Exercising Real Options Sooner or Later? New Insights from Quantile-Preserving Spreads on how to Fasten or Delay Exercise.

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Abstract

Using a general Quantile-Preserving Spread and Stochastic Dominance, this paper studies how modifying a real option's characteristic affects the optimal real option holding value and exercise decision. We study under which conditions an increase in risk may increase the exercise probability and hasten real option exercise. We find that the exercise timing depends on the preserved quantile. Contrary to the conventional wisdom, an increase in risk can decrease the option's holding value if the risk is added through an asymmetric spread. We significantly generalize previously obtained results by developing propositions for an unspecified underlying process and a general call-like payoff function. Our paper enriches a strand of literature in financial economics where agents make a decision to maximize the expected utility of a payoff function. Our paper also adds to literature on the effects of risk on option value. Our results are useful to determine an optimally parsimonious modification of a real option to increase or decrease its exercise probability and have policy implications. Specifically, we find that the modification of strike price tends to be more cost effective than that of the underlying process.

JEL classification: G12, G31, G32

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1 Introduction

Real options can model various real-life decisions under risk. For classical project valuation purposes, a real option modeler compares the present value of future cash flows (underlying process) to the capital investment (strike) and models the value of waiting for a decision by an American call option (see, e.g., McDonald and Siegel 1985, Brennan and Schwartz 1985, and Dixit and Pindyck 1994). A researcher can study capital structure and debt valuation under a real options approach assuming that equity is a call on the firm's assets and creditors are short a put on the assets, as Merton's (1974) seminal paper originally propose. The field of merger and acquisitions takes advantage of the real options approach (see, e.g., Lambrecht 2004, and Bhagwat, Dam, and Harford 2016). The real options approach is applied in studies of management compensation when managers have performance-based incentives which can be modeled by a digital option (see, e.g., Grinblatt and Titman 1989). Furthermore, the real option approach provides useful insights into government policies and the effects on businesses (see, e.g., Julio and Yook 2012 or Gulen and Ion 2016). In fact,"*... real options research has [...] branched out into areas as diverse as management science, strategy, insurance, taxation, environmental economics and engineering*" (Lambrecht 2017).

Real options share similarities with financial options but differ on some critical aspects. For a financial option, writers and buyers are mostly concerned with its value and sensitivities (also known as the Greeks). External stakeholders might use information implied from financial options but will have little interest in the execution of financial options which are zero-sum games. In addition, it is illegal for financial option holders to modify the underlying process because such an action is considered a market manipulation. Real options are different because they can lead to the creation of value and have repercussions onto multiple stakeholders. In many situations, these stakeholders are more concerned with the exercise of the option than its value. For example, a government may want to fasten the exercise of investment options; a debt holder wants to prevent the exercise of the put option; and a manager wants to exercise a compensation option. More importantly, real options are different from financial options because their characteristics can be modified. Both the strike and the dynamics of the underlying process can be modified. Owners, writers, or external stakeholders such as the government can modify a real option specification. If a stakeholder has some control over an option specification, she will be interested in better understanding the relationship between a specification and its effect on a real option's exercise probability and timing. Real option papers addressing specific economic/policy problems do quantify the option exercise probability and timing, but very few aim to propose general statements. Furthermore, the few papers studying the effect of modifications on option exercise narrowly focus on simple risk increases, such as symmetric increases in variance, and convex payoffs, such as typical call and put options. Our results show that the conclusions obtained under such restrictive circumstances may be wrong for more general risk increases and more general option payoffs like digital options.

The objective of this paper is to characterize real options' modifications and their effects on exercise probability and timing by providing results that hold in general. We consider general call-like contingent claims that are options for which the payoff is non-negative when the underlying process is greater or equal to the strike value, and zero otherwise. Our results can be extended to put-like contingent claims. We consider two types of modifications. A modification that adds value to the underlying process, and a modification that adds risk through a quantile-preserving spread (QPS hereafter). We study the effect on exercise probability using European style options, and the effect on exercise timing using American style options.

Our principal findings can be summarized as follows. In a general setting, adding risk to the underlying process of a European style call-like option through a quantile-preserving spread will increase (decrease) the exercise probability if the preserved quantile is below (above) the strike value. In particular, the exercise probability of an out-the-money (in-the-money) call-like option will increase (decrease) with median-preserving spreads. Adding risk to an American style call-like option may hasten or delay exercise. In particular, if risk is added later and the preserved quantile is below the strike value, exercise is always delayed. However, if the preserved quantile is above the strike, exercise may be hasten if the QPS is sufficiently asymmetric. Our results show that an analysis of the real option exercise based exclusively on variance may lead to wrong conclusions. In a general setting, a modification of the underlying process that leads to a First-Order Stochastically Dominant (FSD) distribution will affect the real option exercise probability and timing. In particular, an early FSD modification will hasten the exercise in probability because the owner will want to take advantage of the positive outcome sooner than later, while a late FSD modification, conditional on exercise, will delay the exercise. For example, in a real estate project, the risk of a future municipal regulation that would harm the project constitutes a skewed increase in risk. The modified process has a larger variance from the uncertainty about the regulation, but it is first order stochastically dominated by the original process. Our results show that exercise would be hastened in such a case.

Our paper contributes to the literature by providing the first comprehensive study characterizing the exercise probability and timing of real options. We document a general channel through which an increase in risk may hasten exercise of real options. We re-interpret previously obtained empirical results within our general framework. We also provide several numerical stylized examples. In particular, we answer the question of whether a government should promote investments by offering grants to reduce capital costs, or by increasing the present value of future cash flows by, e.g., offering tax credits. As such, our paper has general policy implications.

Our paper is related to a strand of literature in financial economics where agents make an investment decision to maximize the expected utility of a payoff function. Several papers have studied the effect of risk on that optimal decision. In particular, the definition of mean-preserving spreads by Rothschild and Stiglitz (1970) to model increases in risk was instrumental for several papers in the literature. The objective of those papers is to establish the conditions under which all risk averse investors will react in the same direction to an increase in risk. To that effect, Diamond and Stiglitz (1974) define mean-utility-preserving spread to order agents risk aversion. However, general results cannot be obtained without assumptions on the class of mean-preserving spreads, the type of payoff, or the utility function. In particular, Meyer and Ormiston (1985) define strong-increases in risk, a subset of mean-preserving spreads, and assume the payoff function is three times differentiable and concave to obtain results. Dionne, Eeckhoudt, and Gollier (1993) focus on linear payoffs. With a specific payoff function, they obtain results for a larger subset of mean-preserving spreads which they define as a relatively weak increase in risk. In the field of real options, the payoff does not fit the assumptions made in those papers. Kanbur (1982) studies the effect of mean-preserving spreads for payoffs with a discontinuity in the first derivative. They find that the discontinuity introduces new terms in the optimization and leads to ambiguous results. We differ from all these papers because in our context, the agent already owns the real option and the decisions we study are whether the real option is exercised, and when.

To that effect, our paper is more related to a strand of literature that studies the effect of risk on option value. Cox and Ross (1976) show that under a proportional stochastic process, any European contingent claim inherits qualitative properties of the claims payoff, e.g., convexity. Jagannathan (1984) has similar results, specific to call options, and proposes a sufficient condition under which a call option will always increase in value if risk is added through a mean-preserving spread. Bergman, Grundy, and Wiener (1996) show that if the underlying process follows a diffusion whose volatility depends only on time, a call option always increases in value. We differ from these three papers because we do not limit our model to proportional stochastic processes and mean-preserving spreads. Instead, we do not specify the underlying process and use quantile-preserving spreads, which are more general than mean-preserving spreads. Bliss (2000) shows that if the underlying asset has a simple two-parameter distribution, the call price is always increasing in volatility. We differ from this paper by studying risk modifications which are more general than increases in volatility. Rasmusen (2007) generalizes Bliss (2000) and proposes necessary conditions under which a call value will be increasing in risk modeled by a mean-preserving spread. Huang and Zhang (2013) extend Rasmusen's (2007) results by providing the necessary and sufficient condition under which a call value will be increasing in risk modeled by a mean-preserving spread. Our paper differs from all these papers first because we use quantile-preserving spreads which are more general, second because we study the effect of risk on the exercise probability and timing, and third because our results apply to any call-like option payoff (e.g., calls, digital options, or an option with a step-function payoff). Our results on the effects of risk on option value are instrumental in obtaining our results on option exercise, but are not the main objectives of our paper.

To the best of our knowledge, very few articles investigate the intentional actions of stakeholders to modify a real option's exercise probability in a theoretical perspective. Sarkar (2000) is the closest to our paper. He studies the relationship between volatility and the exercise probability of an infinite-lived real option when the underlying follows a specific model. Alvarez and Stenbacka (2003) also study a deliberate action to influence the exercise of real options. They study a firm that owns a real call option and may decide between two underlying processes with different levels of volatility and show there is a channel through which more volatility may accelerate investment. We differ from these two papers on several aspects. We do not specifically model the underlying process, we study both the exercise probability and exercise timing, we study increases in risk which are more general than a simple increase in volatility, and do not require the real option to be a typical call option.

The rest of the paper is as follows. Section 2 presents the model and general framework to study the exercise probability and timing. Section 3 presents our theoretical results. Section 4 provides numerical examples with simple European and American options. Section 5 concludes.

2 Model

We study the exercise probability and timing of real options. We focus on call-like real options, but our results are easily transposed to put-like options. The real call option is owned by a rational agent that prefers higher positive cash-flows. The exercise decision is based on a strike value that we denote by K, and a discretely observed underlying process $\{X_t : t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Pr)$, $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$. Note that X_t can be discrete or continuous. Let f_{X_t} denote the distribution of $X_t, \forall t \geq 0$, and be from a family of distribution functions defined over an interval I; $I = [a, b]$, where $-\infty \leq a < b \leq +\infty$. The payoff of the call-like real option at exercise, which we denote $v^e(X_t, K)$, does not need to be the typical call option payoff. However, the payoff needs to be positive, weakly-increasing in X_t when the option is exercised and $X_t \geq K$, and 0 otherwise. Payoff examples include, but are not limited to, a classical project where X_t models the present value of future cash-flows from the project and K represents the capital cost of the project, in which case $v^e(X_t, K) = \max(X_t - K, 0)$. Another example is a manager who is paid an incentive when the value of the assets under management beat a benchmark. In this case, X_t models the difference between the value of the assets and the benchmark, and the strike $K = 0$. The payoff to the portfolio manager can be a binary or even a step function when $X_t \geq K$.

To study the exercise probability, we use a European type option. In this case there is only one exercise opportunity and a rational agent will decide to exercise if and only if $X_T \geq K$. Therefore, X_t for $t < T$ is irrelevant in the exercise decision. We define the exercise probability as $Pr(X_T > K)$. We study modifications to the distribution of X_T, irrespective of how and when these modifications are applied.

To study the exercise timing, we use an American type option. We use a discrete model for the underlying process and approximate American style options by Bermudan style options. Throughout the text, we refer to American style options for simplicity. We denote by J the number of early exercise possibilities. We specify the potential exercise points as $t_0 = 0 < t_1 < t_2 < ... < t_J = T$, with t_0 and T corresponding to the current time and maturity of the option, respectively, where it is implicitly assumed that the option cannot be exercised at time t_0. We do not assume a complete market, but assume the existence of a stochastic discount factor, ρ_{t_i,t_j}, that prices all assets between times t_i and t_j, $i \leq j$. We define the value of holding the option by $v^h(X_t, K) = \sup_{\tau \in \mathcal{T}} E[\rho_{t,\tau} \times v^e(X_\tau, K) | X_t = x_t]$, where $\mathcal{T}$ is the set of all stopping times, and $\rho_{t,\tau}$ is the stochastic discount factor (SDF) over the period t to τ. We assume the real option owner is risk averse, and assume the SDF is strictly positive and a decreasing function of X. We do not make further assumptions on the class of utility function, but we assume that $E[\rho_{t,\tau} \times v^e(X_\tau, K) | X_t = x_t]$ is a positive, weakly-increasing function of X, for $X \geq K$. Note that this assumption is general enough to accommodate binary or digital options. This setup is well established in the literature on American option valuation. See, e.g., Clément, Lamberton, and Protter (2002) for a rigorous analysis of the valuation method.

Prior to maturity, there exists an exercise region, for each exercise time, which we denote $\kappa_i(X) = \{x | v^e(x, K) > v^h(x, K)\}$. At any time t_i, the real option is exercised if $X_{t_i} \in \kappa_i(X)$. Note that, for a Markov process, $\kappa_i(X)$ depends on the dynamics of the process X, and the time t_i, but not the current value X_i. We define by $\bar{Pr}$ the $J \times 1$ vector of exercise probabilities. For options with multiple exercise opportunities, we are interested in the timing of the exercise. Ex ante, we cannot determine when the option will be exercised, but we can inspect $\bar{Pr}$. We propose a definition of hastening (slowing) of the exercise of real options. Exercise of a real option is hasten (delayed) by a modification if an early exercise probability increases (decreases) as a result of the modification.

A stakeholder has means to modify K, X_t, or both. The stakeholder can be the owner of the real option, the writer of the real option when he exists, or an external stakeholder. The objective of the modification to the real option characteristics is to either increase or decrease the exercise probability, and hasten or delay the exercise timing. Modifications on the strike K have trivial impacts on the exercise probability, thus we focus on modifications of X_t. However, later, we compare X_t modifications to K modifications in order to determine which has the most impact on the exercise probability, and potentially which is most cost efficient.

We consider two types of modifications. The first type is a modification that increases the value of the underlying process, and the second one is a modification that increases the risk, measured by the variability of X_t. The first type of modification makes the modified distribution first order stochastically dominant. Let Y_t be the modified distribution of the underlying process after a First order Stochastic Dominant modification (hereafter FSD), i.e, $Y_t \underset{FSD}{\ll} X_t$. The modified variable Y_t will be FSD to the variable X_t if $F_{Y_t}(e) \leq F_{X_t}(e), \forall e \in [a,b]$. Example of FSD for real options include offering a tax credit for a specific project. The tax credit will increase the present value of future cash-flows such that $Y_t = X_t + a(X_t)$, where $a()$ is a positive function that represents the present value of the future tax credits.

If a modification of the underlying distribution is not FSD, i.e., if $Z_t \underset{FSD}{\not\ll} X_t$, and $X_t \underset{FSD}{\not\ll} Z_t$, it must be true that $F_{Z_t}(e) \geq F_{X_t}(e)$, for at least one $e \in [a,b]$, and $F_{Z_t}(e) \leq F_{X_t}(e)$, for at least one $e \in [a,b]$. If, $F_{Z_t}(e) \geq F_{X_t}(e)$, for $e \leq c$, where $c \in [a,b]$ and $F_{Z_t}(e) \leq F_{X_t}(e)$, for $e \geq c$ then the modification is a QPS, as defined in Mendelson (1987). A QPS is a general representation of an increase in variability, which, in our context, models a general increase in risk. A QPS is similar in concept to a mean-preserving spread of Rothschild and Stiglitz (1970), but the preserved location measure can be any quantile.

We model the second type of modification by a QPS. First, let us define Z_t, the value of a process Z at time t, with its distribution F_{Z_t} defined on I. Let $F_{X_t}^{-1}(\alpha) = F_{Z_t}^{-1}(\alpha) = \mathcal{A}$. If $F_{X_t}(e) \leq F_{Z_t}(e)$ for $e \leq \mathcal{A}$, $F_{X_t}(e) = F_{Z_t}(e)$ for $e = \mathcal{A}$, and $F_{X_t}(e) \geq F_{Z_t}(e)$ for $e \geq \mathcal{A}$, then Z_t is a QPS of X_t, which we note $F_{X_t} \underset{QPS}{\ll} F_{Z_t}$, and the following five characterizations are equivalent.

1. Z has more weights in the tails than X.[1]

2. Let g be a weakly decreasing-weakly increasing function with its minimum at $\mathcal{A}$. then $E[g(Z_t)] \geq E[g(X_t)]$.

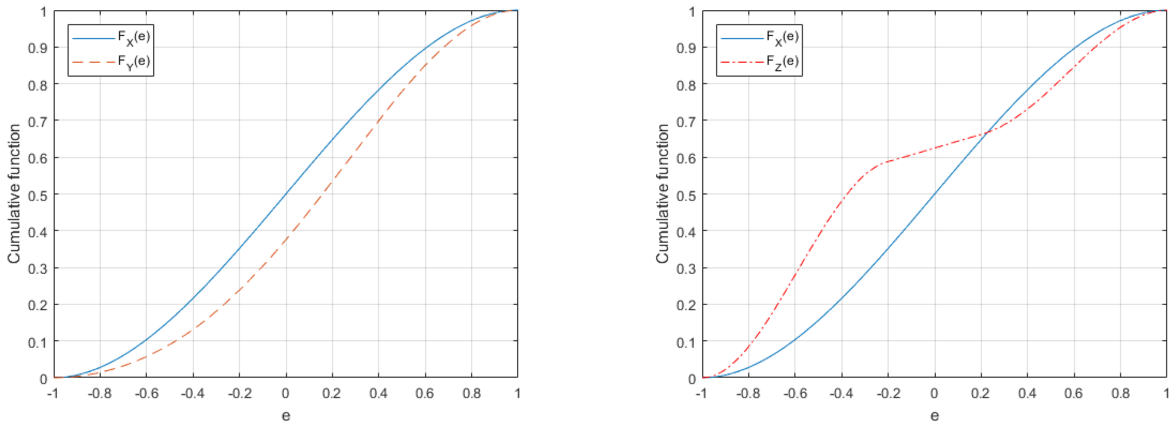
3. Both Z_t and X_t are modifications of the same variable, but Z_t is more stretched, i.e. more risky.

4. The difference generalized Lorenz curve is unimodal, non-negative, and has a maximum at α.

5. Z_t is distributed like X_t plus noise where the noise term is non-positive, zero, and nonnegative, respectively when X_t is below, at, and above the α-quantile, respectively.

For formal proofs, we refer the reader to Mendelson (1987).

[1]The tail being relative to the α-quantile.



(a) First Order Stochastic Dominance.

(b) Quantile Preserving Spread.

Figure 1: Graphical representation of the modifications.

We introduce a special case of QPS, which we call a symmetric QPS. A Symmetric QPS will stretch a distribution with equal proportions on each side the preserved quantile. We define a symmetric QPS as follows. Let $X_t \underset{\alpha}{\ll} Z_t$, and let $QPS(z) = f_z(z) - f_x(z)$. By construction, $\int_a^{\mathcal{A}} \max(QPS(z),0)dz+\int_a^{\mathcal{A}} \min(QPS(z),0)=0$, and $\int_{\mathcal{A}}^b \max(QPS(z),0)dz+\int_{\mathcal{A}}^b \min(QPS(z),0)=0$. We call $X_t \underset{\alpha}{\ll} Z_t$ a symmetric QPS if $\int_a^{\mathcal{A}} \max(QPS(z),0)\,dz = \int_{\mathcal{A}}^b \max(QPS(z),0)\,dz$. An intuitive example of a symmetric QPS is an increase in volatility. In particular, increasing volatility in a log-normal process is a symmetric QPS.

Figure 1 provides a graphical representation of both general modifications of the original random variable at t. In sub-figure (a), we have a typical first-order stochastic dominance. In sub-figure (b), the modified distribution has more weights in the tails, and the preserved quantile is above the median.

3 Results

We first establish general results for a European-style option and the exercise probability. Next, we study interesting special cases for European-style options. Finally, we study the effects of modifications on exercise timing using American-style options.

3.1 European Option Results

In this section, we study the changes in exercise probability of European call-like real options under simple modifications. We consider two major modifications for real options, one that changes the value of the underlying process, and one that changes the risk. A change in value has a trivial effect on the exercise probability, but we need a base case to assess whether it is more efficient to change the value of the underlying process, or the value of the strike. The change in risk that we consider here is very general, and clearly highlights the channel through which an increase in risk may increase the exercise probability. This section also establishes fundamental results that will be useful when studying exercise timing in a later section.

First, we study a change that adds values through a FSD modification at maturity of an European call-like real option. When comparing the distribution of the original process value X_T and the modified process value Y_T, we use the simplified notation: $Y_T \underset{FSD}{\ll} X_T$, to describe the fact that the random variable Y_T is FSD to the original random variable X_T.

Proposition 1 *Adding value to the underlying process of a European call-like real option through a FSD modification will always increase the exercise probability.*

Proof. Let $Y_T \underset{FSD}{\ll} X_T$, where X_T and Y_T represents the original and modified values of the process at $t = T$, respectively. Recall that if $Y_T \underset{FSD}{\ll} X_T$, $F_{X_t}(e) \geq F_{Y_t}$ for $e \in [a, b]$. Thus, $Pr(X_T < K) \geq Pr(Y_T < K)$, $1 - Pr(X_T < K) \leq 1 - Pr(Y_T < K)$, and $Pr(Y_T \geq K) \geq Pr(X_T \geq K)$. ■

Proposition 1 is simple. Increasing the value of the underlying process, which is used to make an exercise decision, always increases the exercise probability. In practice, it is intuitive because the payoff is an increasing function of the underlying process. However, it is not clear whether it is more cost effective to increase the process value or decrease the strike value. We answer this question in the next subsection. Following Proposition 1, here are two lemmas which will be useful in a later section.

Lemma 2 *Let* $Y_T \underset{FSD}{\ll} X_T$, *then* $E[v^e(Y_T, K)] \geq E[v^e(X_T, K)]$.

Proof. Because Y_T is stochastically dominant over X_T, we can reorder members of the sample space $\omega_i \in \Omega$ so that $i \leq j$ implies $X_T(\omega_i) \leq X_T(\omega_j)$ and $Y_T(\omega_i) \leq Y_T(\omega_j)$. Because for all $\omega_i \in \Omega$ $Y_T(\omega_i) \geq X_T(\omega_i)$, and $v^e(x, K)$ is weakly increasing in x, $E[v^e(Y_T(\omega_i), K)] \geq E[v^e(X_T(\omega_i), K)]$.

■

Lemma 3 *Let* $Y_T \underset{FSD}{\ll} X_T$*, then, for an European option,* $v^h(Y_t, K) \geq v^h(X_t, K)$*, for* $t < T$*.*

Proof. Recall that $E[\rho_{t,T} \times v^e(X_T, K) | X_t = x_t]$ is weakly increasing in X_t. Because, for an European option, $v^h(X_t, K) = E[\rho_{t,T} \times v^e(X_T, K) | X_t = x_t]$, it is easy to show that $v^h(Y_t, K) \geq v^h(X_t, K)$ using an argument similar to Lemma 2.

■

Now, let us turn to a study of a change in risk through a QPS modification of the density of the process at maturity of a European call-like real option. We define the original and modified process values at maturity by X_T and Z_T, respectively. To express that the distribution of Z_T is a QPS of the distribution of X_T by $X_T \underset{\alpha}{\ll} Z_T$.

Proposition 4 *Adding risk to the value of the underlying process of a European call-like real option through a QPS will:*

1. *Always increase the exercise probability if and only if* $\mathcal{A} < K$*,*

2. *Always decrease the exercise probability if and only if* $\mathcal{A} > K$*.*

Proof. When $X_T \underset{\alpha}{\ll} Z_T$, and $\mathcal{A} < K$. Then $F_{X_t}(e) \geq F_{Z_t}(e)$, for $e \geq \mathcal{A}$. Thus, $Pr(X_T \geq K) \leq Pr(Z_T \geq K)$. A similar argument can be made to show that $Pr(X_T \geq K) \geq Pr(Z_T \geq K)$, iif $\mathcal{A} > K$.

■

A QPS "stretches" a distribution. Proposition 4 shows that, in general, an increase in risk in the underlying process increases or decreases the exercise probability of a call-type real option, depending on how the underlying distribution at T is stretched. Specifically, two variables, the preserved quantile and strike, completely determine whether the exercise probability increases or decreases. When the preserved quantile is below (above) the strike, stretching the distribution will increase (decrease) the exercise probability of a call-like real option. Recall that α-QPS refers to a transformation that the bottom $\alpha \times 100$ percent of probability mass is stretched to the left in probability, the remaining $(1 - \alpha) \times 100$ percent of the mass is to the right, and the α-quantile is preserved. The preserved α-quantile being less than the strike is the equivalent condition for the increase in the exercise probability. If $\alpha = 0.5$ (i.e., median-preserving spread), such a condition amounts to an "out-of-the-money" status of the real option. In a later section, we will present a figure illustrating this. However, if $\alpha < 0.5$ or $\alpha > 0.5$, the condition for the increase in the exercise probability is not equivalent to the moneyness. Suppose $\alpha < 0.5$: To increase the exercise probability, α-quantile, which is less than the median, must be less the strike. The sufficient condition for increase in the exercise probability is looser than the "out-of-the-money" status of the real option. Even in the case that a call-type real option is in-the-money, a QPS may still increase the exercise probability if α is sufficiently small. If $\alpha > 0.5$, the converse is true: even in the case that a real option is out-of-the-money, a QPS may decrease the exercise probability.

This Proposition generalizes previous results on that topic. In particular, Sarkar (2000) find that the probability of reaching the exercise boundary is not a monotone function of volatility. Proposition 4 shows that Sarkar's (2000) result holds for a more general definition of risk, not limited to volatility, and without assuming a infinite horizon or a geometric Brownian motion. In Subsection 3.4, we discuss how Proposition 4 corresponds to some empirical results in the literature. Following Proposition 4, here a some useful lemmas.

Lemma 5 *Let* $X_T \underset{\alpha}{\ll} Z_T$ *and* $\mathcal{A} < K$*, then,*

1. $E\left[v^e\left(X_T, K\right)\right] \leq E\left[v^e\left(Z_T, K\right)\right]$,
2. $v^h\left(X_t, K\right) \leq v^h\left(Z_t, K\right)$, *for* $t < T$.

Proof. $v^e\left(\cdot\right)$ is a weakly decreasing-weakly increasing function with its minimum at $\mathcal{A}$. If $\mathcal{A} < K$, the second characterization of a QPS completes the proof of Claim 1. Claim 2 is direct from Claim 1, the definition of v^h and the assumption that $E\left[\rho_{t,\tau} \times v^e\left(X_\tau, K\right)|X_t = x_t\right]$ is a positive, weakly-increasing function of X, for $X \geq K$.

■

Lemma 6 *Let* $X_T \underset{\alpha}{\ll} Z_T$*, a symmetric QPS,* $\mathcal{A} > K$*, and* $v^e\left(\cdot\right)$ *is convex. Then,*

1. $E\left[v^e\left(X_T, K\right)\right] \leq E\left[v^e\left(Z_T, K\right)\right]$,
2. $v^h\left(X_t, K\right) \leq v^h\left(Z_t, K\right)$, *for* $t < T$.

Proof. If $v^e(\cdot)$ is convex and risk is added through a symmetric QPS, by convexity and simple function approximations, it is easy to prove that $E[v^e(Z_T, K)] \geq E[v^e(X_T, K)]$ to complete the proof of Claim 1. Claim 2 is direct from Claim 1, the definition of v^h, and the assumption that $E[\rho_{t,\tau} \times v^e(X_\tau, K) | X_t = x_t]$ is a positive, weakly-increasing function of X, for $X \geq K$.

■

Corollary 7 *Let $X_T \underset{\alpha}{\ll} Z_T$, $\mathcal{A} > K$, and $v^e(\cdot)$ be weakly-increasing but not convex. then, it is possible that*

1. $E[v^e(X_T, K)] \geq E[v^e(Z_T, K)]$,
2. $v^h(X_t, K) \geq v^h(Z_t, K)$, *for* $t < T$.

Proof. We prove both Claims by simple opposition to Lemma 6.

■

Corollary 8 *Let $X_T \underset{\alpha}{\ll} Z_T$, an asymmetric QPS, and $\mathcal{A} > K$, then, it is possible that*

1. $E[v^e(X_T, K)] \geq E[v^e(Z_T, K)]$,
2. $v^h(X_t, K) \geq v^h(Z_t, K)$, *for* $t < T$.

Proof. We prove both Claims by simple opposition to Lemma 6.

■

3.2 Special Cases for European Options

The results in the previous subsection are very general. Here, we present special cases. These cases are interesting because they are more intuitive and give insights into designing efficient policies.

Proposition 9 *Let $W_T = (1+e)(X_T - \mathcal{A}) + \mathcal{A}$ where $a \leq A \leq b$ and $e > -1$.*

1. *Suppose $e > 0$. W_T is a quantile-preserving spread of X_T. Furthermore, $Pr(W_T > K) \geq Pr(X_T > K)$ if and only if $K \geq \mathcal{A}$.*
2. *Suppose $e < 0$. X_T is a quantile-preserving spread of W_T. In other words, W_T is a quantile-preserving "contraction" of X_T. Furthermore, $Pr(W_T > K) > Pr(X_T > K)$ if and only if $K < A$.*

Proof. Because the support of X_T is $[a, b]$, there exists at least one $\alpha \in [0, 1]$ such that the α-quantile of X_T is $\mathcal{A}$. Define $g(x) \equiv (1+e)(x-\mathcal{A})+\mathcal{A}$. Because $(1+e)>0$, $W_T = g(X_T)$ strictly increases as X_T increases. Because the α-quantile of X_T is $\mathcal{A}$, the α-quantile of $W_T = g(\mathcal{A}) = (1+e)(\mathcal{A}-\mathcal{A})+\mathcal{A}=\mathcal{A}$. Hence, X_T and W_T share the same α-quantile, $\mathcal{A}$.

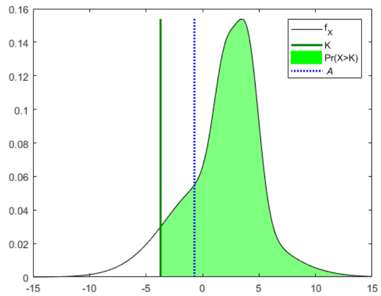
Suppose $e > 0$. Because W_T has more weights in the tails away from $\mathcal{A}$ than X_T does, W_T is a quantile-preserving spread of X_T. Hence, $Pr(W_T \leq K) \leq Pr(X_T \leq K)$ if and only if $K \geq \mathcal{A}$. In other words, $Pr(W_T > K) \geq Pr(X_T > K)$ if and only if $K \geq \mathcal{A}$.

Suppose $e < 0$. Because X_T has more weights in the tails away from $\mathcal{A}$ than W_T does, X_T is a quantile-preserving spread of W_T. Hence, $Pr(X_T \leq K) \leq Pr(W_T \leq K)$ if and only if $K \geq \mathcal{A}$. In other words, $Pr(X_T > K) \geq Pr(W_T > K)$ if and only if $K \geq \mathcal{A}$, and we obtain the required results.

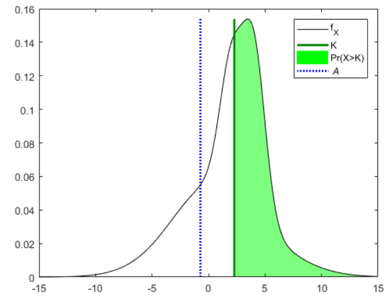
■

Proposition 9 addresses a transformation which changes the risk of the underlying process of a real option. Within Proposition 9, $e > 0$ indicates the increase of the risk, and $e < 0$ the decrease of the risk. The main take-away is that the change in the exercise probability depends on location of the strike relative to the preserved quantile $\mathcal{A}$. Take for example, $\mathcal{A} = Median[X_T]$. The transformation $W_T = (1+e)(X_T-\mathcal{A})+\mathcal{A}$ where $e > 0$ increases the risk of X_T without changing the median (our reference quantile) and increases the probability of exercising the call if $K > Median[X_T]$. In other words, if $Pr(X_T > K) < 50\%$, an increase in risk would increase the exercise probability of the call option.

To get better insights into Proposition 9, we provide a visual representation. Figure 2 shows the effect of increasing (decreasing) the variance of a random variable X on the probability of $Pr(X > K)$. A general distribution is generated for X using a convolution of multiple Gaussian distributions with different means and variances. The resulting distribution has a Mean$(X) = 1.8805$, a Median$(X) = 2.3738$, a Variance$(X) = 11.7798$, a Skewness$(X) = -0.4835$, and a Kurtosis$(X) = 4.1012$. At the top of the figure, the initial distribution is represented, the green shaded area highlights $Pr(X > K)$. We apply to modification $W_T = (1+e)(X-\mathcal{A})+\mathcal{A}$, where $e = 0.25$ ($e = -0.25$). We select $\mathcal{A}$ as the second quintile. We find the proportion of the distribution of X which is above the strike K, and we represent this region using shading on the transformed distribution. This shading helps visualize whether the probability has increased or decreased. On

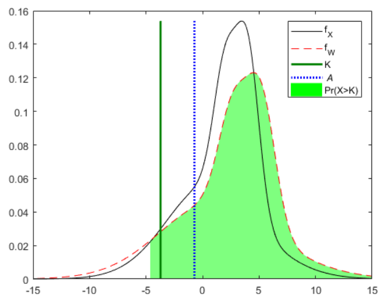


(a) $\mathcal{A} > K$

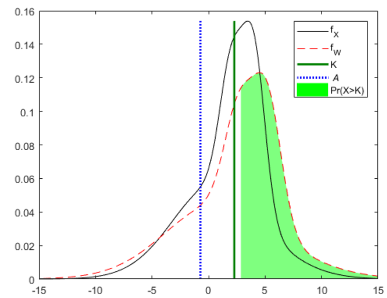


(b) $\mathcal{A} < K$

Initial situation

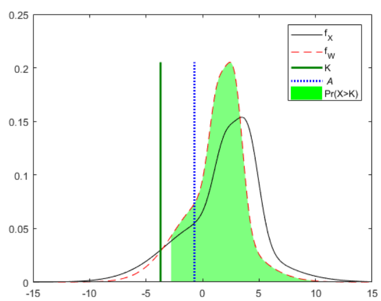


(c) $\mathcal{A} > K$

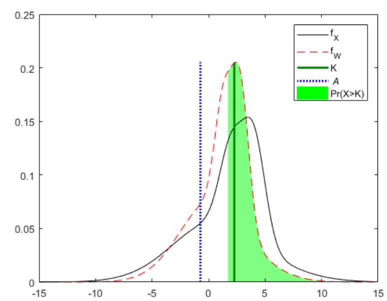


(d) $\mathcal{A} < K$

Increasing Variance



(e) $\mathcal{A} > K$



(f) $\mathcal{A} < K$

Decreasing Variance

Figure 2: Illustration of Proposition 9.

This Figure shows the effect of modifying the variance of a general distribution on the exercise probability of a call option. We set $\mathcal{A}$ as the second quintile. The black solid line represents the original general distribution of X (a convolution of multiple normal distributions). Mean(X) = 1.8805, Median(X) = 2.3738, Variance(X) = 11.7798, Skewness(X) = −0.4835, and Kurtosis(X) = 4.1012. The red dashed lines represents $g\,(X)$. The modification increases (or decreases) the variance using $W_T = (1+e)\,(X-\mathcal{A})+\mathcal{A}$, where $e = 0.25$ ($e = -0.25$). We are interested in how $Pr\,[g\,(X;\mathcal{A},e) > K]$ compares to $Pr\,[X > K]$. The shaded area represent the proportion of $X > K$ projected onto $g\,(X)$ In (c), and (f) the probability decreases. In (d), and (e) the probability increases.

the left side, we have the case where $\mathcal{A} > K$, as in Proposition 9 Claim 1. Observe that when the variance is increased, a portion of the distribution that was above the strike K is now below (see the shaded region below K), thus the probability is decreased by the increase in variance. Whereas the probability increases when the variance decreases. On the right side, we have the case where $\mathcal{A} < K$. The effects of increasing and decreasing the variance are reversed.

A modification of risk as in Proposition 9 applies well to cases where a negligible impact on a reference quantile (e.g., median) of the distribution can be assumed. However, this is not always the case. In addition, if a transformation is a first-order stochastically dominant one, a reference quantile is not even defined. Hence, we consider a transformation which combines a quantile-preserving spread (or contraction) and a first-order stochastically dominant (or dominated) transformation. Furthermore, we decompose the effect of exercise probability into the effect of changing the level of risk (a quantile-preserving spread or contraction) and that of changing the location (stochastically dominant or dominated).

Proposition 10 *Let a real number $\varepsilon > 0$ represent a proportional increase ($\varepsilon > 1$) or decrease ($\varepsilon < 1$) of underlying process X_T. Let a real number $p > 0$ represent a proportional increase ($p > 1$) or decrease ($p < 1$) of strike K.*

1. *Let a transformed process $W_T = \varepsilon X_T$. If $\varepsilon > 1$, W_T is stochastically dominant over X_T. If $\varepsilon < 1$, W_T is stochastically dominated by X_T. In other words, $Pr\left(W_T > K\right) \geq Pr\left(X_T > K\right)$ if and only if $\varepsilon > 1$. Within this proposition, ε stretches or contracts the underlying process X_T in a stochastically dominant or dominated way.*

2. *Let a transformed process $W_T = \varepsilon X_T$. $Pr\left(W_T > pK\right) \geq Pr\left(X_T > K\right)$ if and only if $\frac{\varepsilon}{p} > 1$. Within this proposition, p raises or lowers the strike price. The role of p is similar to that of ε because the increase/decrease of the exercise probability depends on $\frac{\varepsilon}{p}$.*

3. *Let a transformed process $W_T = \varepsilon\left[(1+e)\left(X_T - \mathcal{A}\right) + A\right]$ where $a \leq \mathcal{A} \leq b$ and $e > -1$. Note that W_T combines stochastically dominant (or dominated) transformation by ε with quantile-preserving spread (or "contraction") by e and $\mathcal{A}$. It follows that $Pr\left(W_T > pK\right) \geq Pr\left(X_T > K\right)$ if and only if $K\left((1+e) - \frac{p}{\varepsilon}\right) > e\mathcal{A}$. In other words, if $(1+e) > \frac{p}{\varepsilon}$, $\frac{W_T}{p}$ is a quantile-preserving spread of $\frac{X_T}{p}$ with the preserved quantile of $\frac{eA}{(1+e)-\frac{p}{\varepsilon}}$. On the other hand,*

if $(1+e) < \frac{p}{\varepsilon}$, $\frac{W_T}{p}$ *is a quantile-preserving "contraction" of* X_T *with the preserved quantile of* $\frac{e\mathcal{A}}{(1+e)-\frac{p}{\varepsilon}}$. *This claim nests Proposition 9 because* $\frac{p}{\varepsilon} = 1$ *reduces this claim to Proposition 9.*

Proof. To prove the first claim, let $W_T = \varepsilon X_T$. If $\varepsilon > 1$, $Pr(W_T \leq K) = Pr\left(X_T \leq \frac{K}{\varepsilon}\right) < Pr(X_T \leq K)$ for all K. Hence, W_T is stochastically dominant over X_T. If $\varepsilon < 1$, $Pr(W_T \leq K) = Pr\left(X_T \leq \frac{K}{\varepsilon}\right) > Pr(X_T \leq K)$ for all K. Hence, W_T is stochastically dominated by X_T. Combining these two, we find that $Pr(W_T > K) \geq Pr(X_T > K)$ if and only if $\varepsilon > 1$. A similar logic easily proves the second claim. To prove the third claim, let $W_T = \varepsilon[(1+e)(X_T - \mathcal{A}) + \mathcal{A}]$.

$$\begin{aligned} Pr(W_T > pK) &= Pr(\varepsilon[(1+e)(X_T - \mathcal{A}) + \mathcal{A}] > pK) \\ &= Pr\left(X_T > \frac{\frac{p}{\varepsilon}K - \mathcal{A}}{(1+e)} + \mathcal{A}\right) \\ &= Pr\left(X_T > \frac{\frac{p}{\varepsilon}K + e\mathcal{A}}{(1+e)}\right) \geq Pr(X_T > K) \end{aligned}$$

if and only if $\frac{\frac{p}{\varepsilon}K + e\mathcal{A}}{(1+e)} \leq K$. Furthermore, $\frac{\frac{p}{\varepsilon}K + eA}{(1+e)} \leq K$ is equivalent to $K\left((1+e) - \frac{p}{\varepsilon}\right) \geq e\mathcal{A}$. Consider the case that $(1+e) > \frac{p}{\varepsilon}$. $Pr\left(\frac{W_T}{p} > K\right) \geq Pr[X_T > K)$ if and only if $K \geq \frac{e\mathcal{A}}{(1+e)-\frac{p}{\varepsilon}}$. Consider the case that $(1+e) < \frac{p}{\varepsilon}$. $Pr\left(\frac{W_T}{p} > K\right) \geq Pr(X_T > K)$ if and only if $K \leq \frac{e\mathcal{A}}{(1+e)-\frac{p}{\varepsilon}}$. Hence, we obtain the required results. ■

In Proposition 10, W_T is a modification changing a quantile $\mathcal{A}$ to $\varepsilon\mathcal{A}$ and increasing (decreasing) risk. e represents a risk increase (decrease) separable to a quantile change.

The third claim tells us that increasing (decreasing) the quantile by a factor ε has the same effect on the exercise probability as decreasing (increasing) the strike by a factor $p = 1/\varepsilon$. However, they are not economically equivalent. If $p = \varepsilon$, the third claim of Proposition 10 reduces to the third claim of Proposition 9. In addition, the effect of modifying the location parameter/strike $(K\left(1 - \frac{p}{\varepsilon}\right))$ is additively separate from that of the "pure" modification of risk (e.g., $e(K - \mathcal{A})$). Finally, if the investment opportunity is out of the money, inflating ε (or deflating p) and inflating risk modification e are complimentary; if the investment opportunity is in the money, inflating ε (or deflating p) and deflating risk modification e are complimentary.

Corollary 11 *Assume a European real option as in Proposition 10. Consider a stakeholder who wants to increase the exercise probability by modifying either the underlying process or the strike. Within the context of Proposition 10, the stakeholder either increases* ε *from* 1 *to* $\left(1+\varepsilon'\right)$ *or* *decreases p to $\left(1-p^{'}\right)$, where $\varepsilon^{'} > 0$ and $0 < p^{'} < 1$. We claim that $\varepsilon^{'} = p^{'} + \varepsilon^{'} p^{'} > p^{'}$. The proportional modification of the underlying process $\varepsilon^{'}$ is greater than the proportional modification of the strike $p^{'}$ required to induce the same increase in the exercise probability.*

Proof. According to Claim 2 of Proposition 10, $\left(\frac{\varepsilon}{p}\right)$ determines the increase (decrease) of the exercise probability. Specifically, $\left(\frac{\varepsilon}{p}\right) > 1$ induces an exercise probability increase. Then, it follows that $\left(\frac{1+\varepsilon^{'}}{1}\right) = \left(\frac{1}{1-p^{'}}\right) > 1$. Organizing the terms, we obtain the required results. ■

Policy implications of Proposition 10 are as follows. At least within a European call real option setting, to increase the exercise probability of an out-of-the-money real option, stakeholders should modify the underlying process to a more profitable one and increase the risk of the underlying process. The relationship between giving incentive and raising risk is not a substitute but a compliment. This is in a sharp contrast with the conventional wisdom that a government should decrease risk to stimulate investment.

We may consider a stakeholder with limited budget for modifying an exercise probability. Corollary 11 tells us that if we measure cost-effectiveness of the real option's modification by proportional rates of changes such as $\varepsilon^{'}$ and $p^{'}$, the strike modification is more cost-effective than the underlying process modification. However, it is premature to conclude that the strike modification is preferred to the process modification. We will highlight this in a numerical example.

3.3 American Option Results

To study the effect of modifications on the exercise timing, we now turn to American-style options. Again, we consider modifications that add value (i.e., FSD) or add risk (i.e., QPS). Because the exercise timing depends on the modification type and timing, this leads to a large number of scenarios. For simplicity, we consider a real option where $J = 2$. Thus, we consider modification at one or more times among t_1 and t_2. We assume the strike value K to be constant. To determine whether a modification has a hastening or delaying effect, we inspect the exercise probability at $t = 1$. In particular, an increase (decrease) of the $t = 1$ exercise probability is defined as hastening (delaying). As detailed in the previous section, the exercise boundary is determined by the intersection between the holding value of the option and its exercise value. The holding value at $t = 1$ is $v^h(X_1, K) = E\left[\rho_{t,\tau} \times v^e(X_2, K) \,|X_1 = x_1\right]$, where it is implicit that the density of X_2

is conditional on not exercising at $t = 1$. This is not different from well established American option valuation techniques using backward induction algorithms, such as binomial trees or the Least-Square Monte Carlo method of Longstaff and Schwartz (2001).

We will model two types of modification timing. The first one is a late timing, where the underlying process is modified once at a later period. The second one is an early timing, where the underlying process is modified once at an early period. In all cases, we assume that both processes are started at the same value at $t = 0$. That is, $x_0 = y_0$ for a FSD modification and $x_0 = z_0$ for a QPS modification. Slightly abusing a notation, we use $\{X_j|x_i\}$ to denote the distribution of the random variable X at $t = j$, conditional on the information available at $t = i$, where $i < j$. In particular, we represent the comparison of the distribution of the value of modified process at $t = 1$, conditional on the information at $t = 0$ to the original distribution by $\{X_1|x_0\} \underset{FSD}{\ll} \{Y_1|x_0\}$. We use a similar notation for the QPS modification.

For a case of a late timing, we assume the modification occurs at $t = 2$ for a 2-period option. We will study the effect of this modification on the expected exercise probability at $t = 1$. The distribution at $t = 1$ will be the same for both original and modified processes. The distribution at $t = 2$ will be different for the modified process.

For the second case with an early timing, we identify two types of modifications; a permanent shock and a temporary shock. In the first type, the distribution at $t = 1$ is modified and the modification is permanent. However, there are no more modifications on the process after the initial modification. This means the distribution at $t = 2$ conditional on the information at $t = 1$ will be unchanged from the original one. The distribution at $t = 1$ will be different for the modified processes. Because the shock is permanent, the unconditional distribution at $t = 2$ will also be different. In the second type, the distribution at $t = 1$ is modified, but the modification is temporary, unless the option is exercised. The process is modified and the distribution at $t = 1$ is modified. If the owner decides to execute the option, the payoff is a function of the modified process. However, if the option is held, the modification is eliminated and the process reverts to the original process. In other words, the exercise value at $t = 1$ is modified, but the holding value is not.

3.3.1 Effect of One Late Modification

Here we consider cases where a modification occurs only at a later time $t = 2$. That is, the original and modified processes share the same density at $t = 1$. Again, we consider FSD and QPS modifications.

Proposition 12 *Adding value at a later time through a FSD, such that* $\{Y_1 \equiv X_1|x_0\}$ *and* $\{Y_2 \underset{FSD}{\ll} X_2|x_1\}$ *will always delay exercise of a call-like real option.*

Proof. The proof is straight forward from Lemma 20 in the Appendix. ■

Proposition 12 is intuitive. A modification proposing to increase value only at a later exercise opportunity, always delays real call option's exercise in probability. A rational owner of a real call option will prefer any additional payoff. If that payoff only occurs later, then it is optimal to wait. A policy example would be to announce, a year in advance, a tax break for investments in renewable energy. A non-retroactive tax break. The effect would delay most investments in renewable energy by one year.

Proposition 13 *Adding risk at a later time through a QPS modification, such that* $\{Y_1 \equiv X_1|x_0\}$ *and* $\{X_2 \underset{\alpha}{\ll} Z_2|x_1\}$ *will:*

1. *always delay exercise in probability if* $\mathcal{A} < K$*,*
2. *always delay exercise in probability if risk is added through a symmetric QPS, and* $v^e(\cdot)$ *is convex,*
3. *may hasten exercise if* $\mathcal{A} > K$*, risk is added through a symmetric QPS, but* $v^e(\cdot)$ *is not convex,*
4. *may hasten exercise if* $\mathcal{A} > K$ *and risk is added through an asymmetric QPS.*

Proof. Claim 1 is direct from Lemma 21 Claims 1 to 4 in the Appendix.Claim 2 is direct from Lemma 21 Claim 5. Claim 3 and 4 are proven by opposition to Lemma 21 Claims 4 and 5. ■

Proposition 13 Claims 1 and 2 document sufficient conditions for which a later increase in risk always increases the early holding value of the option, and thus always delays exercise in probability. The second claim is an example of the generally accepted concept that risk (measured by a simple increase in volatility) delays investments. Proposition 13 Claims 3 and 4 document two channels through which general increases in risk later in the life of the real option may hasten exercise. In particular, Claim 4 is intuitive. An event that adds skewness towards lower payoffs at a later date may decrease the value of holding the option and increase an earlier exercise probability.

3.3.2 Effect of One Early Modification

In this subsection, we consider a modification that occurs at time 1, conditional on exercise at $t = 1$. First, we study the effect of temporary modifications that disappear after $t = 1$, i.e., the unconditional distribution at $t = 2$ remains unchanged. Second, we study the effect of permanent modifications that occurs only at $t = 1$. In this case, the conditional distribution at $t = 2$ remains unchanged.

Proposition 14 *A temporarily early modification that adds value through a FSD such that* $\{Y_1|x_0\} \underset{FSD}{\ll} \{X_1|x_0\}$ *and* $\{Y_2|x_0, \tau > 1\} \equiv \{X_2|x_0, \tau > 1\}$ *will always hasten exercise of a call-like real option.*

Proof. The proof is direct from Lemma 22. ■

Proposition 14 is intuitive. Adding value at an early exercise, but not later, hastens exercise in probability. As an example, if a government offers a tax credit for firms investing in solar energy research in this year, but the grant would not be offered in a future year, the probability that a firm would invest in solar energy research would be increased.

Proposition 15 *A temporarily early modification that adds risk through a QPS such that* $\{X_1|X_0 = x_0\} \underset{\alpha}{\ll} \{Z_1|Z_0 = x_0\}$ *and* $\{X_2|X_0 = x_0, \tau > 1\} \equiv \{Z_2|Z_0 = x_0, \tau > 1\}$*:*

1. *Exercise is always delayed in probability if* $\mathcal{A} > \kappa_1(X)$*,*

2. *Exercise is always hasten in probability if* $\mathcal{A} < \kappa_1(X)$*.*

Proof. The proof is direct from Lemma 23 in the Appendix. ■

Proposition 15 follows Proposition 13. An early increase in risk may increase an early exercise probability and hasten the exercise of American style real call options. As an example, imagine a typical project where X_t is the present value of future cash flows, and K the capital cost. At $t = 0$, unexpected elections are announced for $t = 1$. The expected present value of future cash flows at $t = 1$ is more risky. This increase in risk may, or may not, increase the exercise probability at $t = 1$, depending on the relative value of the preserved quantile with respect to the strike value. After the election, the risk of X will be less. However, an early increase in risk may decrease the early exercise value while leaving the holding value the same. This decreases the early exercise probability and delays exercise.

Proposition 16 *A permanent early modification that adds value through a FSD such that* $\{Y_1|x_0\} \underset{FSD}{\ll} \{X_1|x_0\}$ *and* $\{Y_2|x_1\} \equiv \{X_2|x_1\}$ *will always hasten the exercise of call-like real options if the exercise region is a closed set including the upper bound of the domain of* X*.*

Proof. See Lemma 24 in the Appendix. ∎

Proposition 17 *A permanent early modification that adds risk through a QPS such that* $\{X_1|x_0\} \underset{\alpha}{\ll} \{Z_1|x_0\}$ *and* $\{X_2|x_0\} \equiv \{Z_2|x_0\}$ *for an option for which the exercise region is a closed set including the upper bound of the domain of* X*:*

1. *will always hasten exercise in probability if* $\mathcal{A} < \kappa_1(X)$*,*

2. *will always delay exercise in probability if* $\mathcal{A} > \kappa_1(X)$*.*

Proof. See Lemma 25 in the Appendix. ∎

Proposition 17, just like Proposition 15, shows that risk may hasten or delay exercise. However, the mechanism through which the exercise is hasten or delayed is different. Furthermore, when the preserved quantile is below the exercise boundary, exercise is always hastened.

3.3.3 Effect of Multiple Modifications

In this section, we consider modifications that affect the process at each time steps. We do not restrict to modifications which are identical at each time step. However, we consider only modifications which are of the same type. In particular, if a modification adds value at $t = 1$, it also adds value at $t = 2$, though the value can be different. If a modification adds risk at $t = 1$, it also adds risk at $t = 2$, though the risk can be different. For the later, the results depend on several factors and specific conclusions cannot be drawn. However, our framework shows how, in general, risk can hasten or delay the exercise of American style real options.

Table 1: Summary of the Effect of a Modification of Exercise Timing of American Style Options

Modification	When	Type	Exercise
Add Value (FSD)	$t=2$		Always Delay
	$t=1$	Temp. or Perm.	Always Hasten
Add Risk (QPS)			
$A<K$	$t=2$		Always Delay
$A>K$, Symmetric QPS and $v^e(\cdot)$ convex	$t=2$		Always Delay
$A>K$, Asymmetric QPS or $v^e(\cdot)$ non-convex	$t=2$		May Hasten
$A<\kappa_1(X)$	$t=1$	Temp. or Perm.	Always Hasten
$A>\kappa_1(X)$	$t=1$	Temp. or Perm.	Always Delay

Note: This table shows the effect of a basic modification on the exercise timing of American style options with two exercise dates.

Example of a FSD modification offered at each time step includes increasing the risk-free rate and drift under a Geometric Brownian Motion. Under this specific model, exercise would be delayed.

Proposition 18 *Adding the same value at each time step to the underlying process of a call-like real option through a FSD modification may hasten or delay exercise, depending on the payoff function.*

Proof. We prove the theorem by providing an example that delays exercise and an example that hastens exercise. First, imagine an option for which the underlying process follows a geometric Brownian motion, and value is added at each time step by increasing the risk-free rate. It is easy to verify that the continuation value will increase and exercise will be delayed. At the limit, when the risk-free rate becomes larger than the cash distribution rate then a call option is never exercised early. Proving adding value may delay exercise.

Second, imagine a real option modeled by a digital option. In such a case, the exercise boundary is known with certainty and $\kappa_i(\cdot)=K, \forall 0<i<T$. Thus $Pr(Y_1>K)>Pr(X_1>K)$ if $\{Y_1 \underset{FSD}{\ll} X_1\}$. Proving adding value may hasten exercise. ■

Proposition 18 is important even if the result is ambiguous. A modification that only adds value may hasten or delay exercise. A real option modeler, with a complex model under study, cannot conclude that exercise will be delayed by adding value at multiple time steps on the basis that this is what we learn from a typical call real option. It shows that a real-option modeler need to be careful before applying generally accepted results.

Table 1 show the summary of the effect of modifications on the exercise timing of American style options. In general, if value is offered early during the maturity of the American style option, the option holder will prefer receiving this value early and the exercise time will be decreased in probability. However, if value is offered on later exercise dates, then the option holder may delay exercise to profit from the added value. When it comes to adding risk, the results we get in this very general framework confirms empirical results obtained in the literature where risk, sometimes hasten investments.

3.4 Discussion of Propositions and Empirical Evidences

Our results have implications for owners, writers, and stakeholders of real options. In Merton's (1974) structural model framework, a shareholder of a public firm is the owner of a real call option on firm's income, and a creditor is the real put option writer (see Sundaresan 2013 for a literature review on the topic). Both the shareholders and the creditors can modify the underlying process (operating/non-operating income) of the real option by influencing a firm's strategic, production, and marketing decisions. Let us shift our attention to the exercise probability. Because of the costly nature of financial distresses, finance literature has studied the probability of default (PD hereafter). See, e.g., Elkamhi, Jacobs, and Pan (2014), among many others. Our results are relevant in this topic by providing model free conditions for increasing and decreasing physical measure exercise probabilities of European call and put-type real options. To reduce the expected loss (PD times loss given default), both shareholders and creditors are enticed to modify the underlying process in the direction of increasing (decreasing) the physical probability of exercising a call (put) option. So is a government who is responsible for financial stability of national economy. Proposition 4 tells us that owners, writers, and stakeholders want to decrease (increase) risk as long as the preserved quantile is above the strike.

The results also have implications to real option stakeholders such as a government considering incentive (penalty) programs for private sector's investment in certain technologies. Let us take, as an example, investments in green energy. Suppose that the government estimates that the investment in natural gas power plants, as opposed to coal power plants is more likely, i.e., the probability is greater than 50%. Assuming a change in risk would be median-preserving, for simplicity. Based on our proposition, the government should reduce risk by reassuring investors that current tax credits would not change. However, if the government wants to promote investments in cellulosic-ethanol, and that such investments are currently less likely, then the government should both offer new tax credits and let the investors know that the situation may change in the future and become even better. This example combines Proposition 1 to add value and 4 to increase risk in the case where the preserved quantile is below the strike of the real option.

Another example of Proposition 4 is a portfolio manager reallocating his portfolio to change the level of risk. The real option of the portfolio manager can be modeled by a call option where the underlying process is the difference between the portfolio's and benchmark's performances, and the strike is set to zero.[2] Grinblatt and Titman (1989) highlights the potential agency problem related to performance-based contracts for portfolio managers. Carpenter (2000) solves the optimal dynamic investment policy for a risk-averse manager paid with a call option on the assets he controls, which are modeled by a multivariate Brownian motion. She finds that the manager's optimal decision might be to increase or decrease the risk of the assets he controls, depending on the current probability of exercising the option (receiving the incentive). Brown, Goetzmann, and Park (2001) find empirical support for Carpenter's (2000) proposed theoretical result. Good performers in the first half of the year (who have a high probability of exercising their real option linked to their performance based contract) reduce the volatility of the portfolio under management, while bad performers increase the volatility. In particular, for a portfolio manager, we can easily assume a change in variance of the portfolio has minimal impact on the median of the distribution. In such a case, Proposition 9 suggests that when the exercise probability is greater (lower) than 50%, the manager should decrease (increase) the variance to increase the exercise probability. Proposition 9 is confirmed in empirical studies such as Brown, Goetzmann, and Park (2001). Portfolio managers with performance-based contracts for which the embedded real option has a high (low) exercise probability tend to decrease (increase) the risk of the portfolio to increase their exercise probability.

To the best of our knowledge, there are no empirical studies that fully test our Proposition 10. Current empirical researches only show partial applications of the proposition. For example, most papers studying project valuation using real options show an increase in variance causing a decrease in investments, and some papers show the inverse. This is a manifestation of Proposition 10. An empirical test requires the estimation of the exercise probability prior to a policy change, observation

[2]The performance could be a simple return, a Sharpe ratio, or an asset pricing model's alpha.

of a policy change, and estimation of the exercise probability after the policy change. Such an empirical testing of our Proposition 10 deserves a research project on it's own. In the next section, we present numerical evidences for both European and American style real options.

4 Numerical Examples

In this section, we discuss some numerical examples. First, we provide an example comparing the cost of modifying a strike or an underlying process of a real call option. Second, we give a numerical example of an increase in risk, modeled by a QPS that increases the exercise probability. Third, we provide a numerical example for an American style real call option.

4.1 Cost-Effectiveness of Modifying a European Real Option

In this subsection we present a numerical example for Corollary 11. Assume a simple option from Dixit and Pindyck (1994) where the underlying process $X(t)$ follows a Geometric Brownian motion with:

$$dX(t) = \mu X(t)\,dt + \sigma X(t)\,dW(t)$$

where $X(0) = 100$, $\mu = 0.06$, $\sigma = 0.20$, and $W(t)$ is a Brownian Motion. Consider a European call real option with expiration $T = 1$ and strike $K = 100$. Because $S(0) = 100 = K$, this real option is at-the-money. The exercise probability is $Pr(X(t) > K)$. A stakeholder considers two types of modifications to increase the exercise probability at maturity:

1. Process modification: Increase the underlying process from $X(T)$ to $\left(1+\varepsilon^{'}\right)X(T)$, where $\varepsilon^{'} > 0$. Then, the exercise probability increases from:

$$Pr(X(t) > K) \text{ to } Pr\left(\left(1+\varepsilon^{'}\right)X(t) > K\right).$$

2. Strike modification: Decreases the strike price from K to $\left(1-p^{'}\right)K$, where $0 < p^{'} < 1$. Then, the exercise probability increases from $Pr(X(t) > K)$ to $Pr\left(X(t) > \left(1-p^{'}\right)K\right)$.

According to claim 2 of Proposition 10, these two modifications are equivalent if and only if

$\frac{1+\varepsilon^{'}}{1} = \frac{1}{1-p^{'}}$. Let us consider $\varepsilon^{'} = 0.25$, then $p^{'} = 0.20$. As Corollary 11 claims:

$$\varepsilon^{'} = 0.25 = 0.20 + 0.25 \times 0.20 > 0.20 = p^{'}.$$

Before the modification, the exercise probability is:

$$Pr\left(X\left(T\right) > K\right) = 0.58.$$

After the modification, we have:

$$Pr\left(\left(1-p^{'}\right)X\left(T\right) > K\right) = Pr\left(X\left(t\right) > \left(1-p^{'}\right)K\right) = 0.91.$$

To increase the option's exercise probability from 58% to 91%, the stakeholder has two choices: a proportional strike reduction of 20% or a proportional increase in underlying process of 25%. Observe that the proportional strike modification is less than the proportional underlying process modification, conforming to Corollary 11. This finding is relevant to a stakeholder with limited budget.

The proportional rate of modification does not directly speak to the dollar amount of costs for real option modification. There are several ways of quantifying the dollar costs for real option modification depending on when a stakeholder pays for modification costs and how the modification costs are measured.

First, we can measure the modification costs assuming that the stakeholder pays at time zero to modify the underlying process or the stakeholder pays the subsidies for strike price if the real option is exercised at time T. For example, consider adding capital to increase firm value at $t = 0$. Specifically, a stakeholder may increase $X\left(0\right)$ to $\left(1+\varepsilon^{'}X\left(0\right)\right)$ to modify $X\left(T\right)$ to $\left(1+\varepsilon^{'}X\left(T\right)\right)$ or decrease strike price from K to $\left(1-p^{'}\right)K$. In the former case, the costs for option modification incur regardless of whether the real option is exercised or not. The certain costs for modifying $X\left(T\right)$ are \$25 ($= 100 * 0.25$) at time 0. Next, let us consider the modification of the strike price. The stakeholder pays \$20 ($= 100 * 0.20$) at time T only when a real option is exercised. Otherwise, the cost for modification is zero. In either case, the present value of the modification for strike is less costly than the modification of the underlying process. However, the strike modification may end up leaving some "unused budget" that may be politically challenging to a stakeholder if the stakeholder is a government.

Second, we can measure the costs for modifying real options by assuming that the underlying modification is free but strike modification is costly only when the real option is exercised. For example, a government may give tax credits when a company build a factory – such a modification is a process modification. If a company does not build a factory, the company may not pay any tax. If the company builds a factory, the tax revenue of the government may be zero or positive even after the tax credit. Comparing these two cases, we can conclude that the costs for modifying the process is actually zero or even negative. If we measure the costs for real option modification this way, we conclude that the underlying modification is more cost-effective than the strike modification.

Third, we can measure the real option modification costs assuming that the stakeholder pays the subsidies for the underlying process (e.g. proportional tax credit) or strike price at time T only if the real option is exercised. For example, a stakeholder may be a government which compares the tax revenues with and without giving the tax credits. The government may regard such "lost tax revenue" as the costs for option modification. In this case, it turns out that the strike modification is more cost-effective than the underlying modification.

Let a stakeholder pay for the costs of modification only when the real option is exercised. If $\left(1+\varepsilon^{'}\right) X\left(T\right) \leq K$, the costs for the modification is zero. Next, if $\left(1+\varepsilon^{'}\right) X\left(T\right) > K$, the cost for modification is $\varepsilon^{'} X\left(T\right)$. We can express the cost for modification as

$$\varepsilon^{'} X\left(T\right) \mathbf{1}_{\{\varepsilon' X(T)>K\}},$$

where $\mathbf{1}$ is the indicator function. The cost for process modification is either zero or $\varepsilon^{'} X\left(T\right)$. Hence, we can represent the cost range by $\{0\} \cup \left(\frac{\varepsilon^{'}}{1+\varepsilon^{'}} K, +\infty\right)$.

Similarly, the costs for strike modification is

$$p^{'} K \mathbf{1}_{\{\varepsilon' X(T)>K\}}.$$

Differently from the costs for process modification, the cost for strike modification is either 0 or $p^{'} K$. In other words, the range is $\{0, \frac{\varepsilon^{'}}{1+\varepsilon^{'}} K\}$, which is a subset of the range of the cost for process

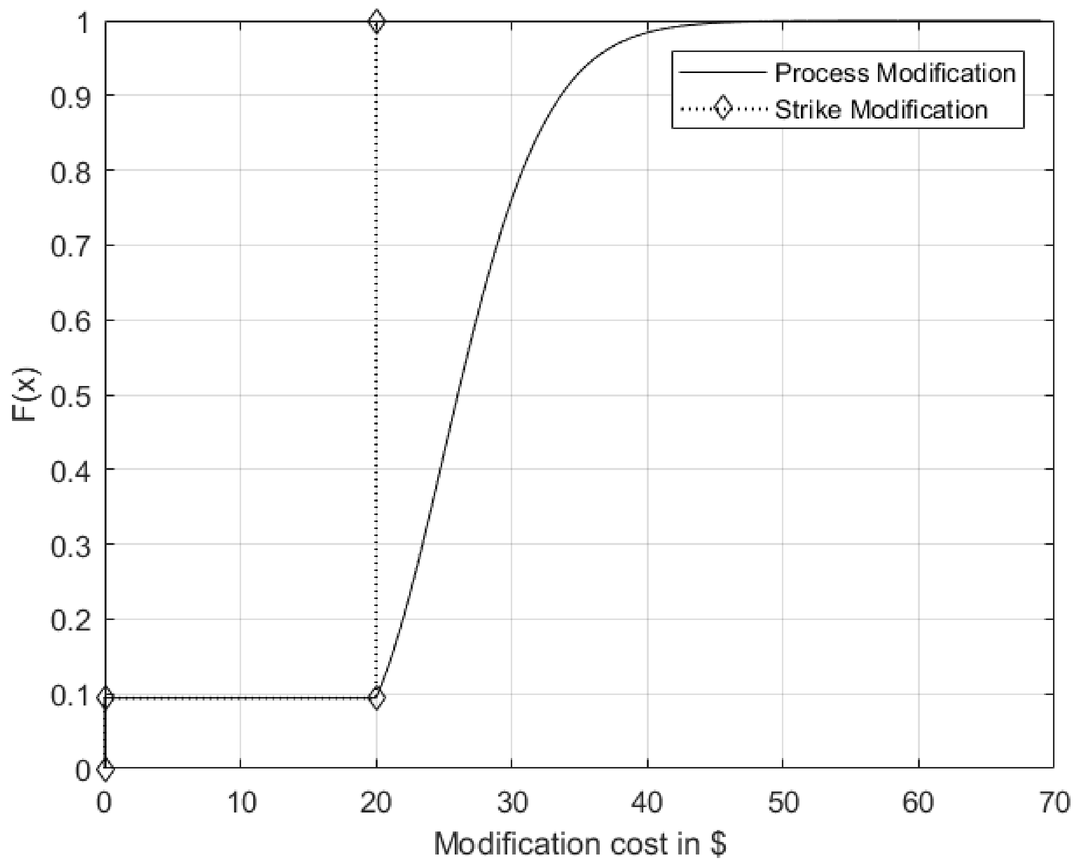


Figure 3: Cumulative distribution function of the modification cost of strike and process.

modification. Specifically, the two elements of the strike modification's range are the lowest portions of the process modification's range.

To compare cost effectiveness of the two types of modification, we report the cumulative distribution functions (CDF). Here, we report the case of an at-the-money real option, but results are similar for in-the-money and out-the-money. Figure 3 shows the CDF for the cost of modifying the strike and the process. Observe that the process modification cost is first-order stochastically dominant over strike modification cost. The bearer of these costs will prefer modifying the strike. Note that results are similar for out-the-money and in-the-money options.

4.2 Example of a QPS Decreasing the Holding Value of an Option.

In this subsection, we provide a numerical example for Proposition 13 by proposing an asymmetric QPS reducing the holding value and fastening the American-type real option exercise.

Consider a holding value function $v^h(z, 40)$ where 40 is the strike value, satisfying

$$\begin{aligned}
\frac{1}{30-20}\int_{20}^{30} v^h(z,40) &= 1; \\
\frac{1}{50-40}\int_{40}^{50} v^h(z,40) &= 10; \\
\frac{1}{60-50}\int_{50}^{60} v^h(z,40) &= 18; \text{and} \\
\frac{1}{80-70}\int_{70}^{80} v^h(z,40) &= 35.
\end{aligned} \tag{1}$$

Next, consider a continuous random variable X which an α-quantile$= 50$ and $f_X(x)$, and $f_X(x) \geq q$ for $X \in [40, 50]$, where q is a strictly positive constant. Now, consider a continuous variable Z with its p.d.f. $f_Z(z)$ defined as

$$f_Z(z) = \begin{cases} f_X(z), & z \leq 20 \\ f_X(z) + q, & 20 < z \leq 30 \\ f_X(z), & 30 < z \leq 40 \\ f_X(z) - q, & 40 < z \leq 50 \\ f_X(z) - q/100, & 50 < z \leq 60 \\ f_X(z), & 60 < z \leq 70 \\ f_X(z) + q/100, & 70 < z \leq 80 \\ f_X(z), & 80 < z \end{cases}$$

Because the α-quantile is still 50 and Z has more weight in the tails than X, Z is a QPS of X. In addition, Z is an asymmetric QPS in the sense that the spread on the left of the preserved quantile if 100 times larger than on the right.

Because the strike $K = 40 < \alpha -$ quantile, claim 1 of Proposition 13 does not apply. Now we show that contrary to the conventional wisdom in finance and real option literature, an increase in risk may reduce the holding value of the option, that is $E\left[v^h(Z, K)\right] < E\left[v^h(X, K)\right]$, even though

Z is riskier than X. To see this:

$$E\left[v^h(Z,K)\right] - E\left[v^h(X,K)\right]$$

$$= \int_{20}^{30} (q)\, v^h(z,40)\, dz + \int_{40}^{50} (-q)\, v^h(z,40)\, dz + \int_{50}^{60} (-q/100)\, v^h(z,40)\, dz + \int_{70}^{80} (q/100)\, v^h(z,40)\, dz$$

$$= q \times 10 \times 1 + -q \times 10 \times 10 + -q/100 \times 10 \times 18 + q/100 \times 10 \times 35$$

$$= q \times 10 \times (1 - 10 - 0.18 + 0.35) < 0.$$

Because the holding value decreases and the exercise value is unchanged, the American-style option exercise is fastened.

The increase in risk has two effects on an ITM option. First, the possibility that an option expires ITM decreases. Hence, the continuation value decreases. Second, the upside potential increases. Hence the continuation value increases. If the increase of risk is "symmetric", the latter always dominates the former by property of convexity and a "simple function approximation." In other words, if a QPS is "symmetric" and a payoff function is convex, the gain on the right side always outperforms than the loss on the left side as claim 2 of Proposition 13 summarizes. However, if the increase in risk is "asymmetric", we cannot use the property of convexity and a "simple function approximation" anymore, and the claim 2 of Proposition 6 does not apply. Furthermore, if the option is in-the-money and the spread of the left side is greater than that of the right side, the negative effect of "turning ITM option to OTM" reduces the holding value.

4.3 Numerical Example for American Style Options with QPS Modifications

In this section, we propose a numerical example where risk is added for a real option with multiple exercise opportunities. The objective is to illustrate the effects of adding early or late risk to the exercise probability at an early date to determine whether exercise is hasten or delayed. To fully illustrate the results from Propositions 13, 15, and 17 we need either a non-convex option payoff, or non-symmetric QPS risk. Though we could come up with examples of non-convex payoffs, they are less intuitive. Thus we propose to use asymmetric QPS modifications. We use a conventional call option to model our real option because it is appropriate for many real option cases.

The option can be exercised at the end of period 1, or the end of period 2, where period 2 is the maturity of the real call option. The option is exercised at maturity if the value of the underlying process $S(t=T=2)$ is above the strike K, and the payoff is $max(0, S(T)-K)$. At $t=1$, the option is exercise if the exercise value is worth more than the continuation value. Under the framework propose in Section 3, we are interested in the effect of a QPS on the exercise probability at $t=1$. If the early exercise probability is increased, we say the exercise is hasten. Otherwise, we say the exercise id delayed.

We model the dynamics of the original process by a diffusion process that follows a geometric Brownian motion. The process has a continuous drift of $\mu = 0.08$, a continuous cash-distribution rate $y = 0.08$, and a diffusion volatility of $\sigma = 0.20$. Under such a dynamics, the continuation value at $t = 1$ is a simple European option and the solution is given by the Black-Scholes-Merton model. The process models the present value of future cash flow from a project. This setup is inspired by the example provided in Dixit and Pindyck (1994). However, we model the real option with a maturity of two years, and one early exercise possible at $t = 1$.

Now, suppose talks about trade agreements bring additional risk to the model. Depending on trade talks, the firm may observe a significant change in input costs. If the negotiations are positive, the present value of future cash flows may jump up, if negotiations are negative, the present value may jump down. We model the arrival of trade talks using a Poisson process with intensity λ. At each jump arrival, the probability of a negative jump is p, and the probability of a positive jump is $(1-p)$. Each negative jump has a constant size of J^-, and each positive jump has a constant size of J^+. In particular, we use the following model:

$$S_t = S_0 \times exp\left(\left(\mu - y - 0.5\sigma^2\right)t + \sigma\sqrt{t}Z_t + \sum_{i=1}^{N_t} \ln D_t\right),$$

where μ is the continuous drift, y the continuous cash-distribution rate, σ the diffusive volatility, N_t is a Poisson process of intensity λ, and $D_t = \begin{cases} J^-, & with\ probability\ p \\ J^+, & with\ probability\ (1-p) \end{cases}$.

To solve the problem, we need to assume the existence of a pricing measure. We assume a risk-free rate $r_f = 3\%$ and assume the risk-neutralized process has a total drift of r_f. This means, we introduce a jump compensation factor as in:

$$S_t = S_0 \times exp\left(\left(r_f - 0.5\sigma^2\right)t + \sigma\sqrt{t}Z_t + \sum_{i=1}^{N_t}\ln D_t - L_t\right),$$

where L_t is the jump compensation factor such that $E\left[S_t\right] = S_0 \times exp\left(r \times t\right)$. To solve for the continuation value of the real option at $t = T - 1$, we use simulation and Longstaff and Schwartz's (2001) Least Squares Monte Carlo method. We need to simulate paths under the physical measure to get the exercise probability, and under the risk-neutral measure to estimate the continuation value. Solving for the effect of the modification on the exercise timing requires 5 steps. First, paths are simulated under the risk-neutral measure for both the original and modified processes. Second, the payoffs at maturity are discounted one period to $t = 1$. Third, the continuation value is approximated from a cross-sectional regression, as detailed in Longstaff and Schwartz (2001). Fourth, the continuation value is then compared to the exercise value to determine the exercise boundary at $t = 1$. Fifth, paths are simulated under the physical measure. Finally, the exercise probability is estimated by the ratio of simulated paths for which the underlying process value is above the exercise boundary. To be able to contrast the numerical results with theoretical propositions, we also need to approximate the preserved quantile.

We present here a small sample of results. Those results are robust to a large range of modeling choices and model parameters. In the sample presented here, risk is added using jumps with a Poisson process of intensity $\lambda = 1$. Jump size are modeled by $J^+ = -J^- \in \{0.01, 0.10\}$, which can model small to important jumps, and can model some asymmetry in the QPS. The probability of a negative jump is set as $p \in \{0.25, 0.50, 0.75\}$. To produce the results, we use $100,000$ simulated paths, and we repeat the experiment 100 times. Results presented in the table are averages over 100 repetitions.

What we can observe in Table 2 is very interesting, and covers the most important cases listed in Table 1. When risk is added later at $t = 2$, the exercise probability at $t = 1$ depends on whether the preserved quantile is below or above the strike. If the preserved quantile is below the strike, like in the first row, exercise should always be delayed. This is what we find. If the preserved quantile is above the strike, exercise is delayed or hastened depending on the asymmetry of the QPS (because we have a convex payoff). We present a case where exercise is delayed and a case

Table 2: Sample of numerical results

When	p	J^{+}	J^{-}	$\mathcal{A}$ $\kappa_i(X)$	$Pr(X > \kappa_i(X))$		$Pr(Z > \kappa_i(Z))$	Theory	Empirical
Late	0.50	0.10	-0.10	96<100	26.88%	>	23.34%	▼	▼
Late	0.75	0.10	-0.10	344>100	26.88%	>	26.44%	-	▼
Late	0.75	0.01	-0.10	333>100	26.88%	<	28.35%	-	▲
Early Temp.	0.25	0.10	-0.10	62<111	26.88%	<	36.86%	▲	▲
Early Temp.	0.75	0.01	-0.10	330>111	26.88%	>	18.47%	▼	▼
Early Perm.	0.25	0.10	-0.10	62<111	26.88%	<	35.47%	▲	▲
Early Perm.	0.75	0.01	-0.10	332>111	26.88%	>	19.78%	▼	▼

Note: This table shows a sample of numerical results when adding risk through a QPS. Metrics for the GBM diffusion model are found under closed-form, and metrics for the jump-diffusion model are solved numerically. For all case, $S(0) = 100$, $K = 100$, the continuous drift $\mu = 0.08$, the continuous cash-distribution rate $y = 0.08$, the diffusion volatility is $\sigma = 0.20$, and the risk-free rate is $r_f = 0.03$. We model the real option with a maturity of two years, and one early exercise possible at $t = 1$. Risk is added using jumps with a Poisson process of intensity $\lambda = 1.0$, representing an average of 1 jump every year. Jumps size are fix and modeled by J^{+} and J^{-}. At $t = 1$, the continuation value of the option is determined using a cross-sectional regression over discounted cash-flows, just like in a regular Least-Square Monte Carlo method. The exercise boundary κ is determined by the intersection between the continuation value and exercise value. The exercise probability at $t = 1$ is determined numerically by $Pr(U_1 > \kappa_1)$, U is either the the original diffusion model, or the jump-diffusion model. The simulation uses 100,000 paths, and is repeated 100 times. The estimates are the average over 100 repetitions. It is not shown in the table, but all changes in probability from adding jumps are statistically significant, based on the statistics from the 100 independent repetitions. κ_i is the exercise boundary of the original diffusion model at the time relevant to the corresponding theorem. $\mathcal{A}$ is the preserved quantile of the QPS, and is shown to compare to κ_i. Finally, ▲, ▼, and $-$, show whether the probability should increases, decreases, or whether the result depends on the asymmetry of the QPS, respectively.

where it is hastened. When risk is added early, with a permanent or temporary effect, the exercise hastening or delaying depends on whether the preserved quantile is below or above the exercise boundary. If it is below, exercised is always hasten. This is what we can observe. Those results are robust to any values we could test for for all parameters. A note though, when risk is added at period 2 only, in the majority of cases exercise is delayed. At least for the particular model we use.

5 Conclusions

The terms and conditions of financial options are contractually binding and do not change after a derivative transaction is made. In contrast, real options can be modified, because real option holders, writers, and stakeholders may have legitimate influence on the strike price and/or the underlying process of the real option. This unique feature of real options has implications to government policy, debt evaluation, and many other areas in the literature and practice of finance and economics. However, literature to date has not paid enough attention to the fundamental theory of this important topic.

In this paper, we study the effect of general modifications of real option on the exercise probability of European-style real options and the exercise timing of American-style real options. Specifically, we study FSD and QPS modifications. We find that a QPS may increase or decrease the exercise probability depending on whether a strike price is above or below the preserved quantile. In addition, if the preserved quantile is greater than the strike price, a QPS may fasten the exercise contrary to the conventional wisdom in real option and finance literature. Our paper also addresses a stakeholder with limited budget. The strike modification may be more cost-effective than the process modification depending on how the stakeholder modify the process and how he/she measures the modification costs.

The results and methods documented in this paper will open avenues for new research. First, our results have the advantage of being very general. However, they only provide general guidance for specific cases. Future research should aim at finding analytic results for more specific cases. For example, one could model the underlying process using a simple stochastic process and add risk through jumps. Such a model would allow for increases in risk, which are more general than variance and more realistic. Second, energy and environmental economists focusing on practical applications may use our approach to investigate policy implications of a green incentive/penalty policy. To do so, a researcher may want the payoff of American-style real options to be more complex than the examples in this paper. Third, corporate finance researchers or practitioners may use our framework to assess the impact of debt forgiveness and many other topics relevant in a corporate's financial decision-making. Forth, motivated by our paper, an empiricist may investigate whether a corporation tries to improve the underlying process, its own earnings, after writing a real option on its firm value.

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A Proofs for exercise timing results

In this Appendix, we present the mathematical results used to prove the propositions related to exercise timing of call-like American style real options. First, we establish some useful general results. Second, we present the results for late modifications. Third, we present the results for early and temporary modifications. Finally, we present the results for early and permanent modifications.

A.1 General Results

[THE FOLLOWING SHOULD BE EDITED.] We study three different cases: a late modification, an early-temporary modification, and an early-permanent modification. The original process is modeled by X, and observed at three time $t \in \{0, 1, 2\}$. Let ω_t be one realization of the process. Let Y and Z be the modified processes with their respective random variables on the observation times. Below is a representation of the modifications for the QPS modification.

In the case of a late modification, we have:

$$\begin{aligned} X(\omega_0) &= Z(\omega_0) = x_0, \\ X(\omega_1) &= Z(\omega_1), \\ X(\omega_2) &\neq Z(\omega_2). \end{aligned}$$

In the case of an early-temporary modification, we have:

$$\begin{aligned} X(\omega_0) &= Z(\omega_0) = x_0, \\ X(\omega_1) &\neq Z(\omega_1), \\ X(\omega_2) &= Z(\omega_2). \end{aligned}$$

In the case of an early-permanent modification, we have:

$$\begin{aligned} X(\omega_0) &= Z(\omega_0) = x_0, \\ X(\omega_1) &\neq Z(\omega_1), \\ X(\omega_2) &\neq Z(\omega_2). \end{aligned}$$

However, for the early-permanent modification, we have that $f_{X(t=2)|X(t=1)} = f_{Z(t=2)|Z(t=1)}$. The conditional distribution for the second period is identical.

When modifying the underlying process and the X_i random variables, this will modify the expected holding and exercise values, and this will change the optimal exercise region(s). To study the impact of the modifications on early exercise probabilities, we need to study how the expected exercise and holding values are affected.

Lemma 19 *Let X be the original random variable, and Z be the modified random variable, where we dropped the time index for simplicity. For simplicity, let $e(u) = E[v^e(X,K)]$, $f(u) = E[v^e(Z,K)]$, $g(u) = E\left[v^h(X,K)\right]$, and $h(u) = E\left[v^h(Z,K)\right]$, Let $A = \{x|e(u) > g(u)\}$ be the exercise region(s) for the original random variable. Let $B = \{z|f(u) > h(u)\}$ be the exercise region(s) for the modified random variable. then:*

- *if $h(u) \geq g(u)$ and $f(u) = e(u)$, then $Pr(Z \in B) \leq Pr(X \in A)$,*
- *if $h(u) = g(u)$ and $f(u) \geq e(u)$, then $Pr(Z \in B) \geq Pr(X \in A)$,*
- *if $h(u) \leq g(u)$ and $f(u) = e(u)$, then $Pr(Z \in B) \geq Pr(X \in A)$,*
- *if $h(u) = g(u)$ and $f(u) \leq e(u)$, then $Pr(Z \in B) \leq Pr(X \in A)$.*

Proof. To prove Claim 1, we need to prove that B is smaller or equal to A. To show that, we need to show $B \subseteq A$, and $A \nsubseteq B$. Pick $u \in B$. Then $e(u) = f(u) > h(u) \geq g(u)$, and $u \in A$. Pick $v \in A$, such that $h(v) \geq f(v) > g(y)$. Then $v \ni B$. Thus, B is smaller or equal to A. Thus, $Pr(v \in B) \leq Pr(v \in A)$. Claims 2, 3 and 4 are proven using similar arguments. ■

A.2 Results for Late Modifications

Lemma 20 *Consider a FSD modification at a later time, such that $\{Y_1 \equiv X_1|x_0\}$ and $\{Y_2 \underset{FSD}{\ll} X_2|x_1\}$. Then,*

1. *$Pr(Y_2 \geq K|x_0) \geq Pr(X_2 \geq K|x_0)$,*
2. *$E\left[v^h(Y_1,K)|x_0\right] \geq E\left[v^h(X_1,K)|x_0\right]$,*
3. *$E[v^e(Y_1,K)|x_0] = E[v^e(X_1,K)|x_0]$,*

4. $\kappa_1(Y) \in \kappa_1(X)$,

5. $Pr(Y_1 \in \kappa_1(Y)|x_0) \leq Pr(X_1 \in \kappa_1(X)|x_0)$.

Proof. The first claim is directly from Proposition 1. Lemma 3 and the law of iterated expectation proves Claim 2. Because $Y_1 \equiv X_1$ and $v^e(Y_1(\omega_i), K) = v^e(X_1(\omega_i), K)$, the third claim is proved. Claims 4 and 5 are easily proven using Lemma 19. ■

Lemma 21 *Consider a QPS modification at a later time such that* $\{X_1 \equiv Z_1|x_0\}$ *and* $\{X_2 \underset{\alpha}{\ll} Z_2|x_1\}$. *Then,*

1. $E[v^e(Z_1, K)] = E[v^e(X_1, K)]$,

2. $E\left[v^h(Z_1, K)\right] \geq E\left[v^h(X_1, K)\right]$, *if* $\mathcal{A} < K$,

3. $\kappa_1(Z, K) \in \kappa_1(X, K)$, *if* $\mathcal{A} < K$,

4. $Pr(Z_1 \in \kappa_1(Z, K)) \leq Pr(X_1 \in \kappa_1(X, K))$, *if* $\mathcal{A} < K$.

5. $Pr(Z_1 \in \kappa_1(Z, K)) \leq Pr(X_1 \in \kappa_1(X, K))$, *if the QPS is symmetric and* $v^e(\cdot)$ *is convex.*

Proof. It is easy to prove Claim 1 knowing $X_1 \equiv Y_1$. The second claim is direct from Lemma 3. The third claim is proven using Lemma 19. Claim 4 is direct from Claim 3 and the fact that $\{X_1 \equiv Z_1|x_0\}$. For the fifth claim, if $\mathcal{A} \leq K$, we get the second claim which is already proven. If $v^e(\cdot)$ is convex and risk is added through a symmetric QPS, by convexity and simple function approximations, it is easy to prove that $E[v^e(Z_2, K)] \geq E[v^e(X_2, K)]$. Thus, $E\left[v^h(Z_1, K)\right] \geq E\left[v^h(X_1, K)\right]$. The rest of the proof is similar to the previous Claims 3 and 4. ■

A.3 Results for Early and Temporary Modifications

Lemma 22 *Consider a temporary early FSD modification such that* $\{Y_1|x_0\} \underset{FSD}{\ll} \{X_1|x_0\}$ *and* $\{Y_2|x_0, \tau > 1\} \equiv \{X_2|x_0, \tau > 1\}$.

1. $Pr(Y_2 \geq K|x_0, \tau > 1) = Pr(X_2 \geq K|x_0, \tau > 1)$,

2. $E\left[v^h(Y_1, K)|x_0\right] = E\left[v^h(X_1, K)|x_0\right]$,

3. $E[v^e(Y_1, K)|x_0] \geq E[v^e(X_1, K)|x_0]$,

4. $\kappa_1(Y) \in \kappa_1(X)$

5. $Pr(Y_1 \in \kappa_1(Y)|x_0) \geq Pr(X_1 \in \kappa_1(X)|x_0)$,

Proof. By construction, $\{\omega_i \in \Omega | Y_2(\omega_i) > K\} = \{\omega_i \in \Omega | X_2(\omega_i) > K\}$ for a given x_0 because $\{Y_2 | Y_0 = x_0, \tau > 1\} \equiv \{X_2 | X_0 = x_0, \tau > 1\}$ for all $\omega_i \in \Omega$. This proves Claim 1. To prove Claim 2, we need to recall that $E\left[v^h(Y_1, K)|x_0\right] = E\left[E\left[\rho_{t,\tau} \times v^e(Y_2, K)|x_1\right]|x_0\right]$. By the law of iterated expectations, we get $E\left[v^h(Y_1, K)|x_0\right] = E\left[\rho_{t,\tau} \times v^e(Y_2, K)|x_0\right]$. Because $\{Y_2|x_0\} \equiv \{X_2|x_0\}$, we get that $E\left[\rho_{t,\tau} \times v^e(Y_2, K)|x_0\right] = E\left[\rho_{t,\tau} \times v^e(X_2, K)|x_0\right]$. Thus, $E\left[v^h(Y_1, K)|x_0\right] = E\left[v^h(X_1, K)|x_0\right]$. To prove Claim 3, we use Lemma 2 and replace time T by 1. We can easily prove Claim 4 and 5 using Lemma 19. ■

Lemma 23 *Consider a temporary early QPS modification such that* $\{X_1 | X_0 = x_0\} \underset{\alpha}{\ll} \{Z_1 | Z_0 = x_0\}$ *and* $\{X_2 | X_0 = x_0, \tau > 1\} \equiv \{Z_2 | Z_0 = x_0, \tau > 1\}$. *Then,*

1. $Pr(Z_1 \in \kappa_1(Z)) \geq Pr(X_1 \in \kappa_1(X))$, *if* $\mathcal{A} < K$,

2. $Pr(Z_1 \in \kappa_1(Z)) \leq Pr(X_1 \in \kappa_1(X))$, *if* $\mathcal{A} > K$.

Proof. First, recall that $E\left[v^h(Z_1, K)|x_0\right] = E\left[E\left[\rho_{t,\tau} \times v^e(Z_2, K)|x_1\right]|x_0\right]$. By the law of iterated expectations, we get $E\left[v^h(Z_1, K)|x_0\right] = E\left[\rho_{t,\tau} \times v^e(Z_2, K)|x_0\right]$. Because $\{Z_2|x_0\} \equiv \{X_2|x_0\}$, we get that $E\left[\rho_{t,\tau} \times v^e(Z_2, K)|x_0\right] = E\left[\rho_{t,\tau} \times v^e(X_2, K)|x_0\right]$. Thus, $E\left[v^h(Z_1, K)|x_0\right] = E\left[v^h(X_1, K)|x_0\right]$. Second, because $\mathcal{A} < K$, for a specific realization, if $X(\omega_1) > K$, by the QPS, we have that $Z(\omega_1) > X(\omega_1)$. Because $v^e(\cdot)$ is weakly-increasing, we have that $E\left[v^e(Z_1, K)|x_0\right] \geq E\left[v^e(X_1, K)|x_0\right]$. Using Lemma 19 we can finish the proof of Claim 1. Similar arguments are used to prove Claim 2. ■

Note, if κ_1 is not a continuous region, results cannot be obtained without further assumptions on the QPS.

A.4 Results for Early and Permanent Modifications

Lemma 24 *A permanent early modification that adds value through a FSD such that* $\{Y_1|x_0\} \underset{FSD}{\ll} \{X_1|x_0\}$ *and* $\{Y_2|x_1\} \equiv \{X_2|x_1\}$ *then*

1. $\kappa_1(Y) = \kappa_1(X)$,

2. $Pr(Y_1 > K) \geq Pr(X_1 > k)$, *if* $\kappa_1(X) = \{x | k \leq x \leq a\}$.

Proof. Because $\{Y_2 | Y_1 = y_1\} \equiv \{X_2 | X_1 = x_1\}$ if $x_1 = y_1$, we have $v^h(Y_1, K)|_{Y_1 = x_1} = v^h(X_1, K)|_{X_1 = x_1}$. Furthermore, $v^e(Y_1, K)|_{Y_1 = x_1} = v^e(X_1, K)|_{X_1 = x_1}$. Thus, $\kappa_1(Y) = \kappa_1(X)$, proving claim 1. Finally, if the exercise region κ_1 is a closed including the upper bound of the domain of X set such that $\kappa_1(X) = \{x | k \leq x \leq a\}$, because $Y(\omega_1) \geq X(\omega_1)$ from the FSD modification, this proves Claim 2. If the exercise region is an open set, then results cannot be obtained without further assumptions. ∎

Lemma 25 *A permanent early modification that adds risk through a QPS such that* $\{X_1 | X_0 = x_0\} \underset{\alpha}{\lll} \{Z_1 | Z_0 = x_0\}$ *and* $\{X_2 | X_1 = x_1\} \equiv \{Z_2 | Z_1 = x_1\}$*:*

1. $\kappa_1(Z) = \kappa_1(X)$,

2. $Pr(Z_1 \in \kappa_1(Z)) \geq Pr(X_1 \in \kappa_1(X))$, *if* $\mathcal{A} < K$ *and* $\kappa_1(X) = \{x | k \leq x \leq a\}$.

3. $Pr(Z_1 \in \kappa_1(Z)) \leq Pr(X_1 \in \kappa_1(X))$, *if* $\mathcal{A} > K$ *and* $\kappa_1(X) = \{x | k \leq x \leq a\}$.

Proof. Because $\{Z_2 | Z_1 = z_1\} \equiv \{X_2 | X_1 = x_1\}$ if $x_1 = z_1$, we have $v^h(Z_1, K)|_{Z_1 = x_1} = v^h(X_1, K)|_{X_1 = x_1}$. Furthermore, $v^e(Z_1, K)|_{Z_1 = x_1} = v^e(X_1, K)|_{X_1 = x_1}$. Thus, $\kappa_1(Z) = \kappa_1(X)$, proving claim 1. When $X_1 \underset{\alpha}{\lll} Z_1$, and $\mathcal{A} < K$ we have $Z(\omega_1) \geq X(\omega_1)$. Thus, $Pr(Z_1 \in \kappa_1(Z)) \geq Pr(X_1 \in \kappa_1(X))$ if $\kappa_1(X) = \{x | k \leq x \leq a\}$. This proves Claim 2. However, if κ_1 is an open set, results cannot be obtained without further assumptions. Claim 3 is proved using similar arguments. ∎