Large Sample Estimators of the Stochastic Discount Factor

Soohun Kim — Georgia Tech

Robert A. Korajczyk — Northwestern University

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Outline

Motivation

- An economy without aribtrages admits stochastic discount factor (SDF) representation:

$$\text{Price}_i = \mathbb{E}\left[m \times \text{Future Payoff}_i\right]$$

- The random variable m summarizes all pricing related information in
    - consumer preference
    - production technology
    - political/international conflicts ...
- Hence, the estimation of m is critical to learn about the pricing rule of an economy

Motivation

- Although SDF representation $\text{Price}_i = \mathbf{E}\left[m \times \text{Future Payoff}_i\right]$ is applicable to every asset in an economy
- In practice, empirical researchers often estimate the SDF by imposing the equilibrium restriction, $\text{Price}_i = \mathbf{E}\left[m \times \text{Future Payoff}_i\right]$, to a small number of assets/portfolios
  - E.g., Jagannathan and Wang (1996) use size/market-beta portfolios
  - Does the choice of portfolios mask interesting features of priced risk factors?
  - If researchers decide portfolios after they observe some patterns in data, this practice may highlight random errors, not true SDF

Goal

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2. We compare our estimators to other candidate SDF estimators and provide simulation evidence on performance
3. We apply our estimators to large panels of U.S. stock return data

Agnostic Estimator

- Pukthuanthong and Roll (PR, 2017) porposed an agnostic SDF estimator using large panels of asset return data
- From the population moment $1 = \mathbb{E}\left[R_{i,t} m_t\right]$, they construct an sample anlogue

$$\underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{1}_N} = \frac{1}{T} \underbrace{\begin{bmatrix} R_{1,1} & \cdots & R_{T,1} \\ \vdots & \ddots & \vdots \\ R_{N,1} & \cdots & R_{T,1} \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} m_1 \\ \vdots \\ m_T \end{bmatrix}}_{\mathbf{m}} + \underbrace{\begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_N \end{bmatrix}}_{\boldsymbol{\varepsilon}}$$

- and treat $\mathbf{m}$ as coefficients in cross-sectional OLS regression with $N > T$

$$\widehat{\mathbf{m}}_{\text{PR}} = \left(\frac{\mathbf{R}'\mathbf{R}}{T^2}\right)^{-1} \frac{\mathbf{R}}{T}' \mathbf{1}_N = T\left(\mathbf{R}'\mathbf{R}\right)^{-1} \mathbf{R}'\mathbf{1}_N$$

Agnostic Estimator

1. Biased for finite T since $\frac{\mathbf{R}}{T}$ is correlated with ε
2. Assumes a large (N and T) balanced panel
   1. survivorship biases
   2. Large T is not adequate for individual stocks (more on this later)
3. Noisy estimates of SDF for typical panel sizes
4. Equivalent to using Asymptotic Principal Components with the number of factors $= T$ and setting the SDF to a linear combination of the factors

Outline

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Set-up

- For the presentation of theory, we focus on pricing gross returns
- We assume that the gross return generating process of each individual security follows a K-factor model:

$$R_{i,t} = \alpha_i + \boldsymbol{\beta}_i'\mathbf{f}_t + e_{i,t}, \text{ for } i = 1, \cdots, N \text{ and } t = 1, \cdots, T, \tag{1}$$

  where $\boldsymbol{\beta}_i$ is the $(K \times 1)$ vector of factor loadings of the i-th asset on the $(K \times 1)$ vector of factor realizations, $\mathbf{f}_t$
- We allow the factor of $\mathbf{f}_t$ to be either traded excess returns, traded gross returns, latent (recovered by statistical factors), or nontraded factors

Set-up

- In an economy without statistical arbitrage opportunities, there exist a scalar λ_0, the gross return on the riskless asset, and a $(K \times 1)$ vector of $\boldsymbol{\lambda}_f$ such that

$$\mathbb{E}\left[R_{i,t}\right] \approx \lambda_0 + \boldsymbol{\beta}_i' \boldsymbol{\lambda}_f$$

- Assuming exact pricing and plugging the above expression into the process of (1) yields

$$R_{i,t} = \lambda_0 + \boldsymbol{\beta}_i' \left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f + \mathbf{f}_t\right) + e_{i,t} \tag{2}$$

Set-up

- The exact factor pricing assumption implies that the SDF is a linear function of the realization of the systematic factors:

$$m_t = \delta_0 + \mathbf{f}_t' \boldsymbol{\delta}_f, \tag{3}$$

which satisfies $\mathbb{E}\left[R_{i,t} m_t\right] = 1$ when the scalar δ_0 and the $(K \times 1)$ vector, $\boldsymbol{\delta}_f$, are functions of parameters of the economy

$$\delta_0 = \frac{1}{\lambda_0}\left(1 + \boldsymbol{\mu}_f' \Sigma_f^{-1} \boldsymbol{\lambda_f}\right), \; \delta_f = -\frac{1}{\lambda_0}\left(\Sigma_f^{-1} \boldsymbol{\lambda_f}\right)$$

- Hence we restrict m_t to be a linear function of $\mathbf{f}_t$ and show how to estimate m_t for various structures of panel data

Outline

RGP

- Matrix representation: $\mathbf{R} = \lambda_0 \mathbf{1}_N \mathbf{1}_T' + \mathbf{B}\left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f\right)\mathbf{1}_T' + \mathbf{B}\mathbf{F}' + \mathbf{E}$
- We make standard assumptions on the systematic factors and factor loadings:

Assumption 1. *As $N \to \infty$, $\frac{1}{N}\mathbf{B}'\mathbf{1}_N \to \boldsymbol{\mu}_\beta$ and $\frac{1}{N}\mathbf{B}'\mathbf{B} \to \mathbf{V}_\beta = \Sigma_\beta + \boldsymbol{\mu}_\beta \boldsymbol{\mu}_\beta'$, where Σ_β is a positive definite matrix. Also, as $T \to \infty$, $\frac{1}{T}\mathbf{F}'\mathbf{1}_T \xrightarrow{p} \boldsymbol{\mu}_f$ and $\frac{1}{T}\mathbf{F}'\mathbf{F} \xrightarrow{p} \mathbf{V}_f = \Sigma_f + \boldsymbol{\mu}_f \boldsymbol{\mu}_f'$, where Σ_f is a positive definite matrix.*

- Also, we need the following assumptions on $\mathbf{E}$. We use $\mathbf{0}_{m\times n}$ to denote the $(m \times n)$ matrix of zeros:

Assumption 2. *As $N, T \to \infty$, $\frac{\mathbf{1}_N'\mathbf{E}\mathbf{1}_T}{NT} \xrightarrow{p} 0$, $\frac{\mathbf{F}'\mathbf{E}'\mathbf{1}_N}{NT}, \frac{\mathbf{B}'\mathbf{E}\mathbf{1}_T}{NT} \xrightarrow{p} \mathbf{0}_K$ and $\frac{\mathbf{B}'\mathbf{E}\mathbf{F}}{NT} \xrightarrow{p} \mathbf{0}_{K\times K}$. Also, there exists a positive constant $M_0 < \infty$ such that the maximum eigenvale of $\frac{\mathbf{E}'\mathbf{E}}{N}$ is smaller than M_0.*

Balanced Panel Estimator

- We know that SDF is a linear function of factors as

$$m_t = \delta_0 + \mathbf{f}_t' \boldsymbol{\delta}_f$$

- Sample analogue of SDF representation is

$$\begin{aligned}
\underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{1}_N} &= \frac{1}{T} \underbrace{\begin{bmatrix} R_{1,1} & \cdots & R_{T,1} \\ \vdots & \ddots & \vdots \\ R_{N,1} & \cdots & R_{T,1} \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} m_1 \\ \vdots \\ m_T \end{bmatrix}}_{\mathbf{m}} + \underbrace{\begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_N \end{bmatrix}}_{\boldsymbol{\varepsilon}} \\
&= \frac{1}{T} \begin{bmatrix} R_{1,1} & \cdots & R_{T,1} \\ \vdots & \ddots & \vdots \\ R_{N,1} & \cdots & R_{T,1} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{f}_1' \\ \vdots & \vdots \\ 1 & \mathbf{f}_T' \end{bmatrix} \begin{bmatrix} \delta_0 \\ \boldsymbol{\delta}_f \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_N \end{bmatrix}
\end{aligned}$$

- This is a CSR equation with $K + 1 \, (\ll T)$ unknowns

Balanced Panel Estimator

- Cross-sectional OLS representation

$$\mathbf{1}_N = \frac{1}{T}\mathbf{R}\mathbf{F}_{\triangle}\boldsymbol{\delta} + \boldsymbol{\varepsilon}$$

Theorem 1. *Under Assumptions 1 and 2, as* $N, T \to \infty$, $\widetilde{\boldsymbol{\delta}} = \left[\widetilde{\delta}_0 \ \widetilde{\boldsymbol{\delta}}_f'\right]' \xrightarrow{p} \boldsymbol{\delta}$ *where* $\widetilde{\boldsymbol{\delta}}$ *is given by*

$$\widetilde{\boldsymbol{\delta}} = \widetilde{\mathbf{D}}^{-1}\widetilde{\mathbf{U}} = \left(\frac{\mathbf{F}_{\triangle}'\mathbf{R}'\mathbf{R}\mathbf{F}_{\triangle}}{NT^2}\right)^{-1}\left(\frac{\mathbf{F}_{\triangle}'\mathbf{R}'\mathbf{1}_N}{NT}\right) \quad (4)$$

Latent Factors

- This approach works even when $\mathbf{F}$ is not directly available

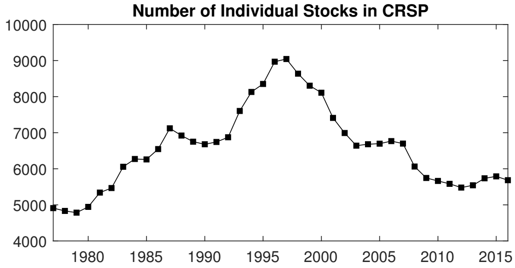
Assumption 3. *We have a factor estimator $\mathbf{F}^*$ that converges to a rotation of $\mathbf{F}$ as N, T increase.*

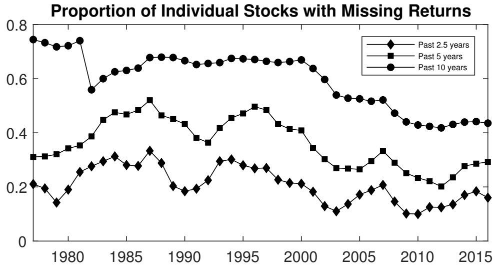
Corollary 1. *Under Assumptions 1, 2, and 3, Theorem 1 holds using the estimated factors.*

- Relation to PR-SDF estimator
  - The Agnostic Estmator by PR is the same as our balanced panel estimator with $K + 1 = T$
  - Empirically, there is a significant gain in performance by imposing a low value for K

Outline

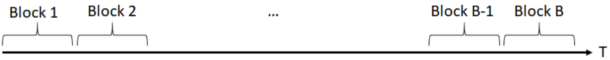
Individual Stock Data





Strategy

- Split the long time series into multiple blocks:



- Balanced within a block of finite size τ but unbalanced across blocks!
- RGP in block b: $\mathbf{R}_{[b]} = \lambda_0 \mathbf{1}_{N_{[b]}} \mathbf{1}'_\tau + \mathbf{B}_{[b]} \left(\boldsymbol{\lambda}_f - \boldsymbol{\mu}_f \right) \mathbf{1}'_\tau + \mathbf{B}_{[b]} \mathbf{F}'_{[b]} + \mathbf{E}_{[b]}$
- Assumptions of the availability of large cross-sectional data in each time block and the time-invariant first two moments in the cross-sectional distribution of factor loadings

Assumption 4. *(i)* $N_{[b]} \to \infty$ *(ii)* $\frac{1}{N_{[b]}} \mathbf{B}'_{[b]} \mathbf{1}_{N_{[b]}} \to \boldsymbol{\mu}_\beta$, $\frac{1}{N_{[b]}} \mathbf{B}'_{[b]} \mathbf{B}_{[b]} \to \mathbf{V}_\beta = \Sigma_\beta + \boldsymbol{\mu}_\beta \boldsymbol{\mu}'_\beta$, $\frac{\mathbf{E}'_{[b]} \mathbf{1}_{N_{[b]}}}{N_{[b]}} \xrightarrow{p} \mathbf{0}_\tau$, $\frac{\mathbf{E}'_{[b]} \mathbf{B}_{N_{[b]}}}{N_{[b]}} \xrightarrow{p} \mathbf{0}_{\tau \times K}$, *and* $\frac{\mathbf{E}'_{[b]} \mathbf{E}_{[b]}}{N} \xrightarrow{p} \mathbf{V}_{e,[b]}$, *where* $\mathbf{V}_{e,[b]}$ *is a* $(\tau \times \tau)$ *diagonal matrix*

Strategy

- Applying the cross-sectional regression idea for a block b

$$\mathbf{1}_{N_{[b]}} = \frac{1}{\tau}\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}\boldsymbol{\delta} + \varepsilon = \left(\frac{1}{\tau}\left(\mathbf{R}_{[b]} - \mathbf{E}_{[b]}\right)\mathbf{F}_{\triangle,[b]} + \frac{1}{\tau}\mathbf{E}_{[b]}\mathbf{F}_{\triangle,[b]}\right)\boldsymbol{\delta} + \varepsilon$$

- With fixed τ, we need to adjust the effect of $\frac{1}{\tau}\mathbf{E}_{[b]}\mathbf{F}_{\triangle,[b]}$ by subtracting $\frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{\triangle,[b]}}{\tau^2}\left(\simeq \frac{\mathbf{F}'_{\triangle,[b]}\mathbf{E}'_{[b]}\mathbf{E}_{[b]}\mathbf{F}_{\triangle,[b]}}{\tau^2}\right)$ in the process of estimation where

$$\widehat{\mathbf{V}}_{e,[b]} \xrightarrow{p} \mathbf{V}_{e,[b]}$$

Analogy to Litzenberger-Ramaswamy (1979)

- Consider a Fama-MacBeth regression of excess returns on a constant and estimated CAMP betas

$$\mathbf{R}_t = [\mathbf{1}_N \ \boldsymbol{\beta}]\,\mathbf{\Gamma}_t + \mathbf{u}_t = \mathbf{Z}\mathbf{\Gamma}_t + \mathbf{u}_t, \mathbf{\Gamma}_t' = [\gamma_0 \ \gamma_1]$$

- But we actually observe $\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta} + \boldsymbol{\varepsilon}$. Let $\mathbf{\Sigma}_Z = \text{plim}\frac{\mathbf{Z}'\mathbf{Z}}{N}$ and $\bar{\sigma}_\varepsilon^2 = \text{plim}\frac{\varepsilon'\varepsilon}{N}$.
- If we simply regress $\mathbf{R}_t$ on $\widehat{\mathbf{Z}} = \left[\mathbf{1}_N \ \widehat{\boldsymbol{\beta}}\right]$

$$\text{plim}\widehat{\mathbf{\Gamma}}_t = \text{plim}\left(\left(\frac{\widehat{\mathbf{Z}}'\widehat{\mathbf{Z}}}{N}\right)^{-1}\frac{\widehat{\mathbf{Z}}'\mathbf{R}_t}{N}\right) = \left(\mathbf{\Sigma}_Z + \begin{bmatrix} 0 & 0 \\ 0 & \bar{\sigma}_\varepsilon^2 \end{bmatrix}\right)^{-1}\mathbf{\Sigma}_Z\mathbf{\Gamma}_t$$

- Solution: subtract a consistent estimate of $\bar{\sigma}_\varepsilon^2$ from the (2,2) element of $\frac{\widehat{\mathbf{Z}}'\widehat{\mathbf{Z}}}{N}$!

Short-τ correction

We utilize the estimator of $\mathbf{V}_{e,[b]}$ proposed by Kim and Skoulakis (2018).

Lemma 1. *Let Assumptions 1, 4 be in effect. Define* $\widehat{\mathbf{V}}_{e,[b]}$ *by*

$$\widehat{\mathbf{V}}_{e,[b]} = \mathit{diag}\left(\left(\mathbf{H}_{[b]} \odot \mathbf{H}_{[b]}\right)^{-1} \mathcal{S}' \mathit{vec}\left(\frac{\widehat{\mathbf{E}}'_{[b]}\widehat{\mathbf{E}}_{[b]}}{N_{[b]}}\right)\right), \tag{5}$$

$$\begin{aligned} \mathbf{H}_{[b]} &= \mathbf{J}_\tau - \mathbf{J}_\tau \mathbf{F}_{[b]} \left(\mathbf{F}'_{[b]} \mathbf{J}_\tau \mathbf{F}_{[b]}\right)^{-1} \mathbf{F}'_{[b]} \mathbf{J}_\tau, \\ \mathbf{J}_\tau &= \mathbf{I}_\tau - \frac{1}{\tau} \mathbf{1}_{\tau\times\tau} \\ \mathcal{S}_{i,j} &= \mathbf{1}(i = (j-1)\tau + j). \end{aligned} \tag{6}$$

Then, it holds that as $N, T \to \infty$, $\widehat{\mathbf{V}}_{e,[b]} \xrightarrow{p} \mathbf{V}_{e,[b]}$ *for each* $b = 1, \cdots, B$.

Unbalanced Panel

- From (i) the moment condition + (ii) short τ adjustment in block b

$$\left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{\triangle,[b]}}{\tau^2} \right) \boldsymbol{\delta} \simeq \frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}\tau}$$

- We aggregate information across blocks as follows
  - Define $d_{[b]}$ and $u_{[b]}$

$$d_{[b]} = \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{F}_{\triangle,[b]}}{\tau} \right)^{-1} \left(\frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{R}_{[b]}\mathbf{F}_{\triangle,[b]}}{N_{[b]}\tau^2} - \frac{\mathbf{F}'_{\triangle,[b]}\widehat{\mathbf{V}}_{e,[b]}\mathbf{F}_{\triangle,[b]}}{\tau^2} \right)$$

$$u_{[b]} = \frac{\mathbf{F}'_{\triangle,[b]}\mathbf{R}'_{[b]}\mathbf{1}_{N_{[b]}}}{N_{[b]}\tau}$$

  - Aggregate across blocks: $\widehat{\mathbf{D}} = \left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T} \right) \frac{1}{B} \sum_{b=1}^{B} d_{[b]}$ and $\widehat{\mathbf{U}} = \frac{1}{B} \sum_{b=1}^{B} u_{[b]}$
  - Finally, we have $\widehat{\mathbf{D}}\boldsymbol{\delta} \simeq \widehat{\mathbf{U}}$, yielding $\widehat{\boldsymbol{\delta}} = \widehat{\mathbf{D}}^{-1}\widehat{\mathbf{U}}$

Unbalanced Panel Estimator

Theorem 2. *Under Assumptions 1, 4, as* $N, T \to \infty$, $\widehat{\boldsymbol{\delta}} = \left[\widehat{\delta}_0 \ \widehat{\boldsymbol{\delta}}'_f\right]' \xrightarrow{p} \boldsymbol{\delta}$ *where* $\widehat{\boldsymbol{\delta}} = \widehat{\mathbf{D}}^{-1}\widehat{\mathbf{U}}$

- Basically, this unbalanced estimator mimics the moment condition of the balanced panel estimator in Theorem 1
- This result can be extended to latent factors (See Corollary 2.2 of the paper)

Asymptotic distribution

- Note that $\widehat{\mathbf{D}} = \left(\frac{\mathbf{F}'_{\triangle}\mathbf{F}_{\triangle}}{T}\right)\frac{1}{B}\sum_{b=1}^{B} d_{[b]}$ and $\widehat{\mathbf{U}} = \frac{1}{B}\sum_{b=1}^{B} u_{[b]}$ where $\widehat{\boldsymbol{\delta}} = \widehat{\mathbf{D}}^{-1}\widehat{\mathbf{U}}$
- Under regularity conditions (Assumptions 5, 7 of the paper), we identify the asmyptotic distribution of $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{U}}$ and apply delta method

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Theorem 3. *Under regularity conditions, as $N, T \to \infty$,*

$$\sqrt{T}\left(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}\right) \xrightarrow{d} N\left(0, \boldsymbol{\Sigma}_{\boldsymbol{\delta}}\right),$$

where $\boldsymbol{\Sigma}_{\boldsymbol{\delta}}$ is given in Theorem 2.3 of the paper

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Theorem 4. *Under regularity conditions, as $N, T \to \infty$,*

$$\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}} = \tau \widehat{\boldsymbol{\Psi}} \left(\frac{1}{B} \sum_{b=1}^{B} \boldsymbol{\eta}_{[b]} \boldsymbol{\eta}'_{[b]} \right) \widehat{\boldsymbol{\Psi}}' \xrightarrow{p} \boldsymbol{\Sigma}_{\boldsymbol{\delta}},$$

where $\widehat{\boldsymbol{\Psi}} = \left[1 \quad -\widehat{\boldsymbol{\delta}}'\right] \otimes \widehat{\mathbf{D}}^{-1}$ and $\boldsymbol{\eta}_{[b]}$ is given by (2.30) of the paper

Outline

Calibration

1. Three return generating processes implied by the CAPM, FF3, FF5
2. To obtain the parameters for a large number of assets in the simulation we exploit 14,277 individual stock returns over 600 months (January 1967 to December 2016) from the CRSP monthly database by regressing the excess returns of $R_{i,t} - R_{f,t}$ on a constant and a vector of factor returns:

$$R_{i,t} - R_{f,t} = \alpha_i + \boldsymbol{\beta}_i' \mathbf{f}_t + e_{i,t}$$

3. The first two moments of factors: $\boldsymbol{\mu}_f = \frac{1}{600} \sum_{t=1}^{600} \mathbf{f}_t$ and $\Sigma_f = \frac{1}{600-1} \sum_{t=1}^{600} (\mathbf{f}_t - \boldsymbol{\mu}_f)(\mathbf{f}_t - \boldsymbol{\mu}_f)'$. The riskless gross return is estimated as the average of the gross realized risk free return over the same period: $\lambda_0 = \frac{1}{600} \sum_{t=1}^{600} R_{f,t}$.

Simulation

1. Simulated economy with $N =$500, 1,000, 2,000, and 4,000.
2. "Monthly" asset returns with $T = $ 60, 120, 240, and 480 (corresponding to 5, 10, 20, and 40 years of data) and $\tau = 30$
3. Returns constructed to obey one of three asset pricing models: CAPM, FF3, FF5
4. Cross-sectional systematic and idiosyncratic risk exposures drawn jointly from the empirical distribution, as in Chen, Connor, and Korajczyk (2018).

Performance metrics

Regress the SDF estimator, $\hat{m}_t$, on the true SDF, m_t: $\hat{m}_t = a + bm_t + u_t$.

1. R^2 =1
2. $a = 0$
3. $b = 1$

Gross returns, FF5 (PR method)

	R^2				intercept(a)				slope(b)			
Panel C: Pukthuanthong and Roll's (2017) Estimator												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.11	0.09	0.05	0.00	0.46	0.29	0.17	0.09	0.55	0.71	0.83	0.92
1000	0.17	0.16	0.11	0.06	0.45	0.29	0.16	0.09	0.55	0.71	0.84	0.91
2000	0.24	0.24	0.20	0.13	0.45	0.28	0.17	0.09	0.55	0.72	0.83	0.91
4000	0.32	0.35	0.31	0.23	0.45	0.28	0.17	0.09	0.56	0.72	0.84	0.91

1. R^2 low
2. R^2 declines in T (estimating SDF with more factors)
3. a and b independent of N improve in T

Gross returns, FF5 (Unbalanced)

	R^2				intercept(a)				slope(b)			
Panel A: Unbalanced Panel Estimator												
A-1: With Observed Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.19	0.20	0.23	0.27	-0.74	1.97	-0.30	-0.05	1.75	-1.10	1.30	1.06
1000	0.21	0.23	0.28	0.37	-0.46	0.10	-0.06	-0.01	1.44	0.91	1.06	1.02
2000	0.26	0.30	0.37	0.49	-0.12	-0.07	-0.02	-0.01	1.14	1.08	1.03	1.01
4000	0.31	0.38	0.49	0.63	-0.10	-0.03	-0.02	-0.01	1.12	1.04	1.02	1.02
A-2: With Estimated Factors												
$N \backslash T$	60	120	240	480	60	120	240	480	60	120	240	480
500	0.19	0.22	0.27	0.36	0.21	0.30	0.30	0.30	0.80	0.70	0.70	0.70
1000	0.23	0.27	0.36	0.47	0.15	0.20	0.20	0.21	0.86	0.80	0.80	0.79
2000	0.26	0.34	0.45	0.59	0.13	0.12	0.13	0.13	0.88	0.88	0.87	0.87
4000	0.32	0.42	0.56	0.71	0.03	0.06	0.07	0.06	0.98	0.95	0.93	0.94

Gross returns, FF5 (Unbalanced)

1. R^2 higher.
2. R^2 increases in T (more data with the fixed number of factors).
3. a and b improve in both N and T.

Gross returns, SDF test

N	1000											
(T, τ)	(450,30)			(750,50)			(600,30)			(1000,50)		
Nominal Size	1	5	10	1	5	10	1	5	10	1	5	10
Panel A: CAPM												
$\delta_{\mathtt{MKT}}$	2.1	7.6	13.7	2.6	8.4	14.1	1.6	6.7	12.4	2.1	7.2	13.3
Panel B: FF3												
$\delta_{\mathtt{MKT}}$	2.1	7.2	13.5	2.6	7.7	13.6	1.5	6.2	11.9	1.8	7.0	12.4
$\delta_{\mathtt{SMB}}$	2.1	6.9	12.9	2.2	8.2	14.2	1.6	6.2	12.0	2.0	7.3	12.7
$\delta_{\mathtt{HML}}$	1.7	6.8	12.6	2.5	8.2	13.9	1.8	7.0	12.8	1.8	6.9	12.5
Panel C: FF5												
$\delta_{\mathtt{MKT}}$	1.5	6.0	11.7	2.1	7.4	13.1	1.4	5.7	10.5	1.7	7.2	12.9
$\delta_{\mathtt{SMB}}$	1.5	6.2	11.5	2.3	7.6	13.5	1.5	5.8	11.0	1.9	7.0	12.3
$\delta_{\mathtt{HML}}$	1.6	6.4	12.0	2.2	7.4	13.2	1.5	6.0	11.3	2.0	6.6	12.4
$\delta_{\mathtt{RMW}}$	1.5	6.5	11.7	2.3	8.2	13.8	1.3	6.0	11.6	2.1	7.2	13.0
$\delta_{\mathtt{CMA}}$	1.7	6.2	11.7	2.1	7.3	13.0	1.4	6.0	11.4	1.7	6.7	12.2

Outline

Data

- CRSP monthly returns
- 600 months from January 1967 to December 2016
- 10,112 individual stocks (five dollar price filter)
- We split 600 months data into 20 blocks with equal length of 30 months
  - The first block is from January 1967 to June 1969
  - The last block is from July 2014 to December 2016
  - The number of stocks is ranged from 1578 to 3443 with the average of 2455 over 20 blocks.

Asset Pricing Models

1. Specific asset pricing model applied to individual asset returns
   1. CAPM, FF3, Hou, Xue, Zhang – HXZ4 (MKT, ME, I/A, ROE), FF5 (FF3+CMA+RMW), Barillas and Shanken – BS6 (HXZ4+MOM+HML_devil), Pástor and Stambaugh – PS5 (FF3+MOM+LIQ)
2. Statistical Factor Models
   1. Connor and Korajczyk (1986, 1991): PCA in each block + Rotation to FF3

Estimate (t-stat)

Panel A: Specific Asset Pricing Models										
	MKT	SMB	HML	I/A	ROE	CMA	RMW	MOM	LIQ	HML(devil)
CAPM	-4.46									
	(-4.38)									
	-2.51									
FF3	-3.42	-2.69	-0.94							
	(-3.49)	(-1.39)	(-0.42)							
	-3.08	-1.90	-5.97							
HXZ4	-5.04	-5.81		-7.85	-12.73					
	(-4.05)	(-4.59)		(-1.73)	(-6.19)					
	-4.79	-4.84		-14.90	-10.32					
FF5	-5.45	-3.31	4.62			-8.48	-11.57			
	(-3.96)	(-1.70)	(1.48)			(-2.38)	(-2.05)			
	-4.83	-3.55	0.40			-8.74	-12.50			
PS5	-5.54	-2.60	-5.37					-8.59	0.19	
	(-6.54)	(-2.71)	(-3.08)					(-6.06)	(0.22)	
BS6	-5.41	-5.73		-5.03	-8.29			-7.43		-5.16
	(-4.28)	(-4.69)		(-1.59)	(-2.86)			(-3.85)		(-2.66)
	-4.90	-5.99		-5.55	-13.25			-5.22		-10.60
All	-5.50	-4.95		-5.79	-9.23	1.91	1.68	-7.96	-0.15	-5.92
	(-4.23)	(-4.12)		(-0.94)	(-2.17)	(0.56)	(0.24)	(-4.05)	(-0.17)	(-2.60)
Panel B: Statistical Factor Model										
	PC1	PC2	PC3							
Single-Factor	-4.58									
	(-4.38)									
Three-Factor	-3.42	-0.96	-2.75							
	(-2.08)	(-0.25)	(-0.68)							

Result

	MKT	SMB	HML	I/A	ROE	CMA	RMW	MOM	LIQ	HML(devil)
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Results

1. The market always enters the SDF significantly - in contrast to many cross-sectional regression estimates of the market risk premium. CAMP with realized moments implies $\delta = -2.5$ while $\widehat{\delta}$ =-4.46
2. SMB usually significant
3. HML mixed while HML(devil) significant
4. ROE and Momentum significant
5. I/A not significant but CMA is
6. Liquidity not significant

Outline

Conclusion

- We propose several SDF estimators exploiting large panel data
- The imposition of a factor structure improves performance over the fully agnostic alternative by PR
- We handle unbalanced panel data with block structure