# On the Variational Approach to the Analysis of Tax Systems: A Cautionary Tale

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**Abstract**

We provide a formal definition for an alternative approach in the analysis of optimal tax systems that has been extensively used in the literature. Additionally, we study optimal income taxation in a class of static one-dimensional Mirleesian economies where preferences do not satisfy the Spence-Mirrlees condition (SMC). We characterize necessary conditions for the optimal taxation using a structural mechanism design approach based on type assignment functions. Because the SMC is violated, local incentive constraints no longer suffice for implementability, and an additional set of global incentive constraints must be explicitly taken into account. When these global constraints bind, they create a tension between infra-marginal types whose ICs are not handled by any local approach. Local perturbations (or small reforms) of the optimal tax schedule may have global (first-order) impacts on welfare, thus invalidating some of the assumptions underlying local variational methods. *JEL Classification*: D82, H21.

## 1 Introduction

The Taxation Principle - Hammond (1979) - guarantees that any incentive feasible allocation can be implemented by a tax schedule. This means, in particular, that deriving optimal taxes is equivalent to deriving optimal allocations. Any method advanced towards characterizing one is an advance towards characterizing the other. In this context, following Piketty's (1997) and Saez's (2001) seminal contributions, optimal income tax theory has geared towards the use of tax perturbation methods as its main approach in the analysis of tax systems, as an alternative to Mirrlees' (1971) structural mechanism design approach.[1,2]

These techniques are based on studying local perturbations (or small reforms) of a baseline nonlinear tax system under some regularity assumptions regarding the optimum. They have the significant advantage of producing formulae based on empirically relevant objects, usually behavioral elasticities, that are very transparent about the economic forces driving the size and shape of the optimal tax schedule. Since these elasticities are often observable and/or may be credibly estimated with available real-world data using quasi-experimental designs, the variational approach brought optimal tax theory closer to actual policy-making.

Another perceived advantage is that it does not require the strong restrictions needed to characterize optimal taxes using a fully structural mechanism design approach as in Mirrlees' (1971) original work. In particular, the Spence-Mirrless condition (SMC) for agents' utility need not be assumed.[3] Hence, the method would be applicable in complex environments where imposing these restrictions seems too strong. All of these gains come at the expense of making technical regularity assumptions on endogenous objects. Golosov, Tsyvinski, and Werquin (2014) provide a systematic analysis of the underlying assumptions necessary for applying the method in the design of optimal tax systems. Intuitively, local perturbations around the optimal tax system should not have any impact on welfare. The required sufficient assumption is the Lipschitz continuity of the agents' decisions with respect to the tax policy, which is an endogenous object. Hence, it is not possible to check its validity in any given application.

The goal of this paper is twofold. First, we present a class of simple Mirrleesian economies that dispense with the SMC and remain tractable enough for the Mirrlees'

[1]Some other terminology has been used to denote these methods. Examples are variational approach, allocations perturbation, tax reforms.

[2]In a similar vain, the so-called sufficient statistics approach proposed by Chetty (2009) uses variational techniques to write the welfare impact of policy changes in terms of few reduced-form elasticities in applications other than the design of income tax systems. The sufficient statistics approach applied to the design of optimal income taxes is, in fact, the main contribution of Piketty (1997) and Saez (2001).

[3]The SMC is often referred to as the single-crossing property. This condition guarantees that agents' marginal utility is ordered with respect to their idiosyncratic characteristic or type. By characterizing incentive compatibility constraints in terms of local conditions, the problem becomes a simple optimal control program.

approach (1971), and we investigate whether the regularity assumptions in Golosov, Tsyvinski, and Werquin (2014) are verified at the optimum tax scheme. Second, we propose a different class of solutions for the optimal design of income tax systems featuring segregation of income groups. These solutions emerge naturally when tackling the mechanism design problem using type assignment functions.[4]

For the class of economies we study, the SMC is replaced by the assumption that the marginal disutility of producing taxable income is increasing in one region and decreasing in another of the parameter space, where the boundary between the regions is described by a monotonic function of income earned. This assumption, first adopted in a monopoly pricing context by Araújo and Moreira (2010), implies that local incentive compatibility constraints are not sufficient to ensure full incentive compatibility. Typically, incentive compatible allocations are not monotonous generating discrete pooling, i.e. the sets of agents for which the same bundles are assigned need not be connected. Building on this work, Araujo, Moreira, and Vieira (2015) exhibit a welfare improvement by relaxing the requirement of convex-valued mechanisms.

Our proposed tax schedule promotes segregation of income groups, which allows the government to more efficiently balance the distortions within each group in an environment where the tax system has to take into account global incentive compatibility constraints. The government handles these constraints by making sure that a pivotal agent is indifferent between migrating from the high to the low-income group.

We assess the properties of schedules that promote segregation of income groups and show that, in particular, they do not satisfy the Lipschitz continuity assumption described in Golosov, Tsyvinski, and Werquin (2014). Hence, we cannot make use of the Gateaux differential to identify the optimum. In fact, when global incentive constraints are binding, local perturbations of the tax system at the optimum have infra-marginal welfare impacts that are not negligible. By distorting the taxation on the pivotal type, global incentive compatibility requires distorting the allocation of a positive measure set of individuals. Thus, the impact on welfare is proportional to the shadow cost of this global incentive constraint measured by the associated Lagrangian multiplier. As a consequence of these complexities, the use of variational methods may fail in providing an optimal income tax system. Moreover, elasticities emerging from the use of variational methods may not even be sufficient statistics

[4]The Mechanism Design Approach typically works in the space of direct mechanism that for each type associates an allocation. Our method uses a type assignment functions which inverts the logic: for each possible bundle a set of types is assigned.

to the welfare analysis.[5]

It is important to stress that the violation of SMC is a necessary but not sufficient condition to the failure of variational methods in characterizing optimal tax systems. It is crucial to have a strong tension between intra-marginal benefits and infra-marginal incentives generated by binding global incentive compatibility constraints. Our class of economies is the most parsimonious environment in which one can generate this tension and still is tractable enough to be solved using mechanism design. It is a direct generalization of a Mirrleesian economy where agents have quasi-linear preferences, and it is a particular case of the economies studied in Golosov, Tsyvinski, and Werquin (2014).

One of the features of the Variational Approach is to be agnostic about the underlying information frictions in the economic environment.[6] However, unless we fully specify the nature of the information friction, we have no hope of assuring the validity of the Variational Approach in identifying the optimum. Apart from its technical contributions, this paper should be thought of as a cautionary tale regarding the usage of variational methods in complex environments where the failure of SMC and the associated impossibility of ordering marginal utilities according to types naturally emerges.[7] Moreover, depending on the characteristics of the underlying information friction the discrepancy between the optimal mechanism and the one we get from the variational approach could be sizable.

The rest of the paper is organized as follows. After this introduction, we briefly discuss the related literature. In Section 2, we present a class of economies where the agent's utility does not satisfy the Spence-Mirrlees condition. In Section 3, we discuss the complications emerging in this environment and characterize the relevant incentive constraints. In Section 4, we propose a novel approach to the optimal income taxation based on type assignment function and compare with the one emerging from the variational approach. Section 5 is reserved for the conclusion and discussion of future steps.

[5] Piketty, Saez, and Stantcheva (2014), Hendren (2013) and Scheuer and Werning (2015) have examples of complex environments where the elasticities that should be considered are not the usual elasticity of taxable income.

[6] The application of the Variational Approach always delivers a candidate, regardless of whether the required conditions on the optimal are satisfied or not.

[7] The possibility of binding global incentive constraints was recognized a long time ago by Mirrlees (1999) in the Moral Hazard context. Unlike the screening environment of Mirrlees (1971), in the Moral Hazard context he couldn't find a natural assumption in terms of fundamentals that guarantees the sufficiency of the First Order Approach.

## Literature review

This paper lies in the intersection of two different branches of the literature: (1) adverse selection models without the SMC and (2) optimal taxation using variational techniques.

Although perturbation methods have been in use since at least Sheshinski (1972), the works by Piketty (1997); Dahlby (1998); Saez (2001) have shown how to extend them from the parametric restrictions on schedules that were imposed in the early literature. These methods proved to be a significant generalization since they allowed us to assess optimal allocations, thus placing them on the same footing as what was accomplished by Mirrlees (1971).[8]

These methods have expanded the scope of optimal tax theory to address migration (as in Lehmann, Simula, and Trannoy (2014) ), dynamics (as in Golosov, Tsyvinski, and Werquin (2014)), general equilibrium effects (as in Sachs, Tsyvinski, and Werquin (2016)) and taxation and political economy (as in Bierbrauer, Tsyvinski, and Werquin (2017)). They have also brought theory closer to applications since tax formulae are usually expressed in elasticities which can be recovered from the data.

On a related literature, the sufficient statistic approach uses variational methods to identify formulas for the welfare impact of policy reforms based on high-level elasticities. For instance, Feldstein (1999) first pointed out that the elasticity of taxable income is a sufficient statistic to welfare analysis. This approach has since been extended by Chetty (2009), Piketty, Saez, and Stantcheva (2014), among others as a middle-ground between structural and reduced form methods. For instance, Saez's (2001) uses variational methods to re-write the Mirrlees' (1971) formula for the optimal income tax rate in terms of labor supply elasticities.

Methodologically, our paper is built upon Araujo, Moreira, and Vieira (2015) and closer to the literature of screening problems without the SMC. Notable examples are Araújo and Moreira (2010) and Schottmüller (2015). The literature of models that do not satisfy SMC is tightly related to the multidimensional screening literature (see Rochet and Choné (1998), Rochet and Stole (2003) and Armstrong and Rochet (1999)). For instance, the failure of the SMC could also emerge naturally due to multidimensionality; for example, in the problem of taxing couples (see Alves, da Costa, and Moreira (2017)).

The use of type assignment functions as an alternative approach to mechanism design dates back to Goldman, Leland, and Sibley (1984) and Noldeke and Samuel-

[8]Recall that under the Taxation Principle – Hammond (1979, 1987) – any constrained efficient allocation can be implemented through a suitable design of budget sets.

son (2007). To the best of our knowledge, we are the first to use this method in the optimal taxation context.

## 2 Environment

We start at a very general level imposing very little structure on the nature of preferences and/or degree of heterogeneity. At this level of generality, we briefly describe the mechanism design program, state the taxation principle and describe the tax perturbation method offering explicit assumptions that justify its use.

The economy is inhabited by a population of agents index by $\theta \in \Theta$ with measure equal to one. Agents have preferences defined over consumption, $c \in \mathbb{R}_+$, and taxable income $z \in Z \subset \mathbb{R}_+$, increasing in the first and decreasing in the latter and represented by $U(c, z, \theta)$.

An allocation is a mapping from $\Theta$ to $\mathbb{R}_+ \times Z$ associating with each $\theta$ a pair $(c(\theta), z(\theta))$.

The revelation principle guarantees that the set of allocations that can be reached, given the information structure and the resource constraint are payoff equivalent to the set of allocations implemented by direct revelation mechanisms. In our context, the strategy space in the direct mechanism is $\Theta$, consisting of a type announcement and an outcome function $(z, c) : \Theta \to Z \times \mathbb{R}_+$ specifying an income-consumption pair for each type $\theta$ reported. The mechanisms induces truthful revelation through the imposition of incentive compatibility constraints. We use interim Bayesian Nash equilibrium as our implementability concept.

**Definition 1.** *An allocation $z, c : \Theta \to Z \times \mathbb{R}_+$ is said to be **incentive-feasible** when:*

*i) the incentive compatibility constraints*

$$U(c(\theta), z(\theta), \theta) \geq U(c(\hat{\theta}), z(\hat{\theta}), \theta), \tag{1}$$

*for all $\hat{\theta}, \theta \in \Theta$, and;*

*ii) the economy's resource constraint,*

$$\int_{\Theta} [z(\theta) - c(\theta)] f(\theta) d\theta \geq G, \tag{2}$$

*are satisfied.*

**Tax schedule and the taxation principle** Under the mechanism design approach one chooses in the space of direct mechanisms the one which implements the allocation leading to maximum welfare. The tax perturbation approach, on the other hand, works directly on the space of tax schedules, often called indirect mechanisms. That is, assume that, instead of designing a truthful direct mechanism, a slightly less sophisticated government defines a common budget set for all agents through the choice of an arbitrary non-linear tax schedule, $T : Z \mapsto \mathbb{R}$.

Facing such budget set a $\theta$ agent chooses $z_\theta(T) = \arg\max_z U(z - T(z), z, \theta)$. The taxation principle guarantees that the set of allocations that are implemented via direct mechanism coincide with the set of allocations that are implemented via taxes.

One of the goals of this paper is to clarify how important phenomena in the space of direct mechanisms translate to tax schedules and vice-versa. We state and prove it in our context for completeness.

**Proposition 1** (**Taxation Principle**). *An allocation $z, c : \Theta \to Z \times \mathbb{R}_+$ is incentive-feasible if and only if it is tax-implementable. In other words, an allocation is incentive-feasible if and only if there exists a tax schedule $T : Z \to \mathbb{R}$ such that for all $\theta \in \Theta$:*

*i)* $T(z(\theta)) = z(\theta) - c(\theta)$*;*

*ii)* $z(\theta) \in \arg\max_{z \in Z(\Theta)} U(z - T(z), z, \theta)$*, and;*

*iii) the government's budget constraint holds:* $\int_{\theta \in \Theta} T(z(\theta)) f(\theta) d\theta \geq G$*.*

*Proof.* See the Appendix. □

Now, instead of prescribing an allocation for the type announcements, the government designs a menu of choices, or budget sets, and lets individuals self-select their income. As we saw in Proposition 1, the allocation generated by this tax system is incentive-feasible. Incentive compatibility in the mechanism design approach translates into Marshallian demands under the tax system. The taxation principle provides the tool to connect direct and indirect mechanisms. It states that any incentive-feasible allocation can also be implemented via an income tax system and conversely any allocation implied by a tax system is incentive-feasible. This result is a powerful tool in the challenging task of formally characterizing the connection between incentives and behavioral responses.

**Tax Perturbation Methods** Tax perturbation methods consist in producing small tax reforms around a candidate optimal schedule and checking whether improvements exist under a pre-specified welfare metric. We are at an optimal when no such reforms exist.

The impact of tax reforms are captured by behavioral responses, usually measured by relevant elasticities. More specifically, when calculating the welfare impacts of a tax reform, the tax designer anticipates the changes in taxable income supply and internalizes it to get the best tax schedule. To make the method workable, assumptions are made that amount to stating that behavioral responses are *well behaved*.

Note that these are assumptions regarding equilibrium objects, taxable income elasticities, which themselves depend on properties of an endogenous object, the optimal tax schedule. In most cases – Piketty (1997); Dahlby (1998); Saez (2001) – these assumptions are never made explicit but alluded to with an appeal to 'sensibility', which makes it difficult to assess the method's generality. A notable exception is Golosov, Tsyvinski, and Werquin (2014). One of the stated goals of this work is to provide a technical foundation for perturbation methods. We shall use their formalization as the main reference of sufficient conditions for the application of tax perturbation methods.

Let us, first re-state Assumption 2 in Golosov, Tsyvinski, and Werquin (2014).

**Assumption 1** (GTW). *The taxable income functional $z_\theta(T)$ is locally Lipschitz continuous in every direction at the initial tax system $T$. That is, for any admissible perturbation $H \in \mathscr{C}^2$, there exists $\bar{\mu} > 0$ and $M$ such that $\mu < \bar{\mu}$ implies $\|z_\theta(T + \mu H) - z_\theta(T + \mu H)\| < M \times \mu$.*

This assumption makes it very transparent the fact that behavioral responses being well behaved depends on properties of the initial schedule $T$. The essence of our question is whether these properties are satisfied *by the optimal tax schedule*. As we shall see, this need not be the case. Maybe Assumption 1 is too stringent, for it requires behavioral responses to be well behaved for all $\theta$. We then consider Assumption 1 in Hendren (2017), which imposes very few restriction on individual responses, but does impose continuity in the aggregate. Our discussion, and numerical example shows that even these weaker restrictions need not be valid at the optimum.

To state Hendren's assumption, first define $R(T) = \mathbb{E}[T(z_\theta(T))]$ as the government's revenue given a tax schedule $T$. Then consider small perturbations around this status quo schedule.

**Assumption 2** (Hendren)**.** *Let* $T_\mu(z) = T(z) + \mu \sum_{j=1}^{N} T^j(z)$ *for some functions* $T^j$*. Then* $R$ *is continuously differentiable in* $\mu$ *and*

$$\left.\frac{d}{d\mu}\right|_{\mu=0} R(T_\mu) = \sum_{j=1}^{N} \left.\frac{d}{d\mu}\right|_{\mu=0} R(T_\mu^j),$$

*where, for* $j = 1, ..N.$, $T_\mu^j(z) = T(z) + \mu T^j(z)$.

## 2.1 A quasi-linear economy

As we have already mentioned, these assumptions pertain to the nature of the planner's program solution. If they are verified at the optimum, then the method does characterize, i.e., conditions are necessary for the optimum. To assess whether these assumptions are warranted we need to be able to actually solve the problem. There is a major issue to be considered in this case. We know that under the SMC, invariably imposed under the mechanism design approach, Assumptions 1 and 2 are valid. Our task is to relax this assumption while maintaining our capacity to solve the associated mechanism design program. To do so, we now specialize the economy to a quasi-linear one.

I.e., we assume

$$U(c, z, \theta) = c - v(z, \theta),$$

with $\Theta = [\underline{\theta}, \bar{\theta}]$.

Function $v(z, \theta)$ is an important piece of our analysis. It represents the utility cost incurred by a type-$\theta$ agent to earn $z$ units of income. We assume $v(z, \theta)$ to be increasing on earnings, convex and three times continuously differentiable on $Z \times \Theta$. The utility function considered here has the advantage of being simple to handle and flexible enough to accommodate several utility functions used in the public economics literature. The interpretation of parameter $\theta$ depends on the application under consideration. For instance, it can be the labor market productivity as in the original Mirrlees's (1971) economy, a labor taste parameter, a discount factor in dynamic models or the amalgam of several variables from a multidimensional model. In Appendix A we present some common environments in the public economics literature and the implied function $v(z, \theta)$ as well as the interpretation for $\theta$. We write it on terms of the taxable income to stress the idea that individuals have margins other than labor supply to adjust in response to taxes.

In autarky, individuals consume all their income and optimally choose the taxable income in order to satisfy the following first-order necessary condition:[9]

$$1 - v_z(z^A(\theta), \theta) = 0. \tag{3}$$

Two properties of the autarky allocation are worth noting. First, all individuals get the same marginal disutility of taxable income supply. They generate income until the marginal cost in terms of the utility of making the extra unit is equal to 1, which is the marginal benefit. Second, when $v_{z\theta} > 0$, higher types get lower incomes in autarky, and when $v_{z\theta} < 0$, higher types get higher incomes.

Our setting is, therefore, very close to that in **?**, with one small difference: we do not impose SMC. Note that in this setting SMC amounts to imposing constancy in the sign of $v_{\theta z}(z, \theta)$. This assumption states that the marginal rate of substitution between consumption and taxable income is everywhere decreasing (or everywhere decreasing) in type. In our model the marginal rate of substitution between consumption and income is given by

$$\left.\frac{dc}{dz}\right|_{V(c,z,\theta)=cte} = v_z(z, \theta). \tag{4}$$

Therefore, the utility function $V(c, z, \theta)$ satisfies the usual Spence-Mirrlees condition if $v_{z\theta} < 0$ (or $v_{z\theta} > 0$), for all $z \in Z$ and $\theta \in \Theta$. Graphically, in a diagram $Z \times C$, the indifference curves of individuals with lower types should be stepper (when $v_{z\theta} < 0$ ) and cross each other at most once.

We depart from SMS by assuming the following.[10]

**Assumption 3.** *The condition $v_{z\theta}(z, \theta) = 0$ defines implicitly a monotonic function $z_0 : \Theta \to Z$ such that $v_{z\theta}(z, \theta) > 0$ for $z < z_0(\theta)$ and $v_{z\theta}(z, \theta) < 0$ for $z > z_0(\theta)$. For simplicity, assume additionally that $v_{zz\theta} > 0$.*

Assumption 3 above, first used in Araújo and Moreira (2010), relaxes the requirement that the utility function satisfies the SMC.[11] Function $z_0(\theta)$, henceforth referred as a *separating curve*, splits the space $\Theta \times Z$ into two regions where the signal of the cross-derivative remains constant. Let us define $CS_- = \{(\theta, z) \in \Theta \times Z : v_{z\theta}(z, \theta) < 0\}$ and $CS_+ = \{(\theta, z) \in \Theta \times Z : v_{z\theta}(z, \theta) > 0\}$ these regions. The function $z_0(\theta)$ can be increasing or decreasing depending on the application.

[9] Note that the second-order sufficient condition is satisfied since we assumed function $v(z, \theta)$ to be convex in $z$.

[10] The assumption $v_{zz\theta} > 0$ is not crucial but will be convenient to guarantee concavity of government objective function.

[11] This assumption was also used in Schottmüller (2015), Choné and Gauthier (2017) and Araujo, Moreira, and Vieira (2015).

See Appendix A for examples of economies generating this particular type of failure of the SMC. Figure 1 plots the separating curve generated in the economy from Example 1.[12]

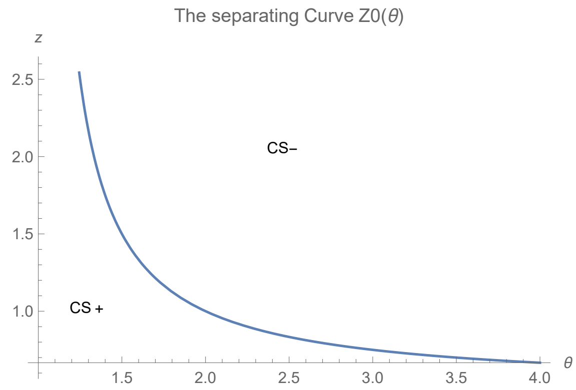


Figure 1: The separating curve

**Example 1** (**Guiding Example**). *Assume that every individual in the economy has two negatively and perfectly correlated characteristics: (i) labor market productivity $\theta$ to generate taxable income (measured in units of the output good) $z = \theta l$; (ii) a cost in terms of utility given by $\chi h(l)$, where $\chi$ represents a taste for labor parameter.*

*Individuals with high $\theta$ generate more income for a given level of labor supply. Individuals with high $\chi$ have higher disutility when supplying a given level $l$ of labor. Assuming $\theta$ and $\chi$ are negatively correlated in the following way: $\chi = \chi(\theta)$ the function $v(z, \theta)$ takes the form:*

$$v(z, \theta) = \chi(\theta) h\left(\frac{z}{\theta}\right) \tag{5}$$

*assuming that $\chi(\theta) = \theta - K$ and $h(\cdot) = \exp(\cdot)$, where $K$ is a constant, we will have the failure of SMC as in Assumption 1. The parameter $\theta$ creates a tension between productivity and tastes for labor. See Example A4 in Appendix A for more details.*

In the region to the left of the separating curve the "laziness" effect dominates the "productivity" effect and in the region to the right the opposite is true. Therefore, there is no clear ordering of the marginal utility (or MRS) according to type $\theta$. In particular, as one can see in Figure 2, indifference curves for different types may cross twice.

In the next section we explore the consequences for optimal tax theory of replacing SMC by Assumption 3.

[12]In Appendix A this example is identified by Example A4.

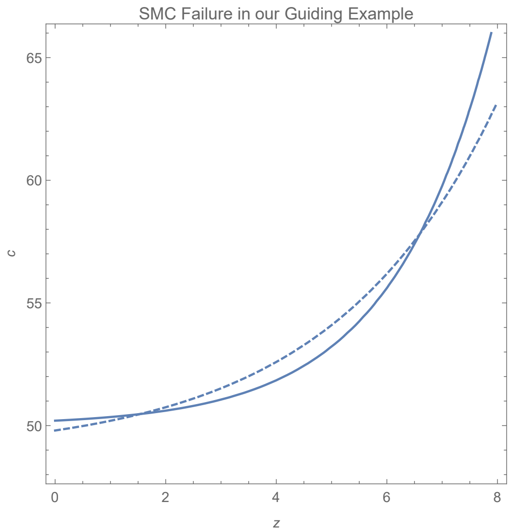


Figure 2: A pair of indifference curves for the economy in Example 1

# 3 The failure of SMC and its implications

Typically taxation models include assumptions which discipline the relationship between marginal rate of substitution and types. In more broad contexts of mechanism design and signaling games, they are known as the Spence-Mirrlees (or single-crossing) condition. They state that the marginal rate of substitution between consumption and taxable income is decreasing in the type. In our model the marginal rate of substitution between consumption and income is given by

$$\left.\frac{dc}{dz}\right|_{V(c,z,\theta)=cte} = v_z(z,\theta). \tag{6}$$

Therefore, the utility function $V(c,z,\theta)$ satisfies the usual Spence-Mirrlees condition if $v_{z\theta} < 0$, for all $z \in Z$ and $\theta \in \Theta$. Graphically, in a diagram $Z \times C$, the indifference curves of individuals with lower types should be stepper (when $v_{z\theta} < 0$ ) and cross each other at most once. The assumption below, first used in Araújo and Moreira (2010), relaxes the requirement that the utility function satisfies the SMC.[13]

**Assumption 4.** *The condition $v_{z\theta}(z,\theta) = 0$ defines implicitly a monotonic function $z_0 : \Theta \to Z$ such that $v_{z\theta}(z,\theta) > 0$ for $z < z_0(\theta)$ and $v_{z\theta}(z,\theta) < 0$ for $z > z_0(\theta)$. For simplicity, assume additionally that $v_{zz\theta} > 0$.*[14]

The function $z_0(\theta)$, henceforth referred as a separating curve, splits the space

[13]This assumption was also used in Schottmüller (2015), Choné and Gauthier (2017) and Araujo, Moreira, and Vieira (2015).

[14]The assumption $v_{zz\theta} > 0$ is not crucial but will be convenient to guarantee concavity of government objective function.

$\Theta \times Z$ into two regions where the signal of the cross-derivative remains constant. Let us define $CS_- = \{(\theta, z) \in \Theta \times Z : v_{z\theta}(z,\theta) < 0\}$ and $CS_+ = \{(\theta, z) \in \Theta \times Z : v_{z\theta}(z,\theta) > 0\}$ these regions. The function $z_0(\theta)$ can be increasing or decreasing depending on the application. See Appendix A for examples of economies generating this particular type of failure of the SMC. Figure 1 plots the separating curve generated in the economy from Example 1.[15]

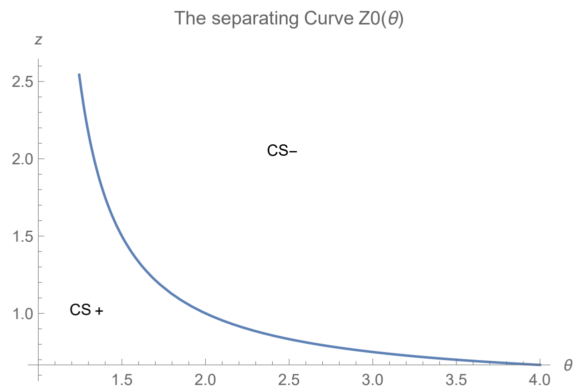


Figure 3: The separating curve

**Example 2 (Guiding Example).** *Assume that every individual in the economy has two negatively and perfectly correlated characteristics: (i) labor market productivity $\theta$ to generate taxable income (measured in units of the output good) $z = \theta l$; (ii) a cost in terms of utility given by $\chi h(l)$, where $\chi$ represents a taste for labor parameter.*

*Individuals with high $\theta$ generate more income for a given level of labor supply. Individuals with high $\chi$ have higher disutility when supplying a given level $l$ of labor. Assuming $\theta$ and $\chi$ are negatively correlated in the following way: $\chi = \chi(\theta)$ the function $v(z,\theta)$ takes the form:*

$$v(z,\theta) = \chi(\theta) h\left(\frac{z}{\theta}\right) \tag{7}$$

*assuming that $\chi(\theta) = \theta - K$ and $h(\cdot) = \exp(\cdot)$, where $K$ is a constant, we will have the failure of SMC as in Assumption 1. The parameter $\theta$ creates a tension between productivity and tastes for labor. See Example A4 in Appendix A for more details.*

In the region to the left of the separating curve the "laziness" effect dominates the "productivity" effect and in the region to the right the opposite is true. Therefore, there is no clear ordering of the marginal utility (or MRS) according to type $\theta$. In

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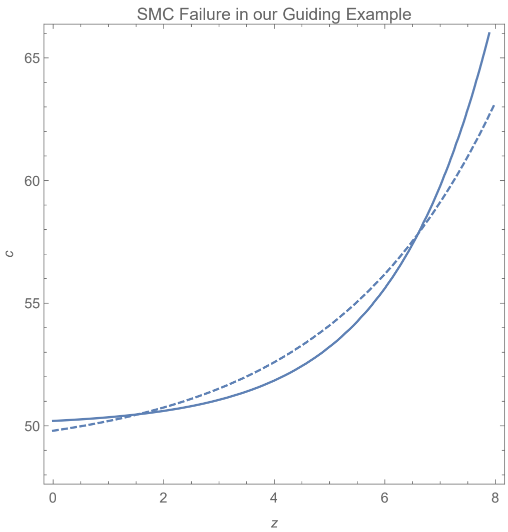



Figure 4: A pair of indifference curves for the economy in Example 1

The natural ordering created by the SMC is an important tool to reduce the complexity of IC constraints that had to be considered. Indeed, when the SMC is satisfied the set of implementable allocations is fully characterized by local conditions. These local incentive compatibility constraints are the usual ones found in the literature, summarized in the following lemma (the necessity of these conditions does not depend on the validity of the SMC).

Let $V : \Theta \to \mathbb{R}_+$ be the informational rent function of the agent in an incentive-feasible mechanism $z, c : \Theta \to Z \times \mathbb{R}_+$. Hence, $V(\theta) = c(\theta) - v(z(\theta), \theta)$.

**Lemma 1.** *Suppose that $z : \Theta \to Z$ is part of a bounded incentive-feasible allocation $z, c : \Theta \to Z \times \mathbb{R}_+$. Then, the following are* ***necessary local conditions for implementability****.*

*(i)* ***(Envelope)*** *The agent's informational rent function, is given by*

$$V(\theta) = V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(z(s), s) ds, \tag{8}$$

*for all $\theta \in \Theta$;*

*(ii)* ***(Monotonicity)*** *$z : \Theta \to Z$ is non-decreasing in the region $CS_-$ and non-increasing in the region $CS_+$.*

*Proof.* See the Appendix. □

Condition (i) of Lemma 1 guarantees payoff equivalence across incentive-feasible mechanisms, and is equivalent to the first-order condition of agents' revelation problems. Condition (ii) is related to the local second-order necessary conditions of these problems. We can recover consumption from the definition of informational rent: $c(\theta) = V(\theta) + v(z(\theta), \theta)$.

When the SMC is imposed, at any allocation, the MRS for different types are ordered in the same way. Under Assumption 1, in contrast, the ordering of MRS between any two types will in general depend on the specific allocation one is considering – see Figure ??. It is, then, natural to ask whether other non-local necessary conditions for implementability must be taken into account since, without a natural ordering, it is possible to have different and "distant" agents choosing the exact same allocation. In this case, a whole new set of global incentive compatibility constraints must be imposed. The necessary condition described in the following proposition is derived from these global incentive compatibility constraints.

**Lemma 2 (Discrete pooling condition).** *Let $z : \Theta \to Z$ be an implementable allocation and let $T(z)$ be the tax schedule that implements it (as in Proposition 1). Suppose that $z : \Theta \to Z$ pools two different types at $z \in Z$ (i.e. $z = z(\theta) = z(\hat{\theta})$) a income level where $T : Z \to \mathbb{R}$ is differentiable. Then, we must have*

$$v_z(z, \theta) = v_z(z, \hat{\theta}). \tag{9}$$

*Proof.* Let $z : \Theta \to Z$ be an allocation. We know from Proposition 1 (taxation principle) that there exists a tax schedule $T : Z \to \mathbb{R}$ such that the problem of the individual is

$$z(\theta) \in \arg\max_{z \in Z(\Theta)} V(z - T(z), z, \theta). \tag{10}$$

Given the differentiability of the tax schedule, the optimum can be characterized by the first-order condition at $z(\theta)$ that is given by

$$1 - T'(z(\theta)) = v_z(z(\theta), \theta). \tag{11}$$

and analogously for $z(\hat{\theta})$.

Since $z(\theta) = z(\hat{\theta}) = z$,

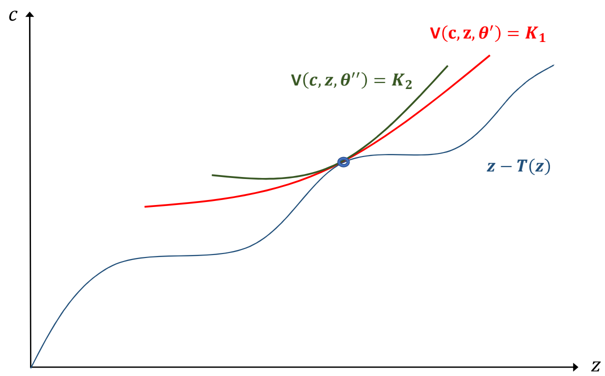
$$v_z(z, \theta) = 1 - T'(z(\theta)) = 1 - T'(z(\hat{\theta})) = v_z(z, \hat{\theta}). \tag{12}$$

□

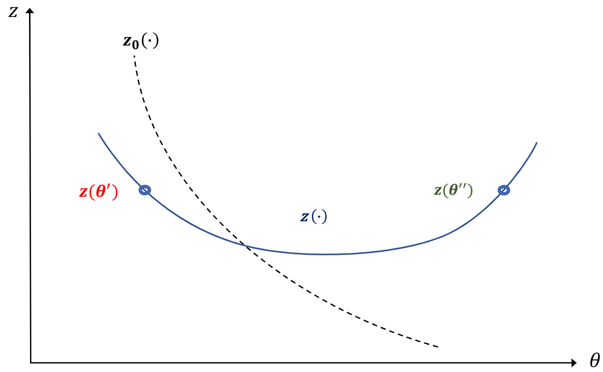
The condition in Lemma 2, henceforth denoted *discrete pooling*, is a direct consequence of the taxation principle and requires that discretely pooled types obtain face the same marginal tax rate despite having 'very different' types.

It is important to distinguish discrete pooling from continuous pooling where all types in a neighborhood receive the same allocation. Continuous pooling usually happens when the monotonicity constraint (as in Lemma 1 (ii)) is binding and an ironing procedure is necessary to guarantee incentive-compatibility.

Figures 5a and 5b show how discrete pooling occurs in the space of indirect and direct mechanisms, respectively. Here we should note that under SMC it is impossible to have this kind of discrete pooling behavior since very different agents will have very different slopes for their indifference curves at any given $(c, z)$.




(a) Indirect Mechanism



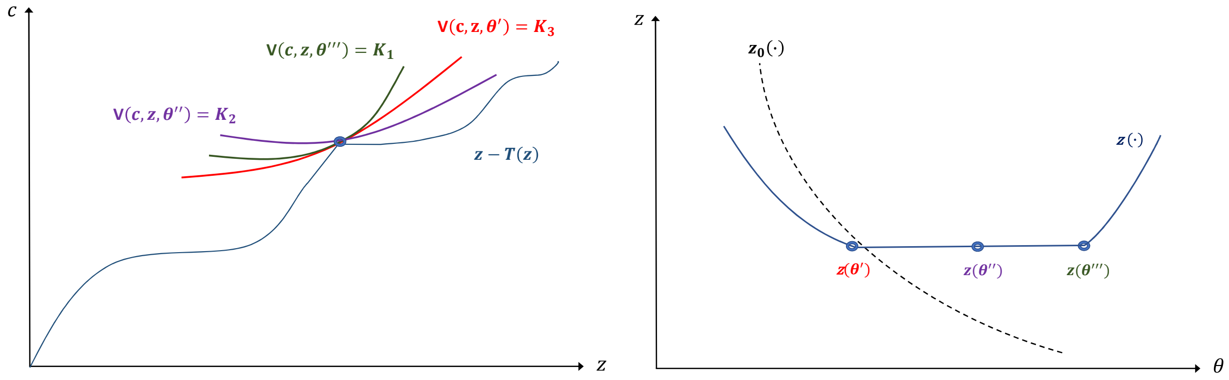

(b) Direct Mechanism

Figure 5: Discrete Pooling

Figures 6a and 6b show how continuous pooling occurs in the space of indirect and direct mechanisms, respectively. Continuous pooling shows up as a flat region in the direct mechanism and a kink in the tax schedule in the indirect mechanism. The kink induces different but close agents to make the same choice even though they do not have the same MRS at their choice. At the kink, the marginal tax rate is not defined, but only the right and left derivatives are defined. Therefore, if type $\theta$ is choosing at the kink (i.e., his/her MRS is in between the right and the left derivatives) all types in a neighborhood will do as well. Note that types $\theta'$ and $\theta''$ are continuously and discretely pooled.

Let $z : \Theta \to Z$ be an arbitrary allocation and define $\psi_b, \psi_s : Z \mapsto \Theta$, the *pseudo-inverse of* $z$, as $\psi_b(z) = \inf\{\theta \in \Theta; z(\theta) \leq z\}$ and $\psi_s(z) = \sup\{\theta \in \Theta; z(\theta) \leq z\}$. Additionally, whenever $\{\theta \in \Theta; z(\theta) \leq z\} = \emptyset$ define $\psi_b(z) = \underline{\theta}$ and $\psi_s(z) = \overline{\theta}$.

**Proposition 2** **(Global IC).** *Let* $z : \Theta \to Z$ *be a bounded allocation and* $\psi_b, \psi_s : Z \to \Theta$ *the pseudo-inverse functions of* $z$*. Therefore, for all* $\theta, \hat{\theta} \in \Theta$*,* $z : \Theta \to Z$ *is*




(a) Indirect Mechanism

(b) Direct Mechanism

Figure 6: Continuous Pooling

*incentive compatible if and only if*

$$\int_{z(\hat{\theta})}^{z(\theta)} [v_z(z, \psi_s(z)) - v_z(z, \psi_b(z))]\, dz \geq 0. \tag{13}$$

*Proof.* Define the *Global Incentive Function* (GIF) as

$$\Phi(\theta, \hat{\theta}; z(\cdot)) := \int_{\hat{\theta}}^{\theta} \left[ \int_{z(\hat{\theta})}^{z(s)} v_{z\theta}(t, s) dt \right] ds. \tag{14}$$

An allocation $z : \Theta \to Z$ is incentive compatible if and only if for all $\theta, \hat{\theta} \in \Theta$, $\Phi(\theta, \hat{\theta}; z(\cdot)) \geq 0$ – see Appendix B.0.1.

Note that

$$\begin{aligned} 0 \leq \Phi(\theta, \hat{\theta}; z(\cdot)) = \int_{\hat{\theta}}^{\theta} \left[ \int_{z(\hat{\theta})}^{z(s)} v_{z\theta}(t, s) dt \right] ds = \\ \int_{z(\hat{\theta})}^{z(\theta)} \left[ \int_{\psi_b(z)}^{\psi_s(z)} v_{z\theta}(t, s) ds \right] dt \\ = \int_{z(\hat{\theta})}^{z(\theta)} [v_z(t, \psi_s(t)) - v_z(t, \psi_b(t))]\, dt, \end{aligned} \tag{15}$$

where the second equality comes from Fubini's theorem and the last from the fundamental theorem of calculus. Change of variables gets the result. □

The next corollary describes a particular global incentive-compatibility constraint that will be shown to be very relevant. It gives the conditions for which a pivotal type $\theta_d$ does not envy the type in the upper bound of type space when the allocation features a U-shape format.

**Corollary 1.** *Let $z : \Theta \to Z$ be a bounded, U-shaped incentive-compatible allocation. Therefore, an arbitrary type $\theta_d \in \Theta$ does not envy $\overline{\theta}$ if*

$$\int_{z_l}^{z_h} [v_z(z, \psi_s(z)) - v_z(z, \theta_d)] \, dz = 0, \tag{16}$$

*where $\psi_b, \psi_s : Z \to \Theta$ the pseudo-inverse functions of $z$, $z_l = z(\theta_d)$ and $z_h = \inf\{z \in Z : \psi_s(z) = \overline{\theta}\}$.*

*Proof.* Note that for any $z \in Z$ such that $\psi_b(z) \neq \psi_s(z)$ $(z \in [\underline{z}, z_l))$ we have discrete pooling and consequently $v_z(z, \psi_s(z)) - v_z(z, \psi_b(z)) = 0$ by Proposition 2 since $z : \Theta \to Z$ is incentive-compatible. □

# 4 Optimal Taxation

## 4.1 Government

In a quasi-linear environment, the government needs a reason to redistribute income across households. We assume that the government maximizes a weighted Utilitarian Social Welfare Function.[16] More specifically, the government wants to maximize a weighted average of individuals' utilities where the weights of types are given by the function $g : \Theta \to \mathbb{R}_+$, which we assume to be non-negative and to integrate to 1. If $G(\theta) = \int_{\underline{\theta}}^{\theta} g(a)da$ first-order stochastically dominates $F$, the government puts higher weight on individuals' utility with lower $\theta$.

The government's problem in the structural mechanism design formulation is to choose an incentive-feasible allocation $c, z : \Theta \to \mathbb{R}_+ \times Z$ to maximize the welfare criterion. We can incorporate the local incentive-compatibility and budget constraints in the objective function and rewrite the planner's mechanism design problem as follows:[17]

$$\max_{z:\Theta \to Z} \int_{\underline{\theta}}^{\overline{\theta}} W(z(\theta), \theta) d\theta, \tag{17}$$

subject to the global incentive constraints defined in equations (9) and (13) from Propositions ?? and 2.

[16]Most papers in optimal taxation assumes the government to follow a unweighted utilitarian or a Rawlsian social welfare criterion. Under the former criterion, only it is the difference between marginal utility of consumption across households due to the concavity of the utility function which generates a desire to redistribute income. However, with quasi-linear utility the marginal utility of consumption is constant and equal to 1 for all individuals and a direct motive for redistribution must be created.

[17]See Appendix C for a detailed derivation.

The social welfare function augmented of incentive-feasibility constraints is given by the function $W : Z \times \Theta \to \mathbb{R}$ defined as

$$W(z,\theta) = \left[ z - v(z,\theta) + v_\theta(z,\theta) \frac{G(\theta) - F(\theta)}{f(\theta)} \right] f(\theta). \tag{18}$$

Let us define some relaxed sub-problems that will help us build the proposed mechanism. All these solutions emerge naturally in our approach based in type assignment functions.[18]

**Relaxed Solution** Let $z^R : \Theta \to Z$ denote the solution of the relaxed problem where all global-incentive compatibility constraints are ignored. Note that this relaxed problem can be solved pointwise with the following Euler equation given by

$$W_z(z^R(\theta), \theta) = 0. \tag{19}$$

For future reference define $T_R : Z \to \mathbb{R}$ the tax schedule that would be implied by this allocation, i.e., $T_R(z(\theta)) = z_R(\theta) - c_R(\theta)$. It is important to note that this tax schedule not necessarily implements this allocation. Indeed, it will not when the allocation is not incentive compatible.

**Discrete Pooling Solution** Let $z^{DP} : \Theta \to Z$ denote the solution of the government problem where the discrete pooling condition as defined in equation (9) and monotonicity constraints as defined in Lemma 1 (ii) are considered. $z^{DP}$ describes the solution when two "very different" types are discretely pooled. The Euler equation for this problem when types $\theta, \hat{\theta}$ are pooled at income level $z$ (i.e., $z = z^{DP}(\hat{\theta}) = z^{DP}(\theta)$) is given by:

$$\frac{W_z(z,\theta)}{v_{z\theta}(z,\theta)} f(\theta) = \frac{W_z(z,\hat{\theta})}{v_{z\theta}(z,\hat{\theta})} f(\hat{\theta}). \tag{20}$$

The mechanism proposed by Araújo and Moreira (2010) has the solutions of these two subproblems as its elements connected through a vertical ironing procedure.[19]

For future reference define $T_{DP} : Z \to \mathbb{R}$ the tax schedule that would be implied by this allocation, i.e., $T_{DP}(z(\theta)) = z_{DP}(\theta) - c_{DP}(\theta)$.

[18] We refer the reader who may be interested in the details of these relaxed problems to Araújo and Moreira (2010) and Araujo, Moreira, and Vieira (2015).

[19] The vertical ironing procedure creates additional distortions on the mechanism to guarantee incentive-compatibility for the vertically pooled types.

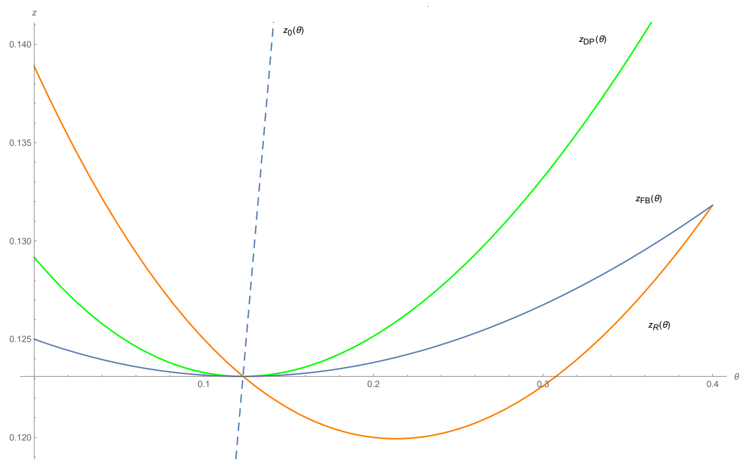



Figure 7: Numerical Solutions for Example A5

**Isoperimetric Solution** The isoperimetric problem, as proposed by Araujo, Moreira, and Vieira (2015), has these same elements but instead of connecting the solutions $z^R$ and $z^{DP}$ through vertical ironing, it allows for discontinuity of the mechanism at a pivotal type $\theta_d$. The introduction of the discontinuity requires us to take into account an additional relevant global IC - that the pivotal type does not envy the allocation prescribed to the highest type as characterized in Corollary 1.[20]

The Euler equation for this iso-perimetric problem when the GIC is binding is given by

$$W_z(z, \psi_s(z)) + \delta v_{z\theta}(z, \psi_s(z)) = 0, \tag{21}$$

where $\delta$ is the Lagrange multiplier associated with the binding GIC.

To illustrate these objects, Figure 7 plots the numerical solutions of these relaxed problems for the economy from Example A5 (see Appendix A).

## 4.2 Variational Approach

The literature has used variational techniques to characterize optimal tax systems since Dahlby (1998), Piketty (1997) and Saez (2001). However, we claim that to use this as a methodology in the task in economic environments other than the traditional Mirrlesian economies a more rigorous assessment of the underlying conditions that justify this method is needed.[21] The first step is to give a definition that is consistent with what has been (loosely) done previously.

[20]For $z \in [z_l, z_h]$ as defined in Corollary 1.

[21]These papers use these variational techniques to write the traditional formula for the marginal income tax rate first derived in Mirrlees (1971) in terms of empirically observable objects.

**Definition 2.** *We define as variational approach the technique to find optima using perturbations within the class of tax schedules in which agents decision can be fully characterized by local conditions (as in agents FONC eq (XXX)).*

This is equivalent to assume that agents demand is always fully determined by the local behavior of the budget set around the choices, and we can work with linearized versions of these sets. Another equivalency, is the assumption that at the optimal an envelope conditions is always met, regardless the characteristics of the baseline mechanism that is being reformed.

Two things are crucial to be noted. First these conditions are imposed on endogenous objects that can never be checked if satisfied or not. Second, these assumptions intrinsically imposes conditions that have to be met by any admissible reforms. Therefore the usage of VA may restrict the search for the optimal in a strict subclass of the a priori available mechanisms, generating then welfare losses.

We argued in the rest of these paper that there are situations where this will be restrictive. The crucial element for it is the existence of binding global incentive compatibility constraints between agents choosing different levels of income/consumption pairs.[22] The literature so far always impose (Explicitly or not) conditions to avoid this possibility. In other words, the VA intrinsically assume that the welfare impact of a reform that canges the tax liability at an income level $z$, is confined to those agent choosing to have income at that same level in the baseline tax system.

Several different assumptions can impose conditions strong enough to guarantee the validity of the variational approach. The literature has not agreed on which conditions would be the "natural" one. For instance Saez (XXX) assumes that the resulting tax schedule is a strictly convex function, Golosov, Tsyvinski, and Werquin (2014) assumes that agents decisions are Lipschitz-continuous with respect to the endogenous tax system, another possibility would be assume that agents decisions (for a given tax system) are singleton. All these assumptions are examples of constraints that guarantees the validity of VA.

We argued that the interpretation of VA as a restriction on the space of admissible mechanism as a natural one since it is more transparent and easier to interpret his economic meaning in terms of incentive constraints. The next proposition describes an important characteristic of mechanisms consistent with the VA.

**Proposition 3.** *The resulting tax system of imposing the variational approach*

[22] Saez (XXX) acknowledge the possibility of binding global incentive constraints. He avoided the complications generated by that by assuming that it would happen at most in a zero measure set of individuals.

*presents convex values.*

*Proof.* ... $\square$

Let's consider as an admissible reform at income level $z \in Z$ a function $h : Z \to \mathbb{R}$, $h \in \mathscr{C}^2$, such that $h(z) = 0$ in $z \in Z/(z - \varepsilon, z + \epsilon)$. Figure XXX illustrates such reforms. These perturbations reforms the tax system only in a neighborhood of the income level $z$, this will be convenient to make clear the impact of this reform in agents that are not choosing income exactly at level $z$. The resulting tax schedule is $T(z) + h(z)$ leaving after tax income to be consumed by the agent that choose income level $z$ to be $c = z - T(z) - h(z)$.

Recall that $z_\theta(T)$ represents the optimal choice (functional) of taxable income of a type $\theta$ individual when facing a tax schedule $T : Z \to \mathbb{R}$, and $\mathscr{V}_\theta(T)$ the indirect utility of this problem. We can write, the objective of government as

$$\tilde{\mathscr{W}}(T) = \frac{1}{\beta}\int_\Theta \mathscr{V}_\theta(T)g(\theta)d\theta + \int_\Theta T(z_\theta(T))f(\theta)d\theta \tag{22}$$

where $\beta$ is the shadow value of public funds, hence this objective is measure in monetary units.

$$g_z(z) = \int\limits_{\{\theta\in\Theta : z_\theta(T)=z\}} g(\theta)d\theta, \quad \text{and} \quad f_z(z) = \int\limits_{\{\theta\in\Theta : z_\theta(T)=z\}} f(\theta)d\theta \tag{23}$$

Now we can change measures to write everything in in terms of income, to get this new objective

$$\tilde{\mathscr{W}}(T) = \frac{1}{\beta}\int_{z\in Z} \mathscr{V}_\theta(T)g_z(z)dz + \int_{z\in Z} T(z)f_z(z)dz \tag{24}$$

From the equation above it is worth noting that the distribution of income and welfare weights are endogenous to the tax schedule.

The welfare impact of a tax change in the direction of a reform $h : Z \to \mathbb{R}$ is calculated by the following Gateaux derivative[23]

$$d\tilde{\mathscr{W}}(T,h) = \frac{1}{\beta}\int_{z\in Z} d\mathscr{V}_\theta(T,h)g(z)dz + \int_{z\in Z} T'(z)dz(T,h)f(z)dz. \tag{25}$$

We shall denote the first term welfare effect and the second revenue effect.

$$\frac{1}{\beta}\int_{z\in Z} d\mathscr{V}_\theta(T,h)g(z)dz \tag{26}$$

[23]For convenience let's consider twice continuously differentiable reforms $h \in \mathscr{C}^2$

and a budget (efficiency) effect

$$\int_{z\in Z} T'(z)dz(T,h)f(z)dz. \tag{27}$$

### 4.3 An approach to the optimal taxation problem based on type assignment functions

The Variational Approach works in the space of indirect mechanisms, where the government proposes a tax schedule and lets individuals self-select. Although it is easier to get first-order conditions using this technique, it is hard to have a clear sense of all objects involved. In particular, it is hard to guarantee sufficiency of these first-order constraints to derive formulae for the optimal taxation.

On the other hand, the traditional mechanism design approach is very clear about the role of each structure but it is also extremely difficult to apply when the characteristics of the problem deviate from the standard model. Our suggestion is to work with the type assignment functions as proposed by Noldeke and Samuelson (2007). We can think of type assignment functions as the pseudo-inverses of a direct mechanism.[24]

The main advantage of using this approach relies on the fact that global incentive constraints are naturally described in terms of income levels instead of types. For the local IC constraints, we have the opposite. They are naturally expressed in terms of types (recall Lemma 1); by using this approach, we take the best characteristics of both.

The first step to implement this approach is to re-write the objective function of the government in equation (17) as follows:

$$\begin{aligned}\int_{\underline{\theta}}^{\overline{\theta}} W(z(\theta),\theta)d\theta &= \int_{\underline{\theta}}^{\overline{\theta}} \left(\int_{\underline{z}}^{z(\theta)} W_z(z,\theta)dz\right)d\theta + \int_{\underline{\theta}}^{\overline{\theta}} W(\underline{z},\theta)d\theta \\ &= \int_{\underline{z}}^{\overline{z}} \left[\mathscr{W}(z,\psi_s(z)) - \mathscr{W}(z,\psi_b(z))\right]dz + \int_{\underline{\theta}}^{\overline{\theta}} W(\underline{z},\theta)d\theta, \end{aligned} \tag{28}$$

where $\mathscr{W} : Z \times \Theta \to \mathbb{R}$ is given by

$$\mathscr{W}(z,\theta) = \int_{\underline{\theta}}^{\theta} W_z(z,s)ds \tag{29}$$

[24]Without SMC, it is common to have non-monotone solutions and the inverse function of any such mechanism is not well defined. However, the pseudo-inverses (to the right and to the left) are well defined.

and $\psi_b(\cdot)$ and $\psi_s(\cdot)$ are the type assignment functions. The first equality follows from the Fundamental Theorem of Calculus and the second from Fubini's Theorem.

Therefore we can rewrite government's optimal taxation problem in terms of type-assignment functions as choosing $\psi_b, \psi_s : Z \to \Theta$ to maximize the functional in Equation 28 subject to all global incentive constraints. Formally, the problem is

$$\max_{\psi_b(\cdot),\psi_s(\cdot)} \int_{\underline{z}}^{\overline{z}} [\mathscr{W}(z, \psi_s(z)) - \mathscr{W}(z, \psi_b(z))]\, dz \tag{30}$$

subject to all global incentive constraints: for all $z_l, z_h \in Z$

$$\int_{z_l}^{z_h} [v_z(z, \psi_s(z)) - v_z(z, \psi_b(z))]\, dz \geq 0. \tag{31}$$

Note that the set of all global incentive constraints includes in particular: (i) discrete pooling constraints, $v_z(z, \psi_b(z)) - v_z(z, \psi_s(z)) = 0$ whenever $\underline{\theta} < \psi_b(\theta) < \psi_s(\theta) < \overline{\theta}$; (ii) monotonicity constraints: $\psi_b$ non-increasing and $\psi_s$ non-decreasing; and (iii) and the local incentive constraints in Lemma 1 (i).

This problem has several convenient properties. First, notice that this problem is entirely described in terms of income levels. The discrete pooling condition is a point-wise constraint and the global incentive constraint is a collection of isoperimetric constraints, which is very common in Calculus of Variations. As a consequence, this formulation disentangles the local IC of global ICs, by allowing us to treat them separately.

Numerical Simulations of Examples A4 and A5 display U-shaped solutions (i.e. $z : \Theta \to Z$ such that there exists $\theta_0 \in int(\Theta)$ with $z(\cdot)$ decreasing for all $\theta < \theta_0$ and $z(\cdot)$ increasing for all $\theta > \theta_0$ ). Inspired by this let's restrict the class of problems to be considered.[25]

**Assumption 5.** *Assume that the relaxed solution is U-shaped and cross the separating curve in its decreasing portion. Assume additionally that $z_R(\underline{\theta}) > z_R(\overline{\theta})$. Lastly, assume that the solutions are uniquely determined.*

Figure 5 illustrates the geometry of our class of problems. The typical geometry has $z_{DP}(\theta) \in [z_0(\theta), z_R(\theta)]$,[26] $z_R(\overline{\theta})$ is equal to the first best value. Te same thing happens in the point where the solutions meet, i.e., $z_R(\theta) = z_{DP}(\theta) = z_0(\theta)$. Therefore, potentially we may have the usual non-distortion at the top result as well as non-distortion at the middle. We will discuss this in greater depth later.

[25]Schottmüller (2015) study a class of monotone solutions where global incentive constraints are binding even though no discrete pooling happens.

[26]With some abuse of notation, we can change the limits of the interval whenever $z_0(\theta) > z_R(\theta)$.

We believe that this class of solutions comprises many interesting applications in public economics and provides a well-behaved solution that has very intuitive and interesting characteristics. In particular, it reduces the set of global ICs to be considered.

### 4.4 A solution featuring segregation

We propose a taxation schedule where the government segments individuals in two groups: low and high income.[27] The intuition for doing such a policy is as follows. By dividing into two groups, the government can further explore the trade-off between equity and efficiency within each group. The only additional difficulty is that the government should guarantee that individuals do not envy the allocations of the other group. These are the global incentive conditions that may be binding, thus helping to broaden the design of tax systems. In particular, when the conditions are binding, the underlying assumptions for the use of variational methods are violated.

Let $\theta_d \in [\underline{\theta}, \overline{\theta}]$ be a pivotal type whose behavior will determine the segregation of groups. Before going into details, let's motivate the solution in some steps. First consider the problem

$$\max_{\psi_b(\cdot), \psi_s(\cdot)} \int_{\underline{z}}^{\overline{z}} \left[ \mathscr{W}(z, \psi_s(z)) - \mathscr{W}(z, \psi_b(z)) \right] dz \tag{32}$$

subject to:

(i) $\left[\overline{\theta} - \psi_s(z)\right] \left[v_z(z, \psi_b(z)) - v_z(z, \psi_s(z))\right] \leq 0$;[28]

(ii) $\left[\psi_b(z) - \underline{\theta}\right] \left[v_z(z, \psi_b(z)) - v_z(z, \psi_s(z))\right] \geq 0$;

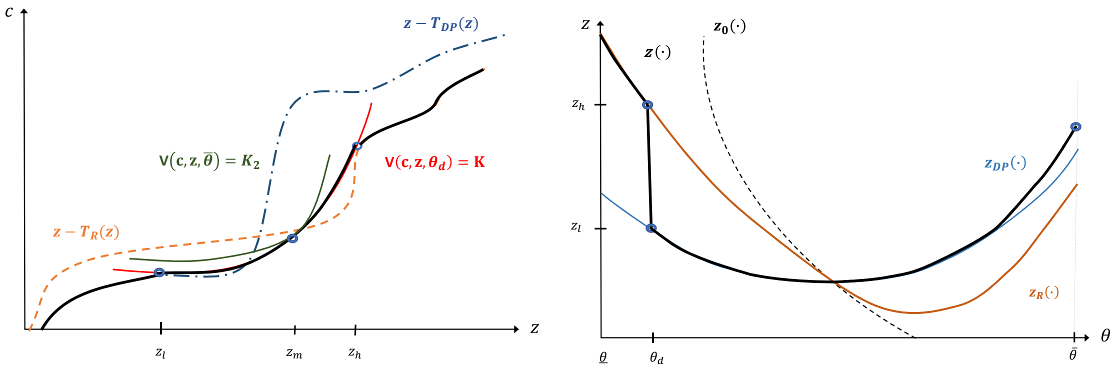
(iii) $\psi_b$ and $\psi_s$ monotonous.

In this exposition we will assume that the only relevant constraint is the condition (i). This problem is the type assignment formulation of the problem considered in Araújo and Moreira (2010). It involves parts of the relaxed solution $z_R(\cdot)$ and the discrete pooling solution $z_{DP}(\cdot)$. The transition between groups is done through a vertical ironing procedure at an endogenously determined pivotal type $\theta_d$. This ironing procedure means that the government commits to offer any income level in the interval $[z(\theta_d^-), z(\theta_d^+)]$. Figures 8a and 8b illustrate the vertical ironing procedure.

**Remark 1.** *The correspondence solution in this problem is convex valued and, therefore, consistent with the variational method.*

[27] The segregation in groups was inspired by Araujo, Moreira, and Vieira (2015).

[28] This is a compact way of writing the condition $v_z(z, \psi_b(z)) - v_z(z, \psi_s(z)) \leq 0$ being valid whenever $\psi_s(z) < \overline{\theta}$.




(a) Indirect Mechanism (b) Direct Mechanism

Figure 8: Vertical Ironing as in Araújo and Moreira (2010)

It turns out that this requirement is very restrictive since several types in the upper part of the type space are discretely pooled with the connected segment. Hence, an additional distortion should be implemented to guarantee incentive compatibility. In other words, the discrete pooling IC becomes binding more often than necessary.

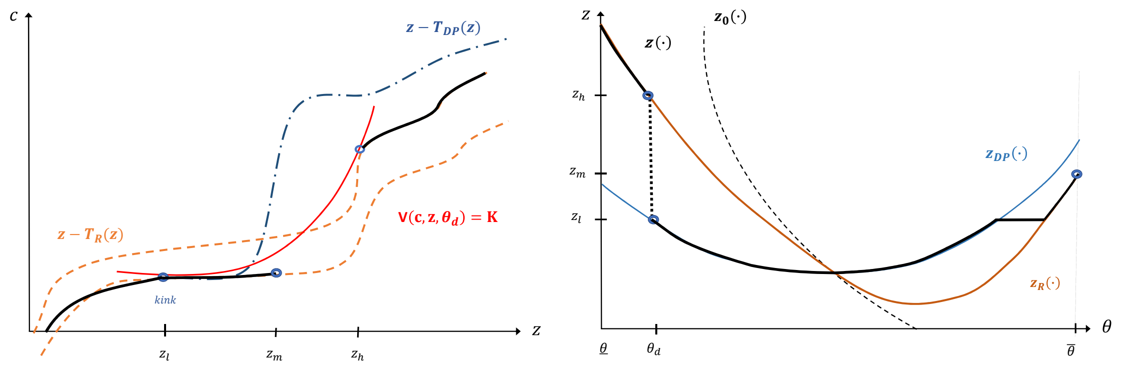
In fact, we can do better. Let us construct our solution in three intuitive (and entertaining) steps. Let $\theta_d$ be the pivotal type where the vertical ironing process occurs. At this point, we can think of this pivotal type as endogenously determined in the problem of (30). Later, this type will parametrize a class of solutions. Define $z_h = z(\theta^-)$ and $z_l = z(\theta^+)$ the right and left limits of the allocation at this pivotal type.

**Step 1** *Jump allows us to return to the relaxed solution benefiting from a lower level of distortion.*

Instead of offering all level of income $z \in [z(\theta_d^-), z(\theta_d^+)]$, the government could offer a discontinuous allocation at $\theta_d$. In this case, the set of agents on the right part of the type space would not be discretely pooled anymore. Hence, the allocation could return to the less distorted relaxed solution. The transition between these solutions is done through continuous pooling to guarantee monotonicity, creating a kink in the tax schedule. Define $z_m = z_R(\overline{\theta})$. This is clearly a welfare improvement since it returns to the less distorted relaxed solution (recall that the relaxed problem is a more constrained problem).

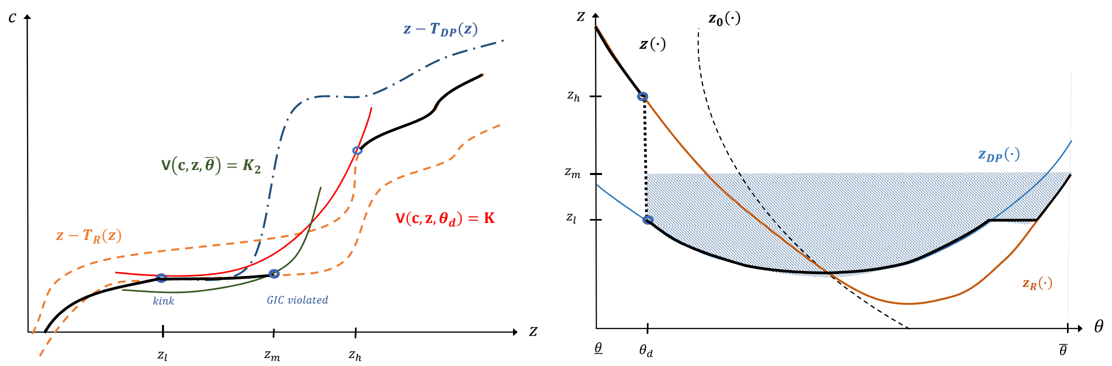
**Step 2** *A global incentive compatibility constraint may be violated.*

In the following Lemma 3 we present the set of potential global incentive com-




(a) Indirect Mechanism (b) Direct Mechanism

Figure 9: Step 1




(a) Indirect Mechanism (b) Direct Mechanism

Figure 10: Step 2

patibility constraints that may be binding. Indeed, the only global constraints that matter are the ones where the pivotal type $\theta_d$ is indifferent between the lowest allocation on the high-income group $z_h$ and the highest allocation in the low-income group $z_l$; and that the highest type $\overline{\theta}$ does not envy the highest allocation at the low-income group $z_l$.

**Lemma 3.** *Under the conditions of Assumption 5 the set of potentially binding global incentive-compatibility constraints reduce only to the following two:*

$$\int_{z_l}^{z_h} \left[v_z(z, \psi_s(z)) - v_z(z, \theta_d)\right] dz = 0 \tag{33}$$

*and*

$$\int_{z_l}^{z_m} \left[v_z(z, \psi_b(z)) - v_z(z, \overline{\theta})\right] dz = 0. \tag{34}$$

*These constraints represent the marginal types that are more willing to get the al-*

*location in the other group.*

These constraints are conflicting i.e., if one is binding the other one will necessary be slack. Therefore, depending on the problem, only one may be binding. For the sake of simplicity, assume that the global IC, given by equation (34), is the one that can potentially be binding.

The shaded area in Figure 8 marks region where this GIC is integrated over. If the area weighted by $v_{z\theta}$ is bigger to the right side of the separating curve, the allocation will not be incentive-compatible.

**Remark 2.** *If both GICs are slack, the solution in the previous step cannot be improved, i.e., it is the optimal in this class.*

**Step 3** *Distort allocations to restore incentive compatibility.*

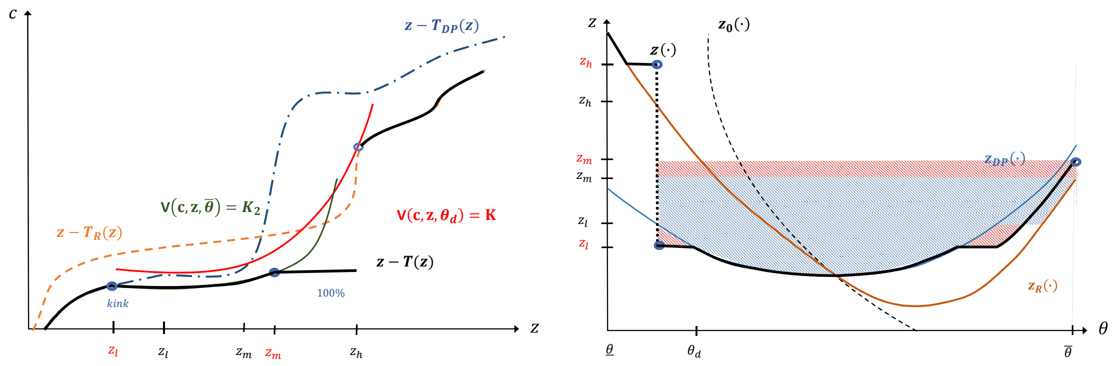
The last step requires distorting the allocation using $z_m, z_h, and z_l$ as margins to restore the GIC. By construction, we have a class of solutions parametrized by the pivotal type. The proposed solution in this paper is to maximize over the parameter $\theta_d$, henceforth referred to as the segregation mechanism. In Appendix C we present the mathematical formulation of the optimization program.

Whenever $\theta_d \in int(\Theta)$ the solution features segregation in low and high income groups. The extreme cases $\theta_d = \underline{\theta}$ and $\theta_d = \overline{\theta}$ are the solutions proposed in Theorem 3 of Araújo and Moreira (2010) (see Figure 4 in the page 1125 for the intuition), which are consistent with the variational method.

As we can see in Figure 11a and 11b the segregation mechanism creates an extra distortion on high types to balance the impact of the binding GIC. Another unusual characteristic is the 100% marginal tax rate at the optimum. This is a empirical phenomenon that do not have any rationalization based on optimality in the taxation literature. At the best of our knowledge, we provided the first theory of why should a government implement 100% marginal tax rate in a optimal tax schedule.

**Remark 3.** *By construction, the monotonicity constraint of $\psi_b$ and $\psi_s$ being increasing and decreasing, respectively, is satisfied.*

**Remark 4.** *If there exists $z \in [\underline{z}, \overline{z}]$ such that $\psi_b^+(z) = \psi_b^-(z) \equiv \theta_d$, the optimal allocation does not present jump. Therefore, the solution in Araújo and Moreira (2010) is a degenerate case of our proposed solution.*




(a) Indirect Mechanism

(b) Direct Mechanism

Figure 11: Step 3

## 4.5 Necessary conditions

The next propositions characterize the optimal marginal tax rate with respect to the type assignment functions.

**Proposition 4.** *Let $\psi_b, \psi_s : Z \to \Theta$ the type assignment functions (correspondences) that solves the segregation mechanism for a given $\theta_d$. The first-order necessary condition for the optimal marginal tax rate is given by:*

*(i) for $z \in (z_h, \overline{z}]$, $\psi_s(z) = \theta_d$ and $\psi_b(z)$ satisfies*

$$T'(z) = 1 - v_z(z, \psi_b(z)) = \left[\frac{G(\psi_b(z)) - F(\psi_b(z))}{f(\psi_b(z))}\right] v_{z\theta}(z, \psi_b(z)); \qquad (35)$$

*(ii) for $z \in (z_m, z_h)$, $\psi_s(z) = \psi_b(z) = \emptyset$ and we have 100% marginal tax rate. This is indeed a sufficient but not necessary condition;*

$$T'(z) = 1; \qquad (36)$$

*(iii) for $z \in [\underline{z}, z_l)$, we have discrete pooling and, $\psi_s(z)$ and $\psi_b(z)$ are defined by*

$$\left[1 - v_z(z, \psi_b(z)) - \left(\frac{G(\psi_b(z)) - F(\psi_b(z))}{f(\psi_b(z))}\right) v_{z\theta}(z, \psi_b(z))\right] \frac{f(\psi_b(z))}{v_{z\theta}(z, \psi_b(z))} =$$
$$\left[1 - v_z(z, \psi_s(z)) - \left(\frac{G(\psi_s(z)) - F(\psi_s(z))}{f(\psi_s(z))}\right) v_{z\theta}(z, \psi_s(z))\right] \frac{f(\psi_s(z))}{v_{z\theta}(z, \psi_s(z))}; \qquad (37)$$

*(iv) for $z \in (z_l, z_m]$, $\psi_b(z) = \theta_d$ and $\psi_s(z)$ satisfies*

$$T'(z) = 1 - v_z(z, \psi_s(z)) = \left[\frac{G(\psi_s(z)) - F(\psi_s(z)) - \delta}{f(\psi_s(z))}\right] v_{z\theta}(z, \psi_s(z)), \quad (38)$$

*where $\delta$ is the Lagrange multiplier associated with the binding GIC (either Equation 33 or 34 from Lemma 3).*

**Remark 5.** *For $z \in \{z_l, z_h\}$ we have continuous pooling and the type assignment function is actually a correspondence.*

**Theorem 1.** *Whenever the segregation mechanism is not degenerate, that is, $\theta_d \in (\underline{\theta}, \overline{\theta})$, the conditions underlying the Variational Method are violated. Moreover, in this case, the segregation mechanism dominates the solution given by the variational method in terms of welfare.*

*Proof.* Given the formulation of our problem, this proof is straightforward. First notice that whenever the segregation mechanism is not degenerate the resulting tax schedule is discontinuous. Moreover, since the GIC constraint is binding in this case, local perturbations of the tax schedules will break the GIC of all types close to the pivotal type (or close to $\overline{\theta}$ if the binding IC is the second equation of Lemma 3). Therefore, local reforms will have discontinuous impact on agents' decisions. Therefore, $z_\theta(T)$ is not Lipschitz-continuous. □

The intuition for the result in Theorem 1 goes as follows. According to the Variational Method, the Gateaux differential of the welfare function, taking the optimal mechanism as the baseline, should be zero in all directions. However, we can find directions where the Gateaux derivative of the welfare function using the segregation mechanism as the baseline will not be zero. As mentioned in the introduction, local perturbations may have first-order impact on the welfare.

**Conjecture 1.** *The taxation scheme proposed in Proposition 4 is the optimal income tax mechanism in the class of càdlag mechanisms.*

**Example 3.** *Suppose that the common prior for the parameter $\theta$ is a uniform distribution over the interval $[0, 1]$. The first-order necessary conditions for the marginal tax rate in Proposition 4 can be written as:*

*(i) for $z \in (z_h, \overline{z}]$, $\psi_s(z) = \theta_d$ and $\psi_b(z)$ satisfies*

$$T'(z) = [G(\psi_b(z)) - \psi_b(z)]\, v_{z\theta}(z, \psi_b(z)) \quad (39)$$

*(iii) for $z \in [\underline{z}, z_l)$, we have discrete pooling and $\psi_s(z)$ and $\psi_b(z)$ are defined by*

$$T'(z)\left[\frac{1}{v_{z\theta}(z,\psi_b(z))} - \frac{1}{v_{z\theta}(z,\psi_s(z))}\right] = [G(\psi_b(z)) - G(\psi_s(z))] - [\psi_s(z) - \psi_b(z)] \quad (40)$$

*(iv) for $z \in (z_l, z_m]$, $\psi_b(z) = \theta_d$ and $\psi_s(z)$ satisfies*

$$T'(z) = [G(\psi_s(z)) - \psi_s(z) - \delta]\, v_{z\theta}(z,\psi_s(z)), \quad (41)$$

*where $\delta$ is the Lagrange multiplier associated with the active GIC.*

# 5 Conclusion

In this paper, we studied the design of income tax schedules in a class of economies where the Spence-Mirrlees condition is violated. We showed that the lack of ordination created by the absence of this condition typically generates allocations featuring non-monotonicities and discontinuities. These properties are novelties in the taxation literature and may invalidate the usage of variational methods. We also proposed a new methodology to tackle the design of income taxes based on type assignment functions.

The next steps in the literature could involve a better understanding of the "correct" elasticities implied by our analysis, as well as an investigation of its behavior, and the possible usage of quasi-experimental designs to estimate them. Our main conclusion is that additional caution should be taken when using variational methods in complex environments, given the potential non-regularity problems.

# A Appendix - examples of the function $v(z,\theta)$

The function $v(z,\theta)$ is flexible enough to summarize a big class of economies used in several applications in the public economics literature.

**Example A1** Diamond (1998) was the first to use quasi-linear utility function in a typical Mirleessian economy. Types represent individual idiosyncratic productivity $\theta$ in the labor market. Individuals generate taxable income $z = \theta l$ By supplying $l$ "units" of labor at a cost in utility terms given by a function $h(l)$ assumed to be sufficiently well behaved. In this model, the function $v(\cdot)$ takes the form

$$v(z,\theta) = h\left(\frac{z}{\theta}\right). \tag{42}$$

Typically $h(\cdot)$ is increasing, convex and twice continuously differentiable function.

**Example A2** Another very common used utility function in taxation literature is

$$v(z,\theta) = \theta h(z), \tag{43}$$

where type $\theta$ represents an idiosyncratic labor taste parameter and $h(\cdot)$ represents the disutility incurred in supplying $l = z$ unities of labor.

**Example A3** In Alves, da Costa, and Moreira (2017) the functional form

$$v(z,\theta) = \min_{z_a,z_b\geq 0}\left\{h\left(\frac{z_a}{\theta_a}\right) + h\left(\frac{z_b}{\theta_b}\right) \text{ s.t. } z_a + z_b = z\right\} \tag{44}$$

is used to denote couples efficient labor supply decision in a unitary model of household. In this application type $\theta = (\theta_a, \theta_b)$ is a two-dimensional vector representing the idiosyncratic labor productivity parameter of each spouse in a couple and $h(l)$ represents the disutility incurred in supplying $l = z/\theta_i$ unities of labor.

The next two examples do not satisfy the Spence-Mirrlees condition property and are helpful in guiding our discussion.

**Example A4** This example mixes the structure of examples A1 and A2 to create a very reasonable economy where agents' utilities do not satisfy SMC. Assume that every individual in the economy has labor market productivity $\gamma$ to generate taxable income (measured in units of the output good) $z = \gamma l$ at a cost in terms of utility given by $\delta h(l)$, where $\delta$ represents a taste labor parameter. Individuals with high $\gamma$ generates more income for a given level of labor supply. Individuals with low $\delta$ have lower disutility when supplying a level $l$ of labor. Assuming $\gamma$ and $\delta$ depend on a common parameter $\theta$ in the following way: $\gamma = \theta$ and $\delta = \psi(\theta)$. In this economy the function $v(z,\theta)$ takes the form[29]

$$v(z,\theta) = \psi(\theta)h\left(\frac{z}{\theta}\right). \tag{45}$$

[29]In the numerical exercises we make $h(l) = \exp(\epsilon l)$ and $\psi(\theta) = \theta - b$.

For these functional forms,

$$v_z(z,\theta) = \frac{\psi(\theta)}{\theta} h'\left(\frac{z}{\theta}\right), \quad v_{zz}(z,\theta) = \frac{\psi(\theta)}{\theta^2} h''\left(\frac{z}{\theta}\right), \tag{46}$$

and

$$v_{z\theta}(z,\theta) = \frac{\psi(\theta)}{\theta^2} h''(l) l \left[ \frac{\frac{d}{d\theta}(\psi(\theta)/\theta)}{\psi(\theta)/\theta} \frac{h'(l)}{h''(l)l}\theta - 1 \right]. \tag{47}$$

Therefore, $v_{z\theta}(z,\theta) \geq 0$ ($\leq 0$) if and only if

$$\frac{\frac{d}{d\theta}(\psi(\theta)/\theta)}{\psi(\theta)/\theta} \frac{h'(l)}{h''(l)l}\theta - 1 \leq 0 \tag{48}$$

or equivalently

$$\left[\frac{\psi'(\theta)}{\psi(\theta)}\theta - 1\right] \frac{h'(l)}{h''(l)l} \leq 1 \quad (\geq 1). \tag{49}$$

Let $\epsilon(l)$ be the elasticity of labor supply,

$$\epsilon(l) = \frac{h'(l)}{h''(l)l}. \tag{50}$$

It is constant equal to $\epsilon$ for the iso-elastic case,

$$h(l) = \frac{1}{1+1/\epsilon} l^{1+1/\epsilon}. \tag{51}$$

It is equals to $\epsilon/l$ for the exponential case,

$$h(l) = \exp(l/\epsilon). \tag{52}$$

Now assume that $\psi(\theta) = \theta - b$ with $b < \underline{\theta}$, the condition defining the function $z_0(\theta)$ simplifies to

$$\left[\frac{\theta}{\theta - b} - 1\right] \leq \frac{1}{\epsilon(l)}. \tag{53}$$

In the iso-elastic case we have a vertical line at

$$\theta = b(\epsilon - 1). \tag{54}$$

In the more interesting case of exponential function we have a decreasing function separating the regions as assumed in Assumption 1. Indeed,

$$z_0(\theta) = \epsilon \frac{b}{1 - b/\theta}. \tag{55}$$

**Example A5** This example is adapted from Araujo, Moreira, and Vieira (2015) to our taxation context. The utility function is

$$v(z,\theta) = z\theta^2 - b^2(\theta+4)\frac{z^2}{2} + 1. \tag{56}$$

In this example type $\theta$ does not have a clear interpretation but it can be thought as a combination of different forces affecting the agent's utility in a non-trivial way. Despite the artificiality, the simplicity of this example is very convenient to allow us to quickly assess the potential pitfalls of using a variational approach in an environment where the SMC condition fails. Indeed, the separating curve $z_0(\theta)$ (Assumption 1) in this particular example is a increasing line given by

$$z_0(\theta) = \frac{\theta}{b^2}. \tag{57}$$

To simplify things even further, we assume that the government follows a Rawlsian social welfare criterion.[30] Assume that types are uniformly distributed on the interval $\Theta = [0,1]$. We simulated this example using Wolfram's Mathematica. We can use the solution in Figure **??** to illustrate the relaxed subproblems in a concrete example.

# B Proofs

### Proof of Proposition 1

*Proof.* Let $z, c : \Theta \to Z \times \mathbb{R}_+$ be an incentive-feasible allocation. Let $T : Z \to \mathbb{R}$ be defined as follows. For all $z \in z(\Theta)$, define $T(z) = z(\theta) - c(\theta)$. For $z \in z(\Theta)^c$ make $T(z) = z$ and $z \in z(\Theta)^c$ make $T(z) = z$. Let $F^*$ to denote the income distribution in the economy. Note that,

$$\begin{aligned}\int_Z T(z)dF^*(z) &= \int_{Z(\Theta)} T(z)dF^*(z) + \int_{Z(\Theta)^c} T(z)dF^*(z)\\ &= \int_{Z(\Theta)} T(z(\theta))dF^*(z(\theta)) + \int_{Z(\Theta)^c} zdF^*(z) \geq \int_\Theta [z(\theta)-c(\theta)]f(\theta)d\theta \geq 0.\end{aligned} \tag{58}$$

where in the first inequality we use that $Z(\Theta) \subset Z \subset \mathbb{R}_+$ and the last inequality comes from the feasibility of the allocation, proving (i).

[30] As in Alves, da Costa, and Moreira (2017), the solution for this problem is equivalent to a solution of a dual program where the government maximizes tax revenue subject to the incentive constraints and a minimum utility requirement for the least well-off individual.

For (ii) take $\theta \in \Theta$ arbitrary, note that for any $z \in z(\Theta)^c$

$$V(z - T(z), z, \theta) = z - T(z) - v(z, \theta) = z - z - v(z, \theta) \leq -v(0, \theta) \tag{59}$$

Therefore, $z \in z(\Theta)^c$ cannot be optimal [31]

Now take any $z \in z(\Theta)$, therefore $z = z(\hat{\theta})$, for some $\hat{\theta} \in \Theta$. By incentive compatibility, we have

$$\begin{aligned} V(z(\theta) - T(z(\theta)), z(\theta), \theta) &= z(\theta) - T(z(\theta)) - v(z(\theta), \theta) \\ &= z(\theta) - [z(\theta) - c(\theta)] - v(z(\theta), \theta) = c(\theta) - v(z(\theta), \theta) \\ &\geq c(\hat{\theta}) - v(z(\hat{\theta}), \theta) = z(\hat{\theta}) - [z(\hat{\theta}) - c(\hat{\theta})] - v(z(\hat{\theta}), \theta) \\ &= z(\hat{\theta}) - T(z(\hat{\theta})) - v(z(\hat{\theta}), \theta) = V(z - T(z), z, \theta), \end{aligned}$$

proving (ii). Therefore, any incentive-feasible allocation can be implemented by a non-linear tax schedule.

Now take a tax schedule $T : Z \to \mathbb{R}$ and let $z, c : \Theta \to Z \times \mathbb{R}_+$ the allocation implemented by this tax schedule where $z(\theta)$ represents the income that type-$\theta$ agent chooses to get when facing the proposed tax schedule. Let us show that it is incentive-feasible. The agent consumes all his income net-of taxes, then $c(\theta) = z(\theta) - T(z(\theta))$ and the government budget constraint implies

$$\begin{aligned} \int_\Theta [z(\theta) - c(\theta)] f(\theta) d\theta &= \int_\Theta z(\theta) - [z(\theta) - T(z(\theta))] f(\theta) d\theta \\ &= \int_\Theta T(z(\theta)) f(\theta) d\theta = \int_Z T(z) dF^*(z) \geq 0, \end{aligned} \tag{60}$$

where the last inequality follows from the budget balance condition. Therefore, the allocation is feasible. To prove the incentive compatibility note that since $z(\theta)$ is the choice facing the tax schedule, for any other $z \in Z$ and in particular $z(\hat{\theta})$. We have

$$\begin{aligned} c(\theta) - v(z(\theta), \theta) &= z(\theta) - T(z(\theta)) - v(z(\theta), \theta) \\ &\geq z(\hat{\theta}) - T(z(\hat{\theta})) - v(z(\hat{\theta}), \theta) = c(\hat{\theta}) - v(z(\hat{\theta}), \theta), \end{aligned}$$

where the inequality follows from the choice. Therefore, the allocation reached with an income tax is incentive compatible. □

[31] Assume for simplicity that $0 \in z(\Theta)$.

### Proof of Lemma 1

These are usual results in the mechanism design literature. We will present a proof in the less general case when the mechanism is differentiable because it is more simple. For more general versions of this result we refer to the classical Milgrom and Segal (2002) and Rochet (1987).

*Proof.* Assume that $z : \Theta \to \mathbb{R}_+$ is differentiable in the interior of $CS_+$ (and $CS_-$ respectively). Let $V(\theta', \theta)$ be the utility that $\theta$-agent gets by announcing to be type $\theta'$ in the mechanism. Incentive compatibility requires $V(\theta, \theta) \geq V(\theta', \theta)$, for all $\theta' \in \Theta$. By differentiability we can characterize the optimum using first and second-order conditions. The first-order condition for incentive compatibility is

$$\frac{\partial}{\partial \theta'} V(\theta', \theta)\bigg|_{\theta'=\theta} = 0; \tag{61}$$

and the second-order condition for incentive compatibility is

$$\frac{\partial^2}{\partial \theta'^2} V(\theta', \theta)\bigg|_{\theta'=\theta} \leq 0. \tag{62}$$

The FOC boils down to

$$\frac{d}{d\theta} c(\theta) - v_z(z(\theta), \theta) \frac{d}{d\theta} z(\theta) = 0. \tag{63}$$

On the other hand, using it, we have

$$\begin{aligned}\dot{V}(\theta) = \frac{\partial}{\partial \theta} V(\theta, \theta) & \\ &= \frac{d}{d\theta} c(\theta) - v_z(z(\theta), \theta) \frac{d}{d\theta} z(\theta) - v_\theta(z(\theta), \theta) \\ &= -v_\theta(z(\theta), \theta) \end{aligned} \tag{64}$$

and applying the fundamental theorem of calculus we have (i). Taking the total derivative of the FOC we have

$$\frac{d^2}{d\theta'^2} V(\theta, \theta) = -\frac{d^2}{d\theta' d\theta} V(\theta, \theta). \tag{65}$$

Then, from the SOC we have

$$\frac{d^2}{d\theta' d\theta} V(\theta, \theta) = v_{z\theta}(z(\theta), \theta) \frac{d}{d\theta} z(\theta) \geq 0. \tag{66}$$

Therefore, $\frac{dz(\theta)}{d\theta}$ and $v_{z\theta}(z,(\theta)\theta)$ should have opposite signs, proving (ii). □

### Proof of Lemma 3

*Proof.* This is a very intuitive result that follows from the function $v(z,\theta)$ being continuously differentiable and the monotonicity constraints of the type assignment functions $\psi_b(\cdot),\psi_s(\cdot)$. □

### B.0.1 Derivation of the global incentive function

Using Lemma 1 (i), it is convenient to write the incentive compatibility constraint using the *Global Incentive Function* (GIF):

$$\Phi(\theta,\hat{\theta};z(\cdot)) := \int_{\hat{\theta}}^{\theta}\left[\int_{z(\hat{\theta})}^{z(s)} v_{z\theta}(t,s)dt\right]ds. \tag{67}$$

Fix a mechanism $z:\Theta\to\mathbb{R}_+$ and take $\theta\in\Theta$. Incentive compatibility requires $V(\theta,\theta)\geq V(\theta',\theta)$, for all $\theta'\in\Theta$. This is equivalent to

$$\begin{aligned} V(\theta)-V(\theta',\theta) = V(\theta)-c(\theta')+v(z(\theta'),\theta)\geq 0 \iff \\ V(\theta)-V(\theta')+v(z(\theta'),\theta)-v(z(\theta'),\theta')\geq 0 \iff \\ \int_{\theta'}^{\theta}\left[v_\theta(z(s),s)-v(z(\theta'),s)\right]ds = \int_{\theta'}^{\theta}\left[\int_{z(\theta')}^{z(s)} v_{z\theta}(t,s)dt\right]ds\geq 0. \end{aligned}$$

Therefore, defining

$$\Phi(\theta,\theta',z(\cdot)) = \int_{\theta'}^{\theta}\left[\int_{z(\theta')}^{z(s)} v_{z\theta}(t,s)dt\right]ds, \tag{68}$$

we have $\Phi(\theta,\theta',z(\cdot))\geq 0$ for all $\theta,\theta'\in\Theta$ if and only if $z:\Theta\to\mathbb{R}_+$ is incentive compatible.

By construction, an allocation $z:\Theta\to\mathbb{R}_+$ is incentive compatible iff, for all $\theta,\hat{\theta}\in\Theta$, $\Phi(\theta,\hat{\theta};z(\cdot))\geq 0$. As we can see above, under the SMC, $v_{z\theta}(t,s)$ would have a constant sign and the necessary monotonicity condition would also be sufficient for incentive compatibility.[32] Without the SMC, we do not have a natural ordering of agents and the impact of $v_{z\theta}(t,s)$ to the right side of the separating curve has to be compensated with the impact to the left side on the GIF to get incentive compatibility.

[32]Notice that for $\theta,\hat{\theta}$ such that $z(\theta)=z(\hat{\theta})$, we have $\Phi(\hat{\theta},\theta)=-\Phi(\theta,\hat{\theta})$.

# C Incorporating local IC's.

The mechanism design problem of the government is to choose the best allocation rule $c, z : \Theta \to \mathbb{R}_+ \times Z$ assigning a pair of income and consumption to each type inducing truthful revelation. Formally,

$$\max_{c(\cdot), z(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} V(c(\theta), z(\theta), \theta) g(\theta) d\theta \tag{69}$$

subject to the budget constraint,

$$\int_{\theta \in \Theta} [z(\theta) - c(\theta)] f(\theta) d\theta \geq 0; \tag{70}$$

and the incentive-compatibility constraints: for all $\theta \in \Theta$,

$$\theta \in \arg\max_{\theta' \in \Theta} V(c(\theta'), z(\theta'), \theta). \tag{71}$$

This formulation is very convenient because it makes easy to incorporate the local incentive-compatibility constraints and the government budget constraints into the objective function.

Recall the definition of the informational rent get by an agent with type $\theta$ in an incentive-compatible mechanism is given by $V(\theta) = c(\theta) - v(z(\theta), \theta)$.[33] Using the envelope condition (Lemma 1 (i)) we can eliminate consumption from the problem.

$$c(\theta) = V(\theta) + v(z(\theta), \theta) = V(\underline{\theta}) + v(z(\theta), \theta) - \int_{\underline{\theta}}^{\theta} v_\theta(z(s), s) ds. \tag{72}$$

Plugging it into the budget constraint we have

$$\begin{aligned} 0 = \int_{\underline{\theta}}^{\overline{\theta}} [z(\theta) - c(\theta)] f(\theta) &= \int_{\underline{\theta}}^{\overline{\theta}} \left[ z(\theta) - V(\underline{\theta}) - v(z(\theta), \theta) + \int_{\underline{\theta}}^{\theta} v_\theta(z(s), s) ds \right] f(\theta) d\theta \\ &= -V(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \left[ z(\theta) - v(z(\theta), \theta) + \int_{\underline{\theta}}^{\theta} v_\theta(z(s), s) ds \right] f(\theta) d\theta \\ &= -V(\underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \left[ z(\theta) - v(z(\theta), \theta) + v_\theta(z(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)} \right] f(\theta) d\theta \end{aligned} \tag{73}$$

Where we used the fact that the budget constraint must be satisfied with equality

[33] Recall that Lemma 1 (i) implies payoff equivalence, up to a constant, for all incentive-compatible mechanism.

since agents utility are strictly increasing in consumption. Re-arranging to have

$$V(\underline{\theta}) = \int_{\underline{\theta}}^{\overline{\theta}} \left[ z(\theta) - v(z(\theta), \theta) - v_\theta(z(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)} \right] f(\theta) d\theta. \tag{74}$$

Equation (78) shows the guaranteed utility level that the lower bound on type space gets in any incentive-feasible allocation. Together with Lemma 1, we see that the government guarantees an utility level that is taxed/subsidized away as type increases increases.

Substituting Equation 72 in the objective function of the government we have

$$\begin{aligned}
\int_{\underline{\theta}}^{\overline{\theta}} g(\theta) \left[ c(\theta) - v(z(\theta), \theta) \right] d\theta &= \int_{\underline{\theta}}^{\overline{\theta}} g(\theta) \left[ V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} v_\theta(z(s), s) ds \right] d\theta \\
= V(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} g(\theta) \left[ \int_{\underline{\theta}}^{\theta} v_\theta(z(s), s) ds \right] d\theta &= V(\underline{\theta}) - \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(z(\theta), \theta) \left[ 1 - G(\theta) \right] d\theta \\
= \int_{\underline{\theta}}^{\overline{\theta}} \left[ z(\theta) - v(z(\theta), \theta) + v_\theta(z(\theta), \theta) \frac{G(\theta) - F(\theta)}{f(\theta)} \right] f(\theta) d\theta \quad (75)
\end{aligned}$$

Therefore, defining the function $W : Z \times \Theta \to \mathbb{R}$ given by

$$W(z, \theta) = \left[ z - v(z, \theta) + v_\theta(z, \theta) \frac{G(\theta) - F(\theta)}{f(\theta)} \right] f(\theta). \tag{76}$$

and the problem of the government can be written as

$$\max_{z(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} W(z(\theta), \theta) d\theta \tag{77}$$

subject to all additional global incentive compatibility constraints.

**Lemma 4.** *Let $z : \Theta \to Z$ be an incentive-feasible allocation. Then the informational rent of the lowest type is given by*

$$V(\underline{\theta}) = \int_{\underline{\theta}}^{\overline{\theta}} \left[ z(\theta) - v(z(\theta), \theta) + v_\theta(z(\theta), \theta) \frac{1 - F(\theta)}{f(\theta)} \right] f(\theta) d\theta. \tag{78}$$

This lemma provides the guaranteed utility level that the lower bound on type space gets in any incentive-feasible allocation. Together with Lemma 1, we see that the government guarantees an utility level that is taxed/subsidized away as type increases. It is given by the average "virtual welfare" of all agents in the economy.

**The Segregation Problem**

Let $\theta_d \in \Theta$ be a pivotal type that parametrizes the program and let $Z = [\underline{z}, \overline{z}]$ to be the set of possible income values. The Segregation Problem is defined as follows.[34]

$$\max_{\psi_b,\psi_s,z_l,z_m,z_h} \int_{\underline{z}}^{z_l} [\mathscr{W}(z,\theta_d) - \mathscr{W}(z,\psi_b(z))]\,dz + \int_{z_l}^{z_m} [\mathscr{W}(z,\psi_s(z)) - \mathscr{W}(z,\theta_d)]\,dz + \int_{z_h}^{\overline{z}} [\mathscr{W}(z,\psi_s(z)) - \mathscr{W}(z,\psi_b(z))]\,dz \quad (79)$$

subject to:

(i) Monotonicity, $\psi_b$ non-increasing and $\psi_s$ non-decreasing.

(ii) For all $z \in [z_h, \overline{z}]$,

$$[\psi_b(z) - \underline{\theta}]\,[v_z(z,\psi_b(z)) - v_z(z,\psi_s(z))] \geq 0, \quad (80)$$

and

$$\underline{\theta} \leq \psi_b(z) \leq \psi_s(z) \leq \theta_d; \quad (81)$$

(iii) For all $z \in [\underline{z}, z_l]$,

$$\left[\overline{\theta} - \psi_s(z)\right]\,[v_z(z,\psi_b(z)) - v_z(z,\psi_s(z))] \leq 0; \quad (82)$$

and

$$\theta_d \leq \psi_b(z) \leq \psi_s(z) \leq \overline{\theta}; \quad (83)$$

(iv) and the GIC from Lemma 3,

$$\int_{z_l}^{z_h} v_z(z,\psi_s(z)) - v_z(z,\theta_d)dz \geq 0. \quad (84)$$

$$\int_{z_l}^{z_m} v_z(z,\psi_b(z)) - v_z(z,\overline{\theta})dz \geq 0. \quad (85)$$

The solution for this problem is parametrized by the pivotal type. We can

[34] Since $\psi_b, \psi_s : Z \to \Theta$ should assign a type for every possible income level, for $z \in (z_m, z_l)$ assign $\psi_b(z) = \theta_d$ and $\psi_s(z) = \overline{\theta}$ completing the definition of the type assignment functions.

choose it in order to maximize the welfare criterion. Let us denote it by the optimal segregation mechanism.

Call $\delta_{\theta_d}$ and $\delta_{\overline{\theta}}$ the Lagrange multiplier associated with GIC in equations (33) and (34). Since by Lemma 3 only one of the GIC will be binding, i.e. $\delta_{\theta_d} > 0$ if and only if $\delta_{\overline{\theta}} = 0$ and vice-versa, we can define the Lagrange multiplier of the active constraint as $\delta = \delta_{\theta_d} + \delta_{\overline{\theta}}$.

There are some very interesting aspects of this formulation. First of all, This program can be solved point-wisely which makes the problem much more tractable than the direct mechanism design and variational approaches.

The usual solutions proposed in the literature are degenerate cases of the segregating mechanism. In particular, whenever $\theta_d \in \{\underline{\theta}, \overline{\theta}\}$ the resulting solution will be consistent with the Variational method as defined in Golosov, Tsyvinski, and Werquin (2014).

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