \tag{50}$$

with initial conditions $x(0) = x_0$ for the predetermined variables and transversality conditions for the non-predetermined variables $v(t)$.

- assume assumptions **(A) – (E)** above to hold
- **additional assumptions (F) – (J) hold:**

A Dynamic Rational Expectations Model

(F) $n = n_1$, that is, there are exactly as many predetermined as non-predetermined variables.

(G)

$$K_{11} = A - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 D_2, \tag{51}$$

$$K_{12} = -B_2(E_2' W_2 E_2)^{-1} B_2', \tag{52}$$

$$K_{21} = -D_2'[I - W_2 E_2(E_2' W_2 E_2)^{-1} E_2'] W_2 D_2, \tag{53}$$

$$K_{22} = rI - A' + D_2' W_2 E_2(E_2' W_2 E_2)^{-1} B_2', \tag{54}$$

$$L_1 = B_1 - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 F_2, \tag{55}$$

$$L_2 = -D_2'[I - W_2 E_2(E_2' W_2 E_2)^{-1} E_2'] W_2 F_2, \tag{56}$$

$$c_1 = -B_2(E_2' W_2 E_2)^{-1} E_2' w_2, \tag{57}$$

$$c_2 = -D_2'[I - W_2 E_2(E_2' W_2 E_2)^{-1} E_2'] w_2. \tag{58}$$

A Dynamic Rational Expectations Model

(H) The exogenous variables $b(t)$ are policy instruments of the government, i. e., $b(t) = u_1(t)$, and there are no further exogenous influences in the rational-expectations model.

(I) The non-predetermined rational-expectations variables $v(t)$ of the private sector are the optimum costate variables $\lambda_2(t)$ of the follower.

(J) The objective function of the government is J_1 from (3), with the objective variable $y_1(t)$ defined as a linear function of all (predetermined and non-predetermined) state variables $x(t)$ and $v(t)$ and of the government's instrument variable $b(t)$.

Under the assumptions (F) – (J), optimal economic policies for a single decision maker (the government) with an economic system characterized by rational expectations are equivalent to the policies for the leader within an open-loop Stackelberg equilibrium solution.

A Dynamic Rational Expectations Model

- Assumption (F) is most restrictive because it implies uniqueness of the solution
- Policies are time-inconsistent, they require pre-commitment and credibility of the government
- Remedies for the time-inconsistency
- Other equilibrium solution concepts: Cohen and Michel (1988) or feedback Stackelberg equilibrium solution (Dockner and Neck (1990))

Thank you for your attention!

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