Aggregate Asymmetry in Idiosyncratic Jump Risk[*]

Huidi Lin[†]and Viktor Todorov[‡]

November 4, 2019

Abstract

We study the structure and pricing of idiosyncratic jumps, i.e., jumps in asset prices that occur outside market-wide jump events. Using options on individual stocks and the market index that are close to expiration as well as local estimates of market betas from returns on the underlying assets, we estimate nonparametrically the asymmetry in the risk-neutral expected idiosyncratic variation, i.e., the difference in variation due to negative and positive returns, which asymptotically is solely attributed to jumps. We derive a feasible Central Limit Theorem that allows to quantify precision in the estimation, with the limiting distribution being mixed Gaussian. We find strong empirical evidence for aggregate asymmetry in idiosyncratic risk which shows that such risk clusters cross-sectionally. Our results reveal the existence and non-trivial pricing of aggregate downside tail risk in stocks during market-neutral systematic events as well as a negative skew in the cross-sectional return distribution during such episodes.

Keywords: Cross-Sectional Tail Risk, Equity Risk Premium, Idiosyncratic Risk, Large Data Sets, Nonparametric Inference, Options, Return Predictability, Time-Varying Skewness.

JEL classification: C51, C52, G12.

[†]Department of Finance, Kellogg School of Management, Northwestern University, Evanston, IL 60208; e-mail: huidi.lin@kellogg.northwestern.edu.

[‡]Department of Finance, Kellogg School of Management, Northwestern University, Evanston, IL 60208; e-mail: v-todorov@northwestern.edu.

1 Introduction

Equity markets experience large downward moves (jumps) and investors are willing to pay in order to avoid such risks. Empirical studies and equilibrium models suggest that the compensation for downside market jump risk demanded by investors is a nontrivial part of the equity risk premium and can help rationalize its dynamics, see e.g., Bates (1991, 1996, 2000), Pan (2002), Bollerslev and Todorov (2011), Drechsler and Yaron (2011), Gabaix (2012) and Wachter (2013), among others. The recent introduction and increased liquidity of markets for short-dated options (options with short time to maturity) facilitate the management of such risks by investors and at the same time provide a powerful way to study them nonparametrically, see e.g., Andersen et al. (2017). To illustrate, in Figure 1, we plot out-of-the-money option prices (i.e., the prices of puts for strikes below the current stock price and calls for strikes above the current stock price) written on the S&P 500 index at market close on November 18, 2016 which expire a week later on November 25, 2016. On this day, the market volatility, inferred by the at-the-money Black-Scholes implied volatility, was 6% which implies that the lowest strike of the displayed put options corresponds approximately to a ten standard deviation downward move on the market. Such moves are essentially impossible to materialize without a significant downward jump on the market, and hence the prices of these very deep out-of-the-money puts are determined largely by the probability of this risk and the price attached to it by investors. The large difference in the option prices for strikes corresponding to positive and negative moves on the market reveals the nontrivial asymmetry in the risk-neutral jump distribution. This large asymmetry is not present in the true return distribution and is thus a manifest of the large risk premium demanded by investors for exposure to downside market risk.

In Figure 1 we compare the S&P 500 option prices with those written on the Apple stock which are recorded at the same time as the index options and have the same expiration date. The option prices are normalized with the corresponding current stock prices, so that they reflect prices of return exceedances and are therefore directly comparable across the two underlying assets. As seen from the figure, the option prices of Apple are more expensive than the market index options and we also have far smaller difference in the pricing of calls and puts with strikes equally distant from the current stock price. Given that the market beta of Apple at the time was very close to one, the reason for the elevated (normalized) option prices of Apple relative to the index options is the presence of jumps in its stock price which do not lead to market jumps. We refer to these jumps as idiosyncratic, i.e., idiosyncratic risk here is relative to market risk. They can be either due to stock-specific news as in the seminal work of Merton (1976) or due to systematic events that do not move the market, e.g., due to exposure of stocks to systematic shocks that do not trigger moves in
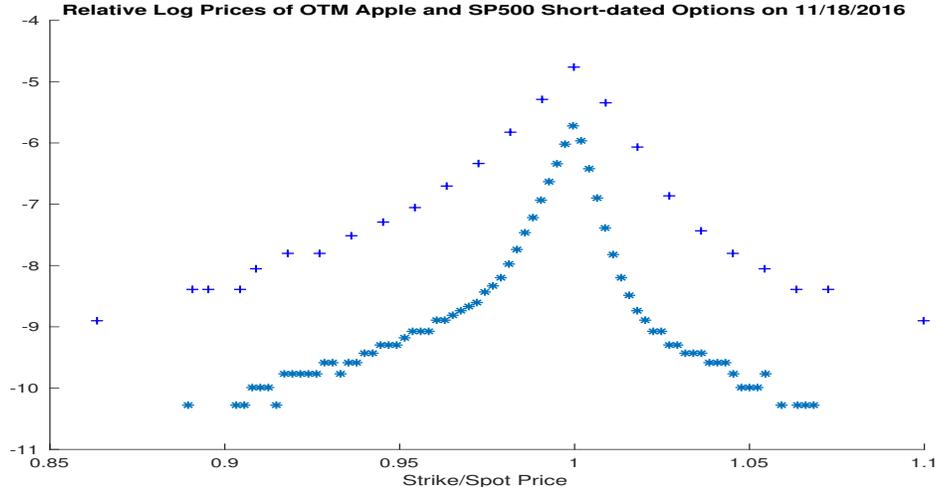
Figure 1: Relative Log Prices of OTM Apple and SPX Short-dated Options. The y-axis is the logarithm of the ratio of the out-of-the-money option prices over the current stock price. Stars correspond to S&P 500 index options and the crosses to options on Apple stock.

the aggregate market index. In the model of Merton (1976), the former type of idiosyncratic jump risk can be diversified away and hence does not appear in the aggregate pricing kernel and does not command risk premium for bearing. This is in general not the case, however, for the second type of idiosyncratic jump risk that is due to the occurrence of market-neutral systematic events.

In this paper we aim to understand the nature of idiosyncratic jump risk in asset prices. We study how it aggregates in the cross-section as well as its pricing. In particular, are idiosyncratic jumps firm-specific only or do they also cluster in the cross-section? Do idiosyncratic jumps share an exposure to a common systematic shock? Is there asymmetry in idiosyncratic jump risk? Does this asymmetry "survive" cross-sectional averaging and if so is this aggregate risk priced like the negative market return skewness? We develop nonparametric tools for studying these questions by taking advantage of the availability of short-dated options written on a cross-section of stocks.

Our estimation strategy is as follows. We first construct model-free measures of asymmetry in the risk-neutral jump distribution from options. We combine these measures with local estimates of market betas from high-frequency returns to identify the asymmetry in the risk-neutral idiosyncratic risk in stocks. Cross-sectionally aggregating these estimates allows us to study clustering of idiosyncratic jump risk across stocks and contrasting them with counterparts of these measures built solely from returns allows us to identify the risk premium component in them.

For the goal of this study, any measure of asymmetry in the risk-neutral jump distribution will do. We use one based on the so-called semivariance, whose estimation from high-frequency

data has been recently studied in Barndorff-Nielsen et al. (2010), see also Bollerslev et al. (2019). Option-based estimates of quantities similar to the semivariances have been also used in empirical applications for studying the downside market variance risk premium by Feunou et al. (2018) and Kilic and Shaliastovich (2018). Formally, the positive and negative semivariance split the total return variation over a given interval into one due to positive and negative, respectively, returns. The positive (negative) semivariance of a semimartingale equals half of the diffusive integrated volatility and the sum of squared positive (negative) jumps. Therefore, by taking the difference of the negative and positive semivariances, we get the difference in the quadratic variation due to negative and positive jumps. We refer to this quantity as the asymmetric quadratic variation. It does not depend on the diffusive risk in the assets and thus it allows us to study directly the asymmetry in the jump distribution. Using the option spanning results of Carr and Madan (2001) as well as the asymptotic behavior of functionals of increments of a continuous time process over shrinking time intervals and approximations in the tails of the conditional distributions based on extreme value theory, which relate directly to the jump tails as in Bollerslev and Todorov (2011), we derive measures of risk-neutral of positive and negative semivariances.

To separately identify the idiosyncratic component of jump risk embedded in the option-based semivarances of the individual assets, we need to account for the exposure of the stock jumps to those on the market, i.e., for their market jump beta. The latter can be estimated using high-frequency return data in a window around the time of observing the option prices, following Li et al. (2017). We derive a Central Limit Theorem (CLT) for the joint behavior of the option and return measures of variation which allows us to conduct formal inference about the pricing of idiosyncratic jump risk. The limit distribution reflects two types of estimation uncertainty: one stems from the martingale component in the price and the other one is due to the measurement error in the observed option prices, with observation errors in the options with strikes in the vicinity of the current stock price playing an asymptotically leading role. Our setup is of joint type, with both the number of high-frequency observations and the number of available short-dated options increasing, and we do not impose restrictions on their relative growth which is convenient for applications.

In addition to the semivariance estimates, we develop alternative measures of asymmetry in the jump distribution which can fully characterize it and can also serve as an important robustness check for our results based on the semivariance. These alternative measures are based on option-implied estimates of the imaginary part of the log-characteristic function of the returns. We show that they have faster rate of convergence than their semivariance counterparts and work in more general asymptotic setups. By varying the value of the characteristic exponent, these measures of

4

asymmetry can shed light on the region of the return distribution that generates its asymmetry.

We estimate the measures of idiosyncratic asymmetric jump variation using return and option data on a cross-section of stocks during the period 2007-2017. Our cross-section consists of stocks that are part of the implied correlation index constructed by the CBOE options exchange. We select the 50 stocks that are most frequently included in the index during our sample period. The total market capitalization of the stocks in our sample is over 40% of the value of the S&P 500 index.[1] In order to focus on the aggregate pricing implications of idiosyncratic jump risk, we average the individual option-based measures in the cross-section and estimate the risk-neutral expectation of the aggregate idiosyncratic asymmetric jump variation.

Using the developed inference theory, we overwhelmingly reject the natural hypothesis that aggregate idiosyncratic asymmetric jump variation is equal to zero, with the empirical results showing that the risk-neutral expectation of the aggregate downside idiosyncratic jump variation risk is significantly higher than its upside counterpart. This finding confirms not only the presence of idiosyncratic jump risk but also sheds light on its nature. In particular, if the idiosyncratic jump risk is solely firm-specific as in Merton (1976), then cross-sectional averaging should remove it as it does not cluster across stocks at any given point in time. The fact that the negative idiosyncratic skewness "survives" cross-sectional aggregation shows that this is not the case empirically and instead the idiosyncratic jump risk is due to the existence of market-neutral systematic events. They can be either due to cross-sectionally uncorrelated jumps arriving at a common point in time or due to exposure of assets to a common systematic shock that does not move the aggregate market. However, for either of this to be a valid explanation of our empirical finding, we crucially also need that the cross-sectional distribution of the asset returns at the market-neutral systematic event to be significantly left skewed. Our finding of the importance of the cross-sectional heterogeneity of idiosyncratic jump risk is related to the recent literature on the granularity of aggregate macro quantities, see e.g., Gabaix (2011) and Acemoglu et al. (2017), where it is shown that the cross-sectional heterogeneity in individual risks can generate aggregate level tail risks. Our empirical result also implies that, conditional on the common shocks in returns, the idiosyncratic jump risk can generate cross-sectional fat tails and asymmetry in the return distribution which have been studied recently in Kelly and Jiang (2014) and Oh and Wachter (2019).

While the option-based measures of idiosyncratic jump risk are under the risk-neutral probability measure, by comparing these quantities with their realized counterparts constructed from

---

[1]In the data appendix, we further extend the empirical results to the wider cross-section of stocks that are part of the S&P 100 index over the last three years of our original sample. The results for the aggregate asymmetry in the idiosyncratic jump risk for the original and the extended cross-section are very similar.

the high-frequency return data with the help of the realized semivariances of Barndorff-Nielsen et al. (2010) and measures of jump variation due to Jacod (2008), we can separate risks from risk premium. Our formal tests reveal that aggregate idiosyncratic downside jump risk is heavily priced and this premium varies significantly over time. To illustrate the economic importance of the latter time variation, we further show that the option-based measure of aggregate idiosyncratic asymmetric jump variation is a strong predictor of future equity returns, performing on par with option-based measures of downside market jump risk. The risk premium for aggregate asymmetry in idiosyncratic jump risk suggests that investors are willing to pay a premium to protect against a scenario where an exposure towards a systematic market-neutral risk can become suddenly big and thus limiting the ability to be diversified away in market-neutral investment strategies which are known to be followed by many hedge funds, see e.g., Khandani and Lo (2007).

Overall, our results show that there is a cross-sectional clustering of jumps outside events that move the aggregate market index. By comparing asymmetry measures on the aggregate market index and its constituents, we find evidence that the cross-sectional distribution of stock returns during such market-neutral systematic events implied by the option data has a left skew. The aggregate asymmetry in idiosyncratic risk commands a nontrivial risk premium and the latter serves as a strong predictor of future equity returns.

The current paper is related to several strands of existing work. First, we build on the earlier result of Carr and Madan (2001) regarding spanning of risk-neutral moments by option portfolios. We use this result and asymptotic expansions of moments of returns over short intervals of time, similar to Qin and Todorov (2018), to derive new measures of asymmetric jump variation as well as a CLT for the option statistics which allows to conduct formal inference. In addition, our measures combine cross-section of option and high-frequency return data and the limit variable in our CLT is governed by both sources of risk. Second, some of our statistics build on semivariance measures which are the object of study in high-frequency context in Barndorff-Nielsen et al. (2010) and Bollerslev et al. (2019), and have been shown to generate better volatility forecasts by Patton and Sheppard (2015). They have been further used by Feunou et al. (2018) and Kilic and Shaliastovich (2018) to separate market variance risk premium into one due to upside and downside moves. Third, there is a lot of work on the pricing of downside negative jump risk both in reduced-form and in equilibrium settings. Examples include Bates (1991, 1996, 2000), Pan (2002), Bollerslev and Todorov (2011), Drechsler and Yaron (2011), Gabaix (2012) and Wachter (2013), among many others. Unlike this strand of work, our focus here is on the structure of jump risk in the cross-section.

Fourth, there is a growing literature that studies cross-sectional asset pricing using information

from individual options. In particular, Bakshi et al. (2003) and Driessen et al. (2009) show that gaps between option-implied second moments and their historical counterparts are significantly smaller than those for the market. Martin and Wagner (2018) presents a model in which expected stock returns are connected with market and individual risk-neutral volatilities. Unlike those studies our focus here is on the asymmetry in the return variation and more specifically on the pricing of downside idiosyncratic jump risk. In addition, Begin et al. (2019) study idiosyncratic jump risk in a parametric setting and assuming constant market betas in a large cross-section of assets. Their focus is in the Merton style firm-specific idiosyncratic jump risk. By contrast, we use nonparametric methods, which are immune to model misspecification. We allow for idiosyncratic risk to cluster cross-sectionally in our analysis and study the implications of that. Another related work from this strand of research is Kelly et al. (2016) who show that there was more compensation for tail risk in financial stocks than for an aggregate financial sector index in the aftermath of the financial crisis. This is in line with our finding regarding the importance of non-market jump risk. Finally, Conrad et al. (2013) and Pederzoli (2018) study skewness risk premium on an individual asset price level. These papers identify idiosyncratic skewness premium as the component in risk-neutral asset skewness that is orthogonal to market (co)skenwess in a time series sense. Thus, orthogonality between systematic and idiosyncratic risk in Conrad et al. (2013) and Pederzoli (2018) is in terms of their risk premium and not the risks per se which is therefore very different from our decomposition of jump risk in assets that is in the (traditional) martingale sense. This difference is highlighted in the ability of our decomposition to identify the existence of market-neutral systematic jump events and cross-sectional skewness in the stock returns during these episodes.

Fifth, our finding of a nontrivial compensation demanded by investors for aggregate idiosyncratic return variation is consistent with recent literature that argues through various economic channels for investors' preference for positive skewness and lottery-like features in individual returns, see e.g., Brunnermeier and Parker (2005), Mitton and Vorkink (2007), Barberis and Huang (2008), Bali et al. (2011) and Boyer and Vorkink (2014).

The paper is organized as follows. We start in Section 2 with introducing the setup and the jump decomposition. In Section 3 we construct semivariance measures on the basis of option and return data, and in Section 4 we develop the necessary limit theory for conducting inference. Section 5 derives the asymptotic properties of alternative measures of asymmetric variation. Section 6 contains our empirical analysis. Section 7 concludes. The assumptions and the proofs are given in Section 8. In Section 9 we provide additional details on the data and report results from various robustness checks.

2   Setup and Notation

2.1   Asset Dynamics and the Decomposition of Jump Risk

We start with introducing our setup and notation. All processes in the paper are defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$. The value of the market portfolio at time t is denoted with $X_t^{(0)}$ while the individual asset prices are denoted with $X_t^{(j)}$, for $j = 1, ..., N$. We will henceforth use lower case letters to denote log-prices. The dynamics of $x_t^{(0)}$ is given by the following general semimartingale

$$x_t^{(0)} = x_0^{(0)} + \int_0^t \alpha_s^{(0)} ds + \int_0^t \lambda_s^{(0)} dW_s + \sum_{s \leq t} \Delta x_s^{(0)}, \tag{1}$$

where W_t is a $N+1 \times 1$ Brownian motion and $\Delta x_s^{(0)} = x_s^{(0)} - x_{s-}^{(0)}$ is the jump in $x^{(0)}$. The dynamics of the log-prices of the individual assets is given by

$$x_t^{(j)} = x_0^{(j)} + \int_0^t \alpha_s^{(j)} ds + \int_0^t \lambda_s^{(j)} dW_s + \sum_{s \leq t} \Delta x_s^{(j)}, \quad j = 1, ..., N, \tag{2}$$

The spot diffusive market variance and the spot diffusive idiosyncratic variances are denoted as

$$V_t^{(0)} = \lambda_t^{(0)}(\lambda_t^{(0)})^\top \quad \text{and} \quad \widetilde{V}_t^{(j)} = \lambda_t^{(j)}(\lambda_t^{(j)})^\top - \frac{\left(\lambda_t^{(j)}(\lambda_t^{(0)})^\top\right)^2}{\lambda_t^{(0)}(\lambda_t^{(0)})^\top}, \quad j = 1, ..., N. \tag{3}$$

Apart from weak regularity conditions, the specification of the asset prices in (1)-(2) puts no restriction on the drift and the continuous martingale part of the prices. For the jumps, we impose the following conditional factor structure:

$$\Delta x_t^{(j)} = \beta_t^{(j)} \Delta x_t^{(0)} + \Delta \widetilde{x}_t^{(j)}, \quad \Delta x_t^{(0)} \Delta \widetilde{x}_t^{(j)} = 0, \quad \forall t \geq 0, \tag{4}$$

where $\beta_t^{(j)}$ is some predictable process. We refer to the jumps $\Delta \widetilde{x}_t^{(j)}$ as idiosyncratic because they happen outside of the times the market jumps. We put no restrictions on those. In particular, we note that we do not rule out the possibility that $\Delta \widetilde{x}_t^{(i)} \Delta \widetilde{x}_t^{(j)} \neq 0$, for $i \neq j$ and $i, j = 1, ..., N$. That is, we do allow for the idiosyncratic jumps to have cross-sectional dependence. In the case when the jumps $\Delta \widetilde{x}_t^{(j)}$ occur at different points in time across the different assets $j = 1, ..., N$, we then have the specification of jump risk introduced in the seminal work of Merton (1976). The critical aspect of the jump specification in (4) is that stock jumps react in a linear way to a market jump but we note that we allow for the loading to be time-varying in an unspecified way. Li et al. (2019) test this specification of jump risk and find empirical support for it. We will further provide evidence for the jump structure in (4) using our short-dated option data.

The compensator of the jumps in $x^{(j)}$, under $\mathbb{P}$, is given by $dt \otimes \nu_{t,j}^{\mathbb{P}}(x)dx$, for $j = 0, 1, ..., N$. The jump compensator associated with the jumps in $x^{(j)}$ which happen outside the jump times of $x^{(0)}$ is denoted with $dt \otimes \widetilde{\nu}_{t,j}^{\mathbb{P}}(x)dx$, for $j = 1, ..., N$. Given the structure of jumps in (4), we have

$$\widetilde{\nu}_{t,j}^{\mathbb{P}}(x) = \nu_{t,j}^{\mathbb{P}}(x) - \nu_{t,0}^{\mathbb{P}}(x/\beta_t^{(j)}), \tag{5}$$

with $\nu_{t,0}^{\mathbb{P}}(\infty) = 0$.

2.2   Option Prices

We continue with introducing notation for the option prices written on the assets. We will assume the existence of a risk-neutral measure, denoted with $\mathbb{Q}$, under which discounted cum-dividend asset prices (including derivatives) are local martingales. Under technical conditions, the probability measure $\mathbb{Q}$ exists provided observed asset prices are free of arbitrage opportunities, see e.g., chapter 6 of Duffie (2001). Local equivalence of $\mathbb{P}$ and $\mathbb{Q}$ implies that the diffusion coefficients in front of the Brownian motions in the dynamics of $x_t^{(j)}$ remain the same under the two measures, and importantly for our analysis the jump decomposition in (4) holds under $\mathbb{Q}$ as well. The reason for the latter result is that the relationship between stock and market jumps holds pathwise and hence is preserved under equivalent change of measure. Note, however, that this does not mean that the jump intensities under $\mathbb{P}$ and $\mathbb{Q}$ are the same. The factor structure of jumps in (4) only implies that the risk premium (the $\mathbb{P}$-$\mathbb{Q}$ wedge) for jumps in the stocks that arrive at the time of market jumps is proportional to the market jump risk premium (with coefficient of proportionality given by the time-varying market beta).

Formally, the risk-neutral dynamics of the assets is given by

$$x_t^{(0)} = x_0^{(0)} + \int_0^t \alpha_s^{(\mathbb{Q},0)}ds + \int_0^t \lambda_s^{(0)}dW_s^{\mathbb{Q}} + \sum_{s \leq t} \Delta x_s^{(0)}, \tag{6}$$

$$x_t^{(j)} = x_0^{(j)} + \int_0^t \alpha_s^{(\mathbb{Q},j)}ds + \int_0^t \lambda_s^{(j)}dW_s^{\mathbb{Q}} + \sum_{s \leq t} \Delta x_s^{(j)}, \quad j = 1, ..., N, \tag{7}$$

where $W_t^{\mathbb{Q}}$ is a $N + 1 \times 1$ Brownian motion. As mentioned above, local equivalence of $\mathbb{P}$ and $\mathbb{Q}$ imply that the jumps in the asset prices continue to satisfy the conditional factor model (4). The counterparts of $\nu_{t,j}^{\mathbb{P}}(x)$ and $\widetilde{\nu}_{t,j}^{\mathbb{P}}(x)$ under $\mathbb{Q}$ are denoted with $\nu_{t,j}^{\mathbb{Q}}(x)$ and $\widetilde{\nu}_{t,j}^{\mathbb{Q}}(x)$. Given (4), exactly as under $\mathbb{P}$, we have

$$\widetilde{\nu}_{t,j}^{\mathbb{Q}}(x) = \nu_{t,j}^{\mathbb{Q}}(x) - \nu_{t,0}^{\mathbb{Q}}(x/\beta_t^{(j)}). \tag{8}$$

We proceed with introducing the notation for the option prices. For simplicity of exposition, as we will consider only short-dated options, we will assume that the risk-free rate and the dividend

9

yield of the assets are all zero in our asymptotic analysis.[2] The theoretical value of out-of-the-money (OTM) European-style option price at time t expiring at time $t + T$ in the future on asset j with strike K is therefore given by

$$O_{t,T}^{(j)}(k) = \begin{cases} \mathbb{E}_t^{\mathbb{Q}}(e^{x_{t+T}^{(j)}} - e^k)^+, & \text{if } f_{t,T}^{(j)} < k, \\ \mathbb{E}_t^{\mathbb{Q}}(e^k - e^{x_{t+T}^{(j)}})^+, & \text{if } f_{t,T}^{(j)} \geq k, \end{cases} \tag{9}$$

where $k = \log(K)$ is the log-strike and $f_{t,T}^{(j)} = \log(F_{t,T}^{(j)})$, with $F_{t,T}^{(j)}$ denoting the price of a futures contract written on asset j at time t and expiring at $t + T$.[3] The option is call if the strike is above the current futures price and is a put otherwise.

For each of the assets, we will consider options that have only one maturity date which is common across all assets. The log-strikes of the observed options are denoted with

$$k_1^{(j)} < ... < k_{S_j}^{(j)}, \quad j = 0, 1, ..., N. \tag{10}$$

We will further set

$$\underline{k} = \max_{j=0,...,N} k_1^{(j)}, \quad \underline{K} = \exp(\underline{k}), \quad \overline{k} = \min_{j=0,...,N} k_{S_j}^{(j)}, \quad \overline{K} = \exp(\overline{k}). \tag{11}$$

Finally, the option prices are observed with error, i.e., we observe

$$\widehat{O}_{t,T}^{(j)}(k_i^{(j)}) = O_{t,T}^{(j)}(k_i^{(j)}) + \epsilon_{t,i}^{(j)}, \quad i = 1, ..., S_j, \quad j = 0, ..., N. \tag{12}$$

The probability space has the product form

$$\Omega = \Omega^{(0)} \times \Omega^{(1)}, \quad \mathcal{F} = \mathcal{F}^{(0)} \times \mathcal{F}^{(1)}, \quad \mathbb{P}(d\omega^{(0)}, d\omega^{(1)}) = \mathbb{P}^{(0)}(d\omega^{(0)})\mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)}),$$

with $\mathbb{P}^{(1)}(\omega^{(0)}, d\omega^{(1)})$ being transitional probability from $\Omega^{(0)}$ to $\Omega^{(1)}$. Th underlying asset price as well as the true (unobserved) option prices are defined on $\Omega^{(0)}$ while the option observation errors are defined on $\Omega^{(1)}$. The observation errors will be $\mathcal{F}^{(0)}$-conditionally independent although not identically distributed, see assumption A6 in the appendix.

3 Nonparametric Option Measures of Variation

We next introduce nonparametric measures of variation constructed from the option data which will allow us to study idiosyncratic jump risk, and more specifically the asymmetry in the idiosyncratic

---

[2]In the empirical application we will account for both dividends and the risk-free interest rate.

[3]With dividend yield and risk-free rate set to zero, we have $f_{t,T}^{(j)} = x_t^{(j)}$.

jump distribution. Using the option spanning results in Carr and Madan (2001), we have

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\left(x_{t+T}^{(j)} - x_t^{(j)}\right)^2 1_{\{x_{t+T}^{(j)} - x_t^{(j)} > 0\}}\right] &= \int_{x_t^{(j)}}^{\infty} 2e^{-k}(1 - k + x_t^{(j)})O_{t,T}^{(j)}(k)dk, \\
\mathbb{E}_t^{\mathbb{Q}}\left[\left(x_{t+T}^{(j)} - x_t^{(j)}\right)^2 1_{\{x_{t+T}^{(j)} - x_t^{(j)} < 0\}}\right] &= \int_{-\infty}^{x_t^{(j)}} 2e^{-k}(1 - k + x_t^{(j)})O_{t,T}^{(j)}(k)dk.
\end{aligned}
\tag{13}
$$

For T small, we can directly relate the signed squared moments of returns with the underlying diffusive volatility and jump risk. At the same time, the integrals on the right-hand sides of the above equalities can be approximated using the available options via Riemann sums and extensions in the tails based on extreme value theory combined with an assumption for regular variation in the tails of the risk-neutral return distributions. More specifically, our estimates of the conditional risk-neutral signed squared moments of returns from the option data are given by

$$
\begin{aligned}
SV_t^{(-,j)} &= \sum_{i=2}^{i^*(j)} h(k_{i-1}^{(j)}, k_i^{(j)}, x_t^{(j)})\widehat{O}_{t,T}^{(a,j)}(k_{i-1}^{(j)}) + h(k_{i^*(j)}^{(j)}, x_t^{(j)}, x_t^{(j)})\widehat{O}_{t,T}^{(j)}(k_{i^*(j)}^{(j)}) + TC_t^{(-,j)}, \\
SV_t^{(+,j)} &= \sum_{i=i^*(j)+2}^{S_j} h(k_{i-1}^{(j)}, k_i^{(j)}, x_t^{(j)})\widehat{O}_{t,T}^{(a,j)}(k_{i-1}^{(j)}) + h(x_t^{(j)}, k_{i^*(j)+1}^{(j)}, x_t^{(j)})\widehat{O}_{t,T}^{(j)}(k_{i^*(j)+1}^{(j)}) + TC_t^{(+,j)},
\end{aligned}
\tag{14}
$$

where we denote the local average option price with

$$
\widehat{O}_{t,T}^{(a,j)}(k_{i-1}^{(j)}) = \frac{1}{2}\left(\widehat{O}_{t,T}^{(j)}(k_{i-1}^{(j)}) + \widehat{O}_{t,T}^{(j)}(k_i^{(j)})\right), \quad i = 2, ..., S_j, \quad j = 0, 1, ..., N,
\tag{15}
$$

the index of the strike immediately preceding the current spot price with

$$
i^*(j) = \sup\{i = 1, ..., S_j : k_i^{(j)} \le x_t^{(j)}\}, \quad j = 0, 1, ..., N,
\tag{16}
$$

and the weight function in the option portfolios in (13) with

$$
h(k_l, k_h, x) = 2e^{-x}\left[(k_h - x)e^{-(k_h - x)} - (k_l - x)e^{-(k_l - x)}\right], \quad x, k_l, k_h \in \mathbb{R}.
\tag{17}
$$

Finally, $TC_t^{(\pm,j)}$ are approximations of the integrals $\int_{k_{S_j}^{(j)}}^{\infty} 2e^{-k}(1-k+x_t^{(j)})O_{t,T}^{(j)}(k)dk$ and $\int_{-\infty}^{k_1^{(j)}} 2e^{-k}(1-k+x_t^{(j)})O_{t,T}^{(j)}(k)dk$ which are based on extreme value theory and are constructed in the following way:

$$
\begin{aligned}
TC_t^{(+,j)} &= \frac{2}{\widehat{\alpha}_t^{(+,j)}}\left(1 - k_{S_j}^{(j)} + x_t^{(j)} - \frac{1}{\widehat{\alpha}_t^{(+,j)}}\right)e^{-k_{S_j}^{(j)}}\widehat{O}_{t,T}^{(j)}(k_{S_j}^{(j)}), \\
TC_t^{(-,j)} &= \frac{2}{\widehat{\alpha}_t^{(-,j)}}\left(1 - k_1^{(j)} + x_t^{(j)} + \frac{1}{\widehat{\alpha}_t^{(-,j)}}\right)e^{-k_1^{(j)}}\widehat{O}_{t,T}^{(j)}(k_1^{(j)}),
\end{aligned}
\tag{18}
$$

and where

$$\widehat{\alpha}_t^{(+,j)} = \operatorname*{median}_{s=t-\tau,...,t} \left\{ \frac{\log(\widehat{O}_{s,T}^{(j)}(k_{i+}^{(j)})/\widehat{O}_{s,T}^{(j)}(k_{S_j}^{(j)}))}{k_{i+}^{(j)} - k_{S_j}^{(j)}} \bigvee 1 \right\},$$

$$\widehat{\alpha}_t^{(-,j)} = \operatorname*{median}_{s=t-\tau,...,t} \left\{ \frac{\log(\widehat{O}_{s,T}^{(j)}(k_{i-}^{(j)})/\widehat{O}_{s,T}^{(j)}(k_1^{(j)}))}{k_{i-}^{(j)} - k_1^{(j)}} \bigvee 1 \right\}, \tag{19}$$

with $i^+ = \max\{i = 1, ..., S_j : k_i^{(j)} \le k_{S_j}^{(j)}/2 + x_t^{(j)}/2\}$ and $i^- = \min\{i = 1, ..., S_j : k_i^{(j)} \ge k_1^{(j)}/2 + x_t^{(j)}/2\}$, and τ being some integer. We slightly simplified notation in the above definitions by not indexing the observed strikes with the point of time the options are observed and further by forcing the time to maturity of the options across the different days to be the same. This is done only for ease of exposition though.

We note that the tail approximations $TC_t^{(\pm,j)}$ are typically small in magnitude because the option prices corresponding to the maximum and minimum available strikes, $\widehat{O}_{t,T}^{(j)}(k_{S_j}^{(j)})$ and $\widehat{O}_{t,T}^{(j)}(k_1^{(j)})$, are in most of the cases very close to zero. Nevertheless, $TC_t^{(\pm,j)}$ help minimize the impact of potential biases when comparing $SV_t^{(\pm,j)}$ for different assets as we will need to do for computing the idiosyncratic variation measures below. The particular form of $TC_t^{(\pm,j)}$ is due to an assumption for regular variation in the tails of the return distribution, see e.g., Definition 1.4.3 in Bingham et al. (1987), which implies linear decay of the log-option prices as $|k| \to \infty$. In Section 9, we illustrate the tail approximation for the deep out-of-the-money options that forms the basis for $TC_t^{(\pm,j)}$.

The semivariance measures in (14) can be viewed as the option-based analogues of the realized semivariances of Barndorff-Nielsen et al. (2010). We denote the difference and the sum of the two option-based semivariance measures as

$$AV_t^{(j)} = SV_t^{(-,j)} - SV_t^{(+,j)}, \quad QV_t^{(j)} = SV_t^{(-,j)} + SV_t^{(+,j)}, \quad j = 0, 1, ..., N, \tag{20}$$

with the first capturing the asymmetry in the return distribution and the second one being a measure of the total return variance. We note that $QV_t^{(j)}$, without the tail approximations used in its construction, is the second-moment option portfolio introduced in Bakshi and Madan (2000) and Bakshi et al. (2003).

Our interest in the paper is in the idiosyncratic risk and in order to identify it from the measures of variation of the assets and the market index, we need to have an estimate for the market jump beta, $\beta_t^{(j)}$. We use high-frequency observations of the underlying assets for its inference. In particular, we assume that for each unit interval (trading day) we have $n + 1$ equidistant price records, resulting in n log-returns denoted as

$$\Delta_i^n x^{(j)} = x_{i/n}^{(j)} - x_{(i-1)/n}^{(j)}, \quad i = 1, 2, ..., \quad j = 0, 1, ..., N. \tag{21}$$

12

Following Li et al. (2017), our high-frequency jump beta estimator is then based on ordinary least squares for the high-frequency increments that identified to contain market jumps:

$$\widehat{\beta}_t^{(j)} = \frac{\sum_{i \in I_t^n} \Delta_i^n x^{(0)} \Delta_i^n x^{(j)} 1_{\{|\Delta_i^n x^{(0)}| > \nu_n^{(0)}\}}}{\sum_{i \in I_t^n} (\Delta_i^n x^{(0)})^2 1_{\{|\Delta_i^n x^{(0)}| > \nu_n^{(0)}\}}}, \quad t \geq \kappa, \ j = 1, ..., N, \ \widehat{\beta}_t = \left(\widehat{\beta}_t^{(1)}, ..., \widehat{\beta}_t^{(N)}\right)', \qquad (22)$$

where $I_t^n = \{\lfloor tn \rfloor - \kappa n + 1, ..., \lfloor tn \rfloor\}$ is a local window around the point in time t, for some integer κ and $\nu_n^{(0)} \asymp n^{-\varpi}$, for some $\varpi \in (0, 1/2)$. Using the high-frequency beta estimates, our measures of idiosyncratic variation are given by

$$\widetilde{AV}_t^{(j)} = AV_t^{(j)} - (\widehat{\beta}_t^{(j)})^2 AV_t^{(0)}, \quad \widetilde{QV}_t^{(j)} = QV_t^{(j)} - (\widehat{\beta}_t^{(j)})^2 QV_t^{(0)}, \quad j = 1, ..., N, \qquad (23)$$

provided of course that the market betas are nonnegative. This is empirically a non-binding assumption and it will therefore be assumed henceforth for ease of exposition.

4 Inference for the Asymmetric Idiosyncratic Variation

We proceed with developing feasible inference for the variation measures introduced above. We note that while the true (unobservable) option prices are computed under the risk-neutral probability measure, the observation equation (12) and the assumption A6 for the observation errors are naturally under the statistical probability measure. Hence all limits results derived here and in the next section are under $\mathbb{P}$ and can be directly used for feasible inference.

We start with a convergence in probability result. For the statement of this result, we need some additional notation which we now introduce. We denote the signed square function with

$$f(x) = x^2 (1_{\{x<0\}} - 1_{\{x>0\}}). \qquad (24)$$

Henceforth, ι_k is a $1 \times N + 1$ vector whose elements are zeros except for the k-th one which is 1. We further set:

$$\iota_{k:N+1} = \left(\iota_k' \cdots \iota_{N+1}'\right)', \quad k = 2, ..., N+1. \qquad (25)$$

In all of the theorems below, we will denote with Δ a reference deterministic sequence that decreases asymptotically to zero and captures the order of magnitude of the mesh of the strike grids of the available options. The precise definition of Δ is given in assumption A5 in Section 8.1. With this notation, we are ready to state our convergence in probability result.

Theorem 1 *Suppose assumptions A1-A7 in Section 8.1 hold. For $\Delta \to 0$, $T \to 0$, $(|\underline{k}| \vee \overline{k}) \to \infty$ and $n \to \infty$, with $\Delta \asymp T^\alpha$ for $\alpha > \frac{1}{2}$, we have for $j = 0, 1, ..., N$:*

$$\frac{1}{T} AV_t^{(j)} \ \overset{\mathbb{P}}{\longrightarrow} \ \int_{\mathbb{R}} f(x) \nu_{t,j}^{\mathbb{Q}}(x) dx, \ \frac{1}{T} QV_t^{(j)} \ \overset{\mathbb{P}}{\longrightarrow} \ \lambda_t^{(0)} (\lambda_t^{(0)})^\top + \widetilde{V}_t^{(j)} + \int_{\mathbb{R}} x^2 \nu_{t,j}^{\mathbb{Q}}(x) dx. \qquad (26)$$

In addition,

$$\frac{1}{T}\widetilde{AV}_t^{(j)} \xrightarrow{\mathbb{P}} \int_{\mathbb{R}} f(x)\widetilde{\nu}_{t,j}^{\mathbb{Q}}(x)dx, \tag{27}$$

and if $\frac{\lambda_t^{(j)}(\lambda_t^{(0)})^\top}{\lambda_t^{(0)}(\lambda_t^{(0)})^\top} = \beta_t^{(j)}$, *then also*

$$\frac{1}{T}\widetilde{QV}_t^{(j)} \xrightarrow{\mathbb{P}} \widetilde{V}_t^{(j)} + \int_{\mathbb{R}} x^2\widetilde{\nu}_{t,j}^{\mathbb{Q}}(x)dx. \tag{28}$$

The asymptotics of Theorem 1 is of joint type: the time to maturity of the options decreases, the mesh of the strike grid shrinks while its span increases, and the sampling frequency of the returns used to estimate betas increases. For the convergence in probability results of the theorem, we need to have $\Delta/\sqrt{T} \to 0$. This condition guarantees that the error associated with the option price with strike closest to the current stock price is of smaller order than the quantity to be estimated via the option portfolios.

Because of the shrinking time to maturity of the options, T, the limits of the option-based measures of variation can be readily linked to the jumps and the volatility of the underlying process. In particular, we note that the asymmetry in the return variation over short intervals is solely due to the asymmetry in the jump distribution of the underlying process. Finally, for $\widetilde{QV}_t^{(j)}$ to measure the risk-neutral idiosyncratic variance, we need the continuous and jump market betas to be the same.

We continue next with deriving Central Limit Theorems for the option-based variation measures. For brevity, given the empirical analysis that follows, we will present only limit results for the asymmetric variation measures. We start with the total asymmetric variation quantities $\{AV_t^{(j)}\}_{j=0,1,\dots,N}$ which are constructed solely from the option data. In the following theorem, $\mathcal{L}|\mathcal{F}^{(0)}$ denotes convergence in probability of the $\mathcal{F}^{(0)}$ laws in the space of probability measures equipped with the weak convergence topology, see VIII.5.26 in Jacod and Shiryaev (2003).

Theorem 2 *Suppose assumptions A2-A7 in Section 8.1 hold. Let $\Delta \to 0$, $T \to 0$ and $(|\underline{k}| \vee \overline{k}) \to \infty$, with $\Delta \asymp T^\alpha$ for $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $(|\underline{k}| \vee \overline{k}) \asymp T^{-\beta}$ for some β satisfying $\min_{j=0,1,\dots,N}(\alpha_j^\pm)\beta > \frac{\alpha}{2} - \frac{1}{4}$, where $\alpha_j^\pm$ are the tail coefficients of assumption A7. Then, we have*

$$\frac{T^{1/4}}{\sqrt{\Delta}}\Omega_{AV,t}^{-1/2} \begin{pmatrix} \frac{1}{T}AV_t^{(0)} - \int_{\mathbb{R}} f(x)\nu_{t,0}^{\mathbb{Q}}(x)dx \\ \vdots \\ \frac{1}{T}AV_t^{(N)} - \int_{\mathbb{R}} f(x)\nu_{t,N}^{\mathbb{Q}}(x)dx \end{pmatrix} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \mathbf{Z}_t, \tag{29}$$

where $\mathbf{Z}_t$ is $N + 1$-dimensional standard normal vector defined on an extension of the original probability space and independent of $\mathcal{F}^{(0)}$ and the $N + 1 \times N + 1$ $\mathcal{F}^{(0)}$-adapted matrix $\Omega_{AV,t}$ is defined in Section 8.3.

There are several errors in the estimation of $\int_{\mathbb{R}} f(x)\nu_{t,j}^{\mathbb{Q}}(x)dx$ via $\frac{1}{T}AV_t^{(j)}$. These are due to the presence of measurement error in the observed option prices, the discreteness and finite range of the observed strike grid as well as the approximation $\frac{1}{T}\mathbb{E}_t^{\mathbb{Q}}(f(x_{t+T}^{(j)} - x_t^{(j)})) - \int_{\mathbb{R}} f(x)\nu_{t,j}^{\mathbb{Q}}(x)dx$. The conditions on the relative order of Δ, T and $(|\underline{k}| \vee \overline{k})$ in Theorem 2 ensure that the leading error term is due to the observation error. More specifically, the error due to the finite range of the strike grid plays an asymptotically negligible role for the limit result whenever $\min_{j=0,1,...,N}(\alpha_j^{\pm})\beta > \frac{\alpha}{2} - \frac{1}{4}$ holds, where $\alpha_j^{\pm}$ control the tail decay of jump compensators, with the condition becoming naturally weaker for a faster tail decay. From a practical point of view, this condition is not binding as typically the strikes of the available options cover the "effective" support of the return distribution (in the sense that the deepest out-of-the-money option quotes we typically observe have asks at the minimum tick size) and we further implement tail approximation on the basis of extreme value theory.

Next, the requirement for the relative magnitude of Δ and T is due to the asymptotic order of the option prices across the different strikes and the approximation error $\frac{1}{T}\mathbb{E}_t^{\mathbb{Q}}(f(x_{t+T}^{(j)} - x_t^{(j)})) - \int_{\mathbb{R}} f(x)\nu_{t,j}^{\mathbb{Q}}(x)dx$. In particular, if we replace $\int_{\mathbb{R}} f(x)\nu_{t,j}^{\mathbb{Q}}(x)dx$ in Theorem 2 with $\frac{1}{T}\mathbb{E}_t^{\mathbb{Q}}(f(x_{t+T}^{(j)} - x_t^{(j)}))$, then the limit result of the theorem will continue to hold but without the requirement for the upper bound on α. On the other hand, the lower bound restriction on α is needed to guarantee that the option error from the option with the strike closest to the current spot price does not dominate the rest of the observation errors (and the quantity to be recovered). In this regard, we note that the option prices with different strikes are of different asymptotic order: the options with strikes in the vicinity of the current spot price are asymptotically larger than those away from it. This carries over to the observation errors attached to the option prices, and therefore the ones corresponding to strikes in the vicinity of the current spot price govern the limit distribution in Theorem 2 and the rate of convergence of the estimator. From a practical point of view, this is a desirable feature due to the higher liquidity of options with strikes that are not very far in the tails. The limit distribution in Theorem 2 is mixed Gaussian, with $\mathcal{F}^{(0)}$-conditional volatility depending on the observed path of the stock prices and the latent true option prices.

We next state a CLT for $\widetilde{AV}_t^{(j)}$. The convergence in law in the next theorem is stable, denoted with $\mathcal{L} - s$, which means that the convergence in law holds jointly with any bounded random variable defined on the original probability space, see e.g., VIII.5.28 in Jacod and Shiryaev (2003).

Theorem 3 *Suppose assumptions A1-A7 in Section 8.1 hold. Let $\Delta \to 0$, $T \to 0$, $(|\underline{k}| \vee \overline{k}) \to \infty$ and $n \to \infty$, with $\Delta \asymp T^{\alpha}$ for $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $(|\underline{k}| \vee \overline{k}) \asymp T^{-\beta}$ for some β satisfying*

$\min_{j=0,1,\ldots,N} (\alpha_j^{\pm}) \beta > \frac{\alpha}{2} - \frac{1}{4}$, *where $\alpha_j^{\pm}$ are the tail coefficients of assumption A7. Set*

$$\Omega_{\widetilde{AV},t} = \frac{\Delta}{\sqrt{T}} (\iota_{2:N+1} \Omega_{AV,t} \iota'_{2:N+1} + diag(\widehat{\beta}_t) \widehat{\beta}_t \iota_1 \Omega_{AV,t} \iota'_1 \widehat{\beta}'_t diag(\widehat{\beta}_t))$$
$$+ \frac{4}{n} \frac{(AV_t^{(0)})^2}{T^2} diag(\widehat{\beta}_t) \Omega_{\beta,t} diag(\widehat{\beta}_t). \tag{30}$$

Then, if $\Omega_{\beta,t}$ is of full rank, we have

$$\Omega_{\widetilde{AV},t}^{-1/2} \begin{pmatrix} \frac{1}{T} \widetilde{AV}_t^{(1)} - \int_{\mathbb{R}} f(x) \widetilde{\nu}_{t,1}^{\mathbb{Q}}(x) dx \\ \vdots \\ \frac{1}{T} \widetilde{AV}_t^{(N)} - \int_{\mathbb{R}} f(x) \widetilde{\nu}_{t,N}^{\mathbb{Q}}(x) dx \end{pmatrix} \xrightarrow{\mathcal{L}-s} \mathbf{Z}_t, \tag{31}$$

where $\mathbf{Z}_t$ is $N+1$-dimensional standard normal vector defined on an extension of the original probability space and independent of $\mathcal{F}$.

The limit distribution of $\widetilde{AV}_t^{(j)}$ is mixed Gaussian, with the limit determined by the option error as well as the error in recovering the betas from the high-frequency return data. There is no requirement for the relative order of Δ and T on one hand and n on the other hand. The rate of convergence is implicitly determined by the smaller of $\sqrt{\Delta}/T^{1/4}$ and $\sqrt{n}$, and we do not need to take a stand on which of them is asymptotically larger.

For making use of the above limit result, we need to replace $\Omega_{\widetilde{AV},t}$ with a consistent estimate for it and further take advantage of the fact that the convergence in Theorem 3 holds stably. The corresponding result is stated in the theorem below, with the explicit construction of the consistent estimate of $\Omega_{\widetilde{AV},t}$ given in Section 8.3.

Theorem 4 *Under the conditions of Theorem 3, the limit result of this theorem continues to hold upon replacing $\Omega_{\widetilde{AV},t}$ with $\widehat{\Omega}_{\widetilde{AV},t}$ constructed from the option and return data and defined in Section 8.3.*

We note that the feasible estimate of the asymptotic variance, $\widehat{\Omega}_{\widetilde{AV},t}$ does not depend on Δ and n.

5   Alternative Option-Based Measures of Asymmetry

While the semivariance measures introduced and analyzed in the previous two sections have easy to interpret limits associated with the variation of jumps in the processes, there is nevertheless a more efficient and complete way of identifying the asymmetry in the jump (and respectively return) distribution from the short-dated options. We introduce and analyze the asymptotic properties of this alternative approach in this section. In the Appendix, as a robustness check, we redo the

empirical analysis of next section by replacing the semivariances with these alternative option-based measures of asymmetry. The measures that we develop here are defined from the following two portfolios of options:[4]

$$\widehat{\mathcal{L}}_{t,T}^{(j)}(u) = 1 - (u^2 + iu)e^{-iux_t^{(j)}} \sum_{l=2}^{S_j} \frac{e^{(iu-1)k_l^{(j)}} - e^{(iu-1)k_{l-1}^{(j)}}}{iu - 1} \widehat{O}_{t,T}^{(a,j)}(k_{l-1}^{(j)}),$$
(32)

$$\widehat{\mathcal{M}}_{t,T}^{(j)} = \sum_{l=2}^{S_j} (e^{-k_{l-1}^{(j)}} - e^{-k_l^{(j)}})\widehat{O}_{t,T}^{(a,j)}(k_{l-1}^{(j)}),$$
(33)

where $\widehat{O}_{t,T}^{(a,j)}(k_{i-1}^{(j)})$ is the average option price defined in equation (15). The first option portfolio $\widehat{\mathcal{L}}_{t,T}^{(j)}(u)$ is a measure of the characteristic function of the return $\mathbb{E}_t^{\mathbb{Q}}(e^{iu(x_{t+T}^{(j)}-x_t^{(j)})})$ (see Qin and Todorov (2018)) while the second one is an estimate of $-\mathbb{E}_t^{\mathbb{Q}}(x_{t+T}^{(j)} - x_t^{(j)})$ and is one-half of the squared volatility VIX index quoted by the CBOE option exchange. Our alternative measure of asymmetry is then given by

$$AM_t^{(j)}(u) = \Im\left(\log(\widehat{\mathcal{L}}_{t,T}^{(j)}(u))\right) + u\widehat{\mathcal{M}}_{t,T}^{(j)},$$
(34)

where the complex logarithm is given by $\log(z) = \log|z| + \text{Arg}(z)$ for $z \in \mathbb{C}$, with $\text{Arg}(z)$ denoting the principal value of the argument (taking values in $(-\pi, \pi]$). The idea behind the above estimator of asymmetry is the following. For small T, using Lévy-Khintchine theorem (Theorem 8.1 in Sato (1999)), we have

$$\mathbb{E}_t^{\mathbb{Q}}(e^{iu(x_{t+T}^{(j)}-x_t^{(j)})}) \approx \exp\left(iuT\alpha_t^{(\mathbb{Q},j)} - \frac{u^2}{2}T(\sigma_t^{(j)})^2 + T\int_{\mathbb{R}}(e^{iuz} - 1 - iuz)\nu_{t,j}^{(\mathbb{Q})}(z)dz\right),$$
(35)

where $\alpha_t^{(\mathbb{Q},j)}$ is the drift term of the log-price under $\mathbb{Q}$ (i.e., the $\mathbb{Q}$ counterpart of $\alpha_t^{(j)}$), and further

$$\mathbb{E}_t^{\mathbb{Q}}(x_{t+T}^{(j)} - x_t^{(j)}) \approx T\alpha_t^{(\mathbb{Q},j)}.$$
(36)

Combining them, we have that for T sufficiently small

$$\Im\left(\log(\mathbb{E}_t^{\mathbb{Q}}(e^{iu(x_{t+T}^{(j)}-x_t^{(j)})}))\right) - u\mathbb{E}_t^{\mathbb{Q}}(x_{t+T}^{(j)} - x_t^{(j)}) \approx T\int_{\mathbb{R}}(\sin(uz) - uz)\nu_{t,j}^{\mathbb{Q}}(z)dz,$$
(37)

and we stress that the above approximation is exact if volatility and jump intensity are constant over $[t, t+T]$. This is not the case for the semivariances, i.e., there will be estimation error in separating volatility from jumps for these measures even when volatility and jump intensity are constant. This

---

[4]For simplicity, here we do not add tail integral approximations like we did in the construction of the semivariance measures.

is the reason why our alternative measure of asymmetry $AM_t^{(j)}(u)$ has better asymptotic properties than $AV_t^{(j)}$ (which is based on the semivariances).

Since the function $\sin(uz) - uz$ is odd, then nonzero values of $AM_t^{(j)}(u)$ correspond to asymmetry in the jump distribution. In fact, asymmetry in $\nu_{t,j}^{\mathbb{Q}}(z)$ is equivalent to $\int_{\mathbb{R}}(\sin(uz) - uz)\nu_{t,j}^{\mathbb{Q}}(z)dz$ being different from zero for some value of u. In that sense $AM_t^{(j)}(u)$ as a function of u can fully characterize the asymmetry in the jump distribution. By contrast, we can have $AV_t^{(j)} = 0$ and the jump distribution being asymmetric. For low values of u, $AM_t^{(j)}(u)$ loads more on the big jumps (indeed $\sin(uz) - uz \asymp -u^3 z^3/6$ for u converging to zero) while for big u, the statistic is dominated by the first moment of the jumps. Compared to $AV_t^{(j)}$, the measure $AM_t^{(j)}(u)$ puts more weight in relative terms to the "moderately-sized" jumps, i.e., for a fixed u, we have $(uz - \sin(uz))/z^2 \to 0$ as either $z \uparrow \infty$ or $z \downarrow 0$.

Using the estimate for the market beta, we can recover the asymmetry in the idiosyncratic jumps via

$$\widetilde{AM}_t^{(j)}(u) = AM_t^{(j)}(u) - AM_t^{(0)}(\widehat{\beta}_t^{(j)}u). \tag{38}$$

In the following theorem we derive the asymptotic order of magnitude of the difference between the measures $AM_t^{(j)}(u)$ and $\widetilde{AM}_t^{(j)}(u)$ and their asymptotic limits.

Theorem 5 *Suppose assumptions A1-A6 in Section 8.1 hold. Let $\Delta \to 0$, $T \to 0$ and $(|\underline{k}| \vee \overline{k}) \to \infty$, with $\sqrt{\Delta}|\log(T)| \to 0$. Then, for a fixed $u \in \mathbb{R}_+$, we have*

$$\frac{1}{T}AM_t^{(j)}(u) = \int_{\mathbb{R}}(\sin(uz) - uz)\nu_{t,j}^{\mathbb{Q}}(z)dz + O_p(\sqrt{\Delta} \vee T \vee e^{-2(|\underline{k}| \wedge |\overline{k}|)}), \quad j = 0, 1, .., N, \tag{39}$$

$$\frac{1}{T}\widetilde{AM}_t^{(j)}(u) = \int_{\mathbb{R}}(\sin(uz) - uz)\widetilde{\nu}_{t,j}^{\mathbb{Q}}(z)dz + O_p(\sqrt{\Delta} \vee T \vee e^{-2(|\underline{k}| \wedge |\overline{k}|)} \vee n^{-1/2}), \quad j = 1, .., N. \tag{40}$$

We note that we can also derive CLT results for $AM_t^{(j)}(u)$ and $\widetilde{AM}_t^{(j)}(u)$ but in order to keep the analysis short, we do not do this here. The difference between $AM_t^{(j)}(u)$ and $\widetilde{AM}_t^{(j)}(u)$ and their asymptotic limits are driven by three sources of error: (1) observation error and discreteness of the strike grid, (2) the finite strike range and (3) the time variation in the volatility and the jump intensity. As mentioned above, $AM_t^{(j)}(u)$ and $\widetilde{AM}_t^{(j)}(u)$ do not contain error in separating volatility from jumps unlike their counterparts $AV_t^{(j)}$ and $\widetilde{AV}_t^{(j)}$. This results in a weaker requirement for the asymptotic relation between Δ and T. In addition, the option portfolios used in forming $AM_t^{(j)}(u)$ and $\widetilde{AM}_t^{(j)}(u)$ load slightly less on the options with strikes in the vicinity of the current spot price than $AV_t^{(j)}$ and $\widetilde{AV}_t^{(j)}$ do. This leads to a smaller impact of the observation error in $AM_t^{(j)}(u)$ and $\widetilde{AM}_t^{(j)}(u)$ in an asymptotic sense.

Overall, $AM_t^{(j)}(u)$ and $\widetilde{AM}_t^{(j)}(u)$ have faster rate of convergence and weaker conditions on Δ and T than the measures $AV_t^{(j)}$ and $\widetilde{AV}_t^{(j)}$ (based on semivariances). Nevertheless, since the asymptotic limits of $AV_t^{(j)}$ and $\widetilde{AV}_t^{(j)}$ are somewhat easier to interpret, we will base our empirical analysis on $AV_t^{(j)}$ and $\widetilde{AV}_t^{(j)}$. In the appendix, we show that the key empirical findings remain intact when switching to $AM_t^{(j)}(u)$ and $\widetilde{AM}_t^{(j)}(u)$ (for low values of u).

6 Empirical Exploration of Asymmetry in Idiosyncratic Risk

6.1 Data

We begin the empirical section with describing the data that we use in our analysis. We obtain daily closing best bid and ask quotes for equity options covering the period 2007-2017 from OptionMetrics. Our sample consists of stocks in the CBOE S&P 500 Implied Correlation Index. Specifically, we select the 50 stocks that are most frequently included in the index during our sample period, all of which are part of the S&P 500 index throughout our sample. Our proxy for the market is the S&P 500 index portfolio and we use options written on the cash index (ticker SPX). We keep out-of-the-money options and take the mid-quote as the observed option price in the construction of the variation measures. On each day, we use options with the shortest available maturity given that it is at least 5 business days. This leads to a median time-to-maturity of the options in the sample of 8 business days. Further details on the data filtering are given in the data appendix, with Table 6 providing summary statistics for the option data. As a robustness check, in Section 9, we extend the empirical analysis to the stocks in the S&P 100 index for the last three years in the sample when the liquidity of the stock option market has increased significantly.

The options on the S&P 500 index are European style while the options on the individual stocks are American, i.e., they allow for early exercise. We ignore the latter in the analysis as the impact of the option to exercise early is negligible.[5]

Following earlier empirical option pricing work, we back out $f_t^{(j)}$ from the option data by making use of put-call parity for European style options. To reduce noise, we use the two strike prices that are closest to the spot price provided by OptionMetrics, and we take the average of the two option-implied futures prices.

Finally, the high-frequency return data for individual stocks is taken from the NYSE Trade and Quote (TAQ) database, and the S&P 500 index exchange traded fund (ticker SPY) is used as a

---

[5]American calls are never optimal to exercise early when the underlying asset does not pay a dividend till expiration (which is the case most of the time). American puts can be optimal to exercise early in order to gain the interest on the strike. However, since the interest rate is rather low and the time to maturity of the options is rather short, these gains are nevertheless small and therefore we ignore them.

proxy for the market index. The sampling frequency is five-minutes during the trading hours. Data cleaning procedures are described in the data appendix.

6.2 Is Aggregate Asymmetry in Idiosyncratic Risk Present?

In our empirical analysis we will focus on cross-sectional averages of the measures of variation. In particular, we will analyze

$$\widetilde{AV}_t = \frac{1}{N} \sum_{j=1}^N \widetilde{AV}_t^{(j)} \quad \text{and} \quad \widetilde{QV}_t = \frac{1}{N} \sum_{j=1}^N \widetilde{QV}_t^{(j)}. \tag{41}$$

Individual stock variation measures $\widetilde{AV}_t^{(j)}$ and $\widetilde{QV}_t^{(j)}$ can reflect temporary increases in expected future stock variation and/or asymmetry in the return distribution, most notably around the times of earnings announcements, which have no aggregate pricing implications. In addition, from an econometric point of view, cross-sectional averaging will reduce estimation error which is of importance as the option coverage for some of the stocks during parts of the sample can be scant. While $\widetilde{AV}_t$ is constructed by putting equal weights to the stocks in the cross-section, in Section 9 we repeat the empirical analysis by weighting the individual $\widetilde{AV}_t^{(j)}$ measures according to the value of the stocks. The results based on the value-weighted statistics are very similar to the ones based on the equally-weighted $\widetilde{AV}$ constructed above.

We start our empirical analysis with conducting tests for deciding whether the aggregate asymmetry in the risk-neutral return distribution of the assets, $\widetilde{AV}_t$, is statistically different from zero. That is, we test whether $\frac{1}{N} \sum_{j=1}^N \int_{\mathbb{R}} f(x) \widetilde{\nu}_{t,j}^{(\mathbb{Q})}(x) dx = 0$. This is a natural hypothesis. Indeed, if jump risk in stocks outside market-wide jump events is perfectly diversifiable as in the seminal work of Merton (1976), then we would expect $\widetilde{\nu}_{t,j}^{(\mathbb{Q})}(x)$ to be averaged out in the cross-sectional aggregation, and therefore in this case $\widetilde{AV}_t$ should not be different from zero statistically.[6]

We implement the test on the option data in our sample, and we note that N differs across days due to data availability (with lower values of N in the early part of the sample). To further gain power, particularly for the early part of the sample, we test the null hypothesis $\sum_{s=t-20}^t \widetilde{AV}_s = 0$. The lower 95% confidence bounds for $\sum_{s=t-20}^t \widetilde{AV}_s$ are plotted in Figure 2 and reveal the latter is statistically different from zero for most of the sample. The null hypothesis of $\sum_{s=t-20}^t \widetilde{AV}_s = 0$ cannot be rejected only in the first half of 2007. However, this is mostly due to the larger estimation uncertainty during that period. Indeed, we have lower N and on average lower number of options per stock in 2007.

---

[6]Our asymptotics is for fixed N. However, it can be extended to the $N \to \infty$ case. For $N \to \infty$, the error associated with measuring the index options will not get averaged out in the cross-sectional aggregation and therefore it will become the leading term asymptotically in the CLT for $\widetilde{AV}_t$.

The results from Figure 2 suggest presence of aggregate idiosyncratic risk, with the positive values of $\widetilde{AV}_t$ being due to a combination of aggregate asymmetry in the idiosyncratic jump distribution under $\mathbb{P}$ and/or asymmetry in the $\mathbb{P} - \mathbb{Q}$ wedge associated with the idiosyncratic jump distribution (i.e., risk premium). We will separate these two components of $\widetilde{AV}_t$ in the next section with the help of return data.


Lower 95% Confidence Bound for $\widetilde{AV}$
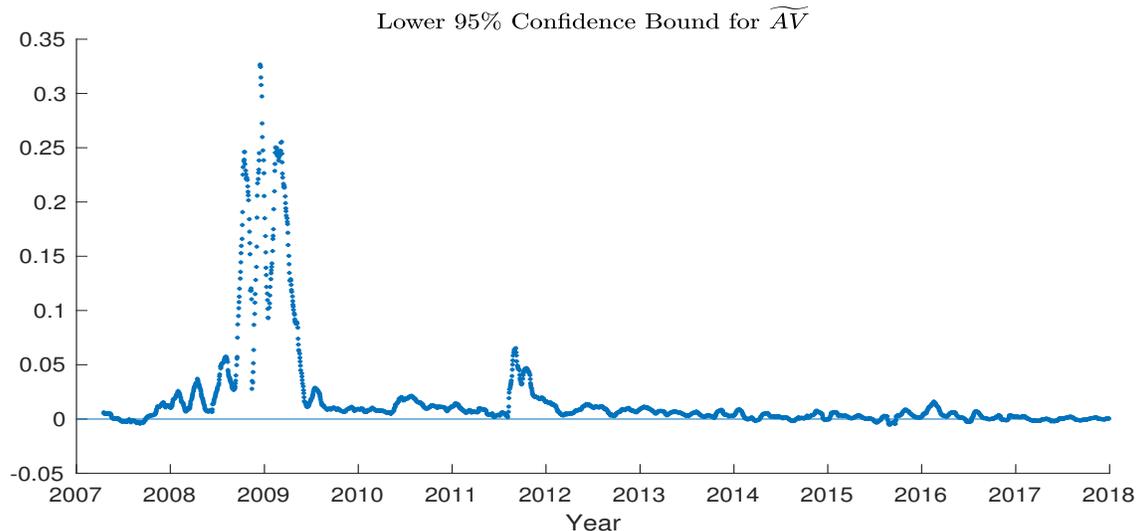
Figure 2: Lower 95% confidence bound for 20-day moving average of $\widetilde{AV}_t$. The confidence interval is constructed on the basis of Theorem 4.

While on majority of the days in the sample after 2007, the null hypothesis of $\sum_{s=t-20}^{t} \widetilde{AV}_s = 0$ is overwhelmingly rejected, there are nevertheless periods where the value of the test statistic is significantly lower. This happens, in particular, before major pre-scheduled economy-wide events in our sample, mainly the Brexit vote in June 2016 and the US elections in November 2016. What is the reason for this? Typically, as the expiration date approaches, the options get cheaper as, intuitively, there is less volatility which "generates" the option value. With our notation, $O_{t,T}(k) \to 0$ as $T \to 0$ almost surely and for any fixed k. This holds in the case when the asset price follows a general semimartingale with no fixed time of discontinuity as is assumed here and in most existing work.

Things are very different, however, when the underlying stock price has a fixed time of discontinuity before the expiration date of the option written on the stock, such as a pre-scheduled economic announcement or an expected major political event. In this case, it is easy to show that $O_{t,T}(k) = O_p(1)$ as $T \to 0$, with the value of the option price solely determined in the limit (as $T \to 0$) by the jump at the known fixed event time. In the case of the Brexit vote and the 2016 US election, the fixed jump event is a market-wide event. Hence, asymptotically for these events our estimate of $\widetilde{AV}_t$ will be zero. This of course does not mean that there is no idiosyncratic jump risk

21

at this points of time. Instead, this is due to the fact that the latter plays only a negligible role in the short-dated options which include the economy-wide jump with fixed arrival time.

Such events, therefore, present an opportunity to test our (conditional) linear factor structure of jumps associated with events that cause the market to jump. More specifically, if the jump structure in (4) is true, then the normalized option prices of the individual stocks should be possible to reconstruct (in the limit as $T \to 0$) from the market index options. For example, if the beta of the stock is 1, then the normalized individual stock and index options should be approximately the same. We illustrate this in Figure 3 with S&P 500 index and Apple stock short-dated options recorded on the day before the 2016 US election (the market beta of Apple on this date was very close to one). Comparing Figures 1 and 3, we see a substantial difference. Mainly, consistent with the conditional factor specification in (4), the normalized Apple options are very close to the market index options on the day before the election. This is because, while idiosyncratic risk is still present in the Apple stock, its importance in relative terms to the anticipated systematic risk is negligible when a prescheduled market-wide event is approaching. This is particularly true for the left tail (corresponding to out-of-the-money puts) for which in relative terms the systematic risk is much bigger than the idiosyncratic one.
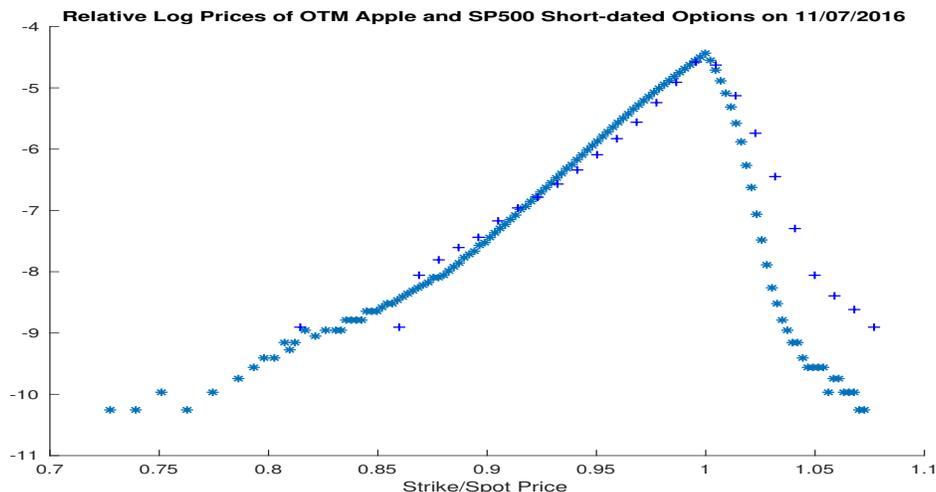
Figure 3: Relative Log Prices of OTM Apple and SPX Short-dated Options before 2016 US Elections. The y-axis is the logarithm of the ratio of the out-of-the-money option prices over the current stock price. Stars correspond to S&P 500 index options and the crosses to options on Apple stock.

While such events are useful for showing the validity of our assumed factor structure at market-wide jump events, they are nevertheless quite rare in the sample and therefore do not invalidate

our inference for the idiosyncratic jump risk.[7]

6.3 Pricing of Asymmetry in Idiosyncratic Jump Risk

6.3.1 Empirical Evidence

Given the strong evidence of the previous section for presence of aggregate asymmetry in the risk-neutral expected idiosyncratic risk, we now study what part of this aggregate asymmetry can be attributed to risk premium. As a reference point, we note that there is a large body of work which has shown that risk-neutral market return skewness is largely explained by time-varying risk premium for downside market risk. Here we investigate whether this is the case for the aggregate idiosyncratic skewness and we further study the implications of this.

In order to isolate the risk premium component in $\widetilde{AV}_t$, we need to construct its counterpart from returns on the assets. To this end, we introduce the realized asymmetric variation measures[8]

$$RAV_t^{(j)} = \sum_{i=\lfloor tn \rfloor + 1}^{\lfloor (t+T)n \rfloor} f(\Delta_i^n x^{(j)}), \quad j = 0, 1, ..., N. \tag{42}$$

Following general results in Barndorff-Nielsen et al. (2010), we have

$$RAV_t^{(j)} \xrightarrow{\mathbb{P}} \sum_{s \in [t,t+T]} f(\Delta x_s^{(j)}), \quad j = 0, 1, ..., N, \tag{43}$$

as $n \to \infty$, i.e., as we sample the asset prices more frequently. Our estimate for the realized idiosyncratic asymmetry risk is then given by

$$\widetilde{RAV}_t^{(j)} = RAV_t^{(j)} - (\widehat{\beta}_t^{(j)})^2 RAV_t^{(0)}, \quad \widetilde{RAV}_t = \frac{1}{N} \sum_{j=1}^{N} \widetilde{RAV}_t^{(j)}, \quad j = 1, ..., N, \tag{44}$$

and under assumption A1 in the appendix, we have:

$$\widetilde{RAV}_t^{(j)} \xrightarrow{\mathbb{P}} \sum_{s \in [t,t+T]} f(\Delta \widetilde{x}_s^{(j)}), \quad j = 1, ..., N. \tag{45}$$

On Figure 4, we plot 20-day moving average of $\widetilde{RAV}_t$ against that of $\widetilde{AV}_t$. As can be seen from the figure, the aggregate realized idiosyncratic asymmetric variation risk is significantly smaller in magnitude than its risk-neutral expectation. Nevertheless, we do note that there are periods in the sample, where this risk is present. Most notably, we do have (negative) aggregate idiosyncratic return skewness in the Fall of 2007, during the Fall of 2008 and in early 2009.

---

[7]That is, even if we were to exclude the days in the sample in which the expiration date of the options is after a large economy-wide event with known arrival time, our empirical findings about $\widetilde{AV}$ will continue to hold.

[8]In computing $RAV_t^{(j)}$, we include the returns covering the period from market close to the the next market open. We also computed $RAV_t^{(j)}$ using daily returns, with the resulting measures being similar (but of course noisier).
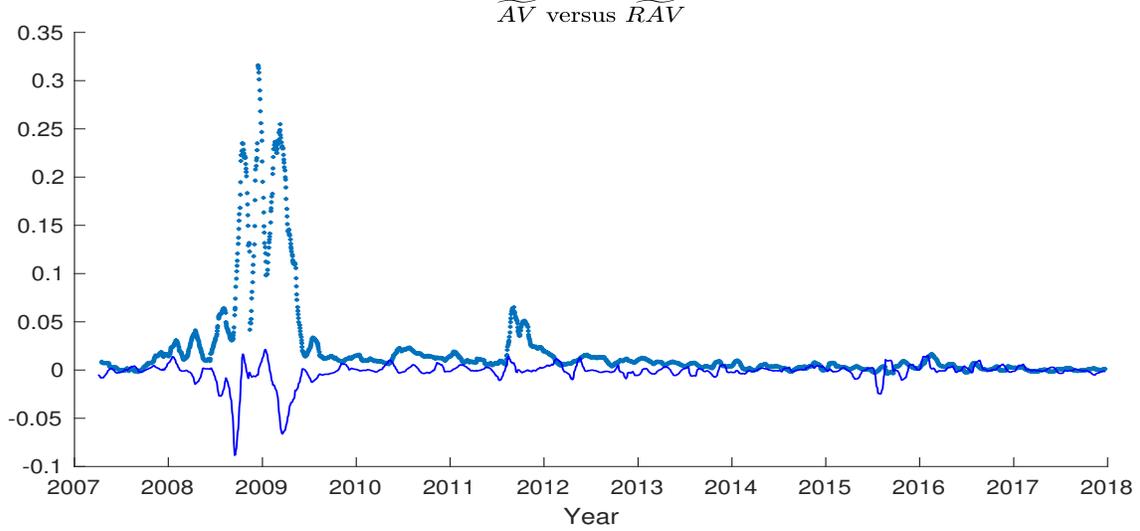
$\widetilde{AV}$ versus $\widetilde{RAV}$

Figure 4: $\widetilde{AV}$ versus $\widetilde{RAV}$. Dotted line is $\widetilde{AV}$ and solid line is $\widetilde{RAV}$ from five-minute returns. Plotted series are 20-day moving averages.

Testing formally the hypothesis $\widetilde{RAV}_t = 0$ will require a feasible CLT for $\widetilde{RAV}_t$ which is in general difficult as the drift term in the asset dynamics will show up in the asymptotic limit as an asymptotic bias.[9] However, a realized counterpart of the asymmetry measure $AM_t^{(j)}$, which is defined in the appendix, has a feasible CLT. Using it, we show formally in the appendix that the aggregate asymmetry in the idiosyncratic risk is statistically different from zero. This is in line with our earlier result for $\widetilde{AV}_t$ being different from zero and formally shows that there is cross-sectional clustering of idiosyncratic risk. This, in turn, means that idiosyncratic jump risk cannot be solely explained with firm-specific shocks as in Merton (1976), and instead idiosyncratic jump arrival is a systematic event.

The wedge between the risk-neutral and statistical probability measures is a manifest of the risk premium. With regards to the risk in $\widetilde{RAV}$, no pricing of the aggregate idiosyncratic asymmetry variation risk at time t means

$$\widetilde{AV}_t = \mathbb{E}_t^{\mathbb{P}}(\widetilde{RAV}_t), \tag{46}$$

and this implies the following conditional moment restriction

$$\mathbb{E}_t^{\mathbb{P}}\left(\widetilde{RAV}_t - \widetilde{AV}_t\right) = 0. \tag{47}$$

Conditional moment restrictions of this type arise naturally from asset pricing models and inference

---

[9]However, we conjecture that a mild truncation of the increments from below (in absolute value) will allow for feasible CLT without changing the probability limit of $\widetilde{RAV}_t^{(j)}$. Such limit result will complement our CLT for $AV_t^{(j)}$ (and can be shown to hold jointly with it). For brevity, we do not pursue this any further here.

procedures for them have been developed following the seminal work on GMM, see e.g., Hansen (1982) and Hansen and Singleton (1982). One can turn the conditional moment restriction into unconditional one by using instruments x_t which are in the information set of the econometrician at time t:

$$\mathbb{E}^{\mathbb{P}}\left[\left(\widetilde{RAV}_t - \widetilde{AV}_t\right) x_t\right] = 0. \tag{48}$$

We implement the above test regarding the pricing of aggregate idiosyncratic asymmetry using the following sample moments

$$\begin{cases} \widehat{m}_{mean} = \frac{1}{\mathcal{T}-\kappa+1} \sum_{t=\kappa}^{\mathcal{T}} \frac{\widetilde{RAV}_t - \widetilde{AV}_t}{RV_t^{(0)}}, \\ \widehat{m}_{auto}^{(\tau)} = \frac{1}{\mathcal{T}-\kappa-\tau+1} \sum_{t=\kappa+\tau}^{\mathcal{T}} \frac{\widetilde{RAV}_t - \widetilde{AV}_t}{RV_t^{(0)}} \frac{\widetilde{RAV}_{t-\tau} - \widetilde{AV}_{t-\tau}}{RV_{t-\tau}^{(0)}}, \quad \tau \in \mathbb{N}, \end{cases} \tag{49}$$

where

$$RV_t^{(i)} = \sum_{i \in I_t^n} (\Delta_i^n x^{(i)})^2, \quad i = 0, 1, ..., N, \tag{50}$$

denotes the realized volatility of the corresponding asset. We scale the differences $\widetilde{RAV}_t - \widetilde{AV}_t$ by the market realized volatility $RV_t^{(0)}$ in order to improve estimation efficiency by accounting for the well-known heteroskedasticity in asset returns. The first of the above moment conditions tests if there is statistical difference in the means of the two (weighted) time series $\widetilde{RAV}_t$ and $\widetilde{AV}_t$. The second set of moment conditions tests for presence of time variation in that gap, i.e., for time-varying risk premium. In calculating the standard errors for the moment conditions in (49) and all subsequent regressions, we use Newey-West standard errors with lag length set following the recommendations in Lazarus et al. (2018). The results from the test are summarized in Table 1. They provide strong evidence for presence of time-varying risk premium for aggregate downside idiosyncratic risk.

Given the above empirical evidence, we now try to link the risk premium component of $\widetilde{AV}_t$ with aggregate priced risk and its time variation. It is natural to conjecture that the risk premium in $\widetilde{AV}_t$ and the one in $\widetilde{AV}_t^{(0)}$ regarding the market index have a common origin, mainly fear of large negative tail events in systematic risk. Therefore, $\widetilde{AV}_t$ should be a good predictor of future equity risk premia as the tail events in returns are part of equity risk. To study this, in Table 2 we report results from running univariate predictive regressions in which the explanatory variable is the future excess return on an aggregate equity portfolio, constructed as an average of the returns of the stocks in our sample, and the explanatory variable being one of the aggregate option variation measures.[10] Our sample is short, so the predictive regression results are obviously noisy.

---

[10]We also conducted predictive regressions for the aggregate market portfolio, using the CRSP Value-Weighted Market Portfolio as a proxy, with very similar results to the ones reported in Table 2.

	t-stat (HF)	t-stat (daily)
$\widehat{m}_{mean}$	-4.7169	-5.3435
$\widehat{m}^{(1)}_{auto}$	2.5801	2.9742
$\widehat{m}^{(5)}_{auto}$	2.3574	2.6127
$\widehat{m}^{(22)}_{auto}$	2.1378	2.0369

Table 1: Tests for Risk Premium in $\widetilde{AV}$. First column corresponds to $\widehat{RAV}$ computed from five-minute intraday data and the second column to $\widetilde{RAV}$ from daily returns. Standard errors are calculated using Newey-West estimator with lag length of $1.3\sqrt{T}$.

	6 Months				9 Months				12 Months			
$AV^{(0)}_{t,T}$	0.8539				1.5029				2.0974			
	(0.5798)				(0.6868)				(0.6861)			
$QV^{(0)}_{t,T} - AV^{(0)}_{t,T}$		0.7592				1.2504				1.7127		
		(0.5320)				(0.5770)				(0.5344)		
$\widetilde{AV}_{t,T}$			0.6080				1.0906				1.4910	
			(0.4862)				(0.4839)				(0.3756)	
$\widetilde{QV}_{t,T} - \widetilde{AV}_{t,T}$				0.3099				0.6040				0.9994
				(1.0305)				(1.2974)				(1.4084)
	0.0481	0.0542	0.0594	0.0040	0.0967	0.0950	0.1239	0.0102	0.1397	0.1314	0.1715	0.0208

Table 2: Predicting Returns of Equally-Weighted Portfolio Constructed by the Stocks in the sample. The explanatory variables are 20-day moving averages. Standard errors are calculated using Newey-West estimator with lag length of $1.3\sqrt{T}$.

However, we note that unlike standard predictors used in prior work on return predictability, our measures, and in particular $\widetilde{AV}_t$, have significantly less time-series persistence (and this obviously helps precision). With these econometric issues in mind, Table 2 reveals that $\widetilde{AV}_t$ is a strong predictor of future returns at the longer horizons of 9 and 12 months. The predictive ability of $\widetilde{AV}_t$ is comparable to that of $AV^{(0)}_t$. Interestingly, the results from the predictive regressions show that once we remove the asymmetric component of $\widetilde{QV}_t$, it no longer has predictive ability for the future equity risk premium. This suggests that significant part of the price for idiosyncratic volatility, see e.g., Herskovic et al. (2016), is due to the risk premium for aggregate downside idiosyncratic jump risk in asset prices.

In order to compare the risk premia dynamics for market and aggregate idiosyncratic downside risks, in Figure 5 we plot the 20-day moving averages of $\widetilde{AV}$ and $AV^{(0)}$ normalized by their sample means. As seen from the figure, the two series have very similar dynamics, becoming elevated in the

aftermath of the financial crisis of 2008 as well as around the time of the European sovereign debt crises of 2010 and 2011. Nevertheless, there are notable differences in their behavior. In particular, $\widetilde{AV}$ increased significantly in the early part of 2009, a period associated with continued market decline and the passage of the stimulus plan aimed at boosting economic growth. This reaction of $\widetilde{AV}$ is in contrast to that of $AV^{(0)}$ which did not increase significantly (in relative terms) and is indicative of concerns among investors regarding the cross-sectional heterogeneity in the impact from the stimulus plan and more generally from the economic turbulence at the time. Another notable example of different behavior of $\widetilde{AV}$ and $AV^{(0)}$ is the flash crash of May 2010 and the subsequent market fears which had a far bigger impact on $AV^{(0)}$ than on $\widetilde{AV}$. Overall, Figure 5 suggests that there are similarities in the pricing of market and aggregate idiosyncratic downside risks but differences can appear particularly during events which can have heterogeneous effect on various sectors of the economy.
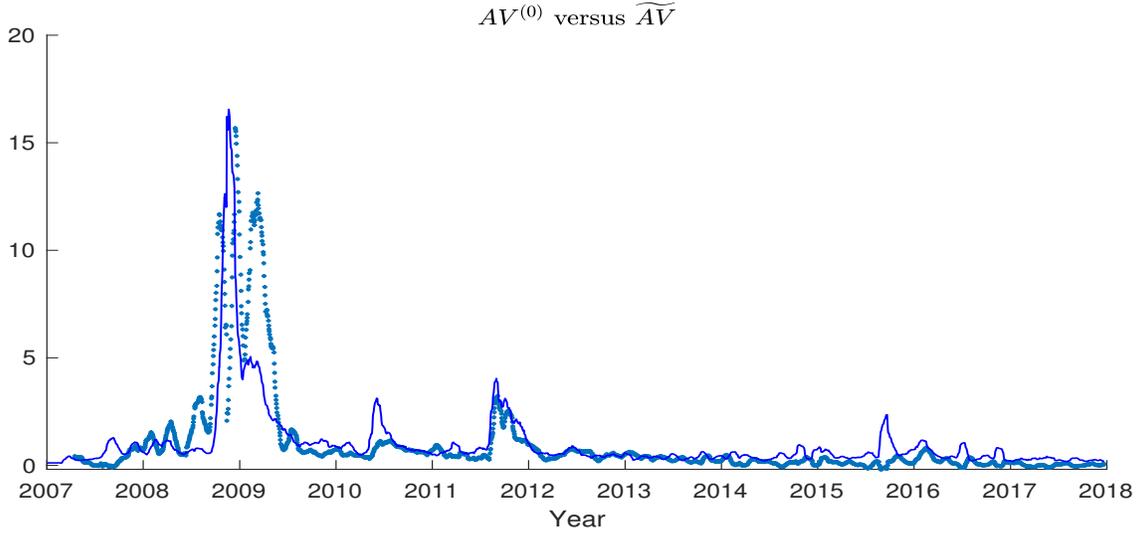

Figure 5: $AV^{(0)}$ versus $\widetilde{AV}$. Dotted and solid lines correspond to 20-day moving averages of $\widetilde{AV}$ and $AV^{(0)}$, respectively. Both series have been normalized by their sample means.

6.3.2 Interpretation of the Results

How can we explain the documented nontrivial price for aggregate idiosyncratic downside risk? As already discussed, if idiosyncratic jumps are in the style of Merton (1976), i.e., arrive at different times with $\Delta \widetilde{x}_t^{(i)} \Delta \widetilde{x}_t^{(j)} = 0$ for $i \neq j$, then at any point in time $\frac{1}{N} \sum_{j=1}^{N} \sum f(\Delta \widetilde{x}_t^{(j)})$ should be approximately zero whenever N is large. Therefore, this type of idiosyncratic jump risk cannot explain $\widetilde{RAV}_t$ being different from zero and the gap $\widetilde{RAV}_t - \widetilde{AV}_t$ documented above. Our empirical results are instead suggestive of cross-sectional clustering of jumps outside market-wide jump

episodes. More specifically, we can justify the findings of the previous section with the following structure for idiosyncratic jump risk:

$$\Delta \widetilde{x}_t^{(j)} = \chi_t^{(j)} Z_t, \quad j = 1, ..., N, \tag{51}$$

where conditional on a "common shock" information set $\mathcal{C}$ (in the terminology of Andrews (2005)), containing the systematic risk information, $\{\chi_t^{(j)}\}_{j=1}^N$ are mean zero, independent from each other and from Z_t, with the latter being adapted to $\mathcal{C}$. In such a specification, we have

$$\frac{1}{N} \sum_{j=1}^N \chi_t^{(j)} \overset{\mathbb{P}}{\longrightarrow} 0, \quad \frac{1}{N} \sum_{j=1}^N f(\chi_t^{(j)}) \overset{\mathbb{P}}{\longrightarrow} \overline{f(\chi)}, \quad \text{as } N \to \infty, \tag{52}$$

provided $\chi_t^{(j)}$ have $\mathcal{C}$-conditional finite fourth moments and where $\overline{f(\chi)} = \mathbb{E}(\chi_t^{(j)}|\mathcal{C})$. In general, $\overline{f(\chi)}$ is a random number that depends on the systematic risks which are in the information set $\mathcal{C}$. Given (52), we have

$$\frac{1}{N} \sum_{j=1}^N \Delta \widetilde{x}_t^{(j)} \overset{\mathbb{P}}{\longrightarrow} 0, \quad \frac{1}{N} \sum_{j=1}^N f(\Delta \widetilde{x}_t^{(j)}) \overset{\mathbb{P}}{\longrightarrow} -\overline{f(\chi)} f(Z_t), \quad \text{as } N \to \infty. \tag{53}$$

That is, the idiosyncratic asymmetry on an individual level "survives" cross-sectional aggregation and generates a systematic source of risk. We note, however, that the probability limit of $\frac{1}{N} \sum_{j=1}^N f(\Delta \widetilde{x}_t^{(j)})$ for N large is different from zero only if $\overline{f(\chi)} \neq 0$, i.e., only if the factor loadings have asymmetry in their cross-sectional distribution. For example, if negative skew in Z carries a positive risk premium, then we need positive skewness in the cross-sectional distribution of $\{\chi_t^{(j)}\}_{j=1}^N$ in order to generate the aggregate (realized) risk premium $\widetilde{RAV}_t - \widetilde{AV}_t$. This will be the case if we have a small number of stocks with very high positive exposure to the risk in Z_t and a majority of stocks with much smaller negative exposure to Z_t. Of course, conditional on the systematic shock Z_t and the market return, the asymmetry and presence of fat-tails in the cross-sectional distribution of the factor loadings $\{\chi_t^{(j)}\}_{j=1}^N$ maps directly into asymmetry and fat tails in the cross-sectional distribution of stock returns, which has been studied in the recent work of Kelly and Jiang (2014) and Oh and Wachter (2019).

There are two very different possibilities regarding the time series behavior of $\chi_t^{(j)}$. The first is the case when $\chi_t^{(j)}$ is only $\mathcal{F}_t$-adapted but is not in the information set $\mathcal{F}_{t-}$. This means that we cannot fully predict the value of the factor loading $\chi_t^{(j)}$ from observing all the information prior to the jump time, captured in $\mathcal{F}_{t-}$. In this case, $\Delta \widetilde{x}_t^{(j)}$ are uncorrelated both in the time series and in the cross-section. In fact, we can even have Z_t being a constant and in this case $\Delta \widetilde{x}_t^{(j)}$ will be cross-sectionally independent. Note, however, that in this scenario the systematic risk consists

of the fact that the jumps in the individual stocks arrive at the same time. When $\chi_t^{(j)}$ is not $\mathcal{F}_{t-}$-adapted, then the econometrician cannot infer $\chi_t^{(j)}$ from the data.

The second situation is one in which $\chi_t^{(j)}$ is $\mathcal{F}_{t-}$-adapted. When this is the case, and under some additional structure, the econometrician can uncover $\chi_t^{(j)}$ from information in $\mathcal{F}_{t-}$, provided Z_t is a known risk factor. In such a situation, one can test for the above specification in (51) by checking whether $AV_t^{(j)}$ line up in the cross-section according to their loadings $\chi_t^{(j)}$. We performed such tests using commonly used risk factors such as the Fama-French ones and we found that the loadings on such risk factors cannot rationalize the observed cross-sectional dispersion in $AV_t^{(j)}$.

7 Conclusion

In this paper we study idiosyncratic jump risk using option and return data on a cross-section of assets. From the option data, we construct nonparametric measures of expected risk-neutral semivariance which split the expected variation into parts due to positive and negative returns as well as additional measures of asymmetric return variation based on the characteristic function of the returns. By combining these option-based quantities with local estimators of market beta from high-frequency returns, we construct nonparametric measures of expected risk-neutral idiosyncratic variation in assets. We derive feasible limit theory for the estimators which builds on in-fill asymptotics in time and in the space of strike grids of the options. Using the developed econometric tools, we show empirically that there is aggregate idiosyncratic downside risk in assets which commands a nontrivial premium with dynamics that yields significant forecast power for future equity returns. Our results suggest cross-sectional clustering of idiosyncratic jump risk as well as cross-sectional negative return skewness during market-neutral systematic jump events.

8 Assumptions and Proofs

In the proofs we will denote with C_t an $\mathcal{F}_t^{(0)}$-adapted finite-valued random variable which can change from line to line.

8.1 Assumptions

Henceforth, we will denote the diffusion coefficient of $x_t^{(j)}$ with

$$\sigma_t^{(j)} = \sqrt{\lambda_t^{(j)}(\lambda_t^{(j)})^\top}, \quad j = 0, 1, ..., N. \tag{54}$$

A1. *The processes* $\{\alpha_t^{(j)}\}_{j=0,...,N}$, $\lambda_t^{(0)}$ *and* $\{\lambda_t^{(j)}\}_{j=1,...,N}$ *are with càdlàg paths. We have* $\Delta x_s^{(0)} \Delta \sigma_s^{(0)} = 0$, $\Delta x_s^{(j)} \Delta \sigma_s^{(0)} = 0$ *and* $\Delta x_s^{(j)} \Delta \sigma_s^{(j)} = 0$, *for* $j = 1, ..., N$ *and* $s \in [t - \kappa, t]$. *Finally,* $\beta_s^{(j)} = \beta_t^{(j)} \geq 0$,

for $j = 1, ..., N$ and $s \in [t - \kappa, t]$.

A2. *We have $|\sigma_t^{(j)}| > 0$ and the process $\sigma^{(j)}$ has the following dynamics under $\mathbb{Q}$:*

$$d\sigma_t^{(j)} = b_t^{(j)}dt + \eta_t^{(j)}dW_t^{\mathbb{Q}} + \overline{\eta}_t^{(j)}d\overline{W}_t^{\mathbb{Q},j} + \int_{\mathbb{R}} \delta^{(\sigma,j)}(t,u)\mu^{(\sigma,j)}(dt,du), \; j = 0,1,...,N, \quad (55)$$

where $\overline{W}^{(\mathbb{Q},j)}$ is a (univariate) Brownian motion independent of $W^{\mathbb{Q}}$; $\mu^{(\sigma,j)}$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator $\nu^{(\sigma,j)}(ds, du) = ds \otimes du$, with the jumps in $x^{(j)}$ having arbitrary dependence with those in $\sigma^{(j)}$; $b^{(j)}$, $\eta^{(j)}$ and $\overline{\eta}^{(j)}$ are processes with càdlàg paths and $\delta^{(\sigma,j)}(s, u) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is left-continuous in its first argument.

A3. *With the notation of A2, there exist $\mathcal{F}_t^{(0)}$-adapted random variables C_t and $\overline{t} > t$ such that for $s \in [t, \overline{t}]$ and $j = 0, 1, ..., N$:*

$$\mathbb{E}_t^{\mathbb{Q}}|\alpha_s^{(\mathbb{Q},j)}|^8 + \mathbb{E}_t^{\mathbb{Q}}|\sigma_s^{(j)}|^{16} + \mathbb{E}_t^{\mathbb{Q}}(e^{8|x_s^{(j)}|}) + \mathbb{E}_t^{\mathbb{Q}}\left(\int_{\mathbb{R}} [e^{4|z|} \vee 1]\nu_{s,j}^{(\mathbb{Q})}(z)dz\right)^8 < C_t, \quad (56)$$

and in addition for some $\iota > 0$

$$\mathbb{E}_t^{\mathbb{Q}}\left(\int_{\mathbb{R}}(|\delta^{(\sigma,j)}(s,z)|^4 \vee |\delta^{(\sigma,j)}(s,z)|)dz\right)^{1+\iota} \leq C_t. \quad (57)$$

A4. *With the notation of A2, there exist $\mathcal{F}_t^{(0)}$-adapted random variables C_t and $\overline{t} > t$ such that for $s, r \in [t, \overline{t}]$ and $j = 0, 1, ..., N$:*

$$\mathbb{E}_t^{\mathbb{Q}}|\alpha_s^{(\mathbb{Q},j)} - \alpha_r^{(\mathbb{Q},j)}|^p + \mathbb{E}_t^{\mathbb{Q}}|\sigma_s^{(j)} - \sigma_r^{(j)}|^p + \mathbb{E}_t^{\mathbb{Q}}||\eta_s^{(j)} - \eta_r^{(j)}||^p + \mathbb{E}_t^{\mathbb{Q}}|\overline{\eta}_s^{(j)} - \overline{\eta}_r^{(j)}|^p$$
$$\leq C_t|s - r|, \; \forall p \in [2, 4], \quad (58)$$

$$\mathbb{E}_t^{\mathbb{Q}}\left(\int_{\mathbb{R}}(e^{z \vee 0}|z| \vee |z|^2)|\nu_{s,j}^{\mathbb{Q}}(z) - \nu_{r,j}^{\mathbb{Q}}(z)|dz\right)^p \leq C_t|s - r|, \; \forall p \in [2, 3], \quad (59)$$

and

$$|\mathbb{E}_t^{\mathbb{Q}}(\alpha_s^{(\mathbb{Q},j)} - \alpha_r^{(\mathbb{Q},j)})| \leq C_t|s - r|, \; |\mathbb{E}_t^{\mathbb{Q}}(\nu_{s,j}^{\mathbb{Q}}(z) - \nu_{r,j}^{\mathbb{Q}}(z))| \leq C_t|s - r|G(z), \quad (60)$$

where the positive-valued function $G(z)$ satisfies $\int_{\mathbb{R}} z^2 G(z)dz < \infty$.

A5. *The log-strike grids $\{k_i^{(j)}\}_{i=1}^{S_j}$, for $j = 0, 1, ..., N$, are $\mathcal{F}^{(0)}$-adapted and we have*

$$c_t\Delta \leq k_i^{(j)} - k_{i-1}^{(j)} \leq C_t\Delta, \quad j = 0, 1, ...N, \; \text{ as } \Delta \downarrow 0, \quad (61)$$

where Δ is a deterministic sequence. In addition, for some arbitrary small $\zeta > 0$:

$$\sup_{i:|k_i^{(j)} - x_t^{(j)}| < \zeta} \left| \frac{k_i^{(j)} - k_{i-1}^{(j)}}{\Delta} - \psi_t^{(j)}(k_{i-1}^{(j)} - x_t^{(j)}) \right| \xrightarrow{\mathbb{P}} 0, \quad j = 0, 1, ..., N, \; \text{ as } \Delta \downarrow 0, \quad (62)$$

where $\psi_t^{(j)}(k)$ are $\mathcal{F}^{(0)}$-adapted functions which are continuous in k at 0.

A6. We have $\epsilon_{t,i}^{(j)} = \nu_t^{(j)}(k_i^{(j)} - x_t^{(j)})\bar{\epsilon}_{t,i}^{(j)}O_{t,T}^{(j)}(k_i^{(j)})$ for $j = 0,1..,N$, where for k in a neighborhood of zero, we have $|\nu_{t,i}^{(j)}(k) - \nu_{t,i}^{(j)}(0)| \leq C_t|k|^\iota$, for some $\iota > 0$; $\{\bar{\epsilon}_{t,i}^{(j)}\}_{i=1}^{S_j}$, for $j = 0,1..,N$, are i.i.d. sequences defined on an extension of $\mathcal{F}^{(0)}$ and independent of it and from each other. We further have $\mathbb{E}(\bar{\epsilon}_{t,i}^{(j)}|\mathcal{F}^{(0)}) = 0$, $\mathbb{E}((\bar{\epsilon}_{t,i}^{(j)})^2|\mathcal{F}^{(0)}) = 1$ and $\mathbb{E}(|\bar{\epsilon}_{t,i}^{(j)}|^\kappa|\mathcal{F}^{(0)}) < \infty$, for some $\kappa \geq 4$ and $j = 0,1,...,N$.

For stating our last assumption, we need to introduce some notation. First, we denote

$$\psi(x) = e^{|x|}, \quad x \in \mathbb{R}. \tag{63}$$

Next, for a generic measure ν, we set the tail integrals

$$\bar{\nu}(x) = \begin{cases} \int_x^\infty \nu(x)dx, & \text{for } x > 0, \\ \int_{-\infty}^x \nu(x)dx, & \text{for } x < 0. \end{cases} \tag{64}$$

We further set

$$\nu_\psi^+(x) = \frac{\nu(\log(x))}{x}, \quad \nu_\psi^-(x) = \frac{\nu(-\log(x))}{x}, \quad \text{for } x > 1. \tag{65}$$

A7. The jump intensities have the following decomposition $\nu_{t,j}^{\mathbb{Q}}(x) = a_{t,j}\nu_j(x)$ for processes $a_{t,j}$ with càglàd paths and the functions $\nu_j(x)$ satisfy the following two conditions:

1. The functions $\bar{\nu}_{j,\psi}^\pm(x)$ are regularly varying at infinity, that is, $\bar{\nu}_{j,\psi}^\pm(x) = x^{-\alpha_j^\pm}L^\pm(x)$, where $\alpha_j^\pm > 1$ and $L_j^\pm(x)$ are slowly varying at infinity, i.e., $\lim_{x\to\infty}\frac{L_j^\pm(tx)}{L_j^\pm(x)} = 1$ for every $t > 0$.

2. The functions $L_j^\pm(x)$ satisfy the condition $\frac{L_j^\pm(tx)}{L_j^\pm(x)} - 1 \sim h(t)g(x)$ as $x \to \infty$, where $g_j^\pm(x) \to 0$ as $x \to \infty$ and $g_j^\pm(x)$ are regularly varying at infinity with index $\rho_j^\pm < 0$.

Finally, we have $\sup_{i=1,...,S_j} \log(1 + \nu_t^{(j)}(k_i^{(j)} - x_t^{(j)})\bar{\epsilon}_{t,i}^{(j)}) = O_p(1)$.

8.2 Auxiliary Results

Lemma 1 Suppose assumptions A2-A4 hold. For x_t being one of $\{x_t^{(j)}\}_{j=0,...,N}$, $\nu_t^{\mathbb{Q}}$ being the corresponding $\{\nu_{t,j}^{\mathbb{Q}}\}_{j=0,...,N}$ and V_t the corresponding diffusive spot volatility, we have as $T \downarrow 0$:

$$\frac{1}{T}\mathbb{E}_t^{\mathbb{Q}}(x_{t+T} - x_t)^2 - V_t - \int_{\mathbb{R}} x^2\nu_t^{\mathbb{Q}}(x)dx = o_p(1), \tag{66}$$

$$\mathbb{E}_t^{\mathbb{Q}}(f(x_{t+T} - x_t)) - T\int_{\mathbb{R}} f(x)\nu_t^{\mathbb{Q}}(x)dx = O_p(T^{3/2}). \tag{67}$$

31

Proof of Lemma 1. We denote with $\widetilde{x}_s$, for $s \geq t$, the same process as x_t but in which the drift coefficient, the diffusive volatility and the jump intensity are frozen at their values at time t. This process can be defined on an extension of the original probability space by making use of Grigelionis representation for jump processes (Theorem 2.1.2 of Jacod and Protter (2012)). The process $\widetilde{x}_s$ is $\mathcal{F}_t^{(0)}$-conditionally a Lévy process.

Using first-order Taylor expansion, we have

$$|f(x_{t+T} - x_t) - f(\widetilde{x}_{t+T})| \leq C|x_{t+T} - x_t - \widetilde{x}_{t+T}||\widetilde{x}_{t+T}| + C|x_{t+T} - x_t - \widetilde{x}_{t+T}|^2. \tag{68}$$

Furthermore, using Burkholder-Davis-Gundy inequalities, we have

$$\mathbb{E}_t^{\mathbb{Q}}|x_{t+T} - x_t - \widetilde{x}_{t+T}|^2 \leq C_t T^2, \quad \mathbb{E}_t^{\mathbb{Q}}|\widetilde{x}_{t+T}|^2 \leq C_t T. \tag{69}$$

From here, by application of Cauchy-Schwartz inequality, we have

$$\mathbb{E}_t^{\mathbb{Q}}|f(x_{t+T} - x_t) - f(\widetilde{x}_{t+T})| \leq C_t T^{3/2}. \tag{70}$$

$\mathcal{F}_t^{(0)}$-conditionally, the jumps of $\widetilde{x}_{t+T}$ are Poisson, therefore

$$\mathbb{Q}_t\left(\# \text{ jumps in } \widetilde{x}_{t+T} \geq 2\right) \leq C_t T^2. \tag{71}$$

From here, applying Cauchy-Schwartz inequality, we have

$$\left|\mathbb{E}_t^{\mathbb{Q}}(f(\widetilde{x}_{t+T})) - T \int_{\mathbb{R}} \mathbb{E}_t^{\mathbb{Q}}(f(x + \sigma_t\sqrt{T}z))\nu_t^{\mathbb{Q}}(x)dx\right| \leq C_t T^{3/2}, \tag{72}$$

where z is a standard normal variable. Finally, applying first-order Taylor series expansion, we have

$$\left|\int_{\mathbb{R}} \mathbb{E}_t^{\mathbb{Q}}(f(x + \sigma_t\sqrt{T}z))\nu_t^{\mathbb{Q}}(x)dx - \int_{\mathbb{R}} f(x)\nu_t^{\mathbb{Q}}(x)dx\right| \leq C_t\sqrt{T}. \tag{73}$$

$\square$

For stating the next lemmas we introduce the following notation:

$$O_{t,T}^{(c,j)}(k) = f\left(\frac{k - x_t^{(j)}}{\sqrt{T}\sigma_t^{(j)}}\right)\sqrt{T}\sigma_t^{(j)} - |k - x_t^{(j)}|\Phi\left(-\frac{|k - x_t^{(j)}|}{\sqrt{T}\sigma_t^{(j)}}\right), \quad j = 0, 1, ..., N, \tag{74}$$

$$\overline{O}(k) = f(k) - |k|\Phi\left(-|k|\right), \tag{75}$$

where f and Φ denote the pdf and cdf, respectively, of the standard normal distribution.

Lemma 2 *Assume A2-A4 hold. There exist $\mathcal{F}_t^{(0)}$-adapted random variables C_t and $\bar{t} > t$, that do not depend on T such that for $T < t + \bar{t}$ and $j = 0, 1, ..., N + 1$, we have*

$$O_{t,T}^{(j)}(k) \le C_t \begin{cases} Te^{3k}, & \text{if } k < -1, \\ Te^{-k}, & \text{if } k > 1, \\ \sqrt{T} \wedge \frac{T}{|k|}, & \text{if } |k| \le 1, \end{cases} \tag{76}$$

$$\left| O_{t,T}^{(j)}(k_1) - O_{t,T}^{(j)}(k_2) \right| \le C_t \left[\frac{T}{k_2^4} \wedge \frac{T}{k_2^2} \wedge 1 \right] |e^{k_1} - e^{k_2}|, \tag{77}$$

where $k_1 < k_2 < x_t^{(j)}$ or $k_1 > k_2 > x_t^{(j)}$. In addition, for $|k - x_t^{(j)}| \le \sqrt{T}|\log(T)|$, we have

$$\left| O_{t,T}^{(j)}(k) - O_{t,T}^{(c,j)}(k) \right| \le C_t T \log^2(T). \tag{78}$$

Proof of Lemma 2. These results follow by applying Lemmas 2-7 in Qin and Todorov (2018). $\square$

To state the following lemma, we need some auxiliary notation. We denote

$$O_{t,T}^{(d,j)}(k) = \begin{cases} \int_k^\infty (e^x - e^k)^+ \nu_{t,j}^{\mathbb{Q}}(x)dx, & \text{for } k \ge x_t^{(j)}, \\ \int_{-\infty}^k (e^k - e^x)^+ \nu_{t,j}^{\mathbb{Q}}(x)dx, & \text{for } k < x_t^{(j)}, \end{cases} \tag{79}$$

and using it, we set

$$\begin{aligned} \widetilde{TC}_t^{(+,j)}(\bar{k}) &= 2 \int_{\bar{k}}^\infty e^{-k}(1 - k + x_t^{(j)}) O_{t,T}^{(d,j)}(k)dk, \\ \widetilde{TC}_t^{(-,j)}(\underline{k}) &= 2 \int_\infty^{\underline{k}} e^{-k}(1 - k + x_t^{(j)}) O_{t,T}^{(d,j)}(k)dk. \end{aligned} \tag{80}$$

Lemma 3 *Suppose assumption A7 holds. We then have:*

$$\begin{aligned} O_{t,T}^{(d,j)}(\bar{k}) &= \frac{a_{t,j}}{\alpha_j^+ - 1} e^{(1 - \alpha_j^+)\bar{k}} + O_p\left(e^{(1 - \alpha_j^+ + \rho_j^+)\bar{k}} \right), & as \quad \bar{k} \to \infty, \\ O_{t,T}^{(d,j)}(\underline{k}) &= \frac{a_{t,j}}{\alpha_j^- + 1} e^{(1 + \alpha_j^-)\underline{k}} + O_p\left(e^{(1 + \alpha_j^- + \rho_j^-)\underline{k}} \right), & as \quad \underline{k} \to -\infty, \end{aligned} \tag{81}$$

and further

$$\begin{aligned} \widetilde{TC}_t^{(+,j)}(\bar{k}) &= \frac{2}{\alpha_j^+} \left(1 - \bar{k} + x_t^{(j)} - \frac{1}{\alpha_j^+} \right) e^{-\bar{k}} O_{t,T}^{(d,j)}(\bar{k}) + O_p\left(\bar{k} e^{(\rho_j^+ - \alpha_j^+)\bar{k}} \right), & as \quad \bar{k} \to \infty, \\ \widetilde{TC}_t^{(-,j)}(\underline{k}) &= \frac{2}{\alpha_j^-} \left(1 - \underline{k} + x_t^{(j)} + \frac{1}{\alpha_j^-} \right) e^{-\underline{k}} O_{t,T}^{(d,j)}(\underline{k}) + O_p\left(|\underline{k}| e^{(-\rho_j^- + \alpha_j^-)\underline{k}} \right), & as \quad \underline{k} \to -\infty. \end{aligned} \tag{82}$$

Proof of Lemma 3. In the proof we suppress the subscript j (the index of the asset) in the notation of various quantities that are indexed by it. We start with establishing the results for the right tail.

Using assumption A7, we can write

$$O_{t,T}^{(d)}(k) = a_t \int_k^\infty (e^x - e^k)\nu(x)dx, \quad k > x_t.$$

Then, by changing the variable of integration and using integration by parts (recall the notation of $\nu_\psi^\pm$ and $\overline{\nu}_\psi^\pm$ of Section 8.1):

$$
\begin{aligned}
O_{t,T}^{(d)}(k) &= a_t \int_{\psi(k)}^\infty (X - \psi(k))\nu(\log(X))d\log(X) \\
&= a_t \int_{\psi(k)}^\infty (X - \psi(k))\frac{\nu(\log(X))}{X}dX \\
&= a_t \int_{\psi(k)}^\infty \overline{\nu}_\psi^+(X)dX.
\end{aligned}
\tag{83}
$$

Using assumption A7 for the regular variation of $\overline{\nu}_\psi^+$, we can write:

$$
\begin{aligned}
O_{t,T}^{(d)}(k) &= a_t\psi(k)\overline{\nu}_\psi^+(\psi(k)) \int_1^\infty \frac{\overline{\nu}_\psi^+(u\psi(k))}{\overline{\nu}_\psi^+(\psi(k))}du \\
&= a_t\psi(k)\overline{\nu}_\psi^+(\psi(k)) \int_1^\infty u^{-\alpha^+}\frac{L^+(u\psi(k))}{L^+(\psi(k))}du \\
&= a_t\psi(k)\overline{\nu}_\psi^+(\psi(k)) \left(\int_1^\infty u^{-\alpha^+}du + g^+(\psi(k)) \int_1^\infty u^{-\alpha^+}h^+(u)du + o(g^+(\psi(k))) \right), \quad \text{as } k \to \infty,
\end{aligned}
$$

where the last step follows from Proposition 2.5.1 in Goldie and Smith (1987) and we note that $\int_1^\infty u^{-\alpha^+}h^+(u)du < \infty$ by results of Section 3.12.1 of Bingham et al. (1987) and our assumption $\alpha^+ > 1$. Thus,

$$
\begin{aligned}
O_{t,T}^{(d)}(k) &= a_t\psi(k)\overline{\nu}_\psi^+(\psi(k)) \left(\int_1^\infty u^{-\alpha^+}du + O(g^+(\psi(k))) + o(g^+(\psi(k))) \right) \\
&= \frac{a_t}{\alpha^+ - 1}\psi(k)\overline{\nu}_\psi^+(\psi(k)) + \psi(k)\overline{\nu}_\psi^+(\psi(k))O_p(g^+(\psi(k))), \quad \text{as} \quad k \to \infty.
\end{aligned}
\tag{84}
$$

Using equation (83) and performing integration by parts, we have:

$$
\begin{aligned}
\widetilde{TC}_t^+ &= 2a_t \int_{\overline{k}}^\infty e^{-k}(1 - k + x_t) \left(\int_{\psi(k)}^\infty \overline{\nu}_\psi^+(X)dX \right) dk \\
&= -2a_t \int_{\overline{k}}^\infty \left(\int_{\psi(k)}^\infty \overline{\nu}_\psi^+(X)dX \right) d\left[e^{-k}(-k + x_t) \right] \\
&= 2a_t \left(\int_{\psi(\overline{k})}^\infty \overline{\nu}_\psi^+(X)dX \right) \left[e^{-\overline{k}}(-\overline{k} + x_t) \right] + 2a_t \int_{\overline{k}}^\infty e^{-k}(-k + x_t)d\left(\int_{\psi(k)}^\infty \overline{\nu}_\psi^+(X)dX \right) \\
&= 2(-\overline{k} + x_t)e^{-\overline{k}}O_{t,T}^{(d)}(\overline{k}) - \underbrace{2a_t \int_{\overline{k}}^\infty (-k + x_t)\overline{\nu}_\psi^+(\psi(k))dk}_{A}.
\end{aligned}
$$

34

We can further decompose the term A as follows:

$$
A = 2a_t \overline{\nu}_\psi^+(\psi(\overline{k})) \int_{\overline{K}}^\infty \frac{(-\log(K) + x_t)}{K} \frac{\overline{\nu}_\psi^+(K)}{\overline{\nu}_\psi^+(\overline{K})} dK
$$

$$
= 2a_t \overline{\nu}_\psi^+(\psi(\overline{k})) \int_1^\infty \frac{(-\log(u\overline{K}) + x_t)}{u} \frac{\overline{\nu}_\psi^+(u\overline{K})}{\overline{\nu}_\psi^+(\overline{K})} du
$$

$$
= 2a_t \overline{\nu}_\psi^+(\psi(\overline{k})) \int_1^\infty \frac{(-\log(u\overline{K}) + x_t)}{u} u^{-\alpha^+} \frac{L^+(u\overline{K})}{L^+(\overline{K})} du
$$

$$
= 2a_t \overline{\nu}_\psi^+(\psi(\overline{k})) \underbrace{\int_1^\infty \frac{(-\log(u\overline{K}) + x_t)}{u} u^{-\alpha^+} du}_{B}
$$

$$
+ 2a_t \overline{\nu}_\psi^+(\psi(\overline{k})) \left(g^+(\psi(\overline{k})) \int_1^\infty \frac{(-\log(u\overline{K}) + x_t)}{u} u^{-\alpha^+} h^+(u) du + o_p(g^+(\overline{k}\psi(\overline{k}))) \right), \quad \text{as } \overline{k} \to \infty,
$$

where we denote $\overline{K} = e^{\overline{k}}$, and in the first and second lines we change the variable of integration, the third line follows from assumption A7, and for the last line we applied Proposition 2.5.1 in Goldie and Smith (1987) and made use of $\alpha^+ > 1$. For the term B, direct integration leads to

$$
B = \int_{\overline{k}}^\infty (-k + x_t) e^{-\alpha^+(k-\overline{k})} dk = \frac{1}{\alpha^+} \left(-\overline{k} + x_t - \frac{1}{\alpha^+} \right).
$$

Thus, for the term A we have

$$
A = \frac{2a_t}{\alpha^+} \left(-\overline{k} + x_t - \frac{1}{\alpha^+} \right) \overline{\nu}_\psi^+(\psi(\overline{k})) + \overline{\nu}_\psi^+(\psi(\overline{k})) O_p(\overline{k} g^+(\psi(\overline{k})))
$$

$$
= \frac{2(\alpha^+ - 1)}{\alpha^+} \left(-\overline{k} + x_t - \frac{1}{\alpha^+} \right) e^{-\overline{k}} O_{t,T}^{(d)}(\overline{k}) + \overline{\nu}_\psi^+(\psi(\overline{k})) O_p(\overline{k} g^+(\psi(\overline{k}))),
$$

where the first line uses assumption A7 and the second line uses equation (84). Altogether, we can write:

$$
\widetilde{TC}_t^+ = \frac{2}{\alpha^+} \left(1 - \overline{k} + x_t - \frac{1}{\alpha^+} \right) e^{-\overline{k}} O_{t,T}^{(d)}(\overline{k}) + \overline{\nu}_\psi^+(\psi(\overline{k})) O_p(\overline{k} g^+(\psi(\overline{k})))
$$

$$
= \frac{2}{\alpha^+} \left(1 - \overline{k} + x_t - \frac{1}{\alpha^+} \right) e^{-\overline{k}} O_{t,T}^{(d)}(\overline{k}) + O_p \left(\overline{k} e^{(\rho^+ - \alpha^+)\overline{k}} \right).
$$

(85)

The results for the left tail follow similar steps. In particular, for $k \to -\infty$, we have:

$$
O_{t,T}^{(d)}(k) = a_t \int_{-\infty}^k (e^k - e^x) \nu(x) dx
$$

$$
= a_t \int_{\psi(k)}^\infty (\psi(-k) - X^{-1}) \nu(-\log(X)) d\log(X)
$$

$$
= a_t \int_{\psi(k)}^\infty X^{-2} \overline{\nu}_\psi^-(X) dX
$$

$$
= a_t e^k \overline{\nu}_\psi^-(\psi(k)) \left(\int_1^\infty u^{-\alpha^- - 2} du + g^-(\psi(k)) \int_1^\infty u^{-\alpha^- - 2} h^-(u) du + o(g^-(\psi(k))) \right).
$$

35

Thus,
$$O_{t,T}^{(d)}(k) = \frac{a_t}{\alpha^- + 1} e^k \overline{\nu}_\psi^-(\psi(k)) + e^k \overline{\nu}_\psi^-(\psi(k)) O_p(g^-(\psi(k))), \quad k \to -\infty.$$

For the left tail integral, we have
$$\widetilde{TC}_t^- = 2 \int_{-\infty}^{\underline{k}} e^{-k}(1 - k + x_t) O_{t,T}^{(d)}(k) dk$$

$$= -2(-\underline{k} + x_t) e^{-\underline{k}} O_{t,T}^{(d)}(\underline{k}) + \underbrace{2a_t \int_{-\infty}^{\underline{k}} (-k + x_t) \overline{\nu}_\psi^-(\psi(k)) dk}_{A},$$

where using assumption A7,
$$A = 2a_t \overline{\nu}_\psi^-(\psi(\underline{k})) \underbrace{\int_1^\infty \frac{(\log(u\underline{K}) + x_t)}{u} u^{-\alpha^-} du}_{B}$$

$$+ 2a_t \overline{\nu}_\psi^-(\psi(\underline{k})) \left(g^-(\psi(\underline{k})) \int_1^\infty \frac{(\log(u\underline{K}) + x_t)}{u} u^{-\alpha^-} h^-(u) du + o_p(g^-(|\underline{k}|\psi(\underline{k}))) \right), \quad \text{as } \underline{k} \to -\infty,$$

and by direct integration
$$B = \frac{1}{\alpha^-} \left(-\underline{k} + x_t + \frac{1}{\alpha^-} \right).$$

Thus, altogether,
$$\widetilde{TC}_t^- = \frac{2}{\alpha^-} \left(1 - \underline{k} + x_t + \frac{1}{\alpha^-} \right) e^{-\underline{k}} O_{t,T}^{(d)}(\underline{k}) + O_p\left(|\underline{k}| e^{(-\rho^- + \alpha^-)\underline{k}} \right). \tag{86}$$

$\square$

Lemma 4 *Suppose assumptions A2-A7 hold, and that $T \to 0$ with $(|\underline{k}| \vee \overline{k}) \asymp T^{-\beta}$ for some $\beta > 0$. Then, for $j = 0, 1, ..., N$, we have*
$$\int_{-\infty}^{k_1^{(j)}} \overline{h}(k, x_t^{(j)}) O_{t,T}^{(j)}(k) dk - \int_{k_{S_j}^{(j)}}^\infty \overline{h}(k, x_t^{(j)}) O_{t,T}^{(j)}(k) dk$$
$$= \widetilde{TC}_t^{(-,j)}(k_1^{(j)}) - \widetilde{TC}_t^{(+,j)}(k_{S_j}^{(j)}) + O_p(T^{3/2} |\log T|^2), \tag{87}$$

where we denote
$$\overline{h}(k, x) = 2e^{-k}(1 - k + x). \tag{88}$$

Proof of Lemma 4. We make use of the notation of $\widetilde{x}^{(j)}$ in the proof of Lemma 1 and we further denote the option price corresponding to $\widetilde{x}_{t+T}^{(j)}$ with $\widetilde{O}_{t,T}^{(j)}(k)$. Then, we have
$$\int_{-\infty}^{k_1^{(j)}} \overline{h}(k, x_t^{(j)})(O_{t,T}^{(j)}(k) - \widetilde{O}_{t,T}^{(j)}(k)) dk - \int_{k_{S_j}^{(j)}}^\infty \overline{h}(k, x_t^{(j)})(O_{t,T}^{(j)}(k) - \widetilde{O}_{t,T}^{(j)}(k)) dk$$
$$= \mathbb{E}_t^{\mathbb{Q}}(f(x_{t+T}^{(j)} - x_t^{(j)}) - f(\widetilde{x}_{t+T}^{(j)} - x_t^{(j)})) - \int_{k_1^{(j)}}^{k_{S_j}^{(j)}} \overline{h}(k, x_t^{(j)})(O_{t,T}^{(j)}(k) - \widetilde{O}_{t,T}^{(j)}(k)) dk \tag{89}$$
$$= O_p(T^{3/2} |\log T|^2),$$

where the last claim follows by making use of Lemma 4 and Lemmas 3 and 4 of Qin and Todorov (2018). Further, by making use of the fact that $\mathcal{F}_t^{(0)}$-conditionally the jumps in $\widetilde{x}_s^{(j)}$ are Poisson (so that $\mathbb{Q}_t\,(\#$ jumps in $\widetilde{x}_{t+T} \geq 2) \leq C_t T^2$) and by applying Cauchy-Shwarz inequality and the integrability conditions of assumption A2, we have

$$
\begin{aligned}
\left| \widetilde{O}_{t,T}^{(j)}(k) - T \int_{\mathbb{R}} \mathbb{E}_t^{\mathbb{Q}}(e^{x+\sigma_t^{(j)}\sqrt{T}z} - e^k)^+ \nu_{t,j}^{\mathbb{Q}}(x)dx \right| &\leq C_t e^{-k} T^{3/2}, \quad k \geq k_{S_j}^{(j)}, \\
\left| \widetilde{O}_{t,T}^{(j)}(k) - T \int_{\mathbb{R}} \mathbb{E}_t^{\mathbb{Q}}(e^k - e^{x+\sigma_t^{(j)}\sqrt{T}z})^+ \nu_{t,j}^{\mathbb{Q}}(x)dx \right| &\leq C_t e^{3k} T^{3/2}, \quad k \leq k_1^{(j)},
\end{aligned}
\tag{90}
$$

where z is a standard normal random variable (under $\mathbb{Q}$). From here, using again the integrability conditions of assumption A2, we have

$$
\begin{aligned}
\left| \int_{\mathbb{R}} \mathbb{E}_t^{\mathbb{Q}}(e^{x+\sigma_t^{(j)}\sqrt{T}z} - e^k)^+ \nu_{t,j}^{\mathbb{Q}}(x)dx - O_{t,T}^{(d,j)}(k) \right| &\leq C_t e^{-k} T^{1/2}, \quad k \geq k_{S_j}^{(j)}, \\
\left| \int_{\mathbb{R}} \mathbb{E}_t^{\mathbb{Q}}(e^k - e^{x+\sigma_t^{(j)}\sqrt{T}z})^+ \nu_{t,j}^{\mathbb{Q}}(x)dx - O_{t,T}^{(d,j)}(k) \right| &\leq C_t e^{3k} T^{1/2}, \quad k \leq k_1^{(j)}.
\end{aligned}
\tag{91}
$$

Combining the above inequalities, we get the result of the lemma. $\qquad\square$

In the next lemma and henceforth we will use the following weights

$$
w_{t,i}^{(j)} =
\begin{cases}
\frac{1}{2} h(k_1^{(j)}, k_2^{(j)}, x_t^{(j)}), & \text{if } i = 1, \\
\frac{1}{2} h(k_{S_j-1}^{(j)}, k_{S_j}^{(j)}, x_t^{(j)}), & \text{if } i = S_j, \\
\frac{1}{2}\left(h(k_{i-1}^{(j)}, k_i^{(j)}, x_t^{(j)}) + h(k_i^{(j)}, k_{i+1}^{(j)}, x_t^{(j)}) \right), & \text{if } i = 2,...,S_j - 1 \setminus \{i^*(j), i^*(j)+1\}, \\
\frac{1}{2} h(k_{i^*(j)+1}^{(j)}, k_{i^*(j)+2}^{(j)}, x_t^{(j)}) + h(x_t^{(j)}, k_{i^*(j)+1}^{(j)}, x_t^{(j)}), & \text{if } i = i^*(j)+1, \\
\frac{1}{2} h(k_{i^*(j)-1}^{(j)}, k_{i^*(j)}^{(j)}, x_t^{(j)}) + h(k_{i^*(j)}^{(j)}, x_t^{(j)}, x_t^{(j)}), & \text{if } i = i^*(j).
\end{cases}
\tag{92}
$$

Lemma 5 *Suppose assumptions A2-A7 hold. Let $T \to 0$, $\Delta \to 0$, $(|\underline{k}| \vee \overline{k}) \to \infty$ with $\Delta \asymp T^\alpha$, for $\alpha \in \left(\frac{1}{2}, \frac{3}{2}\right)$ and $(|\underline{k}| \vee \overline{k}) \asymp T^{-\beta}$ for some β satisfying $\min_{j=0,1,...,N}(\alpha_j^\pm)\beta > \frac{\alpha}{2} - \frac{1}{4}$. Then, we have*

$$
\frac{T^{1/4}}{\sqrt{\Delta}} \left\{ \frac{1}{T} AV_t^{(j)} - \frac{1}{T} \mathbb{E}_t^{\mathbb{Q}}(f(x_{t+T}^{(j)} - x_t^{(j)})) \right\}_{j=0,1,...,N} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \left\{ \sqrt{\omega_{AV,t}^{(j)}} Z_t^{(j)} \right\}_{j=0,1,...,N},
\tag{93}
$$

where $\{Z_t^{(j)}\}_{j=0,1,...,N}$ is a sequence of i.i.d., standard normal random variables defined on an extension of the original probability space and independent of $\mathcal{F}$ and

$$
\omega_{AV,t}^{(j)} = 4 e^{-2x_t^{(j)}} \psi_t^{(j)}(0) \nu_t^{(j)}(0) (\sigma_t^{(j)})^3 \int_{\mathbb{R}} \overline{O}(k)^2 dk, \quad j = 0, 1, ..., N,
\tag{94}
$$

with $\overline{O}(k)$ defined in (75).

Proof of Lemma 5. We note that, following results in Carr and Madan (2001), we have

$$\mathbb{E}_t^{\mathbb{Q}}(f(x_{t+T}^{(j)} - x_t^{(j)})) = \int_{-\infty}^{\infty} \overline{h}(k, x_t^{(j)}) O_{t,T}^{(j)}(k) dk. \tag{95}$$

Using this result, we make the following decomposition

$$AV_t^{(j)} - \mathbb{E}_t^{\mathbb{Q}}(f(x_{t+T}^{(j)} - x_t^{(j)})) = \sum_{i=1}^{3} \eta_{t,i}^{(j)}, \tag{96}$$

$$\eta_{t,1}^{(j)} = \int_{-\infty}^{k_1^{(j)}} \overline{h}(k, x_t^{(j)}) O_{t,T}^{(j)}(k) dk - \int_{k_{S_j}^{(j)}}^{\infty} \overline{h}(k, x_t^{(j)}) O_{t,T}^{(j)}(k) dk - TC_t^{(-,j)} + TC_t^{(+,j)}, \tag{97}$$

$$\eta_{t,2}^{(j)} = \sum_{i=2}^{i^*(j)} h(k_{i-1}^{(j)}, k_i^{(j)}, x_t^{(j)}) O_{t,T}^{(a,j)}(k_{i-1}^{(j)}) + h(k_{i^*(j)}^{(j)}, x_t^{(j)}, x_t^{(j)}) O_{t,T}^{(j)}(k_{i^*(j)}^{(j)})$$

$$- \sum_{i=i^*(j)+2}^{S_j} h(k_{i-1}^{(j)}, k_i^{(j)}, x_t^{(j)}) O_{t,T}^{(a,j)}(k_{i-1}^{(j)}) - h(x_t^{(j)}, k_{i^*(j)+1}^{(j)}, x_t^{(j)}) O_{t,T}^{(j)}(k_{i^*(j)+1}^{(j)}) \tag{98}$$

$$- \int_{k_1^{(j)}}^{x_t^{(j)}} \overline{h}(k, x_t^{(j)}) O_{t,T}^{(j)}(k) dk + \int_{x_t^{(j)}}^{k_{S_j}^{(j)}} \overline{h}(k, x_t^{(j)}) O_{t,T}^{(j)}(k) dk,$$

$$\eta_{t,3}^{(j)} = \sum_{i=1}^{S_j} w_{t,i}^{(j)} \epsilon_{t,i}^{(j)}, \tag{99}$$

where $O_{t,T}^{(a,j)}(k_i^{(j)})$ is defined from $O_{t,T}^{(a,j)}(k_i^{(j)})$ exactly as $\widehat{O}_{t,T}^{(a,j)}(k_i^{(j)})$ is defined from $\widehat{O}_{t,T}^{(a,j)}(k_i^{(j)})$ and recall the definition of $w_{t,i}^{(j)}$ in (92).

We start with $\eta_{t,1}^{(j)}$. First, by making use of Lemma 4, we have

$$\eta_{t,1}^{(j)} = \widetilde{TC}_t^{(-,j)} - \widetilde{TC}_t^{(+,j)} - TC_t^{(-,j)} + TC_t^{(+,j)} + O_p(T^{3/2}|\log T|^2). \tag{100}$$

Next, tedious calculations and making use of the bounds in Lemmas 1-3 in Qin and Todorov (2018) as well as the proof of Lemma 4, lead to

$$|O_{t,T}^{(j)}(k) - O_{t,T}^{(d,j)}(k)| \leq C \begin{cases} e^{-k} T^{3/2}|\log T|, & \text{for } k \geq k_{S_j}^{(j)}, \\ e^{3k} T^{3/2}|\log T|, & \text{for } k \leq k_1^{(j)}. \end{cases} \tag{101}$$

Taking into account this bound and assumptions A6-A7 as well as Lemma 3, we have

$$\widetilde{TC}_t^{(-,j)} - \widetilde{TC}_t^{(+,j)} - TC_t^{(-,j)} + TC_t^{(+,j)} = O_p\left(T\overline{k}e^{-\alpha_j^+\overline{k}} \bigvee T|\underline{k}|e^{-\alpha_j^-|\underline{k}|} \bigvee T^{3/2}|\log T|\right). \tag{102}$$

Thus, altogether

$$\eta_{t,1}^{(j)} = O_p\left(T\overline{k}e^{-\alpha_j^+\overline{k}} \bigvee T|\underline{k}|e^{-\alpha_j^-|\underline{k}|} \bigvee T^{3/2}|\log T|^2\right). \tag{103}$$

38

For $\eta_{t,2}^{(j)}$, we can use the second bound of Lemma 2, and get

$$\eta_{t,2}^{(j)} = O_p(\sqrt{T}\Delta), \quad j = 0, 1, ..., N. \tag{104}$$

We turn to $\eta_{t,3}^{(j)}$. Since $\mathcal{F}^{(0)}$-conditionally, $\{\epsilon_{t,i}^{(j)}\}_{i=1,..,S_j}$ are independent, we will have

$$\frac{1}{T^{3/4}\sqrt{\Delta}}\{\eta_{t,3}^{(j)}\}_{j=0,1,...,N} \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \left\{\sqrt{\omega_{AV,t}^{(j)}}Z_t^{(j)}\right\}_{j=0,1,...,N}, \tag{105}$$

by application of Theorem VIII.5.25 and Remark VIII.5.5.27 of Jacod and Shiryaev (2003), provided we can show the following

$$\frac{1}{T^{3/2}\Delta}\mathbb{E}\left((\eta_{t,3}^{(j)})^2|\mathcal{F}_t^{(0)}\right) \longrightarrow \omega_{AV,t}^{(j)}, \quad \mathcal{F}^{(0)} - \text{a.s.}, \tag{106}$$

$$\frac{1}{T^{(1+\iota/2)3/2}\Delta^{1+\iota/2}}\sum_{i=1}^{S_j}\mathbb{E}\left(|w_{t,i}^{(j)}\epsilon_{t,i}^{(j)}|^{2+\iota}|\mathcal{F}^{(0)}\right) \longrightarrow 0, \quad \mathcal{F}^{(0)} - \text{a.s.}, \tag{107}$$

for some $\iota > 0$. We start with the first of the above convergence results. Application of the first and the third bound of Lemma 2 yields

$$\mathbb{E}\left((\eta_{t,3}^{(j)})^2|\mathcal{F}_t^{(0)}\right) = \sum_{i=1}^{S_j}(w_{t,i}^{(j)})^2 O_{t,T}^{(j)}(k_i^{(j)})^2 v_{t,i}^{(j)}$$

$$= \sum_{i=1,...,S_j:|k_i^{(j)}-x_t^{(j)}|\leq\sqrt{T}|\log T|}(w_{t,i}^{(j)})^2 O_{t,T}^{(j)}(k_i^{(j)})^2 v_{t,i}^{(j)} + O_p\left(\frac{T^{3/2}}{|\log T|}\Delta\right) \tag{108}$$

$$= \sum_{i=1,...,S_j:|k_i^{(j)}-x_t^{(j)}|\leq\sqrt{T}|\log T|}(w_{t,i}^{(j)})^2 O_{t,T}^{(c,j)}(k_i^{(j)})^2 v_{t,i}^{(j)} + O_p\left(\frac{T^{3/2}}{|\log T|}\Delta\right).$$

Further,

$$\mathbb{E}\left((\eta_{t,3}^{(j)})^2|\mathcal{F}_t^{(0)}\right) = \Delta\int_{x_t^{(j)}-\sqrt{T}|\log T|}^{x_t^{(j)}+\sqrt{T}|\log T|}\overline{h}(k, x_t^{(j)})^2 O_{t,T}^{(c,j)}(k)^2\psi_t^{(j)}(k-x_t^{(j)})\nu_t^{(j)}(k-x_t^{(j)})dk$$

$$+ O_p\left(\frac{T^{3/2}}{|\log T|}\Delta \vee T|\log T|\Delta^2\right), \tag{109}$$

$$\mathbb{E}\left((\eta_{t,3}^{(j)})^2|\mathcal{F}_t^{(0)}\right) = 4e^{-2x_t^{(j)}}\psi_t^{(j)}(0)\nu_t^{(j)}(0)\Delta\int_{x_t^{(j)}-\sqrt{T}|\log T|}^{x_t^{(j)}+\sqrt{T}|\log T|}O_{t,T}^{(c,j)}(k)^2 dk + o_p(T^{3/2}\Delta). \tag{110}$$

From here, by change of variable of integration

$$\mathbb{E}\left((\eta_{t,3}^{(j)})^2|\mathcal{F}_t^{(0)}\right) = 4e^{-2x_t^{(j)}}\psi_t^{(j)}(0)\nu_t^{(j)}(0)(\sigma_t^{(j)})^3 T^{3/2}\Delta\int_{-|\log T|/\sigma_t^{(j)}}^{|\log T|/\sigma_t^{(j)}}O(k)^2 dk + o_p(T^{3/2}\Delta)$$

$$= T^{3/2}\Delta\omega_{AV,t}^{(j)} + o_p(T^{3/2}\Delta). \tag{111}$$

39

From here, the first of the two convergence results to be shown follows. For the second one, we can make use of the first bound of Lemma 2 to get

$$\frac{1}{T^{(1+\iota/2)3/2}\Delta^{1+\iota/2}} \sum_{i=1}^{S_j} \mathbb{E}\left(|w_{t,i}^{(j)}\epsilon_{t,i}^{(j)}|^{2+\iota}|\mathcal{F}^{(0)}\right) \leq C_t\Delta^{\iota/2}. \tag{112}$$

Combining (111) and (112), the convergence in (105) follows. This convergence combined with the bounds in (103) and (104) yields the convergence result in (93). □

We define the $N + 1 \times N + 1$ matrix

$$\mathbf{\Sigma}_t = \left((\sigma_t^{(0)})^\top, ..., (\sigma_t^{(N)})^\top\right)^\top. \tag{113}$$

The following lemma follows from Theorem 2.12 of Jacod (2008).

Lemma 6 *Suppose assumption A1 holds and let $\nu_n^{(0)}$ be univariate sequence satisfying $\nu_n^{(0)} \asymp n^{-\varpi}$ for $\varpi \in (0, 1/2)$. Then as $n \to \infty$, we have*

$$\sqrt{n} \begin{pmatrix} \sum_{i \in I_t^n}(\Delta_i^n x^{(0)})^2 1_{\{|\Delta_i^n x^{(0)}|>\nu_n^{(0)}\}} - \sum_{s \in [t-\kappa,t]}(\Delta x_s^{(0)})^2 \\ \sum_{i \in I_t^n} \Delta_i^n x^{(1)}\Delta_i^n x^{(0)} 1_{\{|\Delta_i^n x^{(0)}|>\nu_n^{(0)}\}} - \sum_{s \in [t-\kappa,t]} \Delta x_s^{(0)}\Delta x_s^{(1)} \\ \vdots \\ \sum_{i \in I_t^n} \Delta_i^n x^{(N)}\Delta_i^n x^{(0)} 1_{\{|\Delta_i^n x^{(0)}|>\nu_n^{(0)}\}} - \sum_{s \in [t-\kappa,t]} \Delta x_s^{(0)}\Delta x_s^{(N+1)} \end{pmatrix}$$

$$\xrightarrow{\mathcal{L}-s} \sum_{s \in [t-\kappa,t]} \begin{pmatrix} 2\Delta x_s^{(0)}\iota_1 \mathbf{\Sigma}_s \mathbf{Z}_s \\ (\Delta x_s^{(0)}\iota_2 + \Delta x_s^{(1)}\iota_1)\mathbf{\Sigma}_s \mathbf{Z}_s \\ \vdots \\ (\Delta x_s^{(0)}\iota_{N+1} + \Delta x_s^{(N)}\iota_1)\mathbf{\Sigma}_s \mathbf{Z}_s \end{pmatrix}, \tag{114}$$

where $\{\mathbf{Z}_s\}_{s \in [t-\kappa,t]}$ are standard normal $N + 1$ vectors defined on an extension of the original probability space and independent from $\mathcal{F}$.

A consistent estimate of the $\mathcal{F}$-conditional variance of the limiting distribution, Σ_{hf} is given by

$$\widehat{\Sigma}_{hf} = \sum_{i \in I_t^n:|\Delta_i^n x^{(0)}|>\nu_n^{(0)}} \begin{pmatrix} 2\Delta_i^n x^{(0)}\iota_1 \widehat{\mathbf{\Sigma}}_{i/n} \\ (\Delta_i^n x^{(0)}\iota_2 + \Delta_i^n x^{(1)}\iota_1)\widehat{\mathbf{\Sigma}}_{i/n} \\ \vdots \\ (\Delta_i^n x^{(0)}\iota_{N+1} + \Delta_i^n x^{(N)}\iota_1)\widehat{\mathbf{\Sigma}}_{i/n} \end{pmatrix} \begin{pmatrix} 2\Delta_i^n x^{(0)}\iota_1 \widehat{\mathbf{\Sigma}}_{i/n} \\ (\Delta_i^n x^{(0)}\iota_2 + \Delta_i^n x^{(1)}\iota_1)\widehat{\mathbf{\Sigma}}_{i/n} \\ \vdots \\ (\Delta_i^n x^{(0)}\iota_{N+1} + \Delta_i^n x^{(N)}\iota_1)\widehat{\mathbf{\Sigma}}_{i/n} \end{pmatrix}', \tag{115}$$

where $\widehat{\mathbf{\Sigma}}_{i/n}$ is the square-root of the following matrix

$$\frac{n}{k_n} \sum_{j=1}^{k_n} \Delta_{i-j}^n \mathbf{x}\Delta_{i-j}^n \mathbf{x}' 1_{\{|\Delta_{i-j}^n \mathbf{x}|\leq\nu_n\}}, \tag{116}$$

and ν_n is a $N+1$ vector satisfying $\nu_n \asymp n^{-\varpi}$ for $\varpi \in (0, 1/2)$.

8.3 Notation and Tuning Parameters for Theorems 2-4

For $\Omega_{\widetilde{AV},t}$, we need to define $\Omega_{AV,t}$ and $\Omega_{\beta,t}$. The first of them is defined as

$$\Omega_{AV,t} = \text{diag}\left(\omega_{AV,t}^{(0)}, \omega_{AV,t}^{(1)}, \cdots, \omega_{AV,t}^{(N)}\right). \tag{117}$$

$\Omega_{\beta,t}$ is defined as

$$\Omega_{\beta,t} = \frac{1}{\left(\sum_{s\in[t-\kappa,t]}(\Delta x_s^{(0)})^2\right)^2} \left[\iota_{2:N+1} - (\beta_t \quad \mathbf{0}_{N\times N})\right] \Sigma_{hf} \left[\iota_{2:N+1} - (\beta_t \quad \mathbf{0}_{N\times N})\right]'. \tag{118}$$

For the feasible implementation of the limit result, we construct the counterparts of $\Omega_{AV,t}$ and $\Omega_{\beta,t}$ from the data as follows. First, the tail decay parameters $\widehat{\alpha}_t^{(\pm,j)}$ are computed using a local window of $\tau = 5$ (recall equation (19)). Next, we define

$$\widehat{\epsilon}_{t,i}^{(j)} = \sqrt{\frac{2}{3}} \left[\widehat{O}_{t,T}^{(j)}(k_i^{(j)}) - \frac{1}{2}\left(\widehat{O}_{t,T}^{(j)}(k_{i-1}^{(j)}) + \widehat{O}_{t,T}^{(j)}(k_{i+1}^{(j)})\right)\right], \quad i = 2, ..., S_j - 1, \quad j = 0, 1, ..., N, \tag{119}$$

and $\widehat{\epsilon}_{t,S_j}^{(j)} = \widehat{\epsilon}_{t,S_j-1}^{(j)}$ as well as $\widehat{\epsilon}_{t,1}^{(j)} = \widehat{\epsilon}_{t,2}^{(j)}$, for $j = 0, 1, ..., N$. We then set

$$\widehat{\Omega}_{AV,t} = \text{diag}\left(\widehat{\omega}_{AV,t}^{(0)}, \widehat{\omega}_{AV,t}^{(1)}, \cdots, \widehat{\omega}_{AV,t}^{(N)}\right), \tag{120}$$

where

$$\widehat{\omega}_{AV,t}^{(j)} = \frac{1}{T^{3/2}\Delta} \sum_{i=1}^{S_j} (\omega_{t,i}^{(j)})^2(\widehat{\epsilon}_{t,i}^{(j)})^2, \quad j = 0, 1, ..., N, \tag{121}$$

with $\omega_{t,i}^{(j)}$ defined in (92).

Second, the feasible counterpart of $\Omega_{\beta,t}$ is given by

$$\widehat{\Omega}_{\beta,t} = \frac{1}{\left(\sum_{i\in I_t^n}(\Delta_i^n x_s^{(0)})^2\right)^2} \left[\iota_{2:N+1} - (\beta_t \quad \mathbf{0}_{N\times N})\right] \widehat{\Sigma}_{hf} \left[\iota_{2:N+1} - (\beta_t \quad \mathbf{0}_{N\times N})\right]'. \tag{122}$$

In implementing $\widehat{\Omega}_{\beta,t}$, we set $\nu_n = \alpha n^{-0.49}$ where the elements of the vector α are 3 times daily bipower variation (Barndorff-Nielsen and Shephard (2004)) of the corresponding element of $\mathbf{x}$. Further, the intraday high-frequency interval is of length five-minutes. The length of the window for estimating the high-frequency beta is $\kappa = 66$ (3 months). The local window for estimating the diffusive volatility around a jump time, used in the construction of $\widehat{\Sigma}_{hf}$, is set to $k_n = 30$.

8.4 Proof of Theorems 1 and 2

Theorem 1 is an easy consequence of Lemmas 1, 2 and 6 as well as assumption A6 for the option error. Theorem 2 follows by combining Lemmas 1, 2, 5 and 6.

8.5 Proof of Theorem 3

The convergence results of Lemmas 5 and 6 can be concisely written as

$$\widehat{\Upsilon}_1 \xrightarrow{\mathcal{L}|\mathcal{F}^{(0)}} \Upsilon_1, \quad \widehat{\Upsilon}_2 \xrightarrow{\mathcal{L}-s} \Upsilon_2, \tag{123}$$

with the first one being for $T \to 0$, $\Delta \to 0$ and $(|\underline{k}| \vee \overline{k}) \to \infty$, and the second being for $n \to \infty$. We will show that the above convergence holds jointly, i.e., we will show that

$$(\widehat{\Upsilon}_1, \widehat{\Upsilon}_2) \xrightarrow{\mathcal{L}-s} (\Upsilon_1, \Upsilon_2), \tag{124}$$

where Υ_1 and Υ_2 are $\mathcal{F}$-conditionally independent (the $\mathcal{F}$-conditional marginal laws of Υ_1 and Υ_2 are given in Lemmas 5 and 6 and this together with their $\mathcal{F}$-conditionally independence uniquely determines the law of (Υ_1, Υ_2)). To establish the above result, we need to show the following

$$\mathbb{E}\left(g_1(\widehat{\Upsilon}_1)g_2(\widehat{\Upsilon}_2)Y\right) \to \mathbb{E}\left(g_1(\Upsilon_1)g_2(\Upsilon_2)Y\right), \tag{125}$$

as $T \to 0$, $\Delta \to 0$, $(|\underline{k}| \vee \overline{k}) \to \infty$ and $n \to \infty$, and where Y is $\mathcal{F}$-adapted random variable and $g_i : \mathbb{R}^{N+1} \to \mathbb{R}$, for $i = 1, 2$, are continuous and bounded functions.

First, we consider the case when Y is $\mathcal{F}^{(0)}$-adapted. In this case from Lemma 5, we have

$$\mathbb{E}\left(g_1(\widehat{\Upsilon}_1)|\mathcal{F}^{(0)}\right) \xrightarrow{\mathbb{P}} \mathbb{E}\left(g_1(\Upsilon_1)|\mathcal{F}^{(0)}\right). \tag{126}$$

Therefore, since Y is bounded and g_2 is bounded, we have

$$\mathbb{E}\left((g_1(\widehat{\Upsilon}_1) - g_1(\Upsilon_1))g_2(\widehat{\Upsilon}_2)Y\right) \to 0. \tag{127}$$

Moreover, since $g_1(\Upsilon_1)$ is $\mathcal{F}$-adapted, we have

$$\mathbb{E}\left(g_1(\Upsilon_1)g_2(\widehat{\Upsilon}_2)Y\right) \to \mathbb{E}\left(g_1(\Upsilon_1)g_2(\Upsilon_2)Y\right). \tag{128}$$

This two results show the joint convergence in the case when Y is $\mathcal{F}^{(0)}$-adapted. Suppose now that Y is $\mathcal{F}^{(1)}$-adapted. Since $\mathcal{F}^{(1)}$ is the product sigma algebra of the option observation errors, we can assume that Y depends on a fixed number of them. We now note that the limit result of Lemma 5 will continue to hold even if a fixed number of options is excluded from the construction of the statistics $\{AV_t^{(j)}\}_{j=0,1,\ldots,N}$. Then the above two convergence results will continue to hold for this modification of $\widehat{\Upsilon}_1$ while its remainder term is asymptotically negligible. This shows the joint convergence in the case when Y is $\mathcal{F}^{(1)}$-adapted.

From here the joint stable convergence result in (124) holds. The convergence result of the theorem then follows from the convergence in probability of $\{AV_t^{(j)}\}_{j=0,1,\ldots,N}$ and $\widehat{\beta}_t$, and by making use of the following property of stable convergence: if $X_n \xrightarrow{\mathcal{L}-s} X$ and $Y_n \xrightarrow{\mathbb{P}} Y$ for some random variables X and Y, then $X_n Y_n \xrightarrow{\mathcal{L}-s} XY$.

8.6 Proof of Theorem 4

Given the consistency of $\widehat{\Sigma}_{hf}$, we only need to show the convergence in probability of $\widehat{\omega}_{AV,t}^{(j)}$ to $\omega_{AV,t}^{(j)}$. Given the proof of Lemma 5, we need to show

$$\frac{1}{T^{3/2}\Delta}\sum_{i=1}^{S_j}(\omega_{t,i}^{(j)})^2[(\widehat{\epsilon}_{t,i}^{(j)})^2 - O_{t,T}^{(j)}(k_i^{(j)})^2 v_{t,i}^{(j)}] \xrightarrow{\mathbb{P}} 0, \quad j = 0, 1, ..., N. \tag{129}$$

Using the bounds of Lemma 2 as well as assumption A6 for the $\mathcal{F}^{(0)}$-conditional independence of the option observation errors, we have

$$\frac{1}{T^3\Delta^2}\mathbb{E}\left(\left(\sum_{i=1}^{S_j}(\omega_{t,i}^{(j)})^2[(\widehat{\epsilon}_{t,i}^{(j)})^2 - O_{t,T}^{(j)}(k_i^{(j)})^2 v_{t,i}^{(j)}]\right)^2 \bigg| \mathcal{F}^{(0)}\right) = O_p\left(\frac{\Delta}{\sqrt{T}}\right), \tag{130}$$

and this implies the above convergence in probability in (129) as $\Delta/\sqrt{T} \to 0$ by assumption.

8.7 Proof of Theorem 5

We first introduce some auxiliary notation that we will use throughout the proof. In particular, we denote

$$\mathcal{L}_{t,T}^{(j)}(u) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{iu(x_{t+T}^{(j)} - x_t^{(j)})}\right), \quad \mathcal{M}_{t,T}^{(j)} = \mathbb{E}_t^{\mathbb{Q}}(x_{t+T}^{(j)} - x_t^{(j)}), \tag{131}$$

$$\widetilde{\mathcal{L}}_{t,T}^{(j)}(u) = \exp\left(iuT\alpha_t^{(\mathbb{Q},j)} - \frac{Tu^2}{2}\sigma_t^2 + T\int_{\mathbb{R}}(e^{iux} - 1 - iux)\nu_{t,j}^{(\mathbb{Q})}(x)dx\right). \tag{132}$$

We start the proof with establishing a lemma that bounds the effect on our statistics from "freezing" the characteristics of the semimartingale at time t. This lemma is the counterpart of Lemma 1 for the semivariances.

Lemma 7 *Suppose assumptions A2-A4 hold. Then for $T \downarrow 0$ and $j = 0, 1, ..., N$, we have:*

$$\Re(\mathcal{L}_{t,T}^{(j)}(u)) - \Re(\widetilde{\mathcal{L}}_{t,T}^{(j)}(u)) = O_p(T), \quad \Im(\mathcal{L}_{t,T}^{(j)}(u)) - \Im(\widetilde{\mathcal{L}}_{t,T}^{(j)}(u)) = O_p(T^2), \tag{133}$$

$$\frac{1}{T}\mathcal{M}_{t,T}^{(j)} - \alpha_t^{(\mathbb{Q},j)} = O_p(T). \tag{134}$$

Proof of Lemma 7. The first of the bounds to be proved follows from Todorov (2018). The second one follows upon an application of Itô's lemma and then making use of assumption A2 and A4 as well as the first of the bounds of the current lemma. The last bound to be proved follows similarly. □

Since

$$\Re(\widetilde{\mathcal{L}}_{t,T}^{(j)}(u)) \xrightarrow{\mathbb{P}} 1, \quad \Im(\widetilde{\mathcal{L}}_{t,T}^{(j)}(u)) = O_p(T), \tag{135}$$

we have that for T sufficiently small

$$\text{Arg}(\widetilde{\mathcal{L}}_{t,T}^{(j)}(u)) = \arctan(\Im(\widetilde{\mathcal{L}}_{t,T}^{(j)}(u))/\Re(\widetilde{\mathcal{L}}_{t,T}^{(j)}(u))). \tag{136}$$

Using assumption A6 for the observation error as well as Lemma 2 as well as the relative growth conditions of the theorem, we have

$$|\widehat{\mathcal{L}}_{t,T}^{(j)}(u) - \mathcal{L}_{t,T}^{(j)}(u)| = O_p(T^{3/4}\sqrt{\Delta} \vee \Delta\sqrt{T} \vee Te^{-2(|\underline{k}|\wedge|\overline{k}|)}). \tag{137}$$

Combining this result with the result of Lemma 7, we have that on a set of probability approaching one, denoted with Ω, we have

$$-\iota \leq \Im(\widehat{\mathcal{L}}_{t,T}^{(j)}(u))/\Re(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) \leq \iota, \quad \Re(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) > 1 - \iota, \tag{138}$$

for some arbitrary small and fixed $\iota > 0$. On this set and for T sufficiently small, we have

$$\begin{aligned}
\text{Arg}(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) &= \arctan(\Im(\widehat{\mathcal{L}}_{t,T}^{(j)}(u))/\Re(\widehat{\mathcal{L}}_{t,T}^{(j)}(u))), \\
\text{Arg}(\mathcal{L}_{t,T}^{(j)}(u)) &= \arctan(\Im(\mathcal{L}_{t,T}^{(j)}(u))/\Re(\mathcal{L}_{t,T}^{(j)}(u))).
\end{aligned} \tag{139}$$

Therefore, using Taylor expansion, we have

$$\begin{aligned}
\Big|\text{Arg}&(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \text{Arg}(\mathcal{L}_{t,T}^{(j)}(u)) \\
&- (\Im(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \Im(\mathcal{L}_{t,T}^{(j)}(u)))\Re(\mathcal{L}_{t,T}^{(j)}(u)) + (\Re(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \Re(\mathcal{L}_{t,T}^{(j)}(u)))\Im(\mathcal{L}_{t,T}^{(j)}(u))\Big|1_{\{\Omega\}} \\
&\leq K|\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \mathcal{L}_{t,T}^{(j)}(u))|^2,
\end{aligned} \tag{140}$$

for some positive constant K. Therefore, we have

$$\begin{aligned}
\text{Arg}(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \text{Arg}(\mathcal{L}_{t,T}^{(j)}(u)) &= (\Im(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \Im(\mathcal{L}_{t,T}^{(j)}(u)))\frac{\Re(\mathcal{L}_{t,T}^{(j)}(u))}{|\mathcal{L}_{t,T}^{(j)}(u)|^2} \\
&- (\Re(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \Re(\mathcal{L}_{t,T}^{(j)}(u)))\frac{\Im(\mathcal{L}_{t,T}^{(j)}(u))}{|\mathcal{L}_{t,T}^{(j)}(u)|^2} + O_p(T^{3/2}\Delta \vee T\Delta^2 \vee T^2e^{-4(|\underline{k}|\wedge|\overline{k}|)}).
\end{aligned} \tag{141}$$

Taking into account the bounds of Lemma 7 and the order of probability of $|\widehat{\mathcal{L}}_{t,T}^{(j)}(u) - \mathcal{L}_{t,T}^{(j)}(u)|$ derived above, we further get

$$\text{Arg}(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \text{Arg}(\mathcal{L}_{t,T}^{(j)}(u)) = \Im(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \Im(\mathcal{L}_{t,T}^{(j)}(u)) + O_p\left(T^{7/4}\sqrt{\Delta} \vee T^{3/2}\Delta \vee T^2e^{-2(|\underline{k}|\wedge|\overline{k}|)}\right). \tag{142}$$

Therefore, using Taylor expansion and Lemma 2, we can write

$$\text{Arg}(\widehat{\mathcal{L}}_{t,T}^{(j)}(u)) - \text{Arg}(\mathcal{L}_{t,T}^{(j)}(u)) + u(\widehat{\mathcal{M}}_{t,T}^{(j)} - \mathcal{M}_{t,T}^{(j)})$$

$$= \sum_{l=2}^{S_j} \widetilde{h}(k_{l-1}^{(j)}, x_t^{(j)}, u)(\widehat{O}_{t,T}^{(a,j)}(k_{l-1}^{(j)}) - O_{t,T}^{(a,j)}(k_{l-1}^{(j)}))\Delta_l^{(j)}$$

$$+ \sum_{l=2}^{S_j} \int_{k_{l-1}^{(j)}}^{k_l^{(j)}} \left(\widetilde{h}(k_{l-1}^{(j)}, x_t^{(j)}, u)O_{t,T}^{(a,j)}(k_{l-1}^{(j)}) - \widetilde{h}(k, x_t^{(j)}, u)O_{t,T}^{(j)}(k) \right) dk$$

$$+ O_p \left(T^{7/4}\sqrt{\Delta} \vee T^{3/2}\Delta \vee T\Delta|\log(T)| \vee Te^{-2(|\underline{k}|\wedge|\overline{k}|)} \right), \tag{143}$$

where $\Delta_l^{(j)} = k_l^{(j)} - k_{l-1}^{(j)}$, $O_{t,T}^{(a,j)}(k_{l-1}^{(j)}) = (O_{t,T}^{(j)}(k_{l-1}^{(j)}) + O_{t,T}^{(j)}(k_l^{(j)}))/2$, and we further denote

$$\widetilde{h}(k, x, u) = u(1 - \cos(uk - ux))e^{-k} - u^2 \sin(uk - ux)e^{-k}, \quad u, k_l, k_h, x \in \mathbb{R}. \tag{144}$$

Now, we can note that

$$|\widetilde{h}(k, x, u)| \leq Ce^{-k}|k - x|, \tag{145}$$

for some constant C that does not depend on k and x (but it depends on u). Taking into account this bound, our assumption A6 for the observation error as well as the bounds for the option prices in Lemma 2, we have

$$\sum_{l=2}^{S_j} \int_{k_{l-1}^{(j)}}^{k_l^{(j)}} \left(\widetilde{h}(k_{l-1}^{(j)}, x_t^{(j)}, u)O_{t,T}^{(a,j)}(k_{l-1}^{(j)}) - \widetilde{h}(k, x_t^{(j)}, u)O_{t,T}^{(j)}(k) \right) dk = O_p(T\Delta), \tag{146}$$

$$\sum_{l=2}^{S_j} \widetilde{h}(k_{l-1}^{(j)}, x_t^{(j)}, u)\widehat{O}_{t,T}^{(a,j)}(k_{l-1}^{(j)})\Delta_l^{(j)} = O_p(T\sqrt{\Delta}). \tag{147}$$

Combining the above two results with (143) and Lemma 7, we get the result of the theorem.

9   Data Appendix

9.1   Data Filters

We use the following filters for the high-frequency return data:

1. Delete entries with a time stamp outside the 9:30 – 16:00 window.

2. Delete entries with transaction price equal to zero.

3. Retain entries originating from NYSE, NASDAQ, and AMEX only.

4. Delete entries with corrected trades. (Trades with a Correction Indicator, CORR = 7,8,10,11,12).

45

5. Delete entries with abnormal Sale Condition. (Trades with a Condition Indicator, COND = Z,B,U,T,L,G,W,K,J).

6. If multiple transactions happen within the same second, use the median price.

We use the following filters for the option data:

1. Delete options with maturity less than 5 business days.

2. Delete entries with zero best-bid.

3. Delete entries with best-ask/best-bid greater than 15.

4. For each equity on each day, select the options with the shortest maturity.

5. For each equity, discard the days when there are less than three out-of-money calls or three out-of-money puts.

After filtering, we report in Table 6 for each stock its industry sector, the number of valid trading days in the sample, and the medium number of strikes each day throughout the sample period.

9.2 Tail Approximation

The estimates $TC_t^{(\pm,j)}$ in (18) are approximations of the integrals $\int_{k_{S_j}^{(j)}}^{\infty} 2e^{-k}(1 - k + x_t^{(j)})O_{t,T}^{(j)}(k)dk$ and $\int_{-\infty}^{k_1^{(j)}} 2e^{-k}(1 - k + x_t^{(j)})O_{t,T}^{(j)}(k)dk$ which are based on extreme value theory. In particular, from Lemma 3, we have

$$O_{t,T}^{(j)}(k) \approx \begin{cases} O_{t,T}^{(j)}(k_1^{(j)})e^{1+\alpha_t^{(-,j)}}, & k \leq k_1^{(j)}, \\ O_{t,T}^{(j)}(k_{S_j}^{(j)})e^{1-\alpha_t^{(+,j)}}, & k \geq k_{S_j}^{(j)}. \end{cases} \tag{148}$$

Here we illustrate these "extrapolations" of the option data in the tails in Figure 6 for a representative day and stock in the sample. As seen from the figure, the log-option prices decay approximately linearly in the tails (as implied by extreme value theory) and our tail decay parameter estimates $\widehat{\alpha}_t^{(\pm,j)}$ capture this decay relatively well. In terms of the option price levels, the extrapolation does not appear very big as $\widehat{O}_{t,T}^{(j)}(k_1^{(j)})$ and $\widehat{O}_{t,T}^{(j)}(k_{S_j}^{(j)})$ are already relatively small in value. Nevertheless, such a tail extension ensures that we treat "symmetrically" all stocks and the market index. Furthermore, this way we treat the left and right tail also "symmetrically". As seen from the figure, on this day $\widehat{O}_{t,T}^{(j)}(k_{S_j}^{(j)}) > \widehat{O}_{t,T}^{(j)}(k_1^{(j)})$ and without the tail approximation we can introduce some upward bias in $AV_t^{(j)}$.
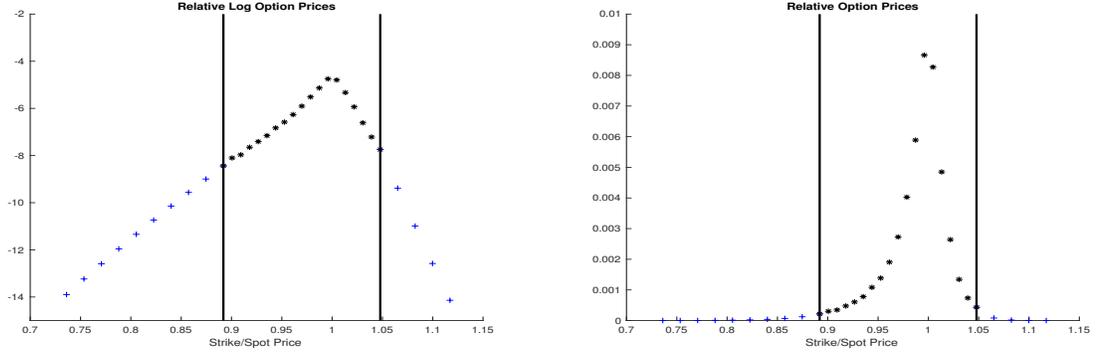
Figure 6: Relative Prices of OTM JNJ Short-Dated Options on 11/16/2016 with Tail Extension. Relative option price stands for option divided by the current stock price. Observed option quotes are denoted with stars while extrapolated ones with crosses.

9.3 Empirical Results using $AM_t^{(j)}$

In this section we perform analysis using the alternative measure of aggregate idiosyncratic asymmetry $\widetilde{AM}_t(u) = \frac{1}{N}\widetilde{AM}_t^{(j)}(u)$. Throughout, we set $u = 5$. The reason for this choice is the following. Too low value of u implies more reliance on the very deep out-of-the-money options which can be (in relative terms) less reliable. On the other hand, a very high value u puts more emphasis on the smoothness of the option price as a function of the strike. Indeed, the characteristic function of the returns (from which $\widetilde{AM}_t(u)$ is derived) is governed by the smoothness of the return density which maps into smoothness of the option price. Given these considerations, we pick u such that for a large support of the jump return distribution of $(-0.3, 0.3)$, we have approximately $|ux| \leq \pi/2$ and for this range of values, $\sin(ux)$ is an increasing function in x.

The realized counterparts of $AM_t^{(j)}$ and $\widetilde{AM}_t^{(j)}$ are naturally defined as

$$RAM_t^{(j)}(u) = \sum_{i=\lfloor tn \rfloor + 1}^{\lfloor (t+T)n \rfloor} (\sin(u\Delta_i^n x^{(j)}) - u\Delta_i^n x^{(j)}), \quad u \in \mathbb{R}, \ j = 0, 1, ..., N, \tag{149}$$

$$\widetilde{RAM}_t^{(j)}(u) = RAM_t^{(j)}(u) - RAM_t^{(0)}(\widehat{\beta}_t^{(j)}u), \quad \widetilde{RAM}_t(u) = \frac{1}{N}\sum_{j=1}^{N} \widetilde{RAM}_t^{(j)}(u), \ u \in \mathbb{R}, \ j = 1, ..., N. \tag{150}$$

Using Theorem 3.3.1 in Jacod and Protter (2012) and under assumption A1, we have

$$\widetilde{RAM}_t^{(j)} \xrightarrow{\mathbb{P}} \sum_{s \in [t, t+T]} (\sin(u\Delta \widetilde{x}_s^{(j)}) - u\Delta \widetilde{x}_s^{(j)}), \quad j = 1, ..., N. \tag{151}$$

Using Theorem 11.1.2 in Jacod and Protter (2012) and using the fact that $\mathbb{E}(\sin(uZ) - uZ) = 0$ for Z a standard normal distribution, we can construct a CLT for $\widetilde{RAM}_t^{(j)}$. Upon implementing it

on the data, we find that on 76% of the days in the sample, $\widetilde{RAM}_t(u)$ is statistically different from zero.[11] This provides strong evidence for clustering of idiosyncratic jumps in the cross-section.

On Figure 7, we compare $\widetilde{AM}$ with $\widetilde{RAM}$. The relationship between the two quantities is very similar to that between $\widetilde{AV}$ with $\widetilde{RAV}$ as seen by comparing Figures 4 and 7. Mainly, while the realized aggregate asymmetry measure $\widetilde{RAM}$ is different from zero in many instances, it is typically much smaller than the risk-neutral expectation for it measured via $\widetilde{AM}$. We next compare $AM^{(0)}$
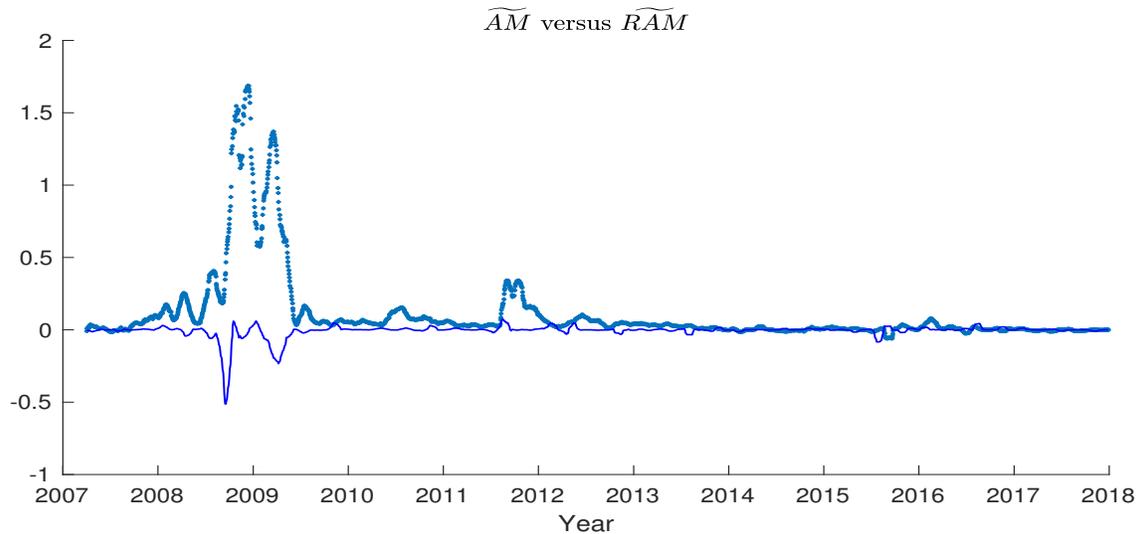
$$\widetilde{AM} \text{ versus } \widetilde{RAM}$$

Figure 7: $\widetilde{AM}$ versus $\widetilde{RAM}$. Dotted line is $\widetilde{AM}$ and solid line is $\widetilde{RAM}$ from five-minute returns. Plotted series are 20-day moving averages.

and $\widetilde{AM}$ in Figure 8. The relationship between these two measures of asymmetry is very similar to that between their counterparts $AV^{(0)}$ and $\widetilde{AV}$. The only notable difference is that now the financial crisis of the Fall of 2008 has a somewhat larger impact on the market asymmetry $AM^{(0)}$.

Finally, in Table 3 we report predictive regression estimates using $AM_{t,T}^{(0)}$ and $\widetilde{AM}_{t,T}$ as the explanatory variables. The aggregate idiosyncratic asymmetry measure $\widetilde{AM}_{t,T}$ has very similar predictive ability as $\widetilde{AV}_{t,T}$. For $AM_{t,T}^{(0)}$, the results are a bit weaker than the corresponding results for $AV_{t,T}^{(0)}$ reported in Table 2. This is likely due to the more extreme reaction of $AM_{t,T}^{(0)}$ during the financial crisis in the Fall of 2008.

Overall, we conclude that the major empirical findings in the text are preserved when using the alternative asymmetry measures $AM_{t,T}^{(0)}$ and $\widetilde{AM}_{t,T}$.

---

[11]Details on the implementation of the test are available upon request.
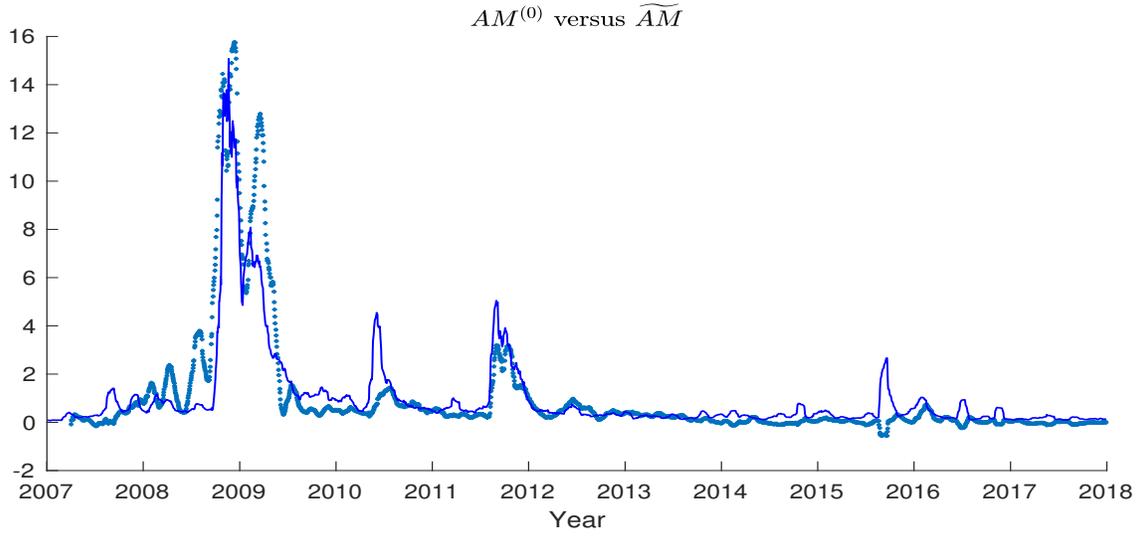
Figure 8: $AM^{(0)}$ versus $\widetilde{AM}$. Dotted and solid lines correspond to 20-day moving averages of $\widetilde{AM}$ and $AM^{(0)}$, respectively. Both series have been normalized by their sample means.

	6 Months		9 Months		12 Months	
$AM_{t,T}^{(0)}$	0.2374		0.3900		0.5101	
	(0.1385)		(0.1558)		(0.1433)	
$\widetilde{AM}_{t,T}$		0.0838		0.1621		0.2344
		(0.0798)		(0.0832)		(0.0698)
R^2	0.0828	0.0419	0.1450	0.1022	0.1835	0.1584

Table 3: Predicting Returns of Equally-Weighted Portfolio Constructed by the Stocks in the Sample. The explanatory variables are 20-day moving averages. Standard errors are calculated using Newey-West estimator with lag length of $1.3\sqrt{T}$.

9.4 Robustness Checks

9.4.1 Value Weighted $\widetilde{AV}$

We check the robustness of our results by computing the following value-weighted counterpart of $\widetilde{AV}$:

$$\widetilde{AV}_t^{vw} = \sum_{j=1}^{N} w_{t,j} \widetilde{AV}_t^{(j)}, \tag{152}$$

where $w_{t,j}$ is the proportion of the value of stock j to the total value of the stocks under consideration at time t. In Figure 9, we compare the equal and value-weighted $\widetilde{AV}$. As seen from the figure, the two series are very close to each other, with $\widetilde{AV}_t^{vw}$ being only slightly lower than $\widetilde{AV}$. This small difference in the two series implies that our test for presence of risk premia for the idiosyncratic downside jump risk and our predictive regressions using $\widetilde{AV}$ will remain largely unchanged when switching to the value-weighted $\widetilde{AV}_t^{vw}$. This is confirmed by the results reported in Tables 4 and 5.
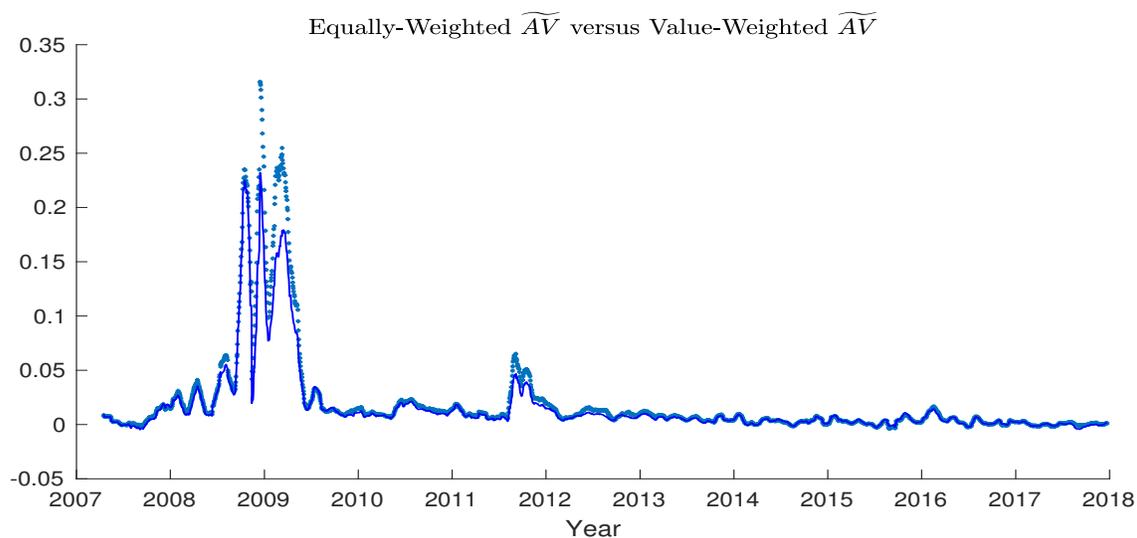
Figure 9: Equally-Weighted versus Value-Weighted $\widetilde{AV}$. Dotted line corresponds to the equally-weighted and the solid line to the value-weighted $\widetilde{AV}$ series. Plotted series are 20-day moving averages.

9.4.2 Extended Cross-Section

We further check the robustness of our results by extending the sample to all stocks in the S&P 100 Index as of December 31, 2017. Due to lower liquidity in the earlier years in our sample, we only focus on the period from 2015 to 2017. Also, we exclude stocks with low option market liquidity, i.e., stocks that do not survive the data filtering procedure. A summary of the extended sample

	t-stat (HF)	t-stat (daily)
$\widehat{m}_{mean}$	-4.8943	-5.4348
$\widehat{m}_{auto}^{(1)}$	2.5849	3.1662
$\widehat{m}_{auto}^{(5)}$	2.3678	2.7119
$\widehat{m}_{auto}^{(22)}$	2.1987	2.1724

Table 4: Tests for Risk Premium in $\widetilde{AV}^{vw}$. Standard errors are calculated using Newey-West estimator with lag length of $1.3\sqrt{T}$.

	6 Months				9 Months				12 Months			
$AV_{t,T}^{(0)}$	0.7768				1.4174				1.9511			
	(0.5448)				(0.6398)				(0.6194)			
$QV_{t,T}^{(0)} - AV_{t,T}^{(0)}$		0.7042				1.1816				1.5896		
		(0.4914)				(0.5162)				(0.4507)		
$\widetilde{AV}_{t,T}$			0.5711				1.1226				1.5675	
			(0.6195)				(0.6363)				(0.4870)	
$\widetilde{QV}_{t,T} - \widetilde{AV}_{t,T}$				0.2067				0.2500				0.4935
				(1.2320)				(1.5790)				(1.7092)
	0.0488	0.0573	0.0390	0.0011	0.1062	0.1047	0.0984	0.0010	0.1524	0.1427	0.1463	0.0037

Table 5: Predicting Returns of Value-Weighted Portfolio Constructed by Stocks in the Sample. Standard errors are calculated using Newey-West estimator with lag length of $1.3\sqrt{T}$.

is reported in Table 7. Overall, the extended cross-section consists of 102 stocks whose market capitalization is over 60% of the value of the S&P 500 index (as of December 31 2017). Comparing Tables 6 and 7, we can notice that the liquidity of the option market have increased significantly over the last few years with the stocks in the original sample having higher number of traded options over the more recent period of 2015-2017. In Figure 10, we compare the value-weighted $\widetilde{AV}$ using our original and extended cross-sections. The two series are very close to each other, with the one based on the larger S&P 100 Index cross-section being slightly larger than the one based on the original cross-section thus providing even stronger evidence for the importance of idiosyncratic downside jump risk.
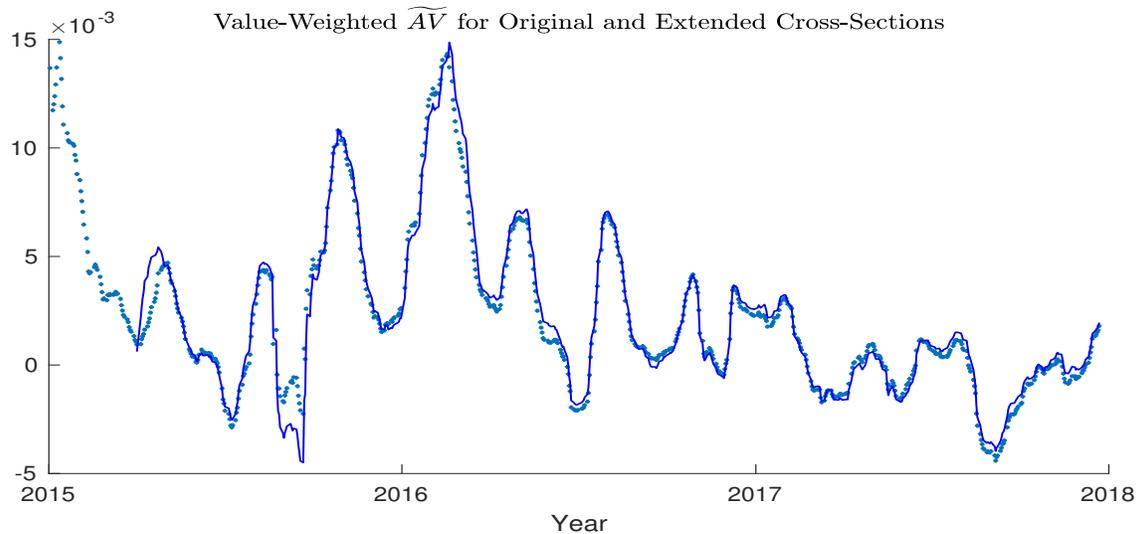
Figure 10: Comparison of Value-Weighted $\widetilde{AV}$ from Original and Extended Cross-Sections. Dotted line corresponds to the original cross-section and the solid line to the extended one. Plotted series are 20-day moving averages.

References

Acemoglu, D., A. Ozdaglar, and A. Tahbaz-Salehi (2017). Microeconomic Origins of Macroeconomic Tail Risks. *American Economic Review 107*, 54–108.

Andersen, T., N. Fusari, and V. Todorov (2017). The Pricing of Short-Term Market Risk: Evidence from Weekly Options. *Journal of Finance 72*, 1335–1386.

Andrews, D. W. K. (2005). Cross-Section Regression with Common Shocks. *Econometrica 73*(5), 1551–1585.

Bakshi, G., N. Kapadia, and D. Madan (2003). Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options. *Review of Financial Studies 16*, 101–143.

Bakshi, G. and D. Madan (2000). Spanning and Derivative-Security Valuation. *Journal of Financial Economics 55*, 205–238.

Bali, T. G., N. Cakici, and R. F. Whitelaw (2011). Maxing Out: Stocks as Lotteries and the Cross-Section of Expected Returns. *Journal of Financial Economics 99*(2), 427–446.

Barberis, N. and M. Huang (2008). Stocks as Lotteries: The Implications of Probability Weighting for Security Prices. *American Economic Review 98*(5), 2066–2100.

Barndorff-Nielsen, O. and N. Shephard (2004). Power and Bipower Variation with Stochastic Volatility and Jumps. *Journal of Financial Econometrics 2*, 1–37.

Barndorff-Nielsen, O. E., S. Kinnebrock, and N. Shephard (2010). Measuring Downside Risk - Realised Semivariance. In T. Bollerslev, J. Russell, and M. Watson (Eds.), *Volatility and Time Series Econometrics, Essays in Honor of Robert F. Engle*. Oxford University Press.

Bates, D. S. (1991). The Crash of '87 – Was It Expected? The Evidence from Options Markets. *Journal of Finance 46*, 1009–1044.

Bates, D. S. (1996). Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options. *Review of Financial Studies 9*, 69–107.

Bates, D. S. (2000). Post-'87 Crash Fears in S&P 500 Future Options. *Journal of Econometrics 94*, 181–238.

Begin, J., C. Dorion, and G. Gauthier (2019). Idiosyncratic Jump Risk Matters: Evidence from Equity Returns and Options. *Review of Financial Studies*.

Bingham, N., C. Goldie, and J. Teugels (1987). *Regular Variation*. Cambridge University Press.

Bollerslev, T., J. Li, A. Patton, and R. Quaedvlieg (2019). Realized Semicovariances. Technical report.

Bollerslev, T. and V. Todorov (2011). Estimation of Jump Tails. *Econometrica 79*(6), 1727–1783.

Bollerslev, T. and V. Todorov (2011). Tails, Fears and Risk Premia. *Journal of Finance 66*, 2165–2211.

Boyer, B. H. and K. Vorkink (2014). Stock Options as Lotteries. *The Journal of Finance 69*(4), 1485–1527.

Brunnermeier, M. K. and J. A. Parker (2005). Optimal Expectations. *American Economic Review 95*(4), 1092–1118.

Carr, P. and D. Madan (2001). Optimal Positioning in Derivative Securities. *Quantitiative Finance 1*, 19–37.

Conrad, J., R. Dittmar, and E. Ghysels (2013). Ex Ante Skewness and Expected Stock Returns. *The Journal of Finance 68*(1), 85–124.

Drechsler, I. and A. Yaron (2011). What's Vol Got to Do with It? *Review of Financial Studies 24*, 1–45.

Driessen, J., P. Maenhout, and G. Vilkov (2009). The Price of Correlation Risk: Evidence from Equity Options. *The Journal of Finance 64*, 1377–1406.

Duffie, D. (2001). *Dynamic Asset Pricing Theory* (3rd ed.). Princeton University Press.

Feunou, B., M. Jahan-Parvar, and C. Okou (2018). Downside Variance Risk Premium. *Journal of Financial Econometrics 16*, 341–383.

Gabaix, X. (2011). The Granular Origins of Aggregate Fluctuations. *Econometrica 79*, 733–772.

Gabaix, X. (2012). Variable Rare Disasters: An Exactly Solved Framework for Ten Puzzles in Macro-Finance. *Quarterly Journal of Economics 127*, 645–700.

Goldie, C. and R. Smith (1987). Slow Variation with Remainder: Theory and Applications. *The Quarterly Journal of Mathematics 38*(1), 45–71.

Hansen, L. (1982). Large Sample Properties of Generalize Method of Moments Estimators. *Econometrica 50*, 1029–1054.

Hansen, L. and K. Singleton (1982). Generalized Instrumental Variables Estimation of Nonlinear Rational Expectations Models. *Econometrica 50*, 1269–1286.

Herskovic, B., B. Kelly, H. Lustig, and S. Van Nieuwerburgh (2016). The Common Factor in Idiosyncratic Volatility: Quantitative Asset Pricing Implications. *Journal of Financial Economics 119*, 249–283.

Jacod, J. (2008). Asymptotic Properties of Power Variations and Related Functionals of Semi-martingales. *Stochastic Processes and their Applications 118*, 517–559.

Jacod, J. and P. Protter (2012). *Discretization of Processes*. Berlin: Springer-Verlag.

Jacod, J. and A. Shiryaev (2003). *Limit Theorems For Stochastic Processes* (2nd ed.). Berlin: Springer-Verlag.

Kelly, B. and H. Jiang (2014). Tail Risk and Asset Prices. *Review of Financial Studies 27*, 2841–2871.

Kelly, B., H. Lustig, and S. Van Nieuwerburgh (2016). Too-Systemic-to-Fail: What Option Markets Imply About Sector-Wide Government Guarantees. *American Economic Review 106*(6), 1278–1319.

Khandani, A. E. and A. W. Lo (2007). What Happened to the Quants in August 2007? *Journal of Investment Management 5*(4), 29.

Kilic, M. and I. Shaliastovich (2018). Good and Bad Variance Premia and Expected Returns. *Management Science 65*(6), 2522–2544.

Lazarus, E., D. J. Lewis, J. H. Stock, and M. W. Watson (2018). HAR Inference: Recommendations for Practice. *Journal of Business & Economic Statistics 36*(4), 541–559.

Li, J., V. Todorov, and G. Tauchen (2017). Jump Regressions. *Econometrica 85*(1), 173–195.

Li, J., V. Todorov, and G. Tauchen (2019). Jump Factor Models in Large Cross-Sections. *Quantitative Economics 10*, 419–456.

Martin, I. and C. Wagner (2018). What is the Expected Return on a Stock? *The Journal of Finance*.

Merton, R. (1976). Option Pricing when Underlying Asset Returns are Discontinuous. *Journal of Financial Economics 3*, 125–144.

Mitton, T. and K. Vorkink (2007). Equilibrium Underdiversification and the Preference for Skewness. *The Review of Financial Studies 20*(4), 1255–1288.

Oh, S. and J. Wachter (2019). Cross-Sectional Skewness. Technical report.

Pan, J. (2002). The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study. *Journal of Financial Economics 63*, 3–50.

Patton, A. J. and K. Sheppard (2015). Good Volatility, Bad Volatility: Signed Jumps and the Persistence of Volatility. *Review of Economics and Statistics 97*(3), 683–697.

Pederzoli, P. (2018). Crash Risk in Individual Stocks. Technical report.

Qin, L. and V. Todorov (2018). Nonparametric Implied Lévy Densities. *Annals of Statistics, forthcoming*.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge, UK: Cambridge University Press.

Todorov, V. (2018). Nonparametric Spot Volatility from Options. *Annals of Applied Probability, forthcoming.*

Wachter, J. A. (2013). Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility? *Journal of Finance 68*, 987–1035.

Table 6: Summary Statistics for Individual Stock Options

Company Name	Ticker	Sector	Sample Days	Med. # of Strikes	Weight (%)
Apple Inc.	AAPL	Information Technology	2233	29	3.62
Abbott Laboratories	ABT	Health Care	982	10	0.42
Amgen Inc.	AMGN	Health Care	2164	12	0.53
Amazon.com & Inc.	AMZN	Consumer Discretionary	2107	22	2.37
American Express Company	AXP	Financials	2070	16	0.36
Boeing Company	BA	Industrials	1996	16	0.74
Bank of America Corp	BAC	Financials	1164	8	1.29
Bristol-Myers Squibb Company	BMY	Health Care	1692	16	0.42
Berkshire Hathaway Inc.	BRK.B	Financials	898	12	1.11
Anheuser-Busch InBev	BUD	Consumer Staples	217	9	0.75
Citigroup Inc.	C	Financials	1909	18	0.83
Comcast Corporation Class A	CMCSA	Communication Services	1381	15	0.78
ConocoPhillips	COP	Energy	1879	15	0.28
Cisco Systems Inc.	CSCO	Information Technology	1514	9	0.80
CVS Health Corporation	CVS	Health Care	1529	16	0.31
Chevron Corporation	CVX	Energy	1600	17	1.00
Walt Disney Company	DIS	Communication Services	1689	16	0.68
Facebook Inc. Class A	FB	Communication Services	1389	29	1.78
General Electric Company	GE	Industrials	1524	9	0.64
Gilead Sciences Inc.	GILD	Health Care	2197	18	0.39
Alphabet Inc. Class C	GOOG	Communication Services	2221	34	1.32
Goldman Sachs Group Inc.	GS	Financials	2550	17	0.40
Home Depot Inc.	HD	Consumer Discretionary	1838	15	0.93
HP Inc.	HPQ	Information Technology	1530	11	0.15
International Business Machines	IBM	Information Technology	2217	13	0.60
Intel Corporation	INTC	Information Technology	1903	10	0.91
Johnson & Johnson	JNJ	Health Care	1288	14	1.58
JPMorgan Chase & Co.	JPM	Financials	2370	18	1.56
Coca-Cola Company	KO	Consumer Staples	1265	9	0.82
McDonald's Corporation	MCD	Consumer Discretionary	1595	14	0.58
3M Company	MMM	Industrials	1259	12	0.59
Altria Group Inc	MO	Consumer Staples	1237	14	0.57
Merck & Co. Inc.	MRK	Health Care	1792	14	0.64
Microsoft Corporation	MSFT	Information Technology	2269	12	2.77
Oracle Corporation	ORCL	Information Technology	1815	11	0.82
Occidental Petroleum Corporation	OXY	Energy	1971	14	0.24
PepsiCo Inc.	PEP	Consumer Staples	1217	14	0.71
Pfizer Inc.	PFE	Health Care	1137	8	0.91
Procter & Gamble Company	PG	Consumer Staples	1447	16	0.98
Philip Morris International Inc.	PM	Consumer Staples	1328	14	0.69
QUALCOMM Incorporated	QCOM	Information Technology	2225	15	0.40
Schlumberger NV	SLB	Energy	2107	15	0.39
AT&T Inc.	T	Communication Services	1734	10	1.00
UnitedHealth Group Incorporated	UNH	Health Care	1786	14	0.90
United Technologies Corporation	UTX	Industrials	1414	14	0.43
Visa Inc. Class A	V	Information Technology	2037	14	0.87
Verizon Communications Inc.	VZ	Communication Services	1790	13	0.91
Wells Fargo & Company	WFC	Financials	2053	14	1.26
Walmart Inc.	WMT	Consumer Staples	1574	16	1.23
Exxon Mobil Corporation	XOM	Energy	1871	14	1.49

Summary statistics are for the period 2007-2017. For each stock, the table reports the sector it belongs to, the number of valid trading days in the sample after filtering, the median number of strikes per day, and its weight in S&P 500 Index as of 12/31/2017.

Table 7: Summary Statistics for Extended Cross-Section of Options

Company Name	Ticker	Sector	Sample Days	Med. # of Strikes	Weight (%)
Apple Inc.	AAPL	Information Technology	729	36	3.62
AbbVie Inc.	ABBV	Health Care	721	18	0.65
Abbott Laboratories	ABT	Health Care	391	11	0.42
Accenture Plc Class A	ACN	Information Technology	709	14	0.41
Allergan plc	AGN	Health Care	574	21	0.23
American International Group Inc.	AIG	Financials	734	19	0.23
Allstate Corporation	ALL	Financials	4	7	0.16
Amgen Inc.	AMGN	Health Care	717	17	0.53
Amazon.com Inc.	AMZN	Consumer Discretionary	678	68	2.37
American Express Company	AXP	Financials	739	25	0.36
Boeing Company	BA	Industrials	728	22	0.74
Bank of America Corp	BAC	Financials	486	8	1.29
Biogen Inc.	BIIB	Health Care	494	29	0.28
Bank of New York Mellon Corporation	BK	Financials	690	10	0.23
BlackRock Inc.	BLK	Financials	108	8	0.35
Bristol-Myers Squibb Company	BMY	Health Care	739	24	0.42
Berkshire Hathaway Inc.	BRK.B	Financials	516	13	1.11
Citigroup Inc.	C	Financials	739	23	0.83
Caterpillar Inc.	CAT	Industrials	734	32	0.39
Celgene Corporation	CELG	Health Care	729	28	0.35
Charter Communications Inc. Class A	CHTR	Communication Services	94	23	0.34
Colgate-Palmolive Company	CL	Consumer Staples	726	15	0.28
Comcast Corporation Class A	CMCSA	Communication Services	730	18	0.78
Capital One Financial Corporation	COF	Financials	705	26	0.20
ConocoPhillips	COP	Energy	732	21	0.28
Costco Wholesale Corporation	COST	Consumer Staples	670	17	0.34
Cisco Systems Inc.	CSCO	Information Technology	475	9	0.80
CVS Health Corporation	CVS	Health Care	723	22	0.31
Chevron Corporation	CVX	Energy	737	22	1.00
Danaher Corporation	DHR	Health Care	7	6	0.27
Walt Disney Company	DIS	Communication Services	737	23	0.68
Emerson Electric Co.	EMR	Industrials	61	8	0.19
Ford Motor Company	F	Consumer Discretionary	111	7	0.20
Facebook Inc. Class A	FB	Communication Services	736	33	1.78
FedEx Corporation	FDX	Industrials	711	15	0.28
Twenty-First Century Fox Inc. Class B	FOX	Consumer Discretionary	11	7	0.11
Twenty-First Century Fox Inc. Class A	FOXA	Communication Services	111	8	0.15
General Dynamics Corporation	GD	Industrials	6	7	0.26
General Electric Company	GE	Industrials	506	10	0.64
Gilead Sciences Inc.	GILD	Health Care	738	33	0.39
General Motors Company	GM	Consumer Discretionary	729	13	0.24
Alphabet Inc. Class C	GOOG	Communication Services	632	46	1.32
Goldman Sachs Group Inc.	GS	Financials	737	20	0.40
Halliburton Company	HAL	Energy	738	21	0.18
Home Depot Inc.	HD	Consumer Discretionary	718	20	0.93
Honeywell International Inc.	HON	Industrials	61	13	0.49
HP Inc.	HPQ	Information Technology	304	11	0.15
International Business Machines	IBM	Information Technology	719	19	0.60
Intel Corporation	INTC	Information Technology	709	12	0.91
Johnson & Johnson	JNJ	Health Care	724	17	1.58
JPMorgan Chase & Co.	JPM	Financials	738	25	1.56
Kraft Heinz Company	KHC	Consumer Staples	15	11	0.40
Kinder Morgan Inc Class P	KMI	Energy	527	10	0.17
Coca-Cola Company	KO	Consumer Staples	535	10	0.82
Eli Lilly and Company	LLY	Health Care	731	19	0.39
Lockheed Martin Corporation	LMT	Industrials	372	12	0.39
Lowe's Companies Inc.	LOW	Consumer Discretionary	426	23	0.32
Mastercard Incorporated Class A	MA	Information Technology	730	26	0.66
McDonald's Corporation	MCD	Consumer Discretionary	708	19	0.58
Mondelez International Inc. Class A	MDLZ	Consumer Staples	434	17	0.27
Medtronic plc	MDT	Health Care	705	21	0.46
MetLife Inc.	MET	Financials	694	17	0.22
3M Company	MMM	Industrials	560	13	0.59
Altria Group Inc	MO	Consumer Staples	734	18	0.57
Monsanto Company	MON	Health Care	398	24	0.22
Merck & Co. Inc.	MRK	Health Care	735	17	0.64
Morgan Stanley	MS	Financials	729	15	0.40
Microsoft Corporation	MSFT	Information Technology	739	19	2.77
NextEra Energy Inc.	NEE	Utilities	2	8	0.31
NIKE Inc. Class B	NKE	Consumer Discretionary	736	19	0.34
NVIDIA Corporation	NVDA	Information Technology	719	23	0.49
Oracle Corporation	ORCL	Information Technology	685	12	0.82
Occidental Petroleum Corporation	OXY	Energy	683	23	0.24
The Priceline Group Inc.	PCLN	Consumer Discretionary	382	90	0.35
PepsiCo Inc.	PEP	Consumer Staples	730	18	0.72
Pfizer Inc.	PFE	Health Care	527	9	0.91
Procter & Gamble Company	PG	Consumer Staples	739	20	0.98
Philip Morris International Inc.	PM	Consumer Staples	717	18	0.69
PayPal Holdings Inc	PYPL	Information Technology	609	15	0.37
QUALCOMM Incorporated	QCOM	Information Technology	733	22	0.40
Raytheon Company	RTN	Industrials	27	8	0.23
Starbucks Corporation	SBUX	Consumer Discretionary	732	18	0.34
Schlumberger NV	SLB	Energy	713	30	0.39
Southern Company	SO	Utilities	34	8	0.20
AT&T Inc.	T	Communication Services	650	11	1.00
Target Corporation	TGT	Consumer Discretionary	735	22	0.15
Time Warner	TWX	Communication Services	582	26	0.30
Texas Instruments Incorporated	TXN	Information Technology	732	19	0.43
UnitedHealth Group Incorporated	UNH	Health Care	681	16	0.90
Union Pacific Corporation	UNP	Industrials	736	27	0.44
United Parcel Service Inc. Class B	UPS	Industrials	715	18	0.34
U.S. Bancorp	USB	Financials	674	13	0.37
United Technologies Corporation	UTX	Industrials	709	15	0.43
Visa Inc. Class A	V	Information Technology	738	25	0.87
Verizon Communications Inc.	VZ	Communication Services	719	15	0.91
Walgreens Boots Alliance Inc	WBA	Consumer Staples	730	25	0.30
Wells Fargo & Company	WFC	Financials	734	18	1.26
Walmart Inc.	WMT	Consumer Staples	738	21	1.23
Exxon Mobil Corporation	XOM	Energy	714	24	1.49

The table summarizes the option data used in the extended cross-section of stocks in the S&P 100 Index with high option liquidity over the period 2015 to 2017. For each stock, the table reports the sector it belongs to, the number of valid trading days in the sample after filtering, the median number of strikes per day, and its weight in S&P 500 Index as of 12/31/2017.