Yield Curve Volatility and Macroeconomic Risk

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Abstract

I show that the relationship between the U.S. Treasury yield curve and macroeconomic risk fluctuates over time. I establish this result by introducing time-varying volatility and variance risk premia in a tractable term structure model. Based on my model, I characterize the joint behavior of the yield curve and macroeconomic risk captured by inflation and unemployment gap from 1971 to 2019. First, I find that the macroeconomic contribution to short-term yield volatility is high in the 1970s, low during the Great Moderation starting in the mid-1980s, and high again after the financial crisis. Second, investors are increasingly anchoring short-rate expectations to macroeconomic risk. Third, deflation fears increase term premia during the financial crisis. Finally, I show that macroeconomic shocks do not explain the yield curve inversion in 2019. My results suggest that the recent inversion is not a warning of an imminent recession and thus should not trigger monetary policy easing.

Keywords: Yield curve, macro-finance term structure model, bond market volatility, multivariate GARCH, variance risk premia.

JEL classification: C32, E43, E44, G12.

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1 Introduction

Policy makers and investors pay close attention to the joint dynamics of the yield curve and macroeconomic variables. These dynamics can, for example, shed light upon how the yield curve is affected by the Federal Reserve's dual mandate of maximum sustainable employment and stable prices. The relationship between macroeconomic risk and the decomposition of yields into short-rate expectations and term premia is also useful for understanding the sources of movements in the yield curve. In light of the recent sharp decline in long-term yields, it is of particular interest to recognize if this development reflects low expectations on future economic activity and, consequently, is likely to anticipate a recession.

I focus on the question, how much variance in the yield curve, short-rate expectations, and term premia can be attributed to macroeconomic risk? I contribute to a large body of literature that uses dynamic term structure models with observed macroeconomic variables to decompose variances into macroeconomic and latent yield-specific shares. Starting with Ang and Piazzesi (2003), the existing literature is centered around Gaussian term structure models (Bikbov and Chernov, 2010, Doshi, Jacobs, and Liu, 2018, Duffee, 2018). These models are celebrated for their tractability and ability to match the yield curve. However, they assume that yield curve volatility is constant, which contradicts the empirical evidence. My paper studies whether time-varying yield curve volatility has implications for the joint behavior of the yield curve and macroeconomic variables.

I build a novel term structure model that has time-varying volatility and variance risk premia, unlike the Gaussian term structure model. Specifically, I bridge the Baba-Engle-Kraft-Kroner (BEKK) multivariate GARCH model proposed by Engle and Kroner (1995) with the exponential-quadratic pricing kernel of Monfort and Pegoraro (2012). My model admits closed-form solutions for no-arbitrage bond yields and their decomposition into short-rate expectations and term premia. I match the U.S. Treasury yield curve at the monthly frequency from 1971 to 2019 using inflation and unemployment gap as macroeconomic variables and three latent yield-specific factors. Model-implied conditional variances and covariances capture realized and rolling measures. In contrast, continuous-time term structure models with stochastic volatility struggle to match empirical volatility proxies with low and often negative correlation between predicted and realized volatilities

(Christensen, Lopez, and Rudebusch, 2014, Collin-Dufresne, Goldstein, and Jones, 2009, Jacobs and Karoui, 2009). Finally, my model implies excess returns that explain the empirical failure of the expectations hypothesis. Thus, my model overcomes a trade-off in the existing literature between modeling time-varying volatility and excess returns as documented by Dai and Singleton (2002).

By embracing the stylized fact that yield curve volatility is time-varying, I show that the relationship between the yield curve and macroeconomic variables fluctuates over time. Gaussian term structure models imply a constant relationship. I find that macroeconomic shocks explain more than half of the variation in yields in some periods, but that yield curve volatility is unrelated to macroeconomic risk in other periods. I argue that large month-to-month fluctuations are related to economic events, for example, the announcement of quantitative easing programs. Thus, macroeconomic news partly drives movements in the yield curve, which is consistent with Andersen, Bollerslev, Diebold, and Vega (2007), Feunou, Fontaine, and Roussellet (2019), and Piazzesi (2005).

I present a novel dynamic characterization of the historical joint behavior of the yield curve and macroeconomic variables. First, I show that the macroeconomic contribution to short-term yield variance has followed a U-shaped pattern since the 1970s. These macroeconomic shares were high in the 1970s, low during the Great Moderation starting from the mid-1980s, and high following the financial crisis. Second, I find an upward trend in the macroeconomic contribution to variance in 10-year short-rate expectations since the Great Moderation. Thus, investors increasingly form expectations on future short rates based on macroeconomic risk. This result possibly reflects a growing belief among investors that the Federal Reserve's commitment to the dual mandate is credible. Third, the macroeconomic share of variance in the 10-year term premium increases during the financial crisis due to inflation shocks. Thus, inflation risk premia are high despite a low-inflation environment. This result is generated by the exponential-quadratic pricing kernel as it allows investors to demand compensation for both positive and negative shocks to inflation as also shown by Roussellet (2018). As an implication, a symmetric inflation target enhances the efficiency of monetary policy during recessions.

In terms of methodology, I show that my model can be estimated by a step-wise approach without relying on filtering methods for estimating the latent yield-specific factors. Specifically, my model is invariant to affine transformations such that latent factors can be rotated into portfolios of synthetic yields. The synthetic yields are constructed by residuals from yields explained with macroeconomic variables only. My estimation method is a generalization of Joslin, Singleton, and Zhu (2011), who provide similar arguments for the Gaussian term structure model. My method is most closely related to Ghysels, Le, Park, and Zhu (2014), who consider a model that differs from my model in two respects. First, my model is consistent with the view that volatility is not priced in Treasury bonds (Andersen and Benzoni, 2010, Collin-Dufresne and Goldstein, 2002, Joslin, 2017). In contrast, Ghysels et al. let volatility affect yields in the spirit of the ARCH-in-mean model in Engle, Lilien, and Robins (1987). Second, I allow for both latent factors and observed macroeconomic variables, whereas Ghysels et al. use only latent factors. Other macro-finance term structure models with GARCH-type volatility in the literature do not allow for a rotation of the latent factors (Campbell, Sunderam, and Viceira, 2017, Haubrich, Pennacchi, and Ritchken, 2012, Koeda and Kato, 2015).

The introduction of time-varying volatility in the yield curve has two implications for model-implied risk compensation in fixed-income markets. First, my model decomposes long-term yields into persistent short-rate expectations and counter-cyclical term premia. The cyclicality is stronger than the term premia implied by the Gaussian term structure model. This difference arises because my model involves both a time-varying price and quantity of risk, whereas the Gaussian model only allows for variation in the price of risk. As an implication, the models disagree about the effectiveness of forward guidance. Specifically, my model is consistent with effective forward guidance as seen in Carlstrom, Fuerst, and Paustian (2015) and McKay, Nakamura, and Steinson (2016), but unlike Hagedorn, Luo, Manovskii, and Mitman (2019).

Second, I find that investors are willing to pay large variance risk premia to hedge macroeconomic uncertainty. As a result, macroeconomic uncertainty may increase trading activity in fixed-income derivative markets. To the extent that the policy rate is determined by the dual mandate, my results complement Cieslak and Povala (2016) who show that fixed-income variance risk premia are particularly related to uncertainty about future monetary policy. My term structure of estimated variance risk premia is downward-sloping in magnitudes, which is consistent with the literature (Choi, Mueller, and Vedolin, 2017, Trolle and Schwartz, 2009, Trolle and Schwartz, 2015). Gaussian term structure models abstract from the presence of variance risk premia by imposing exponential-affine pricing kernels that only allow investors to demand compensation for mean-based risk.

Finally, I show that macroeconomic variables did not drive the movements in the yield curve in the spring 2019. Specifically, I focus on the 50-basis-point decline in the 10-year yield from April to June. This decline led to an inversion of the yield curve, with the 10-year yield below the 3-month yield, in May. Yield curve inversions are sensational because history shows that inversions are strong predictors of recessions. The predictive power reflects that when investors expect a slowing economy, short-rate expectations and hence long-term yields decline. Thus, if the yield curve inversion is likely to predict a recession, movements in short-rate expectations should be driven by macroeconomic variables. My model allows me to zoom in on the relationship between the yield curve and macroeconomic variables specifically during spring 2019. In contrast, the Gaussian term structure model can only draw conclusions that are valid on average across the full sample. I begin by showing that the yield curve inversion in August 2006 that preceded the financial crisis was related to macroeconomic risk. Following this, I show that the macroeconomic contribution to variance in the 10-year yield and specifically in 10-year short-rate expectations decreased in spring 2019. This finding is consistent with the current strength in the U.S. economy. For example, in May 2019 the unemployment rate had been at or below 4 percent for over a year. My results suggest that the recent yield curve inversion is not a warning of an imminent recession. Thus, the Federal Reserve should not launch a monetary policy easing cycle based on the inversion.

The remainder of this paper is organized as follows. First, I present my term structure model with time-varying volatility and variance risk premia in Section 2. Section 3 provides a tractable method for estimating my model. Section 4 details the data and discusses the empirical performance of my model. I analyze the relationship between yield curve volatility and macroeconomic risk in Section 5. Finally, I apply my model to study the 2019 yield curve inversion in Section 6. Conclusions follow in Section 7. Proofs and technical details are provided in Technical Appendix A. The paper is also accompanied by an online appendix.[1]

2 Term Structure Model

In this section, I present my term structure model. My model is distinct from the Gaussian term structure model in two dimensions: First, the dynamics under the physical probability measure exhibit multivariate GARCH volatility. Second, the pricing kernel is exponential-quadratic allowing investors to demand a compensation for exposure to conditional volatility. In the following, I describe these elements separately.

[1]The online appendix is available from my website here.

2.1 Physical Dynamics

The yield curve is driven by a n_X-dimensional state vector, X_t, consisting of n_x latent yield-specific factors, x_t, and n_m observed macroeconomic variables, m_t. Thus, the state vector is $X_t = [x_t', m_t']'$. Assuming that x_t can be filtered from observed bond yields, let $\mathcal{F}_t$ be the filtration given by $(X_t, X_{t-1}, \ldots)$.

Conditional Mean

The physical dynamics of X_t are given by a vector autoregressive model allowing for some lag length, L, in the equation for the macroeconomic variables, m_t. Following the existing literature, latent factors have one lag.[2] The equations for x_t and m_t are given by

$$x_t = \mu_x + \Phi_x^{(1)} x_{t-1} + \Phi_{xm}^{(1)} m_{t-1} + \varepsilon_{x,t}, \tag{1}$$

$$m_t = \mu_m + \Phi_{mx}^{(1)} x_{t-1} + \Phi_m^{(1)} m_{t-1} + \Phi_m^{(2)} m_{t-2} + \ldots + \Phi_m^{(L)} m_{t-L} + \varepsilon_{m,t}, \tag{2}$$

where $\varepsilon_{x,t}$ and $\varepsilon_{m,t}$ are conditionally Gaussian given $\mathcal{F}_{t-1}$ with mean zero and conditional variance matrix, V_t.[3] Equivalently, I write

$$X_t = \mu_X + \Phi_X^{(1)} X_{t-1} + \ldots + \Phi_X^{(L)} X_{t-L} + \varepsilon_t \tag{3}$$

with appropriate zero restrictions in $\Phi_X^{(l)}$ for $l = 2, \ldots, L$ and $\varepsilon_t | \mathcal{F}_{t-1} \sim \mathcal{N}(0, V_t)$.

Conditional Variance Matrix

I model the full variance matrix, V_t, allowing for time-varying conditional covariances between the macroeconomic variables and latent factors. The volatility dynamics are given by the multivariate BEKK GARCH model of Engle and Kroner (1995):

$$V_t = \Sigma_X \Sigma_X' + \sum_{k=1}^{K} A_X^{(k)} \varepsilon_{t-1} \varepsilon_{t-1}' A_X^{(k)\prime} + \sum_{k=1}^{K} B_X^{(k)} V_{t-1} B_X^{(k)\prime}. \tag{4}$$

If Σ_X is lower triangular such that $\Sigma_X \Sigma_X'$ is positive definite, then V_t is also positive definite. Importantly, the model in (4) is invariant to affine transformations. This is a non-trivial property that many multivariate GARCH models do not satisfy.[4] I use this property to propose a tractable estimation method in Section 3.

[2] The model can easily be extended to a full vector autoregressive model of order L. However, allowing for a lag structure in the dynamics of x_t is costly in terms of parameters. Indeed, I show in the online appendix that information criteria prefer models with multiple lags in the equation for m_t only.

[3] It should be noted that V_t denotes the conditional volatility of a process at time t that is adapted to $\mathcal{F}_{t-1}$. This choice follows the GARCH literature, but differs from the literature on stochastic volatility.

[4] For example, GARCH models with asymmetric components and the dynamic conditional correlations model of Engle (2002) are not invariant to affine transformations.

With multiple ARCH and GARCH components, $K > 1$, I allow for rich dynamics of conditional variances and covariances which are typically needed empirically.[5] When K is sufficiently large, the model can obtain a representation equivalent to the vech-GARCH model of Bollerslev, Engle, and Wooldridge (1988).

The full BEKK GARCH model, i.e., the model where $A_X^{(k)}$ and $B_X^{(k)}$ are fully parameterized matrices for some k, suffers from many econometric problems including the curse of dimensionality (Chang and McAleer, 2019). I therefore impose diagonal restrictions on $A_X^{(k)}$ and $B_X^{(k)}$ for all $k = 1, \ldots, K$.

2.2 Pricing Kernel

I relate the state vector to no-arbitrage bond yields by specifying the pricing kernel. Standard term structure models use exponential-affine pricing kernels, see, e.g., Le, Singleton, and Dai (2010). However, recent work suggests that higher-order pricing kernels are needed for capturing the conditional volatility of the yield curve (Creal and Wu, 2017, Ghysels, Le, Park, and Zhu, 2014, Joslin and Konchitchki, 2018). Based on these results, I apply the exponential-quadratic kernel in Monfort and Pegoraro (2012) given by

$$\mathcal{M}_{t+1} = \frac{\exp\left(-r_t + \xi_t' X_{t+1} + X_{t+1}' \Xi_t X_{t+1}\right)}{\mathbb{E}_t\left[\exp\left(\xi_t' X_{t+1} + X_{t+1}' \Xi_t X_{t+1}\right)\right]}, \tag{5}$$

where r_t is the short rate.

I show in Technical Appendix A.1 that exponential-quadratic pricing kernels can be structurally justified in preference-based models. Specifically, the long-run risk model of Bansal and Yaron (2004) implies an exponential-quadratic pricing kernel when solved by the second-order projection developed by Andreasen and Jørgensen (2019). In addition, Hansen and Heaton (2008) propose a long-run risk model based on vector autoregressive dynamics that implies an exponential-quadratic pricing kernel when allowing for a time-varying wealth-consumption ratio.

[5] To illustrate this notion, consider a bivariate model with $K = 1$ in which $A_X^{(1)}$ and $B_X^{(1)}$ are diagonal matrices with elements a_{ii} and b_{ii}, $i = 1, 2$. Then,

$$V_t = \Sigma_X \Sigma_X' + \begin{bmatrix} a_{11}^2 \varepsilon_{1,t-1}^2 & a_{11} a_{22} \varepsilon_{1,t-1} \varepsilon_{2,t-1} \\ a_{11} a_{22} \varepsilon_{1,t-1} \varepsilon_{2,t-1} & a_{22}^2 \varepsilon_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} b_{11}^2 V_{11,t-1} & b_{11} b_{22} V_{12,t-1} \\ b_{11} b_{22} V_{12,t-1} & b_{22}^2 V_{22,t-1} \end{bmatrix}.$$

The parameters a_{ii} and b_{ii} are the ARCH and GARCH effects related to variable i. However, these parameters also determine the conditional covariance between the variables. Thus, there is a tension between modeling conditional variances and covariances simultaneously. This problem is also present when $A_X^{(1)}$ and $B_X^{(1)}$ are full matrices.

The pricing kernel in (5) contains three components. The first component is a time discount, which is determined by the short rate. Following the literature, I impose a short-rate model that is affine in X_t:

$$r_t = \alpha_X + \beta_x' x_t + \beta_m' m_t = \alpha_X + \beta_X' X_t. \tag{6}$$

The remaining components represent compensation for mean-based risk, $\xi_t' X_{t+1}$, and variance-based risk, $X_{t+1}' \Xi_t X_{t+1}$. Thus, the exponential-quadratic pricing kernel allows investors to be averse towards both mean- and variance-based risk. The variables ξ_t and Ξ_t can be interpreted as market prices of risk associated with, respectively, the conditional mean and variance matrix of X_{t+1}. In this sense, the variances of X_{t+1} can be interpreted as quantities of risk. If $\Xi_t = 0$, the pricing kernel in (5) reduces to the standard exponential-affine pricing kernel. Thus, exponential-affine pricing kernels only allow investors to price mean-based risk.

Monfort and Pegoraro (2012) show that if the market prices of risk are chosen by

$$\xi_t = (V_{t+1}^{\mathbb{Q}})^{-1} \mathbb{E}_t^{\mathbb{Q}}(X_{t+1}) - V_{t+1}^{-1} \mathbb{E}_t(X_{t+1}), \tag{7}$$

$$\Xi_t = \frac{1}{2} \left(V_{t+1}^{-1} - (V_{t+1}^{\mathbb{Q}})^{-1} \right), \tag{8}$$

where $\mathbb{E}_t^{\mathbb{Q}}(X_{t+1})$ and $V_{t+1}^{\mathbb{Q}}$ denote the first and second conditional moments of X_{t+1} given $\mathcal{F}_t$ under the risk-neutral probability measure ($\mathbb{Q}$), then $X_{t+1}|\mathcal{F}_t \overset{\mathbb{Q}}{\sim} \mathcal{N}(\mathbb{E}_t^{\mathbb{Q}}(X_{t+1}), V_{t+1}^{\mathbb{Q}})$. Thus, my model allows the risk-neutral conditional variance matrix, $V_t^{\mathbb{Q}}$, to differ from the physical conditional variance matrix, V_t. The discrepancy between the conditional variance under the physical and risk-neutral probability measures is feasible in discrete-time models as they are not restricted by Girsanov's theorem. In contrast, Girsanov's theorem rules out distinct volatility specifications across probability measures in continuous-time models. Specifically, it follows from the market price of variance risk, Ξ_t, that it is the introduction of variance-based risk compensation that facilitates different variance specifications under the physical and risk-neutral probability measures.

The market price of mean-based risk in (7) is akin to that of the standard term structure model with exponential-linear pricing kernel. Specifically, it is given by the differences in the conditional means under physical and risk-neutral probability measures weighted by a conditional variance matrix. In contrast to the standard model, however, I weigh the physical and risk-neutral expectations by different variance matrices, accounting for the different variances across probability measures. The market price of variance-based risk in (8) is diminishing in the quantity of risk under the physical measure, V_{t+1}. This property is consistent with the empirical fact that fixed-income variance risk premia are diminishing in magnitudes with the maturity of bonds (Choi, Mueller, and Vedolin, 2017, Trolle, 2009, Trolle and Schwartz, 2015).

In the following, I specify the risk-neutral moments which pin down the market prices of risk through (7)-(8). This approach contrasts with the standard literature in which the market prices of risk are specified to pin down the risk-neutral moments.

2.3 Risk-Neutral Moments

I specify moments under the risk-neutral probability measure such that my model is tractable and sufficiently flexible to match the yield curve in sample. An autoregressive model of order one with constant volatility satisfies both of these criteria (Joslin, Le, and Singleton, 2013). I also impose independence of the macroeconomic variables and the latent yield-specific factors under the risk-neutral measure. Intuitively, I allow for interdependent dynamics of macroeconomic variables and latent factors through the physical dynamics, but bond market investors do not price these interactions. Failure of this assumption affects the efficiency, but not the consistency, of my proposed estimator. Thus, the risk-neutral conditional mean and variance matrix are given by

$$\mathbb{E}_t^{\mathbb{Q}}\left(X_{t+1}\right) = \begin{bmatrix} \mu_x^{\mathbb{Q}} \\ \mu_m^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} \Phi_x^{\mathbb{Q}} & 0 \\ 0 & \Phi_m^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} x_t \\ m_t \end{bmatrix} = \mu_X^{\mathbb{Q}} + \Phi_X^{\mathbb{Q}} X_t, \tag{9}$$

$$V_{t+1}^{\mathbb{Q}} = \begin{bmatrix} V_x^{\mathbb{Q}} & 0 \\ 0 & V_m^{\mathbb{Q}} \end{bmatrix} = V_X^{\mathbb{Q}}, \tag{10}$$

where $V_X^{\mathbb{Q}}$ is positive definite. The assumption of constant conditional variances and covariances under the risk-neutral measure is likely to be violated. However, recent literature indicates that a violation is unlikely to affect the pricing of bonds. First, Cieslak and Povala (2016) build a flexible term structure model with stochastic volatility under both the physical and risk-neutral probability measures. As a result, no-arbitrage bond yields depend directly on volatility. However, they empirically show that the loading on volatility is close to zero. This result is also supported by a large body of literature that argues that bond market volatility is not spanned by bond yields (Andersen and Benzoni, 2010, Collin-Dufresne and Goldstein, 2002, Joslin, 2017). This phenomenon is known as unspanned stochastic volatility. Second, Joslin and Konchitchki (2018) show that convexity effects under the risk-neutral measure are small. Therefore, the conditional variance under the risk-neutral measure has only small effects on bond pricing.

2.4 No-Arbitrage Bond Pricing

The no-arbitrage bond yields implied by my model are equal to those of the Gaussian term structure model. These are given in the following theorem.

Theorem 1 *Let $P_{t,n}$ denote the no-arbitrage price of a n-period zero-coupon bond. Let $Y_{t,n} = -n^{-1}\log(P_{t,n})$ be the associated yield. Given (6)-(10), $Y_{t,n}$ is given in closed form by*

$$Y_{t,n} = a_n + b'_{x,n}x_t + b'_{m,n}m_t = a_n + b'_n X_t, \tag{11}$$

where $a_n = -n^{-1}\tilde{a}_n$ and $b_{i,n} = -n^{-1}\tilde{b}_{i,n}$ for $i = \{x, m\}$ are given recursively by

$$\tilde{a}_{n+1} = -\alpha_X + \tilde{a}_n + \tilde{b}'_{x,n}\mu_x^{\mathbb{Q}} + \tilde{b}'_{m,n}\mu_m^{\mathbb{Q}} + \frac{1}{2}\tilde{b}'_{x,n}V_x^{\mathbb{Q}}b_{x,n} + \frac{1}{2}\tilde{b}'_{m,n}V_m^{\mathbb{Q}}b_{m,n}, \tag{12}$$

$$\tilde{b}'_{x,n+1} = -\beta'_x + \tilde{b}'_{x,n}\Phi_x^{\mathbb{Q}},, \tag{13}$$

$$\tilde{b}'_{m,n+1} = -\beta'_m + \tilde{b}'_{m,n}\Phi_m^{\mathbb{Q}}, \tag{14}$$

The recursions are initiated at $n = 0$ with $a_0 = 0$, $b_{x,0} = 0_{n_x \times 1}$, and $b_{m,0} = 0_{n_m \times 1}$.

Proof. The solution follows from the no-arbitrage recursion, $P_{t,n} = \mathbb{E}_t\left(\mathcal{M}_{t+1}P_{t+1,n-1}\right)$, see Ang and Piazzesi (2003). □

Remark 1 *I note that the conditional variance matrix, V_t, does not appear in the bond pricing equation, (11). Thus, my model exhibits a form of unspanned volatility, but without introducing restrictions as in the traditional literature on unspanned stochastic volatility.*[6]

Remark 2 *It follows from (11) that conditional yield variance is given by*

$$Var_t(Y_{t+1,n}) = b'_n V_{t+1} b_n. \tag{15}$$

2.5 Risk Compensations

My model distinguishes between two concepts of risk compensation. First, term premia, $\Psi_{t,n}$, are defined by the difference between bond yields and expected future short rates,

[6] Joslin (2017) argues that traditional USV models restrict the cross-section of yield curve volatility in a way that is inconsistent with the data.

$\Upsilon_{t,n}$:

$$\Psi_{t,n} = Y_{t,n} - \Upsilon_{t,n}, \qquad \Upsilon_{t,n} = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{E}_t\left(r_{t+i}\right). \tag{16}$$

The following theorem provides closed-form expressions of short-rate expectations and term premia.

Theorem 2 *Given the physical conditional mean of X_t in (3) and the assumptions of Theorem 1, short-rate expectations, $\Upsilon_{t,n}$, and term premia, $\Psi_{t,n}$, are given by*

$$\Upsilon_{t,n} = a_n^{EH} + (b_n^{EH})'\mathcal{X}_t,$$
$$\Psi_{t,n} = \left(a_n - a_n^{EH}\right) + b_n' X_t - (b_n^{EH})'\mathcal{X}_t.$$

where $\mathcal{X}_t = [X_t,\ X_{t-1},\ \ldots, X_{t-L+1}]$ and

$$a_n^{EH} = \alpha_X + \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} \beta_{\mathcal{X}}' \Phi_{\mathcal{X}}^j \mu_{\mathcal{X}},$$
$$b_n^{EH} = \frac{1}{n}\sum_{i=0}^{n-1} \beta_{\mathcal{X}}' \Phi_{\mathcal{X}}^i.$$

Here, $\beta_{\mathcal{X}} = [\beta_x', \beta_m', 0_{1\times(p+m)(L-1)}]'$ and $\mu_{\mathcal{X}}$ and $\Phi_{\mathcal{X}}$ denote the parameters of the conditional mean associated with the companion form of (3).

Proof. Straightforward. □

Remark 3 *It follows that the conditional variances of short-rate expectations and term premia are given in closed form by*

$$Var_{t-1}(\Upsilon_{t,n}) = (b_n^{EH})'\, Var_{t-1}\left(\mathcal{X}_t\right) b_n^{EH}$$
$$Var_{t-1}(\Psi_{t,n}) = \left(\breve{b}_n - b_n^{EH}\right)'\, Var_{t-1}\left(\mathcal{X}_t\right) \left(\breve{b}_n - b_n^{EH}\right),$$

where $\breve{b}_n = (b_n', 0, 0, \ldots, 0)'$.

The second concept of risk compensation relates to variance risk. While V_t is not priced in bonds, see Theorem 1, investors may demand compensation for variance risk. This compensation is realized in the broader fixed-income market through the pricing of interest-rate derivatives. To analyze these premia and their relation to macroeconomic variables, I follow Bollerslev, Tauchen, and Zhou (2009) and consider the model-implied variance risk premium given by the difference between the expected variance under physical and risk-neutral probabilities. The next theorem provides a closed-form solution for the variance risk premium.

Theorem 3 *Variance risk premia defined by* $\Gamma_{t,n} = \mathbb{E}_t\left[Var_{t+1}(Y_{t+2,n})\right] - \mathbb{E}_t^{\mathbb{Q}}\left[Var_{t+1}(Y_{t+2,n})\right]$ *are given by*

$$\Gamma_{t,n} = b_n' \left(\sum_{k=1}^{K} \left[A_X^{(k)} V_{t+1} A_X^{(k)\prime} - A_X^{(k)} V_X^{\mathbb{Q}} A_X^{(k)\prime} \right] \right) b_n$$

Proof. Straightforward. □

Remark 4 *It follows that* $\mathbb{E}_t^{\mathbb{Q}}\left[Var_{t+1}(Y_{t+2,n})\right]$ *is constant and the dynamics of my variance risk premium are determined solely from the physical dynamics.*

Due to the above remark, I abstract from analyzing time-variation in variance risk premia in my empirical application. Instead, I consider averages, which I believe are accurately estimated by my model.

3 Econometric Method

In this section, I present a simple approach to estimating my model. Specifically, I extend the ideas proposed by Joslin, Singleton, and Zhu (2011) for the Gaussian term structure model with latent yield-specific factors only.

3.1 Estimation Problem

Consider bond yields of N different maturities given by $n_1, \ldots, n_N$ periods. Let $Y_t = (Y_{t,n_1}, \ldots, Y_{t,n_N})'$ denote a vector of these observations. I impose the following assumption on the accuracy of yield measurements:

Assumption 1 *Yields,* Y_t*, are measured with errors given by* $\eta_t \sim i.i.d.\ \mathcal{N}(0, \sigma^2 I_N)$.

I identify latent yield-specific factors, x_t, such that they capture yield curve variation that is not related to the macroeconomic variables. For this purpose, I decompose model-implied yields given in Theorem 1 by

$$Y_{t,n} = a_{m,n} + b_{m,n}' m_t + Y_{t,n}^{\perp m}, \tag{17}$$

where $Y_{t,n}^{\perp m} = Y_{t,n} - a_{m,n} - b_{m,n}' m_t$ and $a_{m,n} = n^{-1}\tilde{a}_{m,n}$ with

$$\tilde{a}_{m,n} = \tilde{a}_{m,n-1} + \tilde{b}_{m,n}' \mu_m^{\mathbb{Q}} + \frac{1}{2}\tilde{b}_{m,n}' V_m^{\mathbb{Q}} \tilde{b}_{m,n}.$$

The synthetic yields, $Y_{t,n}^{\perp m}$, capture the variation in the yield curve that cannot be explained by the macroeconomic variables. I therefore extract latent yield-specific factors from synthetic yields.

I propose a step-wise estimation method to implement this strategy. A first step estimates parameters related to pricing bonds with macroeconomic variables, $\Theta_m^{\mathbb{Q}} = \{\beta_m, \mu_m^{\mathbb{Q}}, \Phi_m^{\mathbb{Q}}, V_m^{\mathbb{Q}}\}$. The marginal likelihood function to be maximized is given by

$$\log \ell_Y \left(Y_t | m_t; \Theta_m^{\mathbb{Q}} \right) = - \sum_{i=1}^{N} \left(Y_{t,n_i} - a_{m,n_i} - b_{m,n_i} m_t \right)^2 .$$

I use these estimated parameters to construct estimates of the synthetic yields, $Y_{t,n}^{\perp m}$.

The remaining parameters can be estimated in a second step by the standard approach, combining maximum-likelihood estimation with a Kalman filter and parameter restrictions for statistical identification. This approach is, however, cumbersome because my model encounters non-linearities in both the conditional variance matrix, V_t, and in the loading recursions, a_n and $b_{x,n}$. In the remainder of this section, I provide a simpler approach that separates the estimation problem such that non-linearities in the physical dynamics are treated separately from the non-linear bond yield equation. Furthermore, my approach avoids Kalman filtering and hence the numerical stability issues that are likely to be associated with filtering methods.

3.2 Rotation and Identifying Restrictions

My model is invariant to affine transformations of the latent factors given by $\rho_t = c_x + C_x x_t$, see Lemma 1 in Technical Appendix A.2. Therefore, the latent factors can be rotated into linear combinations, or portfolios, of the synthetic yields, $Y_{t,n}^{\perp m}$. Let $\rho_t = W Y_t^{\perp m}$ denote a vector of these rotated factors. The matrix of portfolio weights, W, can be chosen freely, e.g., such that the portfolios are the first n_x principal components of the synthetic yield curve. The rotated model is unique given that W has full rank. Theorem 4 formalizes these ideas.

Theorem 4 *The model defined by (3)-(10) with synthetic yields defined in (17) is observationally equivalent to a unique model in which the latent factors are portfolios of synthetic yields given by $\rho_t = W Y_t^{\perp m}$, where the weighting matrix W has full rank. Let $\mathcal{P}_t = (\rho_t', m_t')'$ denote the resulting observable state vector. Then, this unique model is* *given by*

$$\mathcal{P}_t = \mu_{\mathcal{P}} + \sum_{l=1}^{L} \Phi_{\mathcal{P}}^{(l)} \mathcal{P}_{t-l} + \varepsilon_{\mathcal{P},t}, \quad \varepsilon_{\mathcal{P},t} = V_t^{1/2} z_{\mathcal{P},t}, \tag{18}$$

$$V_t = \Sigma_{\mathcal{P}} \Sigma_{\mathcal{P}}' + \sum_{k=1}^{K} A_{\mathcal{P}}^{(k)} \varepsilon_{t-1} \varepsilon_{t-1}' A_{\mathcal{P}}^{(k)\prime} + \sum_{k=1}^{K} B_{\mathcal{P}}^{(k)} V_{t-1} B_{\mathcal{P}}^{(k)\prime}, \tag{19}$$

$$\mathcal{P}_t = \mu_{\mathcal{P}}^{\mathbb{Q}} + \Phi_{\mathcal{P}}^{\mathbb{Q}} \mathcal{P}_{t-1} + \varepsilon_{\mathcal{P},t}^{\mathbb{Q}}, \quad \varepsilon_{\mathcal{P},t}^{\mathbb{Q}} = (V_{\mathcal{P}}^{\mathbb{Q}})^{1/2} z_{\mathcal{P},t}^{\mathbb{Q}}, \tag{20}$$

$$r_t = \alpha_{\mathcal{P}} + \beta_m' m_t + \beta_\rho' \rho_t \tag{21}$$

with $z_{\mathcal{P},t} \sim$ *i.i.d.* $\mathcal{N}(0, I_{n_X})$ *and* $z_{\mathcal{P},t}^{\mathbb{Q}} \sim$ *i.i.d.* $\mathcal{N}(0, I_{n_X})$*.* $V_{\mathcal{P}}^{\mathbb{Q}}$ *is a positive definite matrix.*

Proof. See Technical Appendix A.2. □

Remark 5 *Given the rotation in Theorem 4, model-implied synthetic yields are given by*

$$Y_{t,n}^{\perp m} = a_{\rho,n} + b_{\rho,n}' \rho_t, \tag{22}$$

where $a_{\rho,n} = -n^{-1} \tilde{a}_{\rho,n}$ *and* $b_{\rho,n} = -n^{-1} \tilde{b}_{\rho,n}$ *with*

$$\tilde{a}_{\rho,n+1} = -\alpha_{\mathcal{P}} + \tilde{a}_{\rho,n} + \tilde{b}_{\rho,n}' \mu_\rho^{\mathbb{Q}} + \frac{1}{2} \tilde{b}_{\rho,n}' V_\rho^{\mathbb{Q}} b_{\rho,n}, \tag{23}$$

$$\tilde{b}_{\rho,n+1}' = -\beta_\rho' + \tilde{b}_{\rho,n}' \Phi_\rho^{\mathbb{Q}}. \tag{24}$$

The next step is to impose parameter restrictions such that my model is identified. In a model with a constant conditional variance matrix under the physical measure, these restrictions can be imposed on the risk-neutral dynamics only, see Joslin, Singleton, and Zhu (2011). I adopt these restrictions as formalized in part (i) of Theorem 5, which follows next. In my model, however, I need additional restrictions for identifying the physical dynamics. These are given by part (ii) of Theorem 5. To state the theorem, I define functions that map a set of parameters into a vector of bond yield loadings by $a(\alpha_{\mathcal{P}}, \beta_\rho, \mu_\rho^{\mathbb{Q}}, \Phi_\rho^{\mathbb{Q}}, V_\rho^{\mathbb{Q}}) = (a_{\rho,n_1}, \ldots, a_{\rho,n_N})'$ and $b(\beta_\rho, \Phi_\rho^{\mathbb{Q}}) = (b_{\rho,n_1}, \ldots, b_{\rho,n_N})'$.

Theorem 5 *The model defined by (18)-(21) in Theorem 4 with synthetic yields given by (22) is identified with the following parametrizations:*

(i) The risk-neutral dynamics and the short-rate equation are uniquely parametrized by $\Theta_\rho^{\mathbb{Q}} = \{k_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, V_\rho^{\mathbb{Q}}\}$*, where* $\lambda^{\mathbb{Q}}$ *contains ordered eigenvalues of the transformed risk-neutral autoregressive coefficient matrix given by* $[Wb(\beta_x, \Phi_x^{\mathbb{Q}})]^{-1} \Phi_\rho^{\mathbb{Q}} [Wb(\beta_x, \Phi_x^{\mathbb{Q}})]$*,*

c.f., *Lemma 1 in the Technical Appendix A.2. In particular,*

$$\alpha_{\mathcal{P}} = -\beta_\rho' a_W,$$
$$\beta_\rho = (b_W^{-1})'\iota,$$
$$\mu_\rho^{\mathbb{Q}} = k_\infty^{\mathbb{Q}} b_W e_1 + (I_{n_x} - \Phi_\rho^{\mathbb{Q}}) a_W,$$
$$\Phi_\rho^{\mathbb{Q}} = b_W J(\lambda^{\mathbb{Q}}) b_W^{-1},$$

where $e_1 = (1, 0, \ldots, 0)'$ *and* $J(\lambda^{\mathbb{Q}}) = diag(J_1^{\mathbb{Q}}, J_2^{\mathbb{Q}}, \ldots, J_D^{\mathbb{Q}})$, *where* $J_d^{\mathbb{Q}}$ *for* $d = 1, 2, \ldots, D$ *are Jordan blocks. Also,* $a_W = Wa(0, \iota, k_\infty^{\mathbb{Q}} e_1, J(\lambda^{\mathbb{Q}}), b_W^{-1} V_\rho^{\mathbb{Q}} (b_W^{-1})')$ *and* $b_W = Wb(\iota, J(\lambda^{\mathbb{Q}}))$.

(ii) For identification of the conditional variance matrix under the physical measure, restrict the diagonal elements of $\Sigma_{\mathcal{P}}$ *to be strictly positive and the first entry of* $A_{\mathcal{P}}^{(k)}$ *and* $B_{\mathcal{P}}^{(k)}$ *for* $k = 1, \ldots, K$ *to be non-negative. Finally, a necessary condition for identification is that* $K \leq floor\left(\frac{1}{2}n_X + \frac{1}{2}\right)$.

Proof. See Technical Appendix A.3. □

3.3 Marginal Likelihood for Synthetic Yields

As in Joslin, Singleton, and Zhu (2011), I impose the following assumption to set up the likelihood function for synthetic yields.[7]

Assumption 2 *The portfolios of synthetic yields,* ρ_t, *balance out the measurement errors such that* ρ_t *is observed without errors.*

With ρ_t and hence $\mathcal{P}_t$ observed, the marginal log-likelihood of the synthetic yields can be separated by

$$\log \ell\left(Y_t^{\perp m} | Y_{t-1}^{\perp m}; \Theta_{\mathcal{P}}^{\mathbb{P}}, \Theta_\rho^{\mathbb{Q}}\right) = \log \ell_Y\left(Y_t^{\perp m} | \mathcal{P}_t; \Theta_\rho^{\mathbb{Q}}\right) + \log \ell_{\mathcal{P}}\left(\mathcal{P}_t | \mathcal{P}_{t-1}; \Theta_{\mathcal{P}}^{\mathbb{P}}\right).$$

I note that the two terms of the log-likelihood contribution do not depend on the same parameters. In particular, the likelihood separation clarifies that $\Theta_\rho^{\mathbb{Q}}$ relates to the cross-section of the synthetic yield curve, while $\Theta_{\mathcal{P}}^{\mathbb{P}}$ governs the dynamics of the state vector. It follows that $\Theta_\rho^{\mathbb{Q}}$ and $\Theta_{\mathcal{P}}^{\mathbb{P}}$ can be estimated by solving two unrelated maximum likelihood

[7] Relaxing this assumption requires a filtering technique such as Kalman filtering for estimating the model. Joslin, Singleton, and Zhu (2011) show in the Gaussian term structure model that the parameter estimates do not change significantly when this assumption is relaxed.

problems. The marginal log-likelihood function related to the parameters of the physical dynamics is given up to constants by

$$\log \ell_{\mathcal{P}}\left(\mathcal{P}_t|\mathcal{P}_{t-1};\Theta_{\mathcal{P}}^{\mathbb{P}}\right)$$
$$= -\frac{1}{2}\log|V_t| - \frac{1}{2}\left(\mathcal{P}_t - \mu_{\mathcal{P}} - \sum_{l=1}^{L}\Phi_{\mathcal{P}}^{(l)}\mathcal{P}_{t-l}\right)' V_t^{-1}\left(\mathcal{P}_t - \mu_{\mathcal{P}} - \sum_{l=1}^{L}\Phi_{\mathcal{P}}^{(l)}\mathcal{P}_{t-l}\right), \quad (25)$$

where I recall that V_t depends on the parameters and data through (19). For the remaining parameters, the marginal log-likelihood function to be maximized is given by

$$\log \ell_Y\left(Y_t^{\perp m}|\mathcal{P}_t;\Theta_\rho^{\mathbb{Q}}\right) = \left(Y_t^{\perp m} - a - b\rho_t\right)'\left(Y_t^{\perp m} - a - b\rho_t\right), \quad (26)$$

where $a = a(\alpha_{\mathcal{P}}, \beta_\rho, \mu_\rho^{\mathbb{Q}}, \Phi_\rho^{\mathbb{Q}}, V_\rho^{\mathbb{Q}})$ and $b = b(\beta_\rho, \Phi_\rho^{\mathbb{Q}})$ with $\alpha_{\mathcal{P}}$, β_ρ, $\mu_\rho^{\mathbb{Q}}$, and $\Phi_\rho^{\mathbb{Q}}$ given as explicit functions of $\Theta_\rho^{\mathbb{Q}} = \{k_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, V_\rho^{\mathbb{Q}}\}$ in Theorem 5. It is noteworthy that the log-likelihood function in (26) is identical to that of the Gaussian term structure model. Thus, this part of the estimation is no more difficult to implement than Gaussian term structure models that are celebrated for their tractability.

4 Data and Empirical Performance

4.1 Data

I apply zero-coupon yields of U.S. Treasury bonds provided by Gürkaynak, Sack, and Wright (2007) at the monthly frequency with end-of-month observations. I consider the sample from September 1971 to June 2019, which contains both periods with highly volatile yields, e.g., in the beginning of the 1980s, and periods with low yield volatility, e.g., before the outbreak of the financial crisis and during the zero-lower bound regime. I include maturities of 1, 2, 3, 5, 7, and 10 years. To capture the short end of the yield curve, I also use the 3- and 6-month Treasury bill rates from Federal Reserve Economic Data.

Following the literature, I include two macroeconomic variables: inflation and economic activity. There is no consensus in the literature about which specific data to use for these variables.[8] Economic activity has been defined by both growth and slack measures.

[8]For example, Ang and Piazzesi (2003) construct factors from principal components of a wide range of macroeconomic variables; Joslin, Priebsch, and Singleton (2014) use expected inflation measured from surveys along with the Chicago Fed National Activity Index, which is an estimate of overall economic growth; Bikbov and Chernov (2010) use CPI inflation and the Help Wanted Advertising in Newspapers index.

Bauer and Rudebusch (2016) show that these series are widely different and uncorrelated over the business cycle. They argue that slack variables are appropriate as they are consistent with empirical monetary policy rules. Moreover, they show that the unemployment gap is related to the slope of the yield curve. In light of these results, I adopt the unemployment gap as a measure of economic activity. I construct the unemployment gap by the difference between the unemployment rate from the U.S. Bureau of Labor Statistics and the estimate of natural unemployment from the Congressional Budget Office.[9] I measure inflation by the 12-month change in headline consumer prices from the Bureau of Labor Statistics.

The data are exhibited in Figure 1 and the descriptive statistics are detailed in Table 1. The yield data are highly persistent for all maturities. They are upward-sloping on average with a decreasing term structure of standard deviations. Yield levels peak in the beginning of 1980 and approach the zero-lower bound in the aftermath of the financial crisis. The unemployment gap is weakly and negatively correlated with the yield curve, while the correlations between inflation and yields are higher and positive. The data are cyclical: yields, particularly for short-term maturities, and inflation decrease in recessions, while the unemployment gap increases in recessions. The unemployment gap has decreased steadily during the current expansion.

The dual mandate of the Federal Reserve is related to inflation based on personal consumption expenditures (PCE) rather than CPI inflation. Moreover, it has been argued that core inflation, i.e., change in price indices that exclude food and energy prices, provides a better indicator of the path of future inflation compared to headline inflation (Blinder and Reis, 2005, Mishkin, 2007). I therefore provide results using PCE and core CPI inflation in the online appendix. I also apply my model using a growth measure of economic activity given by the Chicago Fed National Activity Index. The conclusions do not qualitatively change with these choices of macroeconomic variables.

4.2 Model Specification

Number of Latent Factors

Since Litterman and Scheinkman (1991), the literature widely agrees that three latent factors are sufficient for modeling the yield curve. The macroeconomic variables can po-

[9]The natural unemployment estimate is available at the quarterly frequency only. I assume that the natural unemployment rate is constant over the quarters.

tentially reduce the number of latent factors necessary for capturing the residual variation in the synthetic yields. However, I find that three latent factors remain necessary.

As shown in Theorem 4, the latent factors can be rotated into observed portfolios of synthetic yields, $\rho_t = WY_t^{\perp m}$, where W has full rank. I choose W such that the yield portfolios correspond to the first three principal components of the synthetic yield curve. This choice ensures that the portfolios are orthogonal and thus span the entire three-dimensional subspace. As a result, the portfolios capture as much variation in the synthetic yields as possible with three factors.

Lag Length Under the Physical Measure

The order of the vector autoregression under the physical measure determines the lag length of the macroeconomic variables. I implement the model with an order of $L = 12$ corresponding to an annual lag structure, which is a common choice in the macro-finance term structure literature (Ang and Piazzesi, 2003, Jardet, Monfort, and Pegoraro, 2013, Joslin, Le, and Singleton, 2013). The lag length is chosen based on general-to-specific tests and information criteria as documented in the online appendix.

Generality of the Conditional Variance Matrix Under the Physical Measure

With three latent factors and two macroeconomic variables, the state vector has dimension $N_X = 5$, which identifies up to $K = 3$ components in the conditional variance matrix. In fact, for $K = 3$, the BEKK GARCH model is equivalent to the diagonal vech-GARCH model with restrictions for positive definiteness. As already discussed, $K = 1$ does not give the model sufficient freedom to capture both conditional variances and covariances. In contrast, I find that the model can match realized and rolling variances of bond yields closely with $K = 2$, see Section 4.4. In the online appendix, I formally compare models specified with $K = \{1, 2, 3\}$ and show that the data is best modeled with $K = 2$.

4.3 Estimation Results

Parameter estimates are reported in the online appendix. The model is highly persistent under both the physical and risk-neutral measures with maximum eigenvalues of the autoregressive coefficients of respectively 0.995 and 0.997. The conditional variance matrix of the physical dynamics is persistent but remains stationary because

$$\max \left[\sum_{k=1}^{K} (A_{\mathcal{P}}^{(k)} \odot A_{\mathcal{P}}^{(k)} + B_{\mathcal{P}}^{(k)} \odot B_{\mathcal{P}}^{(k)}) \right] = 0.981 < 1,$$

where $\odot$ denotes element-wise Hadamard product.[10]

The estimated latent factors are exhibited in Figure 2 along with the factor loadings $b_{x,n}$ across maturities n. The loadings show that the latent factors capture level, slope, and curvature risk of the synthetic yields, $Y_t^{\perp m}$. The figure shows that these factors are different from the level, slope, and curvature of observed yields, Y_t. In particular, I observe that the macroeconomic variables relate to variations in the level and slope of the yield curve. This is consistent with Rudebusch and Wu (2008), who show that the level and slope of the yield curve have macroeconomic underpinnings, and Bauer and Rudebusch (2016), who show that the unemployment gap is related to the slope.

4.4 Empirical Performance

Goodness of In-Sample Yield Curve Fit

The ability of my model to match the yield curve is summarized by root mean squared errors in Panel A of Table 2. My model fits the data well with an average error across maturities of 8 basis points. This result is not surprising since the risk-neutral dynamics of my model are similar to those of the Gaussian term structure model, which is known to have good in-sample properties in terms of matching the yield curve.

Matching Conditional Yield Curve Volatility

I compare model-implied conditional variances and covariances of the yield curve to two empirical proxies. First, I construct a realized variance matrix using daily yield data.[11] This measure is likely to be a good proxy of conditional variance, because realized variance is a consistent estimator of integrated variance (Barndorff-Nielsen and Shephard, 2004) and conditional variance can be viewed as a noisy measure of integrated variance.[12] As

[10] The stationarity condition for the BEKK GARCH model is given in Engle and Kroner (1995).

[11] I construct the realized variance matrix as follows. Let $y_{t,s,n}$ denote the n-maturity yield on day s in month t and let S denote the total number of daily observations in month t. Following this, I define the realized covariance between yields with maturities n_1 and n_2 by

$$\mathrm{RV}_{t,n_1,n_2} = \sum_{s=1}^{S} \left(y_{t,s,n_1} - y_{t,s-1,n_1}\right)\left(y_{t,s,n_2} - y_{t,s-1,n_2}\right).$$

[12] See Andersen, Bollerslev, Diebold, and Labys, (2001, 2003), Koopman and Scharth (2013), and Hansen, Huang, and Shek (2011).

a second proxy, I use a rolling conditional variance matrix constructed with daily data using a 3-month look-back.

Figure 3 shows model-implied conditional variances and covariances along with the corresponding realized and rolling measures for the 3-month and 10-year maturities. My model captures low-frequency variation of empirical measures closely in both the highly volatile period in the beginning of the 1980s and in the periods with low volatility. Short-lived bursts observed in the empirical proxies that are not reproduced by my model may be due to measurement errors (Andersen, Bollerslev, and Meddahi, 2005) and outliers that should not be predictable ex ante (Cieslak and Povala, 2016).

To provide a formal assessment, I show root mean squared errors in Panels B.1 and B.2 of Table 2. Errors are generally lower for long-term yields, which may reflect that volatility of short-term bond yields is distorted by noise, institutional effects, and factors specific to Treasury bills, see Cieslak and Povala (2016) and references therein. I also run Mincer and Zarnowitz (1969) regressions of the empirical measures on a constant and the model-implied conditional variances and covariances. Table 2 shows the constants, slope coefficients, and adjusted R-squared values from these regressions. I find that a majority of the constants are estimated insignificantly. The estimated slope coefficients are between 0.5 and 0.8 with adjusted R-squared values between 0.4 and 0.6.

Thus, I conclude that my model can match empirical measures of yield curve variances and covariances. This finding contrasts with continuous-time term structure models with stochastic volatility, which result in low and often negative correlation between predicted and realized variances (Christensen, Lopez, and Rudebusch, 2014, Collin-Dufresne, Goldstein, and Jones, 2009, Jacobs and Karoui, 2009). The success of my model can be attributed to the exponential-quadratic pricing kernel. Specifically, the distinction between conditional variance matrices under the physical and risk-neutral probability measures gives my model freedom to capture empirical volatility as seen in Creal and Wu (2017), Ghysels, Le, Park, and Zhu (2014), and Joslin and Konchitchki (2018).

Term Structure Decomposition

Next, I show how my model decomposes yields into short-rate expectations and term premia, see (16). The estimated decomposition is shown for the 10-year yield in the left chart of Figure 4. The expected short rates reflect the dynamics of long-term yields due to the high degree of persistence in yield data. As a result, model-implied term premia are relatively stable. However, the term premia are counter-cyclical as investors demand a higher compensation for risk in times of crisis.

Time-varying volatility can be interpreted as a time-varying quantity of risk. Thus, my model involves time-variation in both the quantity and price of risk, see (7)-(8). In contrast, Gaussian term structure models only allow for a time-varying price of risk. Thus, comparing term premia estimates of my model with those of the Gaussian term structure model provides an insight into the impact of a time-varying quantity of risk on term premia. The right chart in Figure 4 shows the term structure decomposition of the Gaussian term structure model. The time-varying quantity of risk in my model causes stronger counter-cyclicality in term premia compared to the constant quantity of risk in the Gaussian model. Specifically, I observe large differences between the models after the financial crisis. My model implies that term premia decreased slowly and became negative in May 2019. This result signals an increasing optimism among investors, which is consistent with the economic expansion. In contrast, the Gaussian term structure model implies term premia that became negative immediately after the recession. This estimate points to investors regaining optimism shortly after the recession. It is likely that this result arises because the Gaussian model fails to capture the decrease in the quantity of risk following the recession.

Furthermore, my model disagrees with the Gaussian term structure model in regard to the effectiveness of forward guidance. If forward guidance works as intended, short-rate expectations should follow central bank communications. In my model, this is indeed the case for many examples of statements from the Federal Open Market Committee. For example, my model implies increased short-rate expectations as the committee signaled an upward path of the federal funds rate during the lift-off from the zero-lower bound in December 2015. In contrast, the Gaussian term structure model estimates decreasing short-rate expectations during this period, which implies that the forward guidance was not effective. Another example is in December 2008, where the Federal Open Market Committee indicated that the federal funds rate was likely to remain low in the future. My model is consistent with markets reacting to this statement as the short-rate expectations remain low. However, short-rate expectations in the Gaussian term structure model increase over this period, as observed in the right chart in Figure 4. My conclusion that forward guidance is effective is supported by Carlstrom, Fuerst, and Paustian (2015) and McKay, Nakamura, and Steinson (2016), while Hagedorn, Luo, Manovskii, and Mitman (2019) present counter-arguments.

My model also generates time-varying conditional variances of the expected short rates and term premia. These variances are shown at the 10-year maturity in Figure 5. The variance of short-rate expectations is counter-cyclical and mimics the behavior of the conditional yield variances. Specifically, the large burst of volatility in short-term yields in the beginning of the 1980s is transmitted to the volatility of short-rate expectations. The term premium is more stable. Figure 5 also shows the model-implied conditional correlation between the 10-year short-rate expectations and term premium. The correlation is negative throughout the majority of the sample. This is intuitively appealing: whereas a high term premium signals that bond market investors fear risk, i.e., pessimism, high expected short rates reflect an optimism about future economic conditions. The sign, however, tends to reverse during expansions, which may reflect the fact that mid-expansions are typically characterized by a flattening yield curve and low volatility.[13]

Specification Test

The ability of the model to decompose the term structure can be tested by projecting holding period returns, $Y_{t+1,n-1} - Y_{t,n}$, on a constant and the slope of the yield curve, $(Y_{t,n} - r_t)/(n-1)$. Under the expectations hypothesis, such regressions give coefficients equal to one. However, Campbell and Shiller (1991) show that empirically estimated coefficients are negative and exhibit a downward pattern across maturities. I test the ability of the model to generate coefficients that match the empirical failure of the expectations hypothesis, which has been named the LPY-I test by Dai and Singleton (2002).[14] The test results are depicted in the upper-left chart of Figure 6. With the exception of the short maturities, my model produces coefficients that are well within the 95 percent confidence intervals for the empirically estimated coefficients. Thus, the model passes the LPY-I test.

[13] This relationship has been established by Norland (2018) using the VIX index as a measure of volatility. While the VIX index relates to equity markets, it is positively correlated with volatility measures related to bond markets, e.g., the Merill-Lynch Option Implied Volatility Estimate (MOVE).

[14] In particular, I construct yield curve data with maturities $1, 2, \ldots, 120$ months using the method of Gürkaynak, Sack, and Wright (2007). Then, I regress $Y_{t+1,n-1} - Y_{t,n} = c + \phi_{T,n}(Y_{t,n} - r_t)/(n-1) + e_t$, which results in the empirically estimated coefficients $\hat{\phi}_{T,n}$. I compare these with model-implied population coefficients obtained by simulating 100,000 observations from my model.

Dai and Singleton (2002) also propose a LPY-II test which projects the risk-adjusted holding period returns onto a constant and the slope of the yield curve.[15] If the model is well-specified, the resulting coefficients should be equal to one. The lower-left chart on Figure 6 shows that the model-implied 95 percent confidence interval contains one for all maturities. Hence, my model also passes the LPY-II test.

Traditional affine term structure models with stochastic volatility (Dai and Singleton, 2000) fail both the LPY-I and LPY-II tests. In contrast, the Gaussian term structure model passes both tests (Dai and Singleton, 2002), which I confirm in the right panel of Figure 6, but fails to capture time-varying volatility. Thus, the existing literature poses a trade-off between modeling time-varying volatility and passing the LPY tests. My model overcomes this trade-off as I simultaneously match empirical proxies of the conditional yield curve variance matrix and pass both LPY tests.

5 Yield Curve Volatility and the Macroeconomic Risk

In this section, I show that the relationship between the yield curve and macroeconomic risk fluctuates over time. My model provides insights into the patterns and sources of these fluctuations. Specifically, I examine the macroeconomic contribution to the variances of yields, short-rate expectations, and term premia. Finally, I relate the model-implied price of variance risk to macroeconomic variables.

5.1 Macroeconomic Contribution to Yield Curve Variances

I analyze the relationship between the yield curve and macroeconomic risk by decomposing model-implied variances into contributions from macroeconomic variables and latent yield-specific factors. For this purpose, I apply a recursive identification scheme assuming that macroeconomic variables are slow-moving and do not react to contemporary yield-specific shocks. Furthermore, I assume that the unemployment gap responds to inflation shocks with a one-month lag. These assumptions are standard in the structural vector autoregressive literature with Cholesky identification schemes (Bernanke, Boivin, and Eliasz, 2005, Christiano, Eichenbaum, and Evans, 1999, Stock and Watson, 2001). My approach is also standard in the macro-finance term structure literature in which more

[15] The risk-adjusted returns are given by $Y_{t+1,n-1} - Y_{t,n} - (\Psi_{t+1,n-1} - \Psi_{t,n-1}) + \Psi^F_{t,n-1}/(n-1)$, where $Y_{t,n}$ are observed yields, $\Psi_{t,n}$ is the model-implied term premium defined in (16), and Ψ^F is the model-implied forward term premium defined by $\Psi^F_{t,n} = F_{t,n} - \mathbb{E}_t(r_{t+n})$, where $F_{t,n}$ is the forward rate.

sophisticated identification methods are not yet widespread. My results are robust to different orderings of the variables, see the online appendix.

Given these assumptions, conditional yield variances can be decomposed into variation generated by, respectively, macroeconomic variables and latent yield-specific factors. Moreover, the macroeconomic contribution can be decomposed into an inflation and an unemployment-gap component:

$$\mathrm{Var}_{t-1}\left(Y_{t,n}\right) = \mathrm{Var}_{t-1}\left(\epsilon_t^{\text{inflation}}\right) + \mathrm{Var}_{t-1}\left(\epsilon_t^{\text{unemployment gap}}\right) + \mathrm{Var}_{t-1}\left(\epsilon_t^{\text{yield-specific}}\right),$$

where $\epsilon_t^{\text{inflation}}$, $\epsilon_t^{\text{unemployment gap}}$, and $\epsilon_t^{\text{yield-specific}}$ are structural inflation, unemployment-gap, and yield-specific shocks. I define the macroeconomic share of variance in $Y_{t,n}$ by

$$\mathrm{Var}_{t-1}\left(\epsilon_t^{\text{inflation}} + \epsilon_t^{\text{unemployment gap}}\right) / \mathrm{Var}_{t-1}\left(Y_{t,n}\right).$$

Analogously, I construct macroeconomic shares of variance in short-rate expectations and term premia. The novelty of my analysis compared to the previous literature is that the macroeconomic shares are time-varying. I characterize the time-series properties of the term structure of macroeconomic shares in Table 3. I emphasize two results from these descriptive statistics.

First, I observe that the macroeconomic shares vary over wide ranges. For example, the macroeconomic contribution to variance in the 3-month yield varies between 0.02 and 55.56 percent. The ranges are similar for yields of bonds with longer maturities. Thus, in some periods, latent yield-specific factors fully account for movements in the yield curve, whereas macroeconomic shocks explain more than half of the variation in other periods. On average, macroeconomic shocks explain 7.92 percent of the 3-month yield variance. This average is consistent with the constant macroeconomic share obtained with a Gaussian term structure model, which equals 7.72 percent for the 3-month maturity. The models agree on average for all considered maturities. Since the Gaussian term structure model captures the average level of yield curve variance, this result can be interpreted as a validation of my model specification.

Second, the sample kurtosis along with the quantiles suggest that there are occasional large month-to-month fluctuations. Some of these occasions can be tied to meaningful economic events. For example, when the first round of quantitative easing was announced in November 2008, the macroeconomic share of variance in the 10-year yield increased from 5 to 28 percent. The intention of quantitative easing is that central banks impact long-term yields directly by large-scale asset purchases. Thus, successful quantitative easing programs result in a tighter link between long-term yields and the dual mandate. This intuition is indeed consistent with an increased macroeconomic share of variance in the 10-year yield. Another example is when Donald Trump was elected president in the end of 2016, where it was generally believed that Trump's stimulus plans would lead to higher interest rates.[16] During the month following the election, the macroeconomic share of variance in the 10-year yield indeed increased from 5 to 25 percent. More recently, the Federal Reserve announced in January 2019 that the monetary policy tightening cycle would be put on hold due to factors unrelated to the domestic economy.[17] During this month, the macroeconomic share of the 3-month yield variance fell from 16 to 3 percent. This anecdotal evidence is consistent with the widespread idea that macroeconomic announcement effects impact financial markets (Andersen, Bollerslev, Diebold, and Vega, 2007, Balduzzi, Elton, and Green, 2001, Bollerslev, Cai, and Song, 2000, Feunou, Fontaine, and Roussellet, 2019, Fleming and Remolona, 1999, Johannes, 2004, Piazzesi, 2005).

The link between variance in long-term yields and macroeconomic risk can be generated through two channels, namely short-rate expectations and term premia, as shown by the term structure decomposition in (16). Table 4 shows descriptive statistics of the macroeconomic shares of variances in 5- and 10-year short-rate expectations and term premia. Macroeconomic variables account for 0.32 to 75.66 percent of the variance in 10-year short-rate expectations and 41.88 percent on average. The macroeconomic share of variance in the 10-year term premium ranges between 0.36 and 68.58 percent, with an average of 21.30 percent. These averages do not align with the Gaussian term structure model that implies macroeconomic shares of 26.36 and 26.16 percent for the 10-year short-rate expectations and term premium. These differences arise because the Gaussian term structure model obtains a different decomposition of yields by ignoring the time-varying quantity of risk, as discussed in Section 4.4. Thus, even if our interest lies in explaining variation across the full sampling period, it is important to account for time-varying volatility. The results also show that macroeconomic shocks explain more variation in, respectively, short-rate expectations and term premia compared to the yield curve. Hence, investors pay closer attention to macroeconomic risk when forming expectations about future monetary policy compared to when they price bonds. This is consistent with the idea of unspanned macroeconomic risk proposed by Joslin, Priebsch, and Singleton (2014).

[16] See, for example, "How high can US rates go in the Trump era?" Financial Times, December 13, 2016, https://www.ft.com/content/b5551882-c057-11e6-81c2-f57d90f6741a.

[17] "Federal Reserve puts rate rises on hold as global economy slows", Financial Times, January 30, 2019, https://www.ft.com/content/2565e154-24b7-11e9-b329-c7e6ceb5ffdf.

5.2 Historical Joint Behavior of the Yield Curve and Macroeconomic Risk

Next, I characterize the historical relationship between movements in the yield curve and macroeconomic variables throughout my sample. Since investors and policy makers typically care about horizons longer than one month, I consider fluctuations in macroeconomic shares of variances over 5-year periods. Figure 7 shows 5-year moving-average macroeconomic shares of the 3-month and 10-year yield variances along with variances of the 10-year short-rate expectations and term premium.

In the upper-left chart on Figure 7, I observe a U-shaped pattern in the macroeconomic share of the 3-month yield variance. Specifically, macroeconomic variables are important short-term yield factors in the 1970s and 1980s, but cease to be relevant during the Great Moderation starting from the mid-1980s. This result is intuitively appealing as the stabilization of macroeconomic variables reduces the extent to which these shocks can explain volatility in financial markets. In the aftermath of the financial crisis, however, the link between the macroeconomic variables and short-term yields has strengthened. Figure 7 also decomposes the macroeconomic contribution into inflation and the unemployment gap. The role of inflation attenuates over time and only accounts for a small fraction of the short-term yield variance after the Great Moderation. The unemployment gap, on the other hand, accounts for the increasing macroeconomic share after the financial crisis, which coincides with the recovery of the unemployment rate.

The upper-right chart on Figure 7 shows the trend in the macroeconomic share of the 10-year yield variance. This trend can be described by a W-shape with low values during the Volcker period and in the aftermath of the early 1990s recession. The W-shape is driven by inflation, while the contribution from the unemployment gap is stable. The macroeconomic shares of variances in the decomposition of the 10-year yield provides additional insights, see the lower charts of Figure 7.

First, the macroeconomic share of variance in the 10-year short-rate expectations exhibits an upward trend since the Great Moderation. Thus, the expectations hypothesis is an increasingly important channel for how macroeconomic variables shape the yield curve. One interpretation is that there is a growing belief among investors that the Federal Reserve's commitment to the dual mandate is credible. The upward trend is primarily attributed to the unemployment gap with the exception of the financial crisis and its aftermath, where inflation accounts for more variation. This result is consistent with survey data on CPI inflation from the Survey of Professional Forecasters. To show this, I plot the median one-quarter and one-year ahead forecasts of CPI inflation along with the dispersion of these forecasts defined by the difference between the 75th and 25th quantiles in Figure 8. I observe that the median forecasts vary more during recessions and in particular during the financial crisis. Moreover, the dispersions are larger during recessions. I capture this pattern in the model-implied conditional variance of inflation, which is plotted in the lower panel of Figure 8 for forecasting horizons of one quarter and one year.

Second, the macroeconomic share of variance in the 10-year term premium decreases rapidly in the 1980s, but evolves around a constant mean in the remainder of the sample with counter-cyclical fluctuations caused primarily by inflation. Thus, my model is consistent with a counter-cyclical inflation risk premium, as estimated by Buraschi and Jiltsov (2005), Chernov and Mueller (2012), and D'Amigo, Kim, and Wei (2018). The contribution of the inflation risk premium is particularly high during the financial crisis. In my model, a counter-cyclical inflation risk premium is generated by the exponential-quadratic pricing kernel, which allows investors to price the risk associated with both negative and positive inflation shocks. I illustrate this mechanism by simulating 100,000 paths of my model under the risk-neutral ($\mathbb{Q}$) and physical ($\mathbb{P}$) probability measures and estimating kernel density functions of these simulated series. I plot the conditional densities along with their ratio ($\mathbb{Q}/\mathbb{P}$) for the one-month and ten-year horizons in Figure 9. The risk-neutral densities are wider than the physical ones, which reflects the presence of variance risk premia, or, equivalently, different conditional variances under the probability measures. This difference is more pronounced for the long horizon, which involves more uncertainty and thus a larger risk compensation. Differences in the conditional variances result in U-shaped density ratios such that investors are averse to both high and low states of the distribution of inflation. Thus, my model allows investors to demand a premium for deflation fears, which can generate a counter-cyclical inflation risk premium. In this respect my model is similar to Roussellet (2018). An implication of this result is that a symmetric inflation target enhances the efficiency of monetary policy during recessions.

Subsample Analysis: January 1990 to June 2019

Applications in term structure modeling often start the sample in 1990 to avoid issues related to the Volcker period. As a robustness check, I show in Figure 10 that the time-varying patterns of macroeconomic shares of yield curve volatility identified after the Great Moderation for the full sample continue to hold when the model is estimated using the shorter sample. For the 3-month yield, I reproduce half of the U-shaped trend in the 5-year moving-average macroeconomic share of variance with the increase driven by the unemployment gap only. The levels of the ratios are higher than observed from Figure 7, reflecting that the short-maturity loading on unemployment gap is overestimated compared to the full sample. For the 10-year yield, I find that the macroeconomic share of variance is increasing after 1995 due to inflation, which is consistent with the results obtained using the full sample. I also generate macroeconomic shares of the variances in the 10-year short-rate expectations and term premium that are similar to the full-sample results.

5.3 Variance Risk Premia and Macroeconomic Risk

A central distinction of my model from the Gaussian term structure model is the fact that I allow investors to be compensated for variance risk, as defined by Theorem 3. This compensation is not earned by investments in Treasury bonds, as shown in the bond yield equation (11). Rather, the variance risk premium is demanded by investors in the broader fixed-income market through interest-rate derivatives.

Although my model results in time-varying variance risk premia, I abstract from analyzing these dynamics because they – by the assumption of constant risk-neutral volatility – are solely given by the specification of volatility under the physical measure. Instead, I focus on the average variance risk premia.

Following Carr and Wu (2009), I report variance risk premia as ratios of expected variance under the physical measure. Thus, I show the sample average of $\Gamma_{t,n}$ as a fraction of the sample average of $\mathbb{E}_t(\mathrm{Var}_{t+1}(Y_{t+2,n}))$ in the left chart of Figure 11. As expected, the variance risk premia are negative across all maturities, which is consistent with the stylized fact that implied volatilities exceed realized volatilities. The sizes of the variance risk premia are between 30 to 70 percent of the expected variance. This ratio decreases in magnitude across maturities with a small hump at the short end of the yield curve. Thus, fixed-income investors pay large premia for protection against variance risk, particularly in the short end of the yield curve. These results are aligned with existing estimates of variance risk premia in Treasury bond markets (Choi, Mueller, and Vedolin, 2017, Trolle, 2009, Trolle and Schwartz, 2015).

As a novel contribution to the literature, I analyze the link between variance risk premia and the macroeconomic variables. The right chart in Figure 11 decomposes the variance risk premia into contributions from the macroeconomic variables and the latent yield-specific factors. A majority of the variance risk premia, between 80 and near 100 percent, are related to macroeconomic uncertainty. This share is increasing with maturity. Thus, fixed-income investors demand large compensations for exposure to macroeconomic uncertainty, whereas uncertainty related to latent yield-specific factors is less important for variance risk premia. It follows that macroeconomic uncertainty can increase trading activity in fixed-income derivative markets. To the extent that monetary policy is determined by inflation and unemployment gap through the dual mandate, my model is consistent with Cieslak and Povala (2016).

6 The 2019 Yield Curve Inversion and Macroeconomic Risk

The 10-year U.S. Treasury bond yield declined by 50 basis points between April and June 2019. The yield curve inverted with the 10-year yield below the 3-month yield during May. These movements are illustrated in the left chart of Figure 12a. The right chart also shows that my model captures the increased variation in the 10-year yield during the spring 2019.

Yield curve inversions draw attention from policy makers and investors because history shows that inversions are strong predictors of recessions. Indeed, the yield curve has inverted before every recession since 1971. However, the time between inversion and recession varies and history shows examples of yield curve inversions that are not preceded by recessions. A yield curve inversion is therefore subject to debate on whether a recession will occur and when this is likely to happen. Long-term yields may fall below short-term yields because investors expect a slow-down in future economic growth. Such expectations are reflected in the yield curve by low short-rate expectations, and thus low long-term yields for a given term premium. Inversions that are credible warnings of economic slowdowns and potential recessions should therefore be driven by macroeconomic variables through short-rate expectations.

I use my model to show whether the recent yield curve inversion is driven by macroeconomic variables, and thus is likely to pose a warning of an economic slowdown. My model is promising for this purpose because it can analyze time-varying variances conditional on current economic conditions. In contrast, the Gaussian term structure model can only draw conclusions that are valid on average across the full sample.

To support my arguments, I begin by analyzing the yield curve inversion in August 2006 that preceded the financial crisis. Figure 13a shows the yield curve movements leading to this inversion. As in the recent inversion, the yield curve inversion in 2006 was primarily driven by declines in the 10-year yield. Figure 13b attributes these movements to macroeconomic variables. The left chart shows that the macroeconomic variables drive some of the declines in the 10-year yield. The right chart shows a corresponding increase in the macroeconomic share of variance in 10-year short-rate expectations. Thus, the inversion in 2006 is related to macroeconomic risk. I note, however, from the magnitudes of the macroeconomic shares of variances that the relationship between the long-term yield and the macroeconomic variables remains weak despite the increases. This weak link may explain why the financial crisis as defined by the National Bureau of Economic Research began 18 months after the inversion.

Given these insights, I return to the yield curve inversion in 2019. Figure 12b shows in the left chart that the macroeconomic share of the 10-year yield variance declined from 15 percent in February to practically zero percent in June. Specifically, this decline is related to a weakening relationship between 10-year short-rate expectations and the macroeconomic variables. The right chart of Figure 12b shows that the macroeconomic share of variance in 10-year short-rate expectations declined from 71 percent in February to 53 percent in June. The macroeconomic share of variance in the 10-year term premia has been stable over the period. Thus, I find evidence that macroeconomic shocks do not explain the recent large declines in long-term yields and the associated yield curve inversion. In fact, over the period in which the yield curve inverted, investors increasingly base short-rate expectations on information not related to inflation and unemployment gap.

These results indicate that the yield curve inversion in May 2019 is not related to macroeconomic risk. The current strength of the U.S. economy supports this conclusion. For instance, in May 2019 the unemployment rate had been at or below 4 percent for more than a year. Moreover, it has been argued that the relationship between yields and macroeconomic risk is currently weak due to distress in bond markets caused by low rates and quantitative easing programs that are yet to be reversed.[18] What does the yield curve inversion then reflect? Global factors and in particular the trade war between the U.S. and China are likely to play a role. Obviously, these circumstances can impact the domestic economy and contribute to a recession in the future. My results therefore do

[18] Research from Pictet Wealth Management concludes that "the distortions created by extraordinary post-crisis monetary policies have led to the breakdown in the relationship between interest rate expectations and economic growth" ("Why the yield curve is not the economic guide it once was," April 2, 2019, https://www.ft.com/content/15d4048e-552f-11e9-91f9-b6515a54c5b1).

not reject all signs of a recession. However, I emphasize that the yield curve inversion is not currently related to macroeconomic risk and therefore does not signal that the risk of a recession is imminent. Thus, the Federal Reserve should not begin a cycle of monetary policy easing based on the yield curve inversion.

7 Concluding Remarks

Using a novel macro-finance term structure model with time-varying volatility and variance risk premia, I document a time-varying relationship between the yield curve and macroeconomic risk. In particular, I estimate macroeconomic shares of yield curve volatility that fluctuate over time both at the monthly frequency and over longer periods. I relate large month-to-month fluctuations to meaningful economic events. I characterize the historical joint dynamics of yields and macroeconomic risk. First, I show a U-shaped pattern in the link between the short end of the yield curve and macroeconomic risk. Macroeconomic variables are important short-term yield factors in the 1970s-1980s, become less relevant during the Great Moderation, and regain importance after the financial crisis. Second, investors increasingly form expectations on future short rates based on macroeconomic risk. Third, macroeconomic shares of variation in 10-year term premia increase during the financial crisis due to deflation fears.

In conclusion, my results show that the assumption of constant volatility in standard Gaussian term structure models shield information about: (i) fluctuations in the relationship between the yield curve and macroeconomic risk; (ii) counter-cyclicality of term premia; and (iii) the demand of compensation for macroeconomic uncertainty.

My model provides new insights about the predictive power of the 2019 yield curve inversion for a future recession. I find a weak link between the recent inversion and current conditions of the economy. Specifically, the large decline in long-term yield is not driven by inflation and the unemployment gap. I conclude that the yield curve inversion does not reflect low expectations on future economic growth and, consequently, is not likely to predict a recession in the near future.

Bibliography

Andersen, T. G., and L. Benzoni (2010): "Do Bonds Span Volatility Risk in the U.S. Treasury Market? A Specification Test for Affine Term Structure Models," *Journal of Finance*, 65(2), 603–653.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and P. Labys (2001): "The Distribution of Realized Exchange Rate Volatility," *Journal of the American Statistical Association*, 96(453), 42–55.

——— (2003): "Modeling and Forecasting Realized Volatility," *Econometrica*, 71(2), 579–625.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and C. Vega (2007): "Real-Time Price Discovery in Global Stock, Bond and Foreign Exchange Markets," *Journal of International Economics*, 73, 251–277.

Andersen, T. G., T. Bollerslev, and N. Meddahi (2005): "Correcting the Errors: Volatility Forecast Evaluation Using High-Frequency Data and Realized Volatilities," *Econometrica*, 73(1), 279–296.

Andreasen, M. M., and K. Jørgensen (2019): "The Importance of Timing Attitudes in Consumption-Based Asset Pricing Models," *Journal of Monetary Economics*, forthcoming.

Ang, A., and M. Piazzesi (2003): "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables," *Journal of Monetary Economics*, 50(4), 745–787.

Balduzzi, P., E. J. Elton, and T. C. Green (2001): "Economic News and Bond Prices: Evidence from the U.S. Treasury Market," *Journal of Financial and Quantitative Analysis*, 36(4), 523–543.

Bansal, R., and A. Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59(4), 1481–1509.

Barndorff-Nielsen, O. E., and N. Shephard (2004): "Econometric Analysis of Realized Covariation: High Frequency Based Covariance, Regression and Correlation in Financial Economics," *Econometrica*, 72(3), 885–925.

Bauer, M. D., and G. D. Rudebusch (2016): "Resolving the Spanning Puzzle in Macro-Finance Term Structure Models," *Review of Finance*, 21(2), 511–553.

Bernanke, B. S., J. Boivin, and P. Eliasz (2005): "Measuring the Effects of Monetary Policy: A Factor-Augmented Vector Autoregressive (FAVAR)," *Quarterly Journal of Economics*, 120(1), 387–422.

Bikbov, R., and M. Chernov (2010): "No-Arbitrage Macroeconomic Determinants of the Yield Curve," *Journal of Econometrics*, 159, 166–182.

Blinder, A. S., and R. Reis (2005): "Understanding the Greenspan Standard," *Federal Reserve Bank of Kansas City.*

Bollerslev, T., J. Cai, and F. M. Song (2000): "Intraday Periodicity, Long Memory Volatility, and Macroeconomic Announcement Effects in the US Treasury Bond Market," *Journal of Empirical Finance*, 7, 37–55.

Bollerslev, T., R. F. Engle, and J. M. Wooldridge (1988): "A Capital Asset Pricing Model with Time-Varying Covariances," *Journal of Political Economy*, 96(1), 116–131.

Bollerslev, T., G. Tauchen, and H. Zhou (2009): "Expected Stock Returns and Variance Risk Premia," *Review of Financial Studies*, 22(11), 4463–4492.

Buraschi, A., and A. Jiltsov (2005): "Inflation risk premia and the expectations hypothesis," *Journal of Financial Economics*, 75, 429–490.

Campbell, J. Y., and R. J. Shiller (1991): "Yield Spreads and Interest Rate Movements: A Bird's Eye View," *The Review of Economic Studies*, 58(3), 495–514.

Campbell, J. Y., A. Sunderam, and L. M. Viceira (2017): "Inflation Bets or Deflation Hedges? The Changing Risks of Nominal Bonds," *Critical Finance Review*, 6(2), 263–301.

Carlstrom, C. T., T. S. Fuerst, and M. Paustian (2015): "Inflation and Output in New Keynesian Models with a Transient Interest Rate Peg," *Journal of Monetary Economics*, 76, 230–243.

Carr, P., and L. Wu (2009): "Variance Risk Premiums," *Review of Financial Studies*, 22(3), 1311–1341.

Chang, C.-L., and M. McAleer (2019): "The Fiction of Full BEKK: Pricing Fossil Fuels and Carbon Emissions," *Finance Research Letters*, 28, 11–19.

Chernov, M., and P. Mueller (2012): “The Term Structure of Inflation Forecasts,” *Journal of Financial Economics*, 106, 367–394.

Choi, H., P. Mueller, and A. Vedolin (2017): “Bond Variance Risk Premiums,” *Review of Finance*, p. 987–1022.

Christensen, J. H. E., J. A. Lopez, and G. D. Rudebusch (2014): “Can Spanned Term Structure Factors Drive Stochastic Yield Volatility?,” *Federal Reserve Bank of San Fransisco Working Paper Series*.

Christiano, L. J., M. Eichenbaum, and C. L. Evans (1999): “Monetary Policy Shocks: What Have We Learned and to What End?,” in *Handbook of Macroeconomics*, pp. 65–148. Elsevier, Amsterdam.

Cieslak, A., and P. Povala (2016): “Information in the Term Structure of Yield Curve Volatility,” *Journal of Finance*, 71, 1393–1436.

Collin-Dufresne, P., and R. S. Goldstein (2002): “Do Bonds Span the Fixed Income Markets? Theory and Evidence for Unspanned Stochastic Volatility,” *Journal of Finance*, 57(4), 1685–1730.

Collin-Dufresne, P., R. S. Goldstein, and C. S. Jones (2009): “Can Interest Rate Volatility be Extracted from the Cross-section of Bond Yields?,” *Journal of Financial Economics*, 94, 47–66.

Creal, D. D., and J. C. Wu (2017): “Monetary Policy Uncertainty and Economic Fluctuations,” *International Economic Review*, 58(4), 1317–1354.

Dai, Q., and K. J. Singleton (2000): “Specification Analysis of Affine Term Structure Models,” *Journal of Finance*, 55(5), 1943–1978.

——— (2002): “Expectation Puzzles, Time-Varying Risk Premia, and Affine Models of the Term Structure,” *Journal of Financial Economics*, 63, 415–441.

D’Amigo, S., D. Kim, and M. Wei (2018): “Tips from TIPS: the Information Content of Treasury Inflation-Protected Security Prices,” *Journal of Financial and Quantitative Analysis*, 53(1), 395–436.

Doshi, H., K. Jacobs, and R. Liu (2018): “Macroeconomic Determinants of the Term Structure: Long-Run and Short-Run Dynamics,” *Journal of Empirical Finance*, 48, 99–122.

Duffee, G. R. (2018): "Expected Inflation and Other Determinants of Treasury Yields," *Journal of Finance*, 73(5), 2139–2180.

Engle, R. (2002): "Dynamic Conditional Correlation: A Simple Class of Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models," *Journal of Business & Economic Statistics*, 20(3), 339–350.

Engle, R. F., and K. F. Kroner (1995): "Multivariate Simultaneous Generalized ARCH," *Econometric Theory*, 11(1), 122–150.

Engle, R. F., D. M. Lilien, and R. P. Robins (1987): "Estimating Time Varying Risk Premia in the Term Structure : The Arch-M Model," *Econometrica*, 55(2), 391–407.

Epstein, L. G., and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57(4), 937–969.

Feunou, B., J.-S. Fontaine, and G. Roussellet (2019): "Macro-Finance Term Structure Models with External Macro Shocks," *Working Paper*.

Fleming, M. J., and E. M. Remolona (1999): "Price Formation and Liquidity in the U.S. Treasury Market: The Response to Public Information," *Journal of Finance*, 54, 1901–1915.

Ghysels, E., A. Le, S. Park, and H. Zhu (2014): "Risk and Return Trade-off in the U.S. Treasury Market," *Working Paper*.

Gürkaynak, R. S., B. Sack, and J. H. Wright (2007): "The U.S. Treasury Yield Curve: 1961 to the Present," *Journal of Monetary Economics*, 54, 2291–2304.

Hagedorn, M., J. Luo, I. Manovskii, and K. Mitman (2019): "Forward Guidance," *Journal of Monetary Economics*, 102, 1–23.

Hansen, L. P., J. Heaton, J. Lee, and N. Roussanov (2007): "Intertemporal Substitution and Risk Aversion," in *Handbook of Econometrics*, ed. by J. Heckman, and E. Leamer. Elsevier.

Hansen, L. P., and J. C. Heaton (2008): "Consumption Strikes Back? Measuring Long-Run Risk," *Journal of Political Economy*, 116(2), 260–302.

Hansen, P. R., Z. Huang, and H. H. Shek (2011): "Realized GARCH: A Joint Model for Returns and Realized Measures of Volatility," *Journal of Applied Econometrics*, 27(6), 877–906.

Haubrich, J., G. Pennacchi, and P. Ritchken (2012): "Inflation Expectations, Real Rates, and Risk Premia: Evidence from Inflation Swaps," *The Review of Financial Studies*, 25(5), 1588–1629.

Jacobs, K., and L. Karoui (2009): "Conditional Volatility in Affine Term-Structure Models: Evidence from Treasury and Swap Markets," *Journal of Financial Economics*, 91(3), 288–318.

Jardet, C., A. Monfort, and F. Pegoraro (2013): "No-Arbitrage Near-Cointegrated VAR(p) Term Structure Models, Term Premia and GDP Growth," *Journal of Banking & Finance*, 37, 389–402.

Johannes, M. (2004): "The Statistical and Economic Role of Jumps in Continuous-Time Interest Rate Models," *Journal of Finance*, 59(1), 227–260.

Joslin, S. (2017): "Can Unspanned Stochastic Volatility Models Explain the Cross Section of Bond Volatilities?," *Management Science*, 64(4), 1707–1726.

Joslin, S., and Y. Konchitchki (2018): "Interest Rate Volatility, the Yield Curve, and the Macroeconomy," *Journal of Financial Economics*, 128, 344–362.

Joslin, S., A. Le, and K. J. Singleton (2013): "Why Gaussian Macro-Finance Term Structure Models are (Nearly) Unconstrained Factor-VARs," *Journal of Financial Economics*, 109(3), 604–622.

Joslin, S., M. Priebsch, and K. J. Singleton (2014): "Risk Premiums in Dynamic Term Structure Models with Unspanned Macro Risks," *Journal of Finance*, 69(3), 1197–1233.

Joslin, S., K. J. Singleton, and H. Zhu (2011): "A New Perspective on Gaussian Dynamic Term Structure Models," *Review of Financial Studies*, 24(3), 926–970.

Koeda, J., and R. Kato (2015): "The Role of Uncertainty in the Term Structure of Interest Rates: A GARCH-ATSM Approach," *Applied Economics*, 47(34-35), 3710–3722.

Koopman, S. J., and M. Scharth (2013): "The Analysis of Stochastic Volatility in the Presence of Daily Realized Measures," *Journal of Financial Econometrics*, 11(1), 76–115.

Kreps, D. M., and E. L. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice," *Econometrica*, 46(1), 185–200.

Le, A., K. J. Singleton, and Q. Dai (2010): "Discrete-Time Affine Term Structure Models with Generalized Market Prices of Risk," *Review of Financial Studies*, 23(5), 2184–2227.

Litterman, R., and J. Scheinkman (1991): "Common Factors Affecting Bond Returns," *Journal of Fixed Income*, 1(1), 54–61.

McKay, A., E. Nakamura, and J. Steinson (2016): "The Power of Forward Guidance Revisited," *American Economic Review*, 106(10), 3133–3158.

Mincer, J., and V. Zarnowitz (1969): "The Evaluation of Economic Forecasts," in *Economic Forecasts and Expectations: Analysis of Forecasting Behavior and Performance*, ed. by J. Mincer, pp. 3–46. Columbia University Press, New York.

Mishkin, F. S. (2007): "Headline versus Core Inflation in the Conduct of Monetary Policy," Board of Governers of the Federal Reserve System.

Monfort, A., and F. Pegoraro (2012): "Asset Pricing with Second-Order Esscher Transforms," *Journal of Banking & Finance*, 36, 1678–1687.

Norland, E. (2018): "VIX-Yield Curve: At the Door of High Volatility?," CME Group Reports.

Piazzesi, M. (2005): "Bond Yields and the Federal Reserve," *Journal of Political Economy*, 113(2), 311–344.

Pohl, W., K. Schmedders, and O. Wilms (2018): "Higher Order Effects in Asset Pricing Models," *Journal of Finance*, 73(3), 1061–1111.

Roussellet, G. (2018): "The Term Structure of Macroeconomic Risks at the Zero Lower Bound," *Working Paper*.

Rudebusch, G., and T. Wu (2008): "A Macro-Finance Model of the Term Structure, Monetary Policy and the Economy," *Economic Journal*, 118(530), 906–926.

Stock, J. H., and M. W. Watson (2001): "Vector Autoregressions," *Journal of Economic Perspectives*, 15, 101–115.

Trolle, A. B. (2009): "The Price of Interest Rate Variance Risk and Optimal Investments in Interest Rate Derivatives," *EFA 2009 Bergen Meetings Paper*.

Trolle, A. B., and E. S. Schwartz (2009): "A General Stochastic Volatility Model for the Pricing of Interest Rate Derivatives," *Review of Financial Studies*, 22(5), 2007–2057.

——— (2015): "The Swaption Cube," *Review of Financial Studies*, 27(8), 2307–2353.

Weil, P. (1990): "Nonexpected Utility in Macroeconomics," *Quarterly Journal of Economics*, 105(1), 29–42.

A Technical Appendix

A.1 Structural Justification of Exponential-Quadratic Pricing Kernels

I provide two examples from the equilibrium asset pricing literature that result in exponential-quadratic pricing kernels. In both examples, the pricing kernel is determined by the marginal rate of intertemporal substitution of a representative household with Epstein and Zin (1989), Kreps and Porteus (1978), and Weil (1990) recursive preferences.

Long-Run Risk Model with Second-Order Projection Solution

Consider an indirect utility function given by

$$\mathcal{U}_t = \frac{1}{1-\frac{1}{\psi}} C_t^{1-\frac{1}{\psi}} + \beta \left(\mathbb{E}_t \left[\mathcal{U}_{t+1}^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}}$$

and a pricing kernel given by

$$\mathcal{M}_{t+1} = \beta \exp(-r_t) \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left(\frac{\mathcal{U}_{t+1}}{\mathbb{E}_t \left(\mathcal{U}_{t+1}^{1-\alpha} \right)} \right)^{-\alpha},$$

where C_t is consumption. Bansal and Yaron (2004) specify a model with a long-run growth factor, g_t, and a volatility factor, σ_t. The state vector also includes log consumption, $c_t = \log C_t$. The state dynamics are given by

$$\begin{aligned}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_c \sigma_t z_{c,t+1}, \\
g_{t+1} &= \rho_g g_t + \sigma_g \sigma_t z_{g,t+1}, \\
\sigma_{t+1}^2 &= 1 - \rho_\sigma + \rho_\sigma \sigma_t^2 + \sigma_\sigma \varepsilon_{\sigma,t+1},
\end{aligned}$$

where $z_{c,t+1} \sim$ i.i.d. $\mathcal{N}(0,1)$, $z_{g,t+1} \sim$ i.i.d. $\mathcal{N}(0,1)$, and $z_{\sigma,t+1} \sim$ i.i.d. $\mathcal{N}(0,1)$.

Bansal and Yaron (2004) propose an analytical solution using a log-linearization. This approximation results in an exponential-affine pricing kernel. However, Pohl, Schmedders, and Wilms (2018) argue that a first-order approximation is likely to generate large numerical errors. In response, Andreasen and Jørgensen (2019) solve the model using a second-order projection in which

$$\begin{aligned}
u_t &= \gamma_0^u + \gamma_X^u X_t + X_t' \gamma_{XX}^u X_t, \\
\tilde{u}_t &= \gamma_0^{\tilde{u}} + \gamma_X^{\tilde{u}} X_t + X_t' \gamma_{XX}^{\tilde{u}} X_t,
\end{aligned}$$

where $u_t = \log \mathcal{U}_t$, $\tilde{u}_t = \log \mathbb{E}_t\left(\mathcal{U}_{t+1}^{1-\alpha}\right)$, and $X_t = [c_t', g_t', \sigma_t^{2'}]'$. The implied log pricing kernel is then given by

$$\begin{aligned}\log \mathcal{M}_{t+1} &= \log \beta - r_t - \frac{1}{\psi}\Delta c_{t+1} - \alpha u_{t+1} + \alpha \tilde{u}_{t+1} \\ &\propto \xi' X_{t+1} + X_{t+1}' \Xi X_{t+1}\end{aligned}$$

with $\xi = -\frac{1}{\psi} - \alpha\gamma_X^u + \alpha\gamma_X^{\tilde{u}}$ and $\Xi = \gamma_{XX}^u + \gamma_{XX}^{\tilde{u}}$. This pricing kernel is a special case of (5) with constant prices of risk.

Long-Run Risk Model with Time-Varying Wealth-Consumption Ratio

Monfort and Pegoraro (2012) note that the exponential-quadratic kernel can be justified by a long-run risk model with a time-varying wealth-consumption ratio as considered in Hansen and Heaton (2008) and Hansen, Heaton, Lee, and Roussanov (2007). Here, I derive the details to clarify this justification. Indirect utility is specified by

$$\mathcal{U}_t = \left\{(1-\beta)C_t^{1-\rho} + \beta\left[\mathbb{E}_t\left(\mathcal{U}_{t+1}^{1-\gamma}\right)\right]^{\frac{1-\rho}{1-\gamma}}\right\}^{\frac{1}{1-\rho}}.$$

The log-consumption dynamics are defined by

$$\Delta c_{t+1} = \mu_c + \rho_{cx} x_t + \sigma_c z_{t+1},$$

where x_t is the state vector that follows first-order vector autoregressive dynamics given by

$$x_{t+1} = \Phi x_t + \Sigma z_{t+1}$$

with $z_t \sim$ i.i.d. $\mathcal{N}(0, \mathrm{I})$. Indirect utility is related to the ratio of wealth to consumption by

$$\frac{W_t}{C_t} = \frac{1}{1-\beta}\left(\frac{\mathcal{U}_t}{C_t}\right)^{1-\rho}.$$

When the wealth-consumption ratio is constant, i.e., when $\rho = 1$, the model implies a linear-exponential pricing kernel given by

$$\log \mathcal{M}_{t+1,1} = \mu_{\mathcal{M}} + \rho_{\mathcal{M}x} x_t + \sigma_{\mathcal{M}} z_{t+1}.$$

In the more realistic case of $\rho \neq 1$, Hansen and Heaton (2008) derives an approximate solution by expanding around the case of $\rho = 1$:

$$\log \mathcal{M}_{t+1,1} \approx \log \mathcal{M}_{t+1,1}|_{\rho=1} + (\rho - 1)\left[\frac{1}{2} z_{t+1}' \Gamma_1 z_{t+1} + z_{t+1}' \Gamma_1 x_t + \varrho_0 + \varrho_{\mathcal{M}x} x_t + \varrho_{\mathcal{M}z} z_{t+1}\right].$$

Thus, by allowing for a time-varying wealth-consumption ratio, the pricing kernel becomes quadratic as in (5) with prices of risk given by $\xi_t = \varrho_{\mathcal{M}z} + \Gamma_1 x_t$ and $\Xi = \frac{1}{2}\Gamma_1$.

A.2 Proof of Theorem 4

The proof follows the steps in Joslin, Singleton, and Zhu (2011) closely. I rely on invariant affine transformations of the state vector, X_t, given by $\mathcal{P}_t = c + CX_t$, where transformation of the observed macroeconomic variables can be prevented by restricting

$$c = \begin{bmatrix} c_x \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_x & 0 \\ 0 & \mathrm{I}_{n_m} \end{bmatrix}. \tag{27}$$

The following lemma gives the model parameters resulting from this rotation.

Lemma 1 *Consider the affine transformation $\mathcal{P}_t = c + CX_t$ restricted by (27). Applying this transformation to the model in (3)-(10) gives an observationally equivalent model with state vector $\mathcal{P}_t = (\rho_t', m_t)'$ and parameters given by*

$$\begin{aligned}
\mu_{\mathcal{P}} &= c + C\mu_X - \sum_{l=1}^{L} C\Phi_X^{(l)} C^{-1} c, \\
\Phi_{\mathcal{P}}^{(l)} &= C\Phi_X^{(l)} C^{-1}, \quad l = 1, \ldots, L \\
\Sigma_{\mathcal{P}} &= C\Sigma_X, \\
A_{\mathcal{P}}^{(k)} &= CA_X^{(k)} C^{-1}, \quad k = 1, \ldots, K \\
B_{\mathcal{P}}^{(k)} &= CB_X^{(k)} C^{-1}, \quad k = 1, \ldots, K \\
\alpha_{\mathcal{P}} &= \alpha_X - \beta_x' C_x^{-1} c_x, \\
\beta_{\mathcal{P}}' &= \beta_X' C^{-1}, \\
\mu_{\mathcal{P}}^{\mathbb{Q}} &= c + C\mu_X^{\mathbb{Q}} - C\Phi_X^{\mathbb{Q}} C^{-1} c, \\
\Phi_{\mathcal{P}}^{\mathbb{Q}} &= C\Phi_X^{\mathbb{Q}} C^{-1}, \\
V_{\mathcal{P}}^{\mathbb{Q}} &= CV_X^{\mathbb{Q}} C'.
\end{aligned}$$

Proof. Straightforward. □

To prove uniqueness, I apply the following lemma, which is identical to Proposition 1 in JSZ. Although my model differs from the Gaussian term structure model considered in JSZ, the lemma remains valid because the model involves Gaussian-affine risk neutral dynamics, which is the only part of the model invoked in the proof of the result.

Lemma 2 *The model defined by (3)-(10) with synthetic yields defined in (17) is observationally equivalent to a model in real ordered Jordan form with $r_t^{\perp m} \equiv r_t - \alpha_m - \beta_m' m_t = \iota' x_t$, where ι is a vector of ones. The parameters determining the $\mathbb{Q}$-dynamics* *of this model are given by* $\mu_x^{\mathbb{Q}} = (k_\infty^{\mathbb{Q}}, 0, \ldots, 0)'$*, a positive definite matrix* $V_x^{\mathbb{Q}}$*, and* $\Phi_x^{\mathbb{Q}} = diag(J_1^{\mathbb{Q}}, J_2^{\mathbb{Q}}, \ldots, J_D^{\mathbb{Q}})$*, where for* $d = 1, \ldots, D$*,*

$$J_d^{\mathbb{Q}} = \begin{bmatrix} \lambda_d^{\mathbb{Q}} & 1 & \ldots & 0 \\ 0 & \lambda_d^{\mathbb{Q}} & \ldots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & \ldots & 0 & \lambda_d^{\mathbb{Q}} \end{bmatrix}.$$

Proof. See JSZ. □

Given these lemmas, I prove Theorem 4. The portfolios of synthetic yields given by $\rho_t = WY_t^{\perp m}$ are affine transformation of the latent factors because

$$\rho_t = WY_t^{\perp m} = Wa_x + Wb_x x_t = a_W + b_W x_t.$$

Thus, $\mathcal{P}_t = c + CX_t$ with

$$c = \begin{bmatrix} a_W \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} b_W & 0 \\ 0 & \mathrm{I}_{n_m} \end{bmatrix}. \tag{28}$$

By an application of Lemma 1, the model can be rotated into an equivalent model with state vector given by $\mathcal{P}_t$. I prove by contradiction that this model is unique. Suppose that two models with state vector $\mathcal{P}_t$ exist. Assume that these models are parametrized by, respectively, Θ_1 and Θ_2. By Lemma 2, each model is observationally equivalent to the model in real ordered Jordan form with $r_t^{\perp m} = \iota' x_t$, whose parametrization I denote by Θ_J. It follows that $\Theta_1 = \Theta_J = \Theta_2$.

A.3 Proof of Theorem 5

By Lemma 2, the model defined by (3)-(10) with synthetic yields defined in (17) can be rewritten in real ordered Jordan form and the rotation only affects parameters of the $\mathbb{Q}$-dynamics and the short-rate equation. These rotated parameters are given by

$$\left\{\alpha_X, \ \ \beta_x, \ \ \mu_x^{\mathbb{Q}}, \ \ \Phi_x^{\mathbb{Q}}, \ \ V_x^{\mathbb{Q}}\right\} = \left\{0, \ \ \iota, \ \ k_\infty^{\mathbb{Q}} e_1, \ \ J(\lambda^{\mathbb{Q}}), \ \ V_x^{\mathbb{Q}}\right\},$$

where e_1 is a vector of zeros except with the first entry equal to one and $V_x^{\mathbb{Q}}$ is positive definite. Given the model in Jordan form from Lemma 2, I apply the invariant transformation $\mathcal{P}_t = c + CX_t$ with c and C given by (28). By an application of Lemma 1, the parameters of the rotated model are given by $\Theta_\rho^{\mathbb{Q}} = \{k_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, V_\rho^{\mathbb{Q}}\}$ and

$$\begin{aligned}\Theta_{\mathcal{P}}^{\mathbb{P}} &\equiv \left\{\mu_{\mathcal{P}},\ \ \{\Phi_{\mathcal{P}}^{(l)}\}_{l=1}^{L},\ \ \Sigma_{\mathcal{P}},\ \ \{A_{\mathcal{P}}^{(k)}, B_{\mathcal{P}}^{(k)}\}_{k=1}^{K}\right\} \\ &= \left\{c + C\mu_X - \textstyle\sum_{l=1}^{L} C\Phi_{\mathcal{P}}^{(l)}C^{-1}c,\ \ \{C\Phi_X^{(l)}C^{-1}\}_{l=1}^{L},\ \ C\Sigma_X,\ \ \{CA_{\mathcal{P}}^{(k)}C^{-1}, CB_{\mathcal{P}}^{(k)}C^{-1}\}_{k=1}^{K}\right\}.\end{aligned}$$

Since $\Theta_X^{\mathbb{P}} = \left\{\mu_X, \{\Phi_X^{(l)}\}_{l=1}^{L}, \Sigma_X, \{A_X^{(k)}, B_X^{(k)}\}_{k=1}^{K}\right\}$ is not involved in the rotation into the Jordan form, restricting $\Theta_{\mathcal{P}}^{\mathbb{P}}$ is not necessary to preclude rotations of the state vector given $\Theta_\rho^{\mathbb{Q}} = \{k_\infty^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, V_\rho^{\mathbb{Q}}\}$.

Restrictions are, however, necessary to identify parameters of the conditional mean under the $\mathbb{P}$-measure. To ensure a unique Cholesky decomposition, $\Sigma_{\mathcal{P}}\Sigma_{\mathcal{P}}'$, I restrict the diagonal of the lower triangular matrix $\Sigma_{\mathcal{P}}$ to be strictly positive. Moreover, since $A_{\mathcal{P}}^{(k)}$ and $B_{\mathcal{P}}^{(k)}$ enter the model in quadratic form, I require the first elements in these matrices to be non-negative. The following lemma derives an upper bound for K that is necessary for identification.

Lemma 3 *Consider the variance specification given by*

$$V_t = \Sigma_{\mathcal{P}}\Sigma_{\mathcal{P}}' + \sum_{k=1}^{K} A_{\mathcal{P}}^{(k)}\varepsilon_{t-1}\varepsilon_{t-1}'A_{\mathcal{P}}^{(k)'} + \sum_{k=1}^{K} B_{\mathcal{P}}^{(k)}V_{t-1}B_{\mathcal{P}}^{(k)'}, \tag{29}$$

where $\Sigma_{\mathcal{P}}$ is lower triangular with strictly positive elements on the diagonal. Let $A_{\mathcal{P}}^{(k)}$ and $B_{\mathcal{P}}^{(k)}$ be diagonal matrices for $k = 1, \ldots, K$. Then, a necessary condition for the parameters to be identified is $K \leq \text{floor}\left(\frac{1}{2}n_X + \frac{1}{2}\right)$, where n_X is the dimension of V_t.

Proof. I give the proof for a trivariate model, but the arguments can be extended to higher dimensions. I assume that $K = \text{floor}\left(\frac{1}{2}n_X + \frac{1}{2}\right) = 2$ and show that this degree of generalization gives the maximum number of identified parameters. The ARCH and GARCH terms of the BEKK GARCH model can be treated separately. Furthermore, the arguments for the two terms are identical. I focus on the ARCH term below. Let

$$A_{\mathcal{P}}^{(1)} = \begin{bmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{1,2} & 0 \\ 0 & 0 & a_{1,3} \end{bmatrix} \quad \text{and} \quad A_{\mathcal{P}}^{(2)} = \begin{bmatrix} a_{2,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{2,3} \end{bmatrix}.$$

Then,

$$A_{\mathcal{P}}^{(1)}\varepsilon_t\varepsilon_t' A_{\mathcal{P}}^{(1)'} + A_{\mathcal{P}}^{(2)}\varepsilon_t\varepsilon_t' A_{\mathcal{P}}^{(2)'} =$$

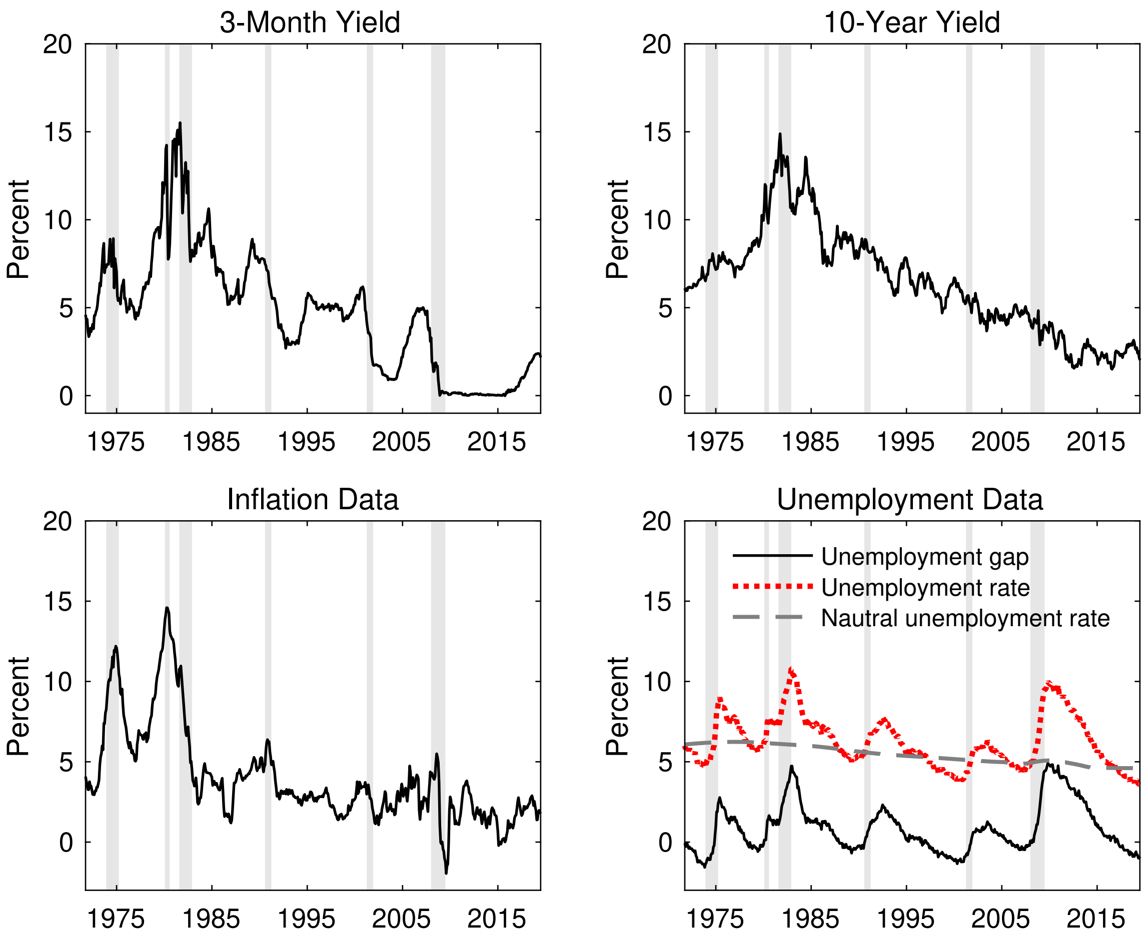
$$\begin{bmatrix} \left(a_{1,1}^2 + a_{2,1}^2\right)\varepsilon_{1,t}^2 & \left(a_{1,1}a_{1,2} + a_{2,1}a_{2,2}\right)\varepsilon_{1,t}\varepsilon_{2,t} & \left(a_{1,1}a_{1,3} + a_{2,1}a_{2,3}\right)\varepsilon_{1,t}\varepsilon_{3,t} \\ \left(a_{1,1}a_{1,2} + a_{2,1}a_{2,2}\right)\varepsilon_{1,t}\varepsilon_{2,t} & \left(a_{1,2}^2 + a_{2,2}^2\right)\varepsilon_{2,t}^2 & \left(a_{1,2}a_{1,3} + a_{2,2}a_{2,3}\right)\varepsilon_{2,t}\varepsilon_{3,t} \\ \left(a_{1,1}a_{1,3} + a_{2,1}a_{2,3}\right)\varepsilon_{1,t}\varepsilon_{3,t} & \left(a_{1,2}a_{1,3} + a_{2,2}a_{2,3}\right)\varepsilon_{2,t}\varepsilon_{3,t} & \left(a_{1,3}^2 + a_{2,3}^2\right)\varepsilon_{3,t}^2 \end{bmatrix}$$

Suppose that $a_{2,1}$, $a_{2,2}$, and $a_{2,3}$ are identified. Then, $a_{1,1}$, $a_{1,2}$, and $a_{1,3}$ follows from the coefficients on respectively $\varepsilon_{1,t}^2$, $\varepsilon_{2,t}^2$, and $\varepsilon_{3,t}^2$. This leaves the following three equations to show that $a_{2,1}$, $a_{2,2}$, and $a_{2,3}$ are indeed identified:

$$\begin{aligned} a_{2,1}a_{2,2} &= c_1 \\ a_{2,1}a_{2,3} &= c_2 \\ a_{2,2}a_{2,3} &= c_3. \end{aligned}$$

This system is obviously identified. Thus, the model is identified with $K = \text{floor}\left(\frac{1}{2}n_X + \frac{1}{2}\right)$. Naturally, it follows that the model is also identified with $K < \text{floor}\left(\frac{1}{2}n_X + \frac{1}{2}\right)$. Now, suppose that K exceeds $\text{floor}\left(\frac{1}{2}n_X + \frac{1}{2}\right) = 2$, say $K = 3$. For the ARCH equation, this leaves $n_X(n_X + 1)/2 = 6$ equations to identify $Kn_X = 9$ parameters, which is not feasible. $\square$

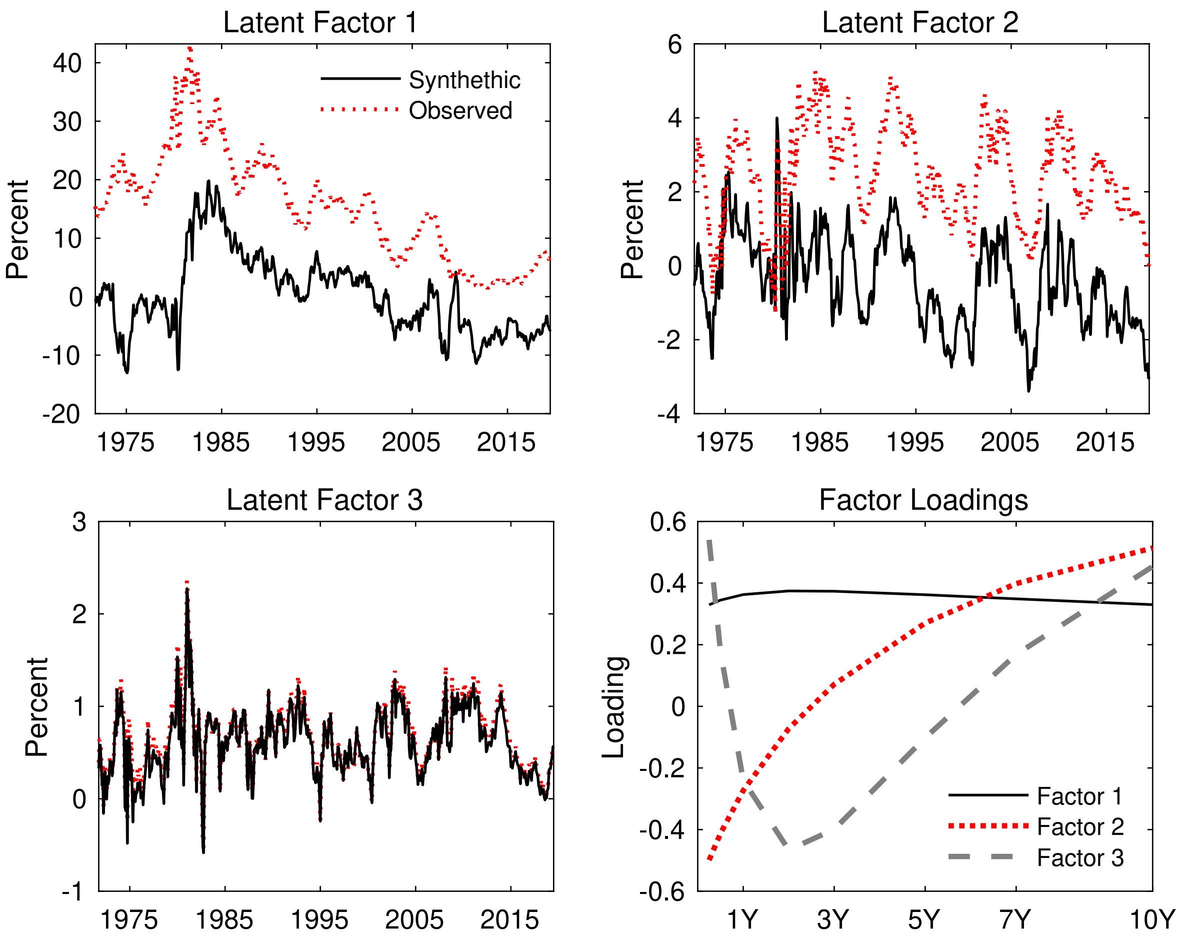
Figure 1: Yield and Macroeconomic Data



Notes: The figure shows 3-month and 10-year bond yields (upper panel) and the macroeconomic data (lower panel). The yields are from U.S. Treasury bills and bonds provided by Federal Reserve Economic Data and Gürkaynak, Sack, and Wright (2007). Inflation is the 12-month change in consumer prices provided the Bureau of Labor Statistics. Unemployment gap is the unemployment rate from the Bureau of Labor Statistics minus the natural unemployment rate from the Congressional Budget Office. The data are from September 1971 to June 2019. Shaded areas represent recessions, as defined by the National Bureau of Economic Research.

(Back to text)

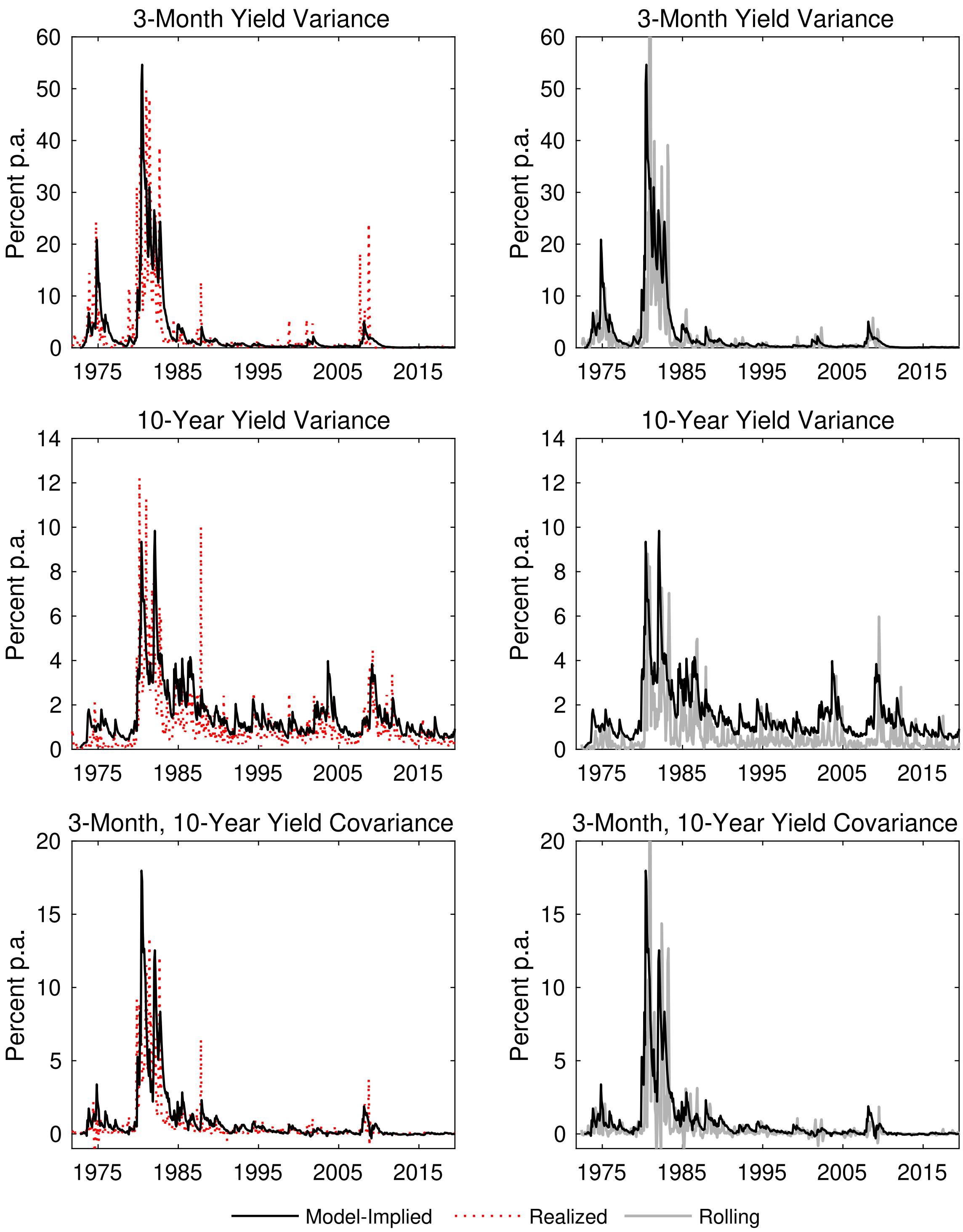
Figure 2: Estimated Latent Factors and Factor Loadings



Notes: The figure shows the estimated latent factors in the upper panel and in the lower left chart (solid lines). These are estimated by the first three principal components of the synthetic yields given by the residuals from projecting yields on the macroeconomic variables. The figure compares these with the principal components of the observed yield curve (dotted lines). The lower-right chart shows the estimated factor loadings across maturities as given in Theorem 1.

(Back to text)

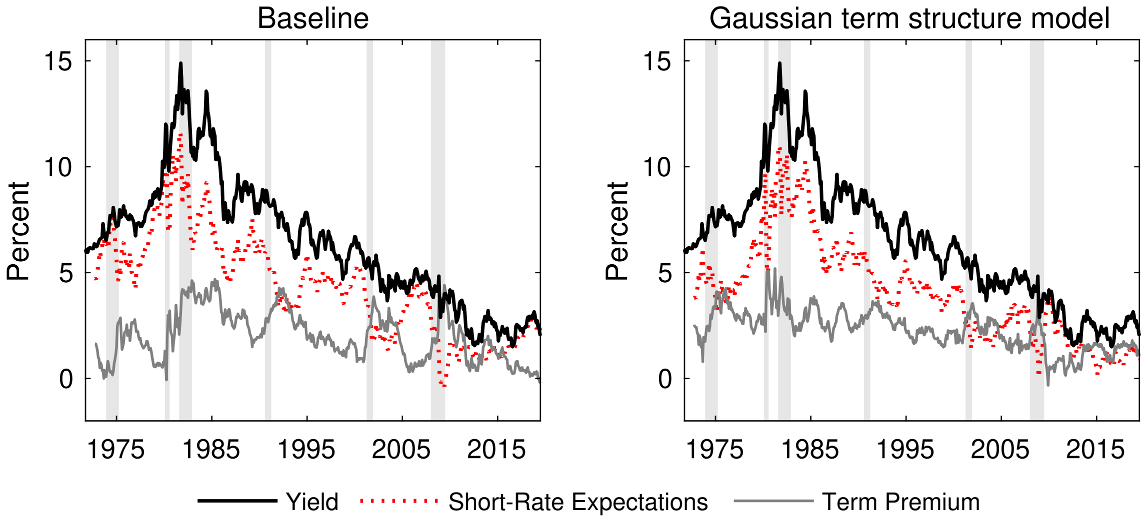
Figure 3: Model-Implied, Realized, and Rolling Variances and Covariances



Notes: The figure shows model-implied conditional variances and covariances along with empirical measures given by: (i) realized variances constructed from daily data (left panel), and (ii) rolling variances of daily data with a 3-month look-back (right panel). Results are shown results for the 3-month and 10-year bond yields.

(Back to text)

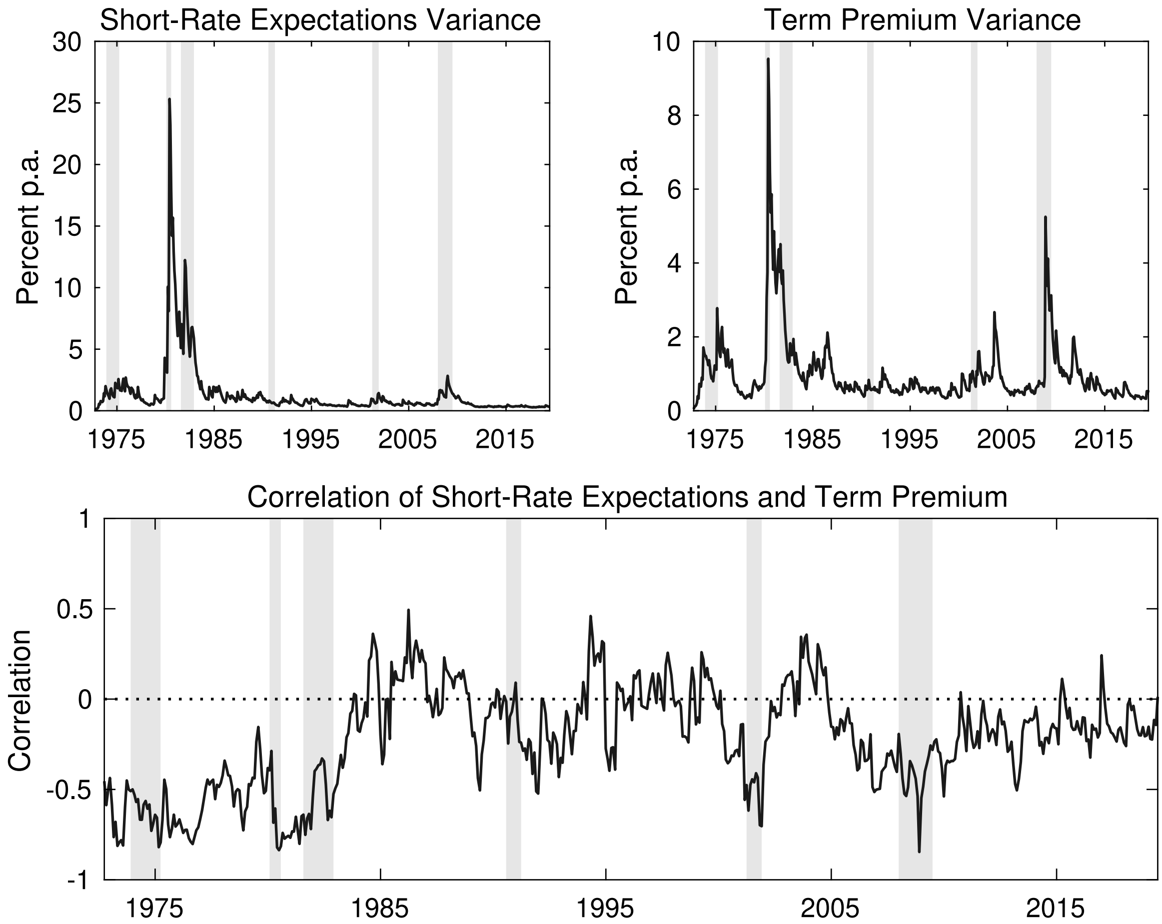
Figure 4: Decomposition of 10-Year Yield into Short-Rate Expectations and Term Premia



Notes: The figure shows the model-implied decomposition of the 10-year yield into short-rate expectations and a term premium. The left chart shows the decomposition implied by my model, whereas the right chart shows the decomposition implied by the Gaussian term structure model. Shaded areas represent recessions, as defined by the National Bureau of Economic Research.

(Back to text)

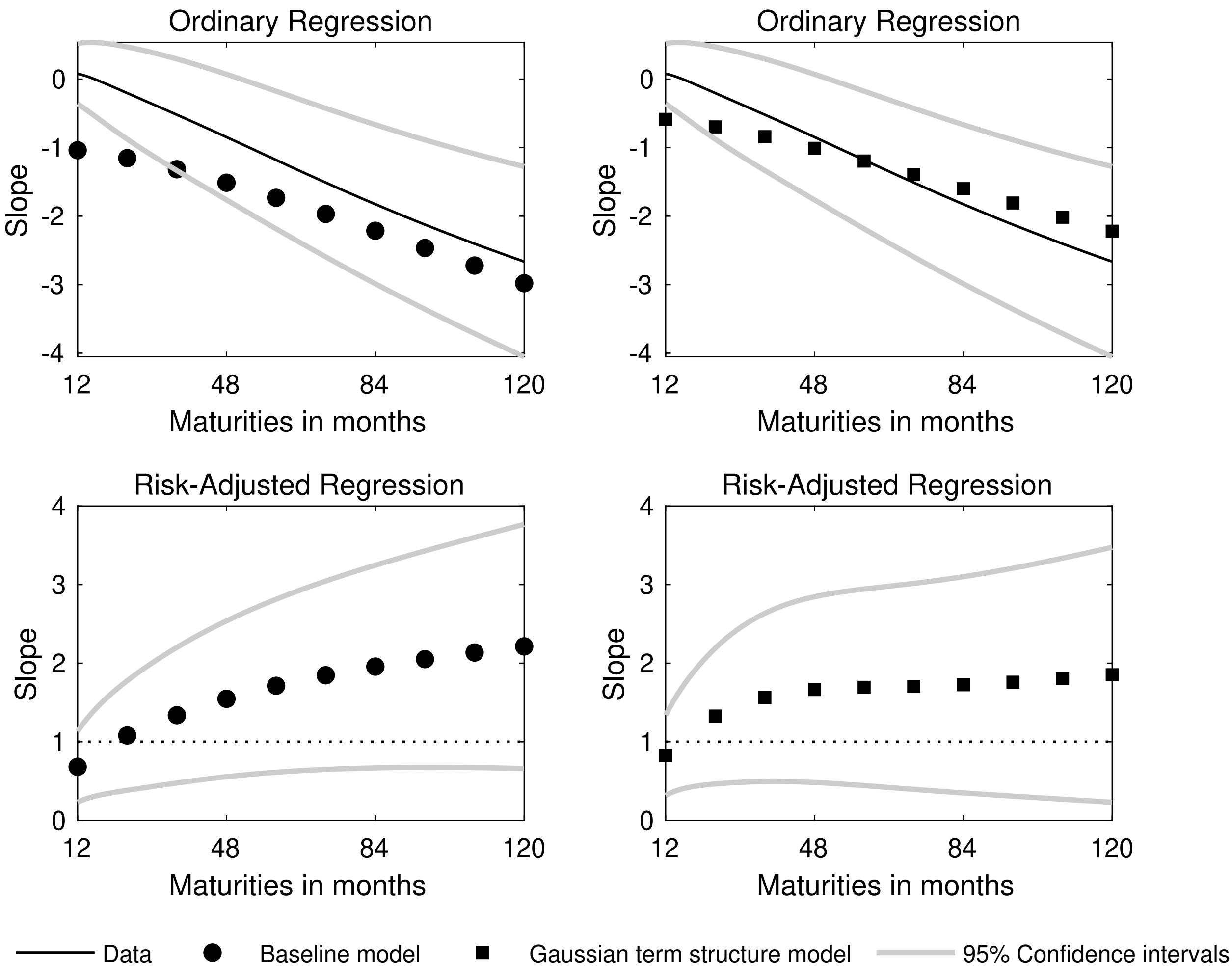
Figure 5: Conditional Variances of and Correlation Between 10-Year Short-Rate Expectations and Term Premium



Notes: The figure shows model-implied conditional variances of the 10-year short-rate expectations and term premium (upper panel). The lower panel shows their conditional correlation implied by the model. Shaded areas represent recessions, as defined by the National Bureau of Economic Research.

(Back to text)

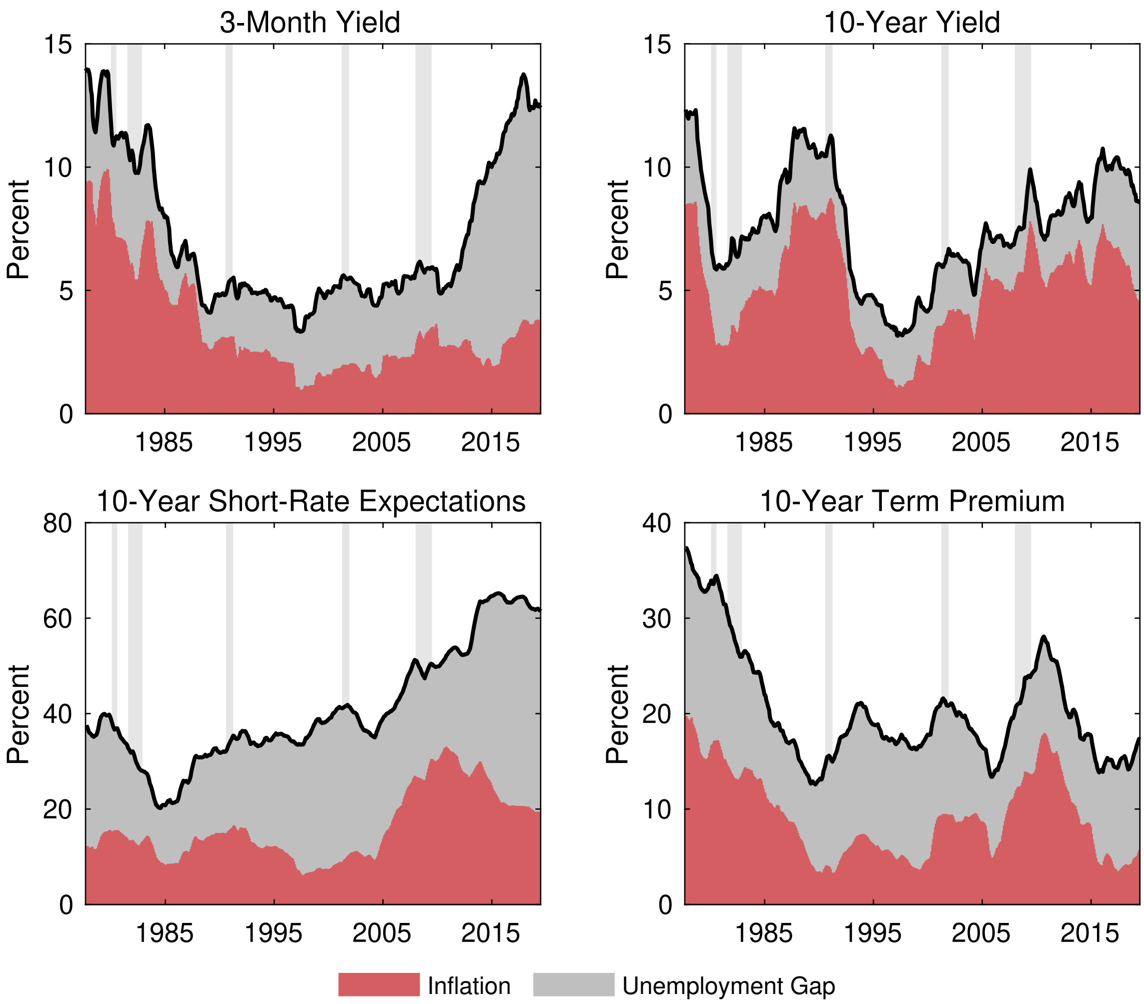
Figure 6: Ordinary and Risk-Adjusted Campbell-Shiller Regression Slope Coefficients



Notes: The figure shows slope coefficients from ordinary and risk-adjusted Campbell and Shiller (1991) regressions. For the ordinary regressions, model-implied slope coefficients are obtained by simulating 100,000 observations in my model (left panel) and the Gaussian term structure model (right panel). Confidence intervals are estimated with Newey-West robust standard errors with automatically selected lag length.

(Back to text)

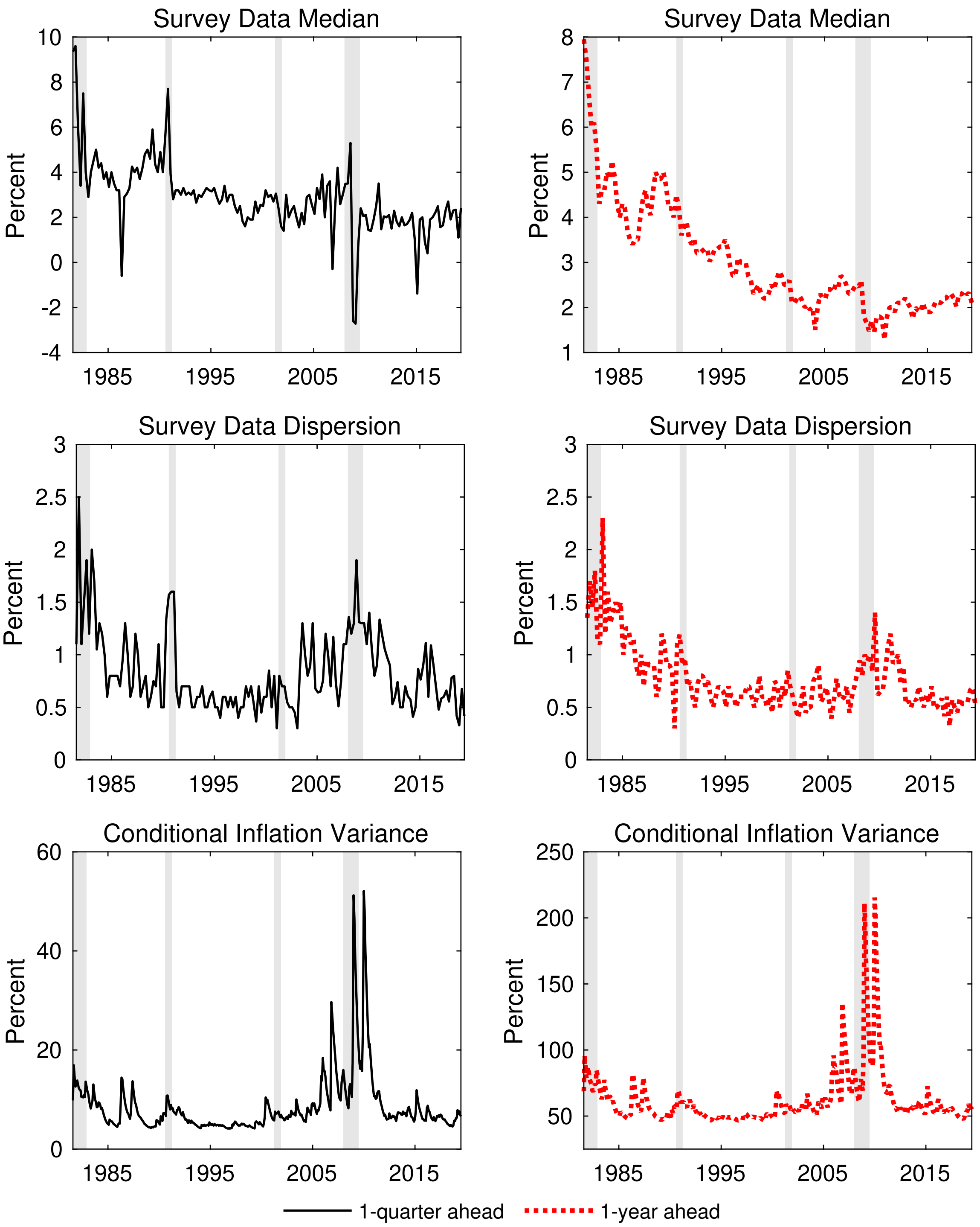
Figure 7: 5-Year Moving-Average Macroeconomic Shares of Variances



Notes: The figure shows 5-year moving averages of the macroeconomic contribution to variances in the 3-month and 10-year yields, 10-year short-rate expectations, and the 10-year term premium. Shaded areas represent recessions, as defined by the National Bureau of Economic Research.

(Back to text)

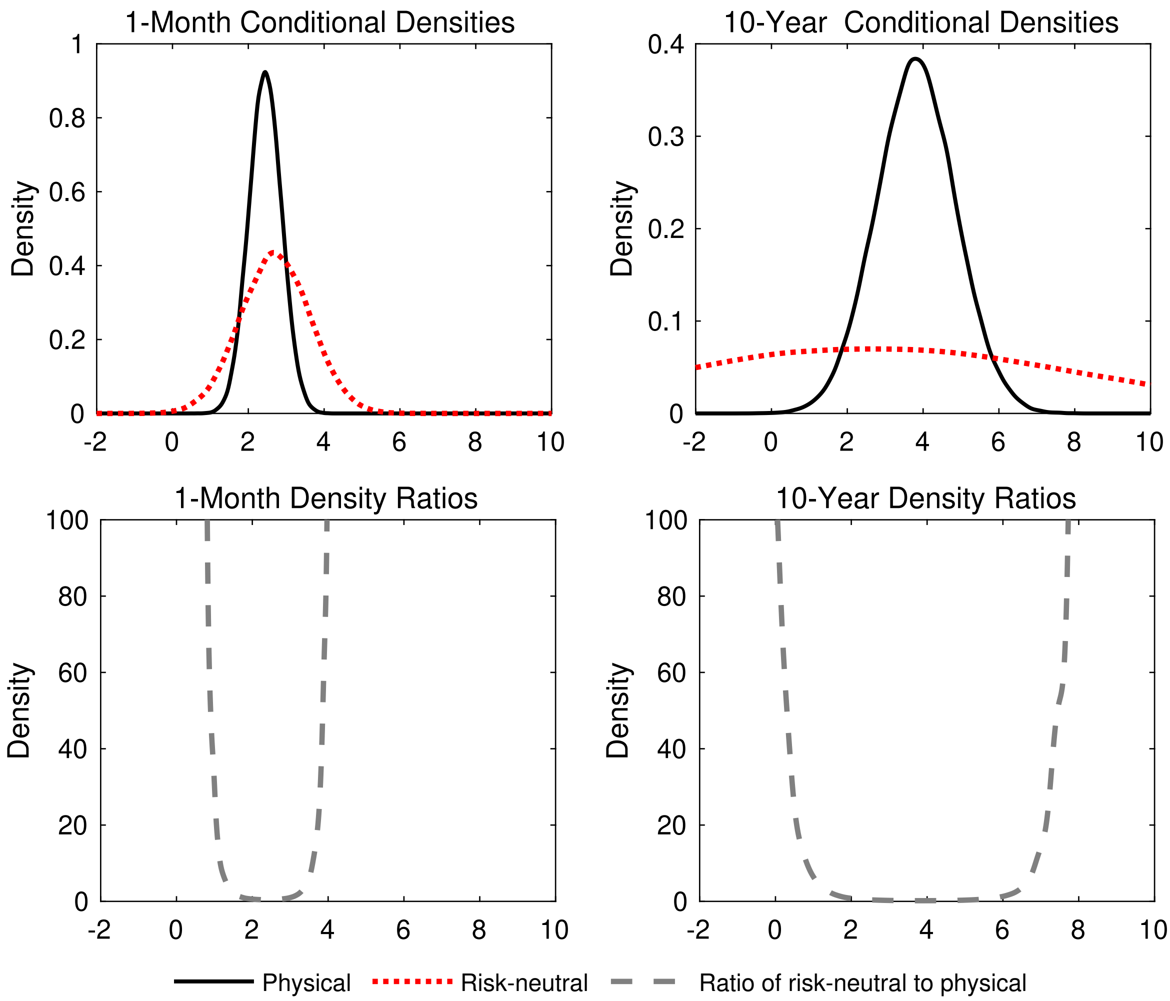
Figure 8: Inflation Survey Expectations and Model-Implied Conditional Inflation Variance



Notes: The figure shows the median of inflation survey forecasts from the Survey of Professional Forecasters in the upper panel. The middle panel shows forecast dispersions defined by the difference between the 75th and 25th quantiles. These data are available at quarterly frequency from the third quarter of 1981. The lower panel shows the model-implied conditional inflation variance from July 1981. Shaded areas represent recessions, as defined by the National Bureau of Economic Research.

(Back to text)

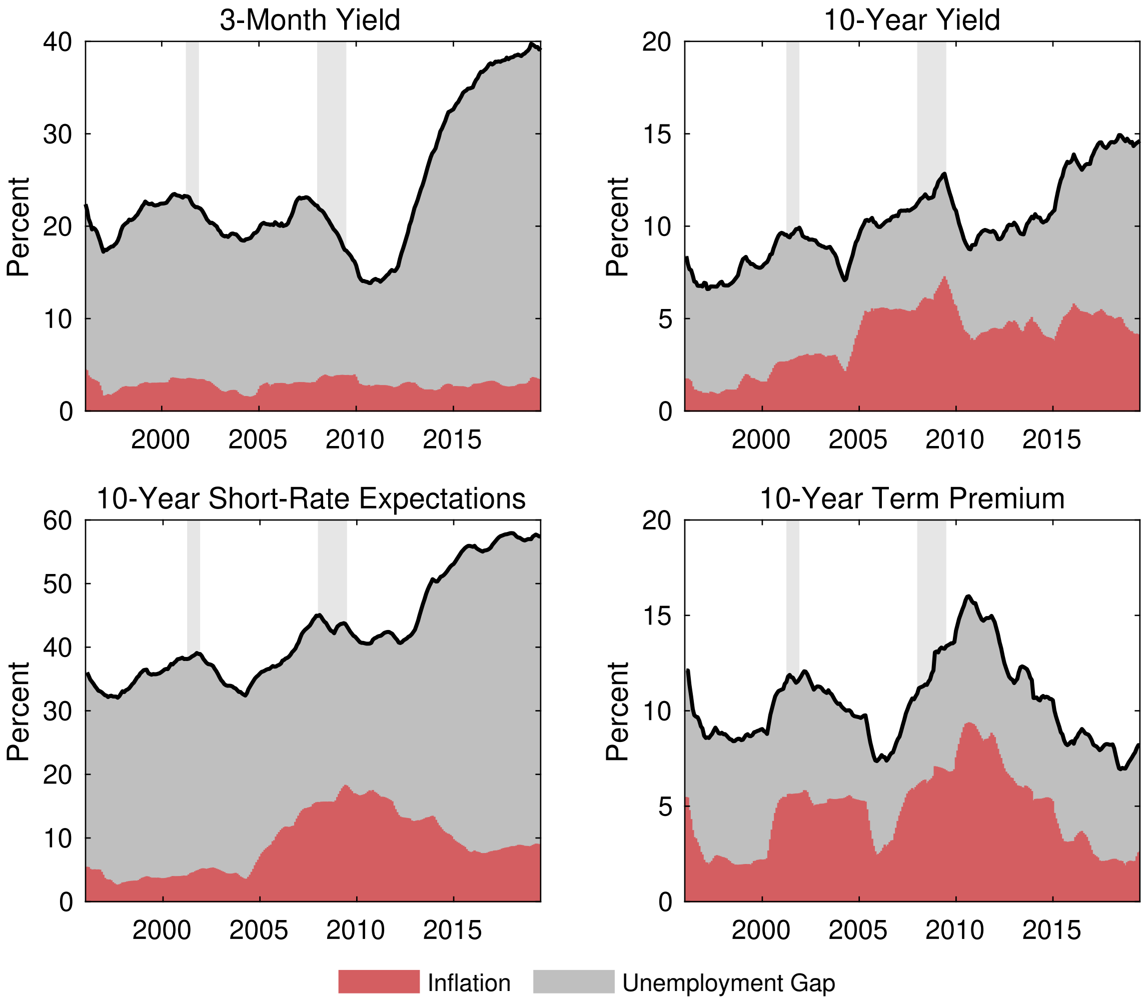
Figure 9: Conditional Physical and Risk-Neutral Inflation Densities



Notes: The figure shows model-implied conditional densities under the physical and risk-neutral probability measures (upper panel). The lower panel shows ratios of the risk-neutral to physical densities. The densities are estimated by the Epanechnikov kernel function applied to 100,000 simulated paths with horizon of one month and 10 years. The simulations are initiated at the sample average from January 1990. The bandwidth is selected by the optimal choice for estimating normal densities.

(Back to text)

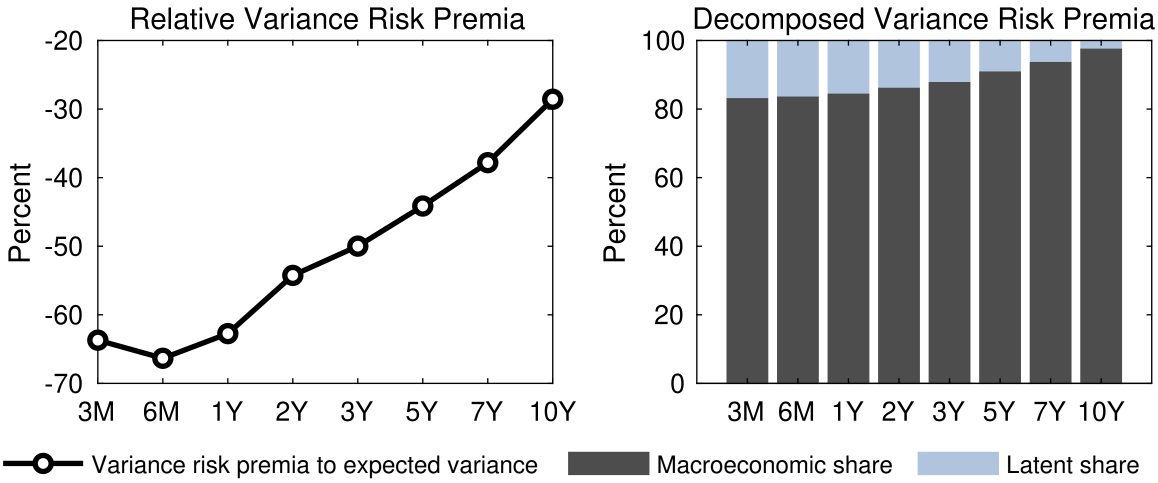
Figure 10: 5-Year Moving-Average Macroeconomic Shares of Variances from 1990-2019



Notes: The figure shows 5-year moving averages of the macroeconomic contribution to variances in the 3-month and 10-year yields, 10-year short-rate expectations, and the 10-year term premium. The model is estimated using data from January 1990 to June 2019. Shaded areas represent recessions, as defined by the National Bureau of Economic Research.

(Back to text)

Figure 11: Variance Risk Premia

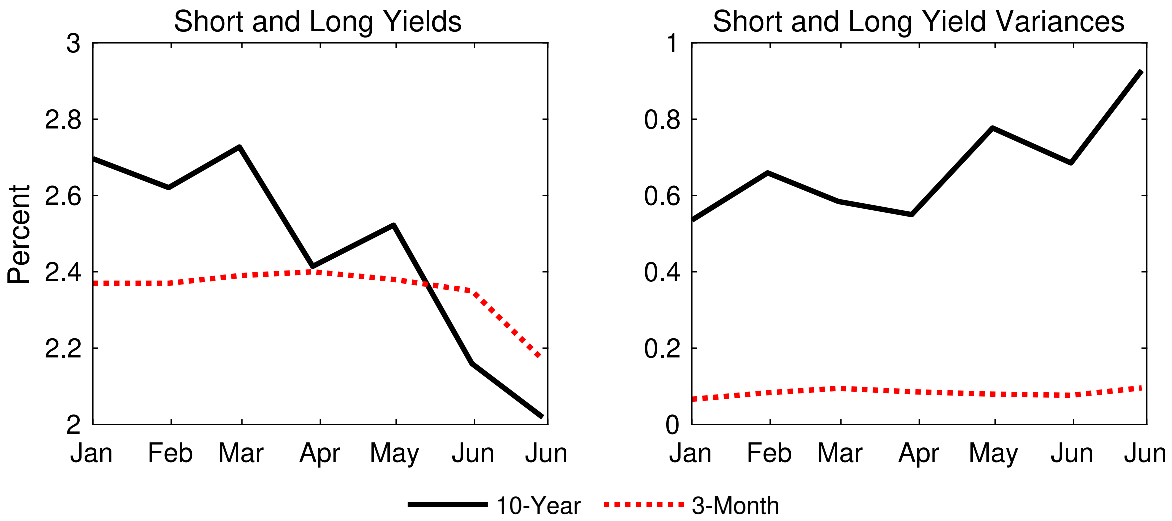


Notes: The figure shows model-implied variance risk premia. The left panel presents the ratios of average variance risk premia to the average expected variance under the physical measure. The right panel decomposes the average variance risk premia into macroeconomic variables and latent factors. Results are shown for different maturities.

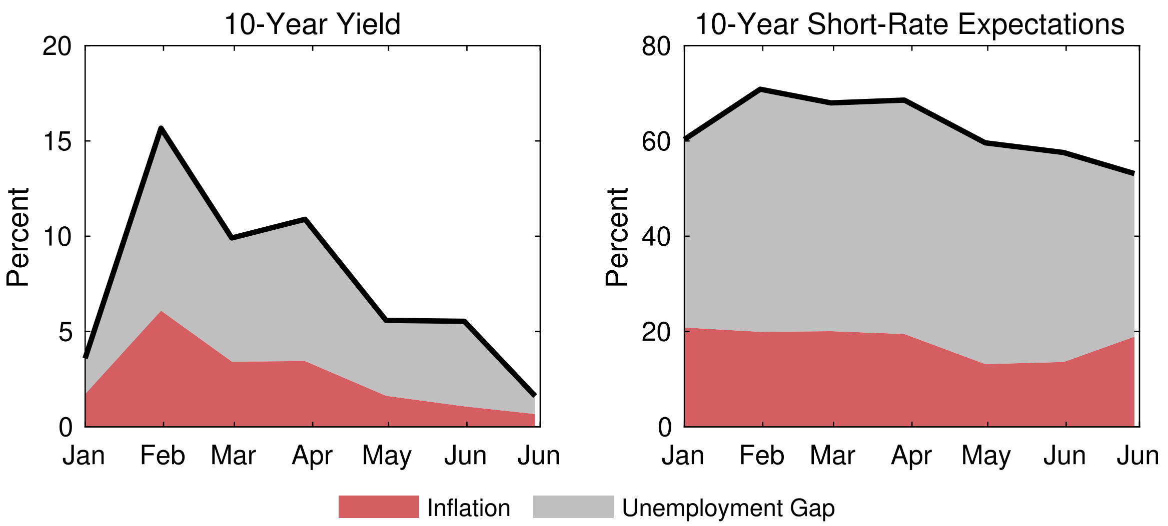
(Back to text)

Figure 12: Analysis of the 2019 Yield Curve Inversion

(a) Yield Curve Inversion and Model-Implied Yield Variances



(b) Macroeconomic Shares of Variances

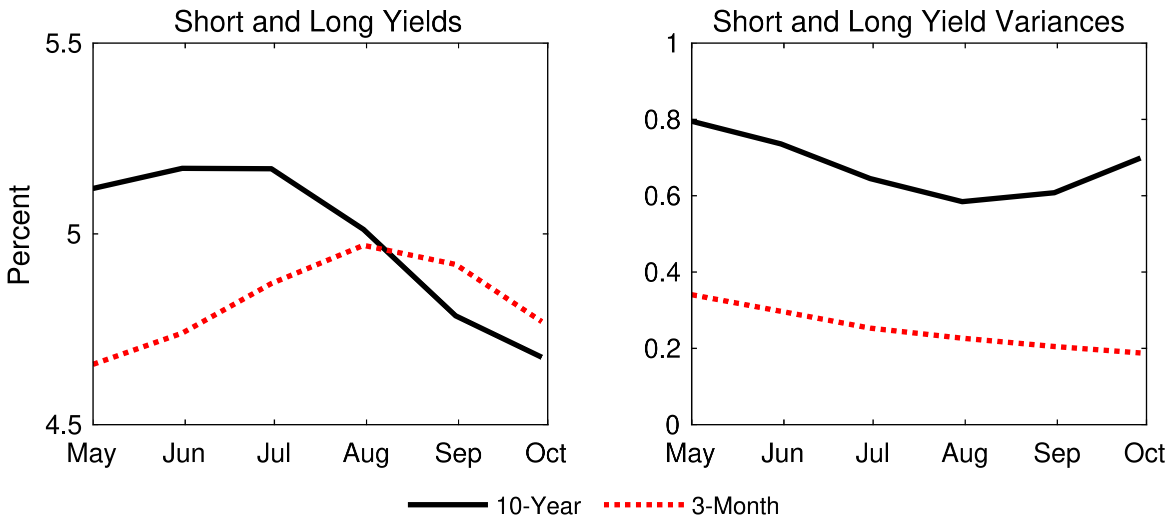


Notes: The figure shows in panel (a) the 3-month and 10-year bond yields (left chart) and the model-implied conditional variances these yields (right chart). Panel (b) shows macroeconomic shares of the variances in 10-year yield and short-rate expectations.

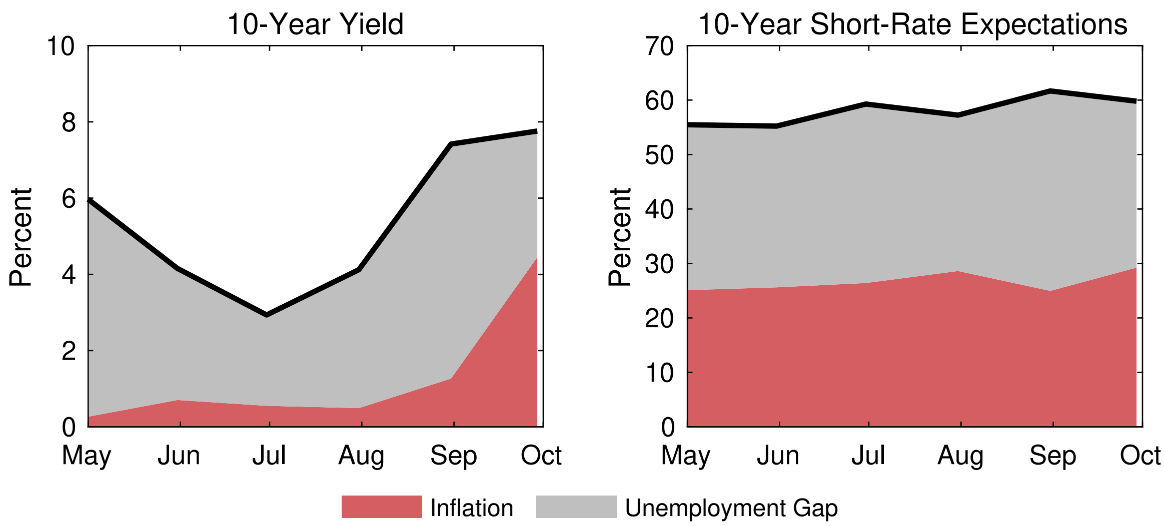
(Back to text)

Figure 13: Analysis of the 2006 Yield Curve Inversion

(a) Yield Curve Inversion and Model-Implied Yield Variances



(b) Macroeconomic Shares of Variances



Notes: The figure shows in panel (a) the 3-month and 10-year bond yields (left chart) and the model-implied conditional variances these yields (right chart). Panel (b) shows macroeconomic shares of the variances in 10-year yield and short-rate expectations.

(Back to text)

Table 1: Descriptive Statistics of the Yield and Macroeconomic Data

	Yields with maturities in years								Inflation	Unemployment gap
	0.25	0.5	1	2	3	5	7	10		
Average	4.63	4.76	5.13	5.37	5.56	5.88	6.12	6.39	3.98	0.82
Std.dev.	3.42	3.42	3.56	3.49	3.41	3.23	3.09	2.94	3.00	1.49
Skewness	0.58	0.52	0.46	0.38	0.37	0.39	0.41	0.44	1.49	0.93
Kurtosis	3.17	3.00	2.83	2.69	2.64	2.63	2.67	2.74	4.97	3.24
Autocorrelations										
Lag 1	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
Lag 2	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.97	0.98
Lag 3	0.96	0.97	0.97	0.97	0.97	0.98	0.98	0.98	0.96	0.97
Lag 6	0.93	0.94	0.94	0.95	0.95	0.96	0.96	0.96	0.90	0.90
Lag 12	0.87	0.88	0.89	0.9	0.91	0.92	0.92	0.91	0.75	0.73

Note: The table shows descriptive statistics of bond yield data and macroeconomic data in percent per annum. The yield data are U.S. Treasury bond yields from Gürkaynak, Sack, and Wright (2007) and Treasury bill rates from Federal Reserve Economic Data. The macroeconomic data are given by the 12-month change in consumer prices and the unemployment gap defined by the unemployment rate subtracted the natural unemployment rate estimated by the Congressional Budget Office. CPI inflation and unemployment rates are provided by the Bureau of Labor Statistics. The data are sampled monthly from September 1971 to June 2019.

(Back to text)

Table 2: In-Sample Performance

Panel A. Yield Curve Fit

	Maturities in years								
	0.25	0.5	1	2	3	5	7	10	Average
RMSE (basis points)	8.74	10.90	12.42	2.97	5.64	7.86	3.64	8.35	8.18

Panel B.1. Fit to Realized Variances and Covariances

	Maturities in years								
	0.25	1	2	5	10	(0.25,5)	(0.25,10)	(5,10)	Average
RMSE	4.83	4.53	3.77	2.04	1.23	3.37	1.60	1.47	2.85
Constant	0.03	0.05	0.06	0.04	-0.01	0.03	0.02	0.02	0.03
	(0.02)	(0.02)	(0.02)	(0.02)	(0.02)	(0.01)	(0.01)	(0.02)	
Slope	0.77	0.50	0.53	0.63	0.80	0.59	0.69	0.69	0.65
	(0.06)	(0.05)	(0.06)	(0.05)	(0.06)	(0.05)	(0.06)	(0.06)	
$\bar{R}^2$	0.63	0.56	0.52	0.49	0.50	0.63	0.63	0.48	0.56

Panel B.2. Fit to Rolling Variances and Covariances

	Maturities in years								
	0.25	1	2	5	10	(0.25,5)	(0.25,10)	(5,10)	Average
RMSE	3.91	3.13	2.16	1.68	1.38	2.78	1.82	1.58	2.31
Constant	-0.01	-0.01	-0.01	-0.00	-0.04	0.02	0.02	-0.03	-0.01
	(0.02)	(0.03)	(0.03)	(0.02)	(0.02)	(0.02)	(0.01)	(0.02)	
Slope	0.67	0.63	0.62	0.65	0.70	0.64	0.71	0.67	0.66
	(0.09)	(0.09)	(0.09)	(0.08)	(0.08)	(0.08)	(0.09)	(0.08)	
$\bar{R}^2$	0.53	0.49	0.44	0.42	0.41	0.50	0.48	0.43	0.46

Notes: The table shows the in-sample performance of my model. Panel A shows root mean squared errors (RMSEs) of model-implied against observed yields in basis points per annum. Panels B.1 and B.2 show RMSEs of model-implied against realized and rolling variances and covariances. The table also shows constants and slope coefficients with standard errors in parenthesis and adjusted R-squared values from Mincer and Zarnowitz (1969) regressions of realized and rolling variances and covariances onto a constant and model-implied variances and covariances.

(Back to text)

Table 3: Descriptive Statistics of Macroeconomic Shares of Variances in Yields

	Maturities in years							
	0.25	0.5	1	2	3	5	7	10
Average	7.92	8.51	7.61	7.14	7.33	7.91	8.14	7.91
Std. dev.	7.94	7.95	7.27	6.87	7.00	7.51	7.67	7.59
Skewness	2.07	2.01	2.58	2.33	2.06	1.88	1.80	1.84
Kurtosis	9.07	8.84	12.79	10.41	8.44	7.48	7.11	7.43
Min	0.02	0.05	0.03	0.03	0.02	0.00	0.01	0.00
Max	55.56	56.24	54.03	45.05	45.91	46.39	50.84	53.86
Median	5.23	6.40	5.78	5.16	5.27	5.48	5.66	5.49
Autocorrelation	0.71	0.71	0.68	0.64	0.65	0.67	0.69	0.69
Quantiles								
10 percent	1.06	1.20	1.18	0.99	0.97	1.03	1.05	0.95
25 percent	2.53	2.97	3.05	2.82	2.73	2.9	2.84	2.74
75 percent	10.85	11.74	9.76	9.11	9.25	10.57	11.17	10.87
90 percent	17.68	19.34	16.02	15.96	16.39	17.89	18.26	17.73
Gaussian term structure model	7.72	8.29	8.39	8.03	8.01	8.17	7.93	6.92

Note: The table shows descriptive statistics of the estimated shares of yield variances determined by the macroeconomic variables. The statistics are reported in percent. The table also shows the constant macroeconomic share of variances estimated using the Gaussian term structure model.

(Back to text)

Table 4: Descriptive Statistics of Macroeconomic Shares of Variances in the Decomposition of 5- and 10-Year Yields

	Short-rate expectations		Term premia	
Maturity	5-year	10-year	5-year	10-year
Average	40.24	41.88	34.08	21.30
Std. dev.	18.12	18.28	14.91	12.44
Skewness	-0.12	-0.15	0.38	0.88
Kurtosis	1.94	1.96	2.85	3.78
Min	0.23	0.32	0.73	0.36
Max	75.8	75.66	75.84	68.59
Median	40.29	42.04	32.33	19.06
Autocorrelation	0.89	0.90	0.67	0.69
Quantiles				
10 percent	15.49	16.67	15.23	6.55
25 percent	26.25	27.64	23.60	12.11
75 percent	56.56	58.60	43.63	28.80
90 percent	64.51	65.57	54.52	38.32
Gaussian term structure model	33.36	26.36	48.36	26.16

Note: The table shows descriptive statistics of the estimated shares of variances in short-rate expectations and term premia determined by the macroeconomic variables. The statistics are reported in percent. The table also shows the constant macroeconomic share of variances estimated using the Gaussian term structure model.

(Back to text)