A Framework for Debt-Maturity Management*

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Abstract

We characterize the optimal debt-maturity management of an impatient government in a small open-economy. The Government issues a continuum of finite-life bonds that differ in their maturity. It takes into account the price impact of each issuance. The optimal issuance of any bond equals the product of the liquidity coefficient times the difference between the market price and the domestic valuation. This property holds in presence of income or interest-rate risk and the option to default. The model sheds light on the different forces that shape an optimal maturity distribution.

Keywords: Debt maturity; Debt management; Liquidity costs.

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1 Introduction

Any government faces a large-stakes problem, to design a strategy for the quantity and maturity of its debt. This paper presents a new framework to think about that design.

The framework makes two innovations. First, it puts forth the importance of liquidity frictions, the notion that an abrupt adjustment of debt at a given maturity can impact its price. This is a common consideration by practitioners, but has been neglected by normative theory. The second innovation is technical. Debt-management problems have been studied under a rich set of shocks, but the curse of dimensionality quickly restricts the number and class of bonds that can be considered. Quantitative studies typically model bonds that mature exponentially and work with two maturities.[1] In practice, governments issue in many maturities and never issue bonds that mature exponentially. The framework here flips things around. Shocks can only occur once, but the Government can issue simultaneously a continuum of bonds of arbitrary coupon structure. The framework is tractable, portable, and easy to compute.

The goal of building this framework is to make a step forward towards a better comprehension of an optimal debt-maturity design. To this end, we lay out a continuous-time small-open economy environment. An impatient government chooses the issuance or (re-)purchase of fixed-coupon bonds within a continuum of maturities. Its financial counterparts are risk-neutral international investors. The Government's objective is to smooth consumption. The Government faces income and interest risk and can default when those risks materialize. Liquidity costs emerge because bonds are auctioned to primary dealers that take time to liquidate bond holdings after an auction, as in Duffie et al. (2005). The larger the auction, the lower the price. Bond markets are specific to a maturity, as in Vayanos and Vila (2009).

A general principle emerges. The problem can be studied as if the Government delegates issuances to a continuum of subordinate traders, each in charge of a single maturity. Each trader computes a domestic valuation of the bond of his maturity. To compute its valuation, the trader uses a common discount designated by the Government. Each trader then applies a simple rule to determine how much to issue of his maturity:

$$\Delta\% \text{ issuance/GDP } = \text{liquidity coefficient } \cdot \Delta\%\text{value gap.}$$

This simple rule says that the optimal issuance of a bond of a given maturity is the product of a value gap and a liquidity coefficient. The value gap is the difference between the price and the domestic valuation at a given maturity, as a percentage of the price. A positive value gap indicates that the trader would otherwise want to issue as much debt as possible. Liquidity costs

[1] This limitation is easily understood with a simple example. If we want to construct a yearly model where the government issues a single 30 year, zero-coupon bond, we need at least 30 state variables: a 30 year bond becomes a 29 year bond the following year, and a 28 year bond the year after, and so on. By contrast, a bond that matures by 5 per cent every year is still a bond that matures by 5 per cent the year after its issuance.

contain that desire through as reflected by the liquidity coefficient. The liquidity coefficient can be read off from data on turn-over rates and intermediation spreads. The lower the liquidity coefficient, the greater the issuance for a positive value gap. The nature of risk or default alters the details behind the valuations, but the overall principle is the same.

The principle delivers the optimal solution as long as the Government designates the correct discount factor. This discount factor should emerge from a fixed-point problem in the path of consumption: An imputed consumption path maps into a Government discount factor. This discount factor, delivers a path for debt through the issuance rule. Ultimately, the path for debt produces a new consumption path. In the optimal solution, both consumption paths must coincide. This fixed-point problem is solved through a numerical algorithm and allows the computation of rich transitional dynamics.

The paper exploits this principle to shed light on optimal maturity management. The paper is built in layers. The first layer is perfect foresight. Under perfect foresight, steady-state issuances are tilted towards longer horizons because longer maturities have a larger value gap. The amount of outstanding debt decreases with maturity because long-term bonds become short-term bonds as they mature, but not the other way around. Without liquidity costs, the maturity profile is indeterminate, but remains determinate as liquidity costs vanish.

Two economic forces govern the transitional dynamics of the maturity structure, *consumption smoothing* and *yield management.* We investigate these forces by calculating the perfect-foresight transitional dynamics that follow unexpected shocks. Both forces are uncovered only thanks to the liquidity costs. Without liquidity costs, the domestic discount factor and short-term rate coincide, so even if the Government smooths consumption the maturity structure is indeterminate. With liquidity costs, the maturity structure is determinate because the discount factor is not equal to the short-term rate, and that difference compounds with maturity. The consumption smoothing force is activated when the consumption path changes domestic discounts and, as a result, modifies domestic valuations. Yield management operates when short-term rates affect bond prices. Consumption smoothing lengthens the maturity during downturns: when a shock produces a drop in consumption, the domestic discount remains temporarily high. This reduces domestic valuations, particularly for longer maturities. The economic intuition is that the payment streams are delayed by long-term bonds, and as a result, with a higher discount, these are valued less. Thus, the Government bears a higher price impact upon the issuance of long-term debt during a downturn. By constrast yield management shortens the maturity when short-term rates increase temporarily. The intuition is that there is little discount in short-term bonds to begin with, so at short horizons valuation gaps are always small. When rates increase, long-term bonds are priced with a higher discount, so the valuation gap decreases more for long maturity bonds.

The second layer incorporates risk. To make the problem tractable, we study an economy where interest or income shocks occur once. The anticipation of risk introduces an extra term into the domestic valuations: coupon payments are adjusted by the ratio of post- to pre-shock *marginal utilities*. We characterize the risky steady state (RSS). The RSS is the steady state reached when the government expects the shock, but the shock has not yet materialized.[2] The debt distribution at the RSS reveals how risk introduces a new force, *insurance*. Insurance shapes the maturity profile in two ways. First, the Government tries to hedge by generating capital gains from its portfolio in adverse states. Liquidity costs make a perfect hedge too costly, even if it is available. Second, the Government self insures. Self insurance shows up in the ratio of marginal utilities because a future drop in consumption increases that ratio and, thus, raises valuations. By raising valuations, self insurance reduces the stock of debt. Self insurance also lengthens the average maturity because coupon payments are inflated by the ratio of marginal utilities, and this increases valuations especially for long-term debt. The intuition is that long-term bonds are less likely to have expired by the time a shock arrives, and this reduces the expected costs of debt repayments under a negative shock.

The final layer incorporates strategic default. We allow the Government to default the instant it is hit by the shock. Default implies an output loss and the permanent exclusion from financial markets, but allows the Government to erase all debts. Default is exercised only if it renders a higher value to the Government. Prior to default, the Government commits to a debt program. With default, prices and valuations are corrected by a default-risk premium. Valuations are also adjusted by an additional term which we dub the *revenue-echo effect*. The echo effect appears only in the valuations because traders anticipate that a marginal increase in their issuances marginally raise the chances of default in the future. The increase in default risk echoes back in time through market prices that anticipate the risk. This anticipation reduces revenue collections from issuances. In terms of the maturity structure, the default premium and the echo effect are a force that shortens debt maturity. Both effects close the value gap. The paper ends with a list of potential applications and extensions of the framework.

Related literature. Maturity management appears in various areas of finance: international, public, and corporate. The framework here captures forces that have been previously identified. An advantage of the framework is that it allows for a large number of securities of realistic coupon structure.

The framework also incorporates liquidity costs. There is a large literature that studies both, theoretically and empirically, the sources and magnitudes of liquidity costs. The formulation of liquidity costs here builds on two ideas. As in Vayanos and Vila (2009), markets for bonds of different maturity are segmented. As in Duffie et al. (2005), issuances are intermediated by

[2] The concept of RSS is equivalent to the one that appears in Coeurdacier et al. (2011). The analysis of the RSS is the only tractable solution that does not rely on approximations. Transitional dynamics prior to the shock is not analytically tractable. Furthermore, if instead of a single jump, we consider multiple jumps, complexity escalates. An analogous challenge appears in models of incomplete markets which is why the literature uses the approximation in Krusell and Smith (1998) or similar approaches. To our knowledge, this is the first paper to employ the RSS in a model with an aggregate shock when the state variable is a distribution.

dealers that face a high discount and need time to reallocate assets. In Vayanos and Vila (2009), a finite mass of long-lived investors demand bonds of a specific maturity. Investors trade with intermediaries that trade in all maturities. The price impact in that model depends on the overall outstanding amount of bonds of a specific maturity. In our framework, a large flow of customers contact dealers, independently of the outstanding amount at a maturity. Therefore, in our case the price impact depends on the issued amount, because it affects the time taken by dealers to offload bonds to investors. Liquidity costs introduce the yield management force that shapes the optimal maturity profile under risk neutrality. The framework, notwithstanding, can also be used to study maturity management without liquidity costs.

Beyond the theoretical appeal, there is substantial evidence various forms of liquidity costs in different asset classes, as surveyed by Vayanos and Wang (2013b) or Duffie (2010). In the specific case of fixed-income securities, studies that provide evidence suggestive of the presence of liquidity costs are Cammack (1991), Duffee (1996), Spindt and Stolz (1992), Fleming (2002), Green (2004), Pasquariello and Vega (2009), and Krishnamurthy and Vissing-Jorgensen (2012) for US Treasury markets, by Duffie et al. (2003), Naik and Yadav (2003b) and Cheung et al. (2005) for sovereign bond markets, Edwards et al. (2007) and Feldhutter (2011) for US corporate bond markets, and Green et al. (2007) for US municipal bond markets. Our calibration is based on inventory-depletion evidence in the US Treasuries market documented by Fleming and Rosenberg (2008).

International finance stresses several forces that interact with liquidity costs in our framework. One force is *insurance.* A small-open economy effectively has access to complete markets when income shocks are correlated with changes to the yield curve in a way that allows complete asset spanning, as discussed in Duffie and Huang (1985). With full spanning, the maturity design is governed by the formation of a hedge that perfectly insures the country. Maturity plays no role when income shocks are uncorrelated with changes in rates, and in that case, debt dynamics are governed only by self insurance (as in Chamberlain and Wilson (2000) or Wang et al. (2016)). In between these extremes, there is are a range of cases without full asset spanning, but with a role for both hedging and self insurance. In our framework, liquidity costs make perfect hedges prohibitively costly, so self-insurance and partial hedging shape the debt-maturity profile under risk regardless of the asset structure. We explain how the insurance force is captured in the risk premium.

Another force is *incentives*. When the Government can default, it should take into consideration how its current issuances affects the likelihood of a future default, in the future and that which affects current prices (Eaton and Gersovitz, 1981). The effect of incentives is captured by the revenue-echo effect. In terms of maturity choice the literature has identified other important trade offs. Bulow and Rogoff (1988) identified that without commitment to a debt program, long-term debt is prone to debt dilution. The literature on sovereign default without commitment has identified two important channel though which incentives affect the maturity choice. First, is debt dilution. Debt dilution occurs when the price of outstanding bonds fall as a result of a new issuance. The expectation of debt dilution is a problem for a Government that cannot commit because it raises the cost of long-term debt. Debt dilution is a force towards the concentration of issuances at short maturities. Calvo (1988) and Cole and Kehoe (2000) identified a force that operates in the opposite direction, roll-over risk. A solvent government faces roll-over risk if it cannot refinance a large principal and, as a result, is forced to default. Roll-over risk is a force towards spreading debt services.[3]

Restrictions on the number of bonds and coupon structure are a serious limitation to sovereign debt models. For example, a country could, in principle, construct a hedge with two bonds of very short maturities. Thus, any trade off between debt dilution and hedging can be mitigated if the country is able to choose its maturity structure. Another example of a limitation is that roll-over risk can be mitigated if the country can issue a perpetuity. To our knowledge, this is the first paper to study a debt-management problem where the control is the entire distribution of debt.[4] Although our model does not feature roll-over risk and abstracts from debt dilution, it lays the foundations of a framework that can, later on, be enriched to address these issues.

Maturity choice is also a classic theme in public finance models. Angeletos (2002) and Buera and Nicolini (2004) show how to implement optimal taxes under complete-markets (in a Lucas and Stokey, 1983) with a menu of bonds that differ in maturity. Buera and Nicolini (2004) argue that portfolios that provide insurance are very volatile. Liquidity costs can make debt profiles more realistic. Recently, Bhandari et al. (2017a), Bhandari et al. (2017b), Debortoli et al. (2017) and Debortoli et al. (2018) study maturity choice when markets are incomplete as in (in an Aiyagari et al., 2002, economy). In corporate finance, Diamond and He (2014) and DeMarzo and He (2016) build versions of Leland and Toft (1996) where maturity is a dynamic choice. There is no reason why our framework cannot be adapted to answer questions in either area.

On the technical front, the framework employs infinite-dimensional optimization techniques similar to those applied in heterogeneous agent models. Lucas and Moll (2014) study a problem with heterogeneous agents that allocate time between production and technology generation. Nuño and Moll (2018) study a constrained-efficient allocation in a model with heterogeneous agents and incomplete markets. Nuño and Thomas (2018) study optimal monetary policy in a heterogeneous-agent model with incomplete markets and nominal debt. The contribution rel-

[3] Aguiar et al. (Forthcoming) show analytically that under debt dilution, once the country has issued long-term debt, it should leave it untouched until maturity. Chatterjee and Eyigungor, 2012, Chatterjee and Eyigungor, 2015, and Hatchondo et al., 2016 are quantitative models of these effects. Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012), and Hatchondo et al. (2016) are quantitative studies that study the interaction between insurance, debt dilution, and roll-over risk. Passadore and Xu (2018) study a model with a feedback loop between default risk and secondary-market liquidity. Dovis et al. (Forthcoming) studies a risk-sharing problem and shows that the problem can be decentralized.

[4] There are two alternative frameworks that allow for richer debt structures, Sánchez et al. (2018) and Bocola and Dovis (2018). In Sánchez et al. (2018) the maturity of the entire stock of debt is a choice, but to get around the curse of dimensionality, the Government is allowed to change maturity by one period every period. In Bocola and Dovis (2018), instead, debt is a consol but its maturity can be changed at a quadratic cost.

ative to those studies is that this paper introduces the risky steady-state approach to study the impact of aggregate risk. This feature is novel, and allows for an exact characterization that can be used to analyze models with aggregate risk.

2 Maturity management with liquidity costs

2.1 Model setup

Time is continuous. The model features two exogenous state variables, $\{y(t), r^*(t)\}$, respectively representing an endowment process and a short-term interest rate process. The vector of exogenous states is $X(t) \equiv [y(t), r^*(t)]$. For any variable, endogenous or exogenous, we employ the notation $x_\infty \equiv \lim_{t\to\infty} x(t)$ to refer to its asymptotic value and x_{ss} to refer to its steady state value, if it exists.[5] All of the partial-differential equations (PDE) that we present here have exact solutions which are presented in Table A in Appendix A.

Households. We consider a small open economy. There is a single freely-traded consumption good. The economy is populated by a representative household with preferences over expenditure paths, $\{c\,(t)\}$, given by

$$\mathcal{V}_0 = \int_0^\infty e^{-\rho t} U\left(c\,(t)\right) dt,$$

where $\rho \in (0, 1)$ is the discount factor and $U\,(\cdot)$ an increasing and concave utility function.

Government. The economy features a benevolent Government that issues bonds to foreign investors on behalf of households. Bonds differ by their time-to-maturity. Denote by τ the time to maturity of a given bond. Issuances are chosen among a continuum of maturities, $\tau \in [0, T]$, where T is an exogenous maximum maturity. The maturity of a given bond falls with time, $\frac{\partial \tau}{\partial t} = -1$, and the bond is retired once $\tau = 0$. Prior to maturity, the bond pays an instantaneous constant coupon δ and one unit of good when the principal matures. The outstanding stock of bonds with a time-to-maturity τ at t is denoted by $f\,(\tau, t)$. We call $\{f\,(\tau, t)\}_{\tau\in[0,T]}$ the debt profile at time t. The law of motion of $f\,(\tau, t)$ follows a Kolmogorov-Forward equation (KFE):

$$\frac{\partial f}{\partial t} = \iota\,(\tau, t) + \frac{\partial f}{\partial \tau}, \tag{2.1}$$

with boundary conditions $f(T^+, t) = f(0^-, t) = 0$. The intuition behind the equation is that, given τ and t, the change in the quantity of bonds of $\tau-$maturity, $\partial f/\partial t$, equals the issuances at that maturity, $\iota\,(\tau, t)$, plus the netflow of bonds, $\partial f/\partial \tau$. The latter term captures the instantaneous outflow towards smaller maturities and the instantaneous inflow from bonds of higher maturities. Issuances, $\iota\,(\tau, t)$, are chosen from a space of functions $\mathcal{I} : [0, T] \times (0, \infty) \to \mathbb{R}$ that

[5] In the case of the exogenous variables $\{y, r^*\}$ we use both subscripts indistinctly.

meets some technical conditions.[6] When negative, issuances are interpreted as bond purchases or repurchases. The initial stock of τ maturity debt is given, $f(\tau,0) = f_0(\tau)$. The Government's budget constraint is:

$$c(t) = y(t) - f(0,t) + \int_0^T \left[q(\tau,t,\iota)\,\iota(\tau,t) - \delta f(\tau,t) \right] d\tau, \tag{2.2}$$

where $y(t)$ is the (exogenous) output endowment, with steady state value y_{ss}. In this budget, $-f(0,t)$ is the principal repayment, $\delta \int_0^T f d\tau$ are total coupon payments from all outstanding bonds, and $\int_0^T q\iota d\tau$ the funds received from debt issuances at all maturities. Finally, $q(\tau,t,\iota)$ is the issuance price of a bond of maturity τ at date t in the primary market. The price depends on the issuance amount.

International investors. The Government trades bonds with competitive risk-neutral international investors. The issuance price, q, has two components, a frictionless market price and a liquidity cost :

$$q(\tau,t,\iota) = \underbrace{\psi(\tau,t)}_{\text{market price}} - \underbrace{\lambda(\tau,t,\iota)}_{\text{liquidity costs}}. \tag{2.3}$$

The first component, $\psi(\tau,t)$, is the arbitrage-free market price of the domestic bond. This price depends on the path of the international risk-free interest rate, $r^*(t)$. Given this risk-free rate, the market price $\psi(\tau,t)$ has a PDE representation:

$$r^*(t)\,\psi(\tau,t) = \delta + \frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial \tau}, \tag{2.4}$$

with a boundary condition $\psi(0,t) = 1$.[7] Except if indicated otherwise, we assume that $\delta = r^*_{ss}$.

Liquidity costs. The second component in the issuance price, $\lambda(\tau,t,\iota)$, represents a *liquidity cost* associated with the issuance (or purchase) of ι bonds of maturity τ at date t. In Appendix B we present a wholesale-retail model of the bond market, whose solution yields the formulation of the liquidity cost that we employ throughout the paper. The spirit is similar to models that feature OTC frictions like Duffie et al. (2005). The main virtue of this friction is that it produces a price impact, which has been extensively documented in bonds markets.

In particular, we assume that to issue bonds of the maturity τ at date t, the Government decides to auction $\iota(\tau,t)$ bonds. The participants of that auction are a continuum of investment bankers. Bankers participate in the auction and buy large vintages of bonds. This is the wholesale market. Then, bankers offload bonds to international investors in a retail (secondary) market. As in Duffie et al. (2005), bankers have higher costs of capital than investors. In par-

[6] In particular $\mathcal{I}$ is the space of functions $g(\tau,t)$ on $[0,T] \times [0,\infty)$ such that $e^{-\rho t} g$ is square Lebesgue-integrable.

[7] The solution to this PDE is $\psi(\tau,t) = e^{-\int_t^{t+\tau} r^*(u)du} + \delta \int_t^{t+\tau} e^{-\int_t^s r^*(u)du} ds$. It is a continuous-time net present value formula, which can be recovered from the PDE via the method of characteristics or as an immediate application of the Feynman-Kac formula.

ticular, the bankers' cost of capital is $r^*(t)+\eta$, where $\eta > 0$ is a spread. In the retail market, bankers are continuously contacted by investors. The contact flow is μy_{ss} per instant. Each contact results in an infinitesimal bond purchase by investors from a banker's bond inventory. Thus, in an interval Δt, the stock of bonds sold by the banker is $\mu y_{ss}\Delta t$. We assume that bankers extract all the surplus from the international investors.

The key friction that translates into a liquidity cost for the Government is that it takes time for bankers to liquidate their bond portfolios. This, together with the fact that bankers have an inventory holding cost due to the finance premium, implies that the larger the auction, the longer the waiting time to resell a bond and, thus, the lower the price the banker is willing to offer in an auction. As auction size vanishes, the opposite occurs, and the price converges to the market price $\psi(\tau,t)$. These properties are common to OTC models, e.g., Duffie et al. (2005). In Appendix B we present an exact solution to the auction price as a function of the market price and the issuance size. Here, we present an approximation, a first-order Taylor expansion around $\iota=0$, which yields a convenient linear expression for the auction price:

$$q(\iota,\tau,t) \approx \psi(\tau,t) - \frac{1}{2}\underbrace{\frac{\eta}{\mu y_{ss}}}_{\bar{\lambda}}\psi(\tau,t)\iota. \tag{2.5}$$

Thus, the approximate liquidity cost function is $\lambda\left(\tau,t,\iota\right) \approx \frac{1}{2}\bar{\lambda}\psi(\tau,t)\iota$. The term $\bar{\lambda}$ is a liquidity cost coefficient which increases with the spread and decreases with the contact volume.[8]

We now return to the Government's problem and make no further reference to the source of the liquidity costs.

Government problem. The Government maximizes the utility of the households given by

$$V\left[f\left(\cdot,0\right)\right] = \max_{\{\iota(\cdot)\}\in\mathcal{I}} \int_0^\infty e^{-\rho t} U\left(c\left(t\right)\right) dt, \tag{2.6}$$

subject to the law of motion of debt (2.1), the budget constraint (2.2), the initial condition f_0, and debt prices (2.3). The object V is the *value functional*, which maps the initial debt profile given f_0, into a real number. It is a functional because the state variable is infinite-dimensional.

2.2 Solution: the debt issuance rule

We can employ infinite-dimensional optimization techniques to solve the Government's problem. This section presents a gist of the approach. A full proof is found in Appendix D.1. The

[8] An implicit assumption behind this formulation is that there are no congestion externalities: the contact rate is independent of the outstanding debt at a given maturity. A departure from this assumption, produces a strategic behavior similar to the case with default, as we discuss in the extensions of Section 5.

main idea is to formulate a Lagrangian:

$$
\begin{aligned}
\mathcal{L}\left[\iota,f\right] &= \int_0^\infty e^{-\rho t} U\left(y\left(t\right)-f\left(0,t\right)+\int_0^T\left[q\left(t,\tau,\iota\right)\iota\left(\tau,t\right)-\delta f\left(\tau,t\right)\right]d\tau\right)dt \\
&\quad+\int_0^\infty\int_0^T e^{-\rho t} j\left(\tau,t\right)\left(-\frac{\partial f}{\partial t}+\iota\left(\tau,t\right)+\frac{\partial f}{\partial \tau}\right)d\tau dt,
\end{aligned}
$$

where we substitute out consumption from the objective function by use of the budget constraint (2.2). The necessary conditions are obtained by a classic variational argument. The idea is that at the optimum the optimal issuance and debt paths cannot be improved. A first condition is that no infinitesimal variation around the control ι can produce an increase in the Lagrangian. This implies that:

$$
U'\left(c(t)\right)\left(q\left(t,\tau,\iota\right)+\frac{\partial q}{\partial \iota}\iota\left(\tau,t\right)\right)=-j\left(\tau,t\right). \tag{2.7}
$$

This necessary condition is intuitive: the issuance of a (τ,t)-bond produces a marginal cost and a marginal benefit and both margins must be equal at an optimum. The marginal benefit is the marginal utility of the marginal increase in consumption; the marginal increase in consumption equals the average price of that bond, q, plus the price impact of an additional issuance, $\frac{\partial q}{\partial \iota}\iota$. The marginal cost of the issuance is summarized in the Lagrange multiplier, $-j$, which captures the forward-looking information on the bond repayment.

A second condition is that no infinitesimal variation over the state f can yield an improvement in value. The solution cannot be improved as long as the Lagrange multipliers j satisfy the following partial differential equation (PDE):

$$
\rho j\left(\tau,t\right)=-U'\left(c\left(t\right)\right)\delta+\frac{\partial j}{\partial t}-\frac{\partial j}{\partial \tau},\ \ \tau\in(0,T], \tag{2.8}
$$

with a boundary condition: $j\left(0,t\right)=-U'\left(c\left(t\right)\right)$.

Each Lagrange multiplier is forward-looking because it captures future repayment costs in the form of a continuous-time present-value formula. The first term, $-U'\delta$, is a disutility flow associated with the future marginal cost of coupon payments. The second and third terms, $\partial j/\partial t$ and $\partial j/\partial \tau$, capture the change in flow utility as time and the maturity of the bond progress. For interpretation purposes, it is convenient to translate the multiplier j from utiles into consumption units. This change of units is useful to transform each multiplier into a financial cost. Define a transformed multiplier via $v\left(\tau,t\right)\equiv -j\left(\tau,t\right)/U'\left(c\left(t\right)\right)$. We refer to v as the *domestic valuation* of a (τ,t)-bond. Aided with this definition, we re-express the first-order conditions (2.7) and (2.8) as

$$
\frac{\partial q}{\partial \iota}\iota\left(\tau,t\right)+q\left(t,\tau,\iota\right)=v\left(\tau,t\right), \tag{2.9}
$$

and the PDE,

$$r\left(t\right)v\left(\tau,t\right)=\delta+\frac{\partial v}{\partial t}-\frac{\partial v}{\partial\tau},\text{ if }\tau\in\left(0,T\right],\tag{2.10}$$

with terminal condition $v\left(0,t\right)=1$. The rate $r\left(t\right)$ is

$$r\left(t\right)\equiv\rho-\frac{U''\left(c\left(t\right)\right)c(t)}{U'\left(c\left(t\right)\right)}\frac{\dot{c}(t)}{c(t)}.\tag{2.11}$$

Different from $r^{*}\left(t\right)$, the rate $r\left(t\right)$ is the infinitesimal discount factor. Under CRRA utility, $U(c)=\frac{c^{1-\sigma}-1}{1-\sigma}$, the internal discount factor satisfies the classical formula $r(t)=\rho+\sigma\dot{c}(t)/c(t)$. There is a remarkable connection between the domestic valuation (2.10) and the market-price equations (2.4). Both, the domestic valuation and the market price, are net-present values of the cash-flows associated with each bond. The only difference between them is the interest used to discount. In the case of the market price the bond is discounted at r^{*} whereas the domestic valuation uses r. The optimal issuances given by (2.9) depend on the spread between the two valuations. The next proposition summarizes the discussion into a full characterization of the problem's solution:

Proposition 1. *(Necessary conditions) If a solution $\{c(t),\iota\left(\tau,t\right),f\left(\tau,t\right)\}_{t=0}^{\infty}$ to (2.6) exists then domestic valuations satisfy equation (2.10),optimal issuances $\iota\left(\tau,t\right)$ are given by the issuance rule (2.9) and the evolution of the debt distribution can be recovered from the law of motion for debt, (2.1), given the initial condition f_0. Finally, $c(t)$ and $r(t)$ must be consistent with the budget constraint (2.2) and the formula for the internal discount, (2.11).*

Proof. See Appendix D.1. □

There are two noteworthy features of the solution. The first feature is a *decentralization* result: We can interpret the solution as if the Government designates a continuum of traders, one for each τ, to give an internal price to its debt, given the internal discount factor r. Each trader then issues debt according to the rule in (2.9). Of course, the discount factor must be internally consistent with the consumption path produced by issuances.

The second feature is that issuances are given by a simple debt *issuance rule*. Consider the liquidity cost function (2.5), the optimal-issuance condition (2.9) can be written as:

$$\iota(\tau,t)=\underbrace{\frac{1}{\bar{\lambda}}}_{\text{liquidity coefficient}}\underbrace{\frac{\psi(\tau,t)-v(\tau,t)}{\psi(\tau,t)}}_{\text{value gap}}.\tag{2.12}$$

The rule states that the optimal issuance of a (τ,t)-bond should equal the product of the inverse liquidity coefficient and a value gap. This value gap is the difference between the market price ψ, and the domestic valuation of a bond v, relative to the market price. When the value gap is positive, a trader would wants to issue as much debt as possible because the market price exceeds the valuation of that debt. A force contains the desire to issue: the liquidity cost that captures the reduction in price as the issuance becomes larger. This force appears as the coefficient $1/\bar{\lambda}$. The lower $\bar{\lambda}$, the greater the issuance.

The long-run behavior of the solution can be characterized analytically. In some instances, the solution reaches a steady state and in others the solution converges asymptotically to zero consumption. Whether there is a well-defined steady state with positive consumption depends on the value of $\bar{\lambda}$. In Appendix D.3 we characterize the asymptotic behavior of the solution. In particular, we obtain an expression for a *threshold* $\bar{\lambda}_o$. A steady state exists if and only if $\bar{\lambda} > \bar{\lambda}_o$. If $\bar{\lambda} \leq \bar{\lambda}_o$, there is no steady state: consumption decreases asymptotically at the exponential rate $r^*_{ss} - \rho$ and $r(t)$ converges to a limit value $r_\infty(\bar{\lambda})$.[9] Naturally, $\bar{\lambda}_o$ depends on the Government's relative impatience.

In the case where $\bar{\lambda} > \bar{\lambda}_o$, the solution renders an analytic expression for the steady state, with market prices and valuations:

$$\psi_{ss}(\tau) = \delta\frac{1 - e^{-r^*_{ss}\tau}}{r^*_{ss}} + e^{-r^*_{ss}\tau} \text{ and } v_{ss}(\tau) = \delta\frac{1 - e^{-\rho\tau}}{\rho} + e^{-\rho\tau}. \tag{2.13}$$

Since, $\delta = r^*_{ss}$, we have that all bonds are issued at par $\psi_{ss}(\tau) = 1$ and $v_{ss}(\tau) = r^*_{ss}\frac{1-e^{-\rho\tau}}{\rho} + e^{-\rho\tau}$. Issuances at steady state, $\iota_{ss}(\tau)$ follow from (2.12), and the outstanding debt satisfies $f_{ss}(\tau) = \int_\tau^T \iota_{ss}(s)ds$, so their expressions are given by:

$$\iota_{ss}(\tau) = \frac{\rho - r^*_{ss}}{\rho\bar{\lambda}}\left(1 - e^{-\rho\tau}\right) \text{ and } f_{ss}(\tau) = \frac{\rho - r^*_{ss}}{\rho\bar{\lambda}}\int_\tau^T \left(1 - e^{-\rho s}\right) ds. \tag{2.14}$$

We can investigate these expressions to learn about the forces that govern the steady state debt profile. In a deterministic environment, the entire maturity structure is determined by the desire to spread out issuances to minimize the liquidity costs. In steady state, *issuances are increasing in maturit*y—in fact, increasing and concave in maturity. For the shortest maturity, $\iota_{ss}(0) = 0$ because domestic valuations coincide with market prices. We can see that issuances increase in maturity, which can be verified through the derivative of the issuance rule with respect to maturity,

$$\frac{\partial \iota_{ss}}{\partial \tau} = \frac{\rho - r^*_{ss}}{\bar{\lambda}}e^{-\rho\tau} > 0.$$

Intuitively, differences in valuations are higher for longer maturities, as the Government discounts future cash-flows at a compounding rate greater than r^*_{ss} whereas investors value all bonds at 1. This means that at the margin, the Government is willing to receive a lower price on a longer maturity bond, which is why it tolerates a higher price impact on longer bonds.

[9] The asymptotic discount factor $r_\infty(\bar{\lambda})$ is increasing and continuous in $\bar{\lambda}$ with bounds $r_\infty(\bar{\lambda}_o) = \rho$ and $r_\infty(0) = r^*_{ss}$.

By contrast to the maturity distribution of issuances, the distribution of the maturity of the stock of debt is decreasing in maturity. The reason is simple, there is always a bigger stock of short-term debt because there is always a constant flow of long-term bonds that become short-term bonds at steady state. In other words, the outstanding amount of short term debt is the sum of past flows of longer-term debt, something obvious from the expression for $f_{ss}(\tau)$. *Ceteris paribus*, as we increase the spread $(\rho - r^*_{ss})$ the maturity profile shifts towards longer maturities and the overall stock of debt increases. The liquidity cost coefficient decreases issuances in equal proportion.

A final property is that the Government's problem has a *dual* cost-minimization problem. The dual is the problem of minimizing the net-present value of financial expenses, given the discount $r(t)$ constructed out of a desired consumption path $c(t)$. Formally, the dual problem is

$$\min_{\{\iota(\cdot)\}\in\mathcal{I}} \int_0^\infty e^{-\int_0^t r(s)ds} \left(f(0,t) + \int_0^T \delta f(\tau,t)\, d\tau - \int_0^T q(\tau,t,\iota)\iota(\tau,t) d\tau \right) dt \tag{2.15}$$

where $r(t)$ is given by (2.11), and the minimization is subject to the law of motion of debt (2.1), the initial condition f_0, and debt prices satisfy (2.3). The object in parenthesis is the net flow of financial receipts. This dual is consistent with a mandate for a debt-management office should minimize the financial expenses of a given expenditure path. The proof is in Appendix D.2.

2.3 No liquidity costs and vanishing liquidity costs

Proposition 1 characterizes the optimal debt profile in presence of liquidity costs. We characterize the solution both without liquidity costs and at the limit when liquidity costs vanish. As expected, at the case without liquidity costs, the maturity profile is indeterminate. By contrast, the limit solution as liquidity costs vanish features an uniquely determined debt profile.

Consider first the case without liquidity costs, i.e., $\bar{\lambda} = 0$. The necessary conditions for a solution are still summarized in Proposition 1. Since the issuance rule (2.9) still holds, any interior solution must satisfy an equality between domestic valuations and prices, $v(\tau,t) = \psi(\tau,t)$ for any τ. This requires $r^*(t) = r(t)$.

This feature of the limit solution should be familiar: internal discount-factors are equal to the interest rate in standard consumption-savings problems. This observation is enough to characterize the solution at the limit. Denote the market value of the Government's debt as:

$$B(t) \equiv \int_0^T \psi(\tau,t)\, f(\tau,t)\, d\tau. \tag{2.16}$$

Proposition 7 in Appendix D.4 shows that any solution of the Government problem with $\bar{\lambda} = 0$ yields the same value as a consumption-savings problem with a single instantaneous bond in amount $B(t)$, a budget constraint $\dot{B}(t) = -r^*(t)B(t) + y(t) - c(t)$ and an initial condition given by $B(0) = \int_0^T \psi(\tau, 0) f_0(\tau) d\tau$. Hence, there are infinitely many solutions that satisfy (2.16) for a $B(t)$ that solves the problem with an instantaneous bond. The intuition is simple, given that the yield curve is arbitrage free and the discount factor coincides with the interest rate there is no way to structure debt to reduce the internal cost of debt. All bonds are redundant but the path of consumption is consistent with $r^*(t) = r(t)$ and an intertemporal budget.

Now consider the limit solution as $\bar{\lambda} \to 0$. In Appendix D.5, Proposition 8, we present a general formula for the limiting distribution. In the particular case where $\delta = r^*_{ss}$, the *limiting issuance distribution* is determinate and equal to:

$$\lim_{\bar{\lambda} \to 0} \iota_\infty(\tau) = \frac{1 - e^{-r^*_{ss}\tau}}{1 - e^{-r^*_{ss}T}}\kappa,$$

where κ is a positive constant that guarantees that the budget constraint is consistent with zero-consumption at the limit. The debt profile shares the qualitative features of the case with positive liqudity costs of Proposition 1: debt issuances are also tilted toward long term bonds, but now ρ plays no role. The result shows that there is a discontinuity in the limit as liquidity costs vanish, since for any arbitrarily small cost the distribution is determined. This limiting solution can be employed as a s*election device* that determines the maturity structure.

2.4 Quantitative analysis

Computational method. Despite the analytical expressions for the steady state, the computation of the transitional dynamics produced by the solution to the problem requires a numerical algorithm. The idea follows directly from Proposition 1: Given a guess for $\{c(t)\}$, we obtain the domestic discount factor $r(t)$ through equation (2.11). This discount factor produces valuations $\{v(\tau, t)\}$ according to (2.10), which, in turn, determine the issuances $\iota(\tau, t)$ through the optimal issuance rule (2.9). Issuances produce a path of debt profiles $\{f(\tau, t)\}$ obtained from the law of motion of debt, (2.1). Given the produced debt profiles and the issuance policies, the budget constraint, equation (2.2), determines a new consumption path. A transition to the steady state is a fixed point problem in $\{c(t)\}$, where the guess and resulting paths consumption paths should coincide. All in all, the solution boils down to a fixed point in $\{c(t)\}$. We present the details of this numerical algorithm in Appendix E.

Calibration. To provide further insights, we calibrate the model to the Spanish economy. The objective of the calibration is to illustrate the ability of the model to generate realistic debt profiles, and to study the qualitative and quantitative responses to unexpected income and interest rate shocks. Following Aguiar and Gopinath (2006), we set the coefficient the value of the coefficient of the utility function to, σ, to 2, and the long-run annual risk-free rate, r^*_{ss}, to 4 percent. We normalize income y_{ss} to 1 and the maximum maturity, T, to 20 years, as roughly 90 percent of Spanish debt has a maturity below 20 years. The coupon, δ, is set to 4 percent so

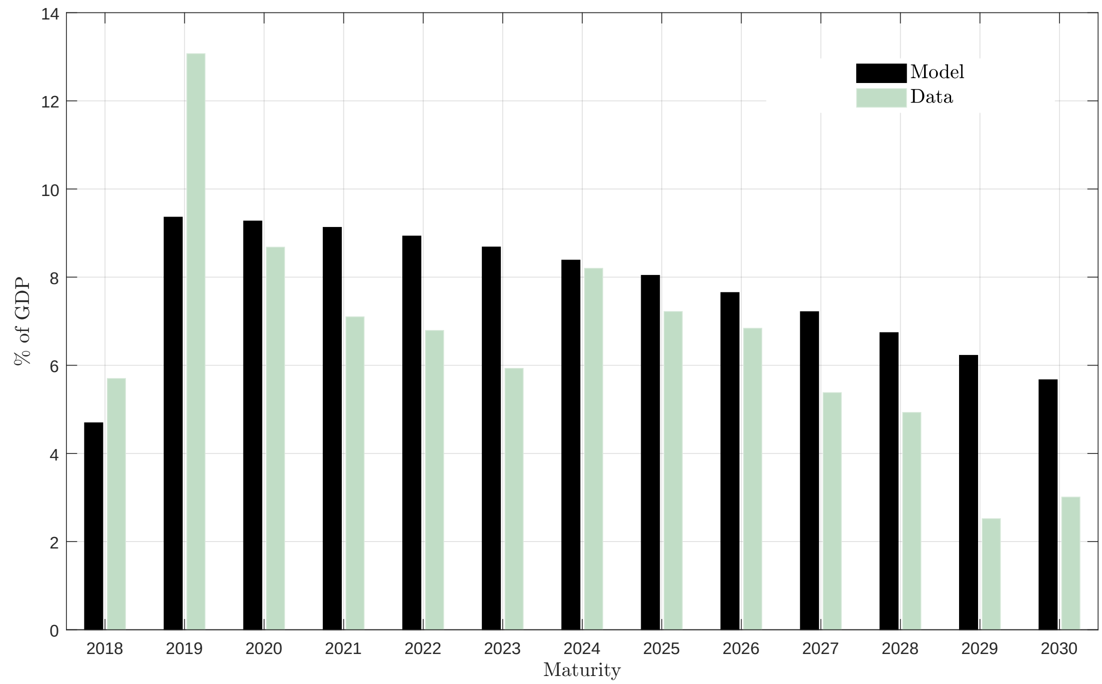

Figure 2.1: Maturity distribution. Data for Spain in July 2018 and model in steady state.

the market price equals one at all maturities. Steady state output is normalized to one. In order to calibrate the price impact of bonds, $\bar{\lambda}$, we need to set the values to the arrival rate, μ, and the spread, η. In the model, η represents the cost of capital for market makers. We calibrate this spread to 150 bp. As a reference, we obtain an approximation of the cost of capital of the biggest five US banks in terms of their assets and compute the average spread.

We jointly calibrate the discount factor, ρ, and the arrival rate, μ, to match the average level of Spanish public debt and to replicate the average time that market makers require to exhaust bond inventories. In particular, the calibration tries to match to two targets: First, the Spanish Government Net Debt, obtained from the IMF WEO, was 46 percent of GDP over the period 1985-2016. Second, according to Fleming and Rosenberg (2008), US primary dealers exhaust 60 percent of their inventory one week after they participated in the US treasury primary market.These two targets imply that, at steady state,

$$\frac{\mu T}{\int_0^T \iota(\tau) d\tau} = 0.6, \text{ and } \int_0^T f(\tau) d\tau = 0.46.$$

Accordingly, we set the value of μ to 0.0011 and the value of ρ to 0.0416. The implied value of $\bar{\lambda}$ is 7.08. Note that $\frac{\partial q}{\partial \iota} \frac{\iota}{q} \approx \frac{\partial q}{\partial \iota} \frac{\iota}{\psi} = -\frac{1}{2} \bar{\lambda} \iota$. We can compute the elasticity of auction prices with respect to changes in the size of the issue that is produced by this calibration, to get a sense of the magnitude of price impact. The maximum value of issuances is 0.003, for a maturity of 20 years, and hence the elasticity of the auction price equals $-\frac{1}{2} \times 7.08 \times 0.003 = 0.011$. This means that if the Government duplicates the size of its steady-state issuance of the 20 year bond, its price falls by 1.1 percent at the time of the auction.

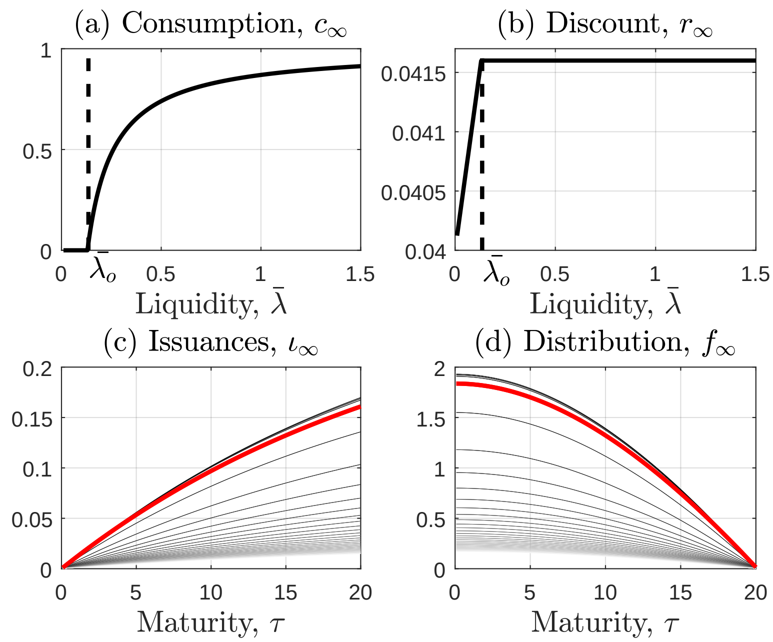

Figure 2.2: Steady-state equilibrium objects as a function of the liquidity cost $(\bar{\lambda})$. Note: The thick red line in panels (c) and (d) indicates the values at the threshold $\bar{\lambda}_o$.

Steady state. Figure 2.1 presents the steady state debt profile generated by the calibrated model against the data corresponding to Spanish debt profile of July 2018. Admittedly, the comparison has some limitations: we compare a steady state object of the model with an empirical counterpart at a particular period. Nevertheless, the figure shows that albeit the simplicity, the model reproduces the decreasing maturity profile of Spanish debt.

Under the calibration, the model has a proper steady state because the calibration satisfies $\bar{\lambda} > \bar{\lambda}_o$. Figure 2.2 displays steady-state objects as functions of $\bar{\lambda}$. The value of the threshold $\bar{\lambda}_o$ is 0.14, well below the calibrated value of $\bar{\lambda}$ which is 7.08. Panels (a) and (b) show how, for liquidity costs below the threshold, asymptotic consumption c_∞ is zero as no steady state exists. Similarly, as liquidity costs vanish, the discount factor r_∞ converges to the annual risk-free rate of 4 percent. The thick red line, panels (c) and (d) indicates the values at the threshold $\bar{\lambda}_o$. The black lines above it are the values for $\bar{\lambda} \leq \bar{\lambda}_o$. The limiting distribution is approached as $\bar{\lambda}$ approaches zero, but is relatively similar to the distribution at the threshold. This implies that, even in the case of liquidity cost values such that no steady state exists, the debt and issuance profiles are approximately the same as those at $\bar{\lambda} = \bar{\lambda}_o$.

Responses to unexpected shocks. We now study transitions after unexpected shocks to income or the short-term rate. Transitions are initiated from a steady state debt profile. In the experiments we reset either $y(0)$ or $r^*(0)$ to an initial value, and then let the variable revert back to steady state. Once either variable is reset, the entire path is anticipated. We study the dynamics of the debt profile. These dynamics are governed by two forces: *consumption* *smoothing* and *yield management.* Both forces are counterbalanced by liquidity costs.

Consumption smoothing refers to the management of the debt profile with the purpose of smoothing consumption along a transition. Consumption smoothing operates when the Government is risk averse, $\sigma > 0$. This force operates even without liquidity costs, but the force has an effect on the debt profile only with positive liquidity costs, $\bar{\lambda} > 0$. Without liquidity costs, permanent income and the path of interest rates determine the consumption path, but the debt maturity profile is indeterminate, as we discuss above. By contrast, when $\bar{\lambda} > 0$, consumption smoothing influences the debt profile: this force is entirely summarized by the internal discount $r(t) = \rho + \sigma \dot{c}(t)/c(t)$. In fact, when r^* is constant ($\psi = 1$) the value gap equals $1 - v$. In such cases, consumption growth shapes the maturity profile directly from the solution to v presented in Proposition 1: if a shock produces a growing consumption path, the domestic discount increases and the debt distribution tilts towards longer maturities, as valuations at longer horizons are more affected. Notice how this mechanism also induces an increase in the volume of issuances.

Yield management is a more subtle force. This force determines how the debt profile changes with bond prices, $\psi(\tau, t)$. The force is present even when we shut down consumption smoothing, $\sigma = 0$. This force is more subtle because we can only uncover it in presence of liquidity costs, $\bar{\lambda} > 0$. Without liquidity costs, the internal discount coincides with the interest rate, $r(t) = r^*(t)$, and since bond prices are arbitrage free, the Government is indifferent between issuing at any two maturities. With liquidity costs, even if consumption smoothing is shut down, the Government is no longer indifferent. The reason is that liquidity costs allow for a gap between $r(t)$ and $r^*(t)$. Any gap between these two rates gets compounded differently along different horizons, and thus the Government is not indifferent between issuing at two different maturities. We can again observe how this force operates through the value gap. When we shut down consumption smoothing, valuations are constant and equal to their steady state value v_{ss}. By contrast, a shock to rates moves the market price. The solution to (2.4) immediately reveals that a shock that temporally increases short-term rates creates a force that reduces debt issuances and shortens the maturity distribution by reducing bond prices, specially at long maturities. When consumption smoothing is turned on, there is a race between both forces.

To gather further insights on these two forces, we analyze the responses to unexpected shocks to $y(t)$ and $r^*(t)$ separately. Consider first the response to an income shock, displayed in Figure 2.3. In the experiment, income follows $dy(t) = \alpha_y (y_{ss} - y(t))\, dt$, the continuous-time counterpart of an $AR(1)$ deterministic process. We set $y(0)$ to a level 5 percent under its steady state value, corresponding to a major recession in Spain. The reversion coefficient, α_y, is set to 0.2 in line with Spanish data. In this case, the only active force is consumption smoothing, as the interest rate remains constant.

Figure 2.3 displays the transition. Panels (a) and (b) show how the fall in income produces a decline in consumption on impact followed by a recovery. The expected consumption-

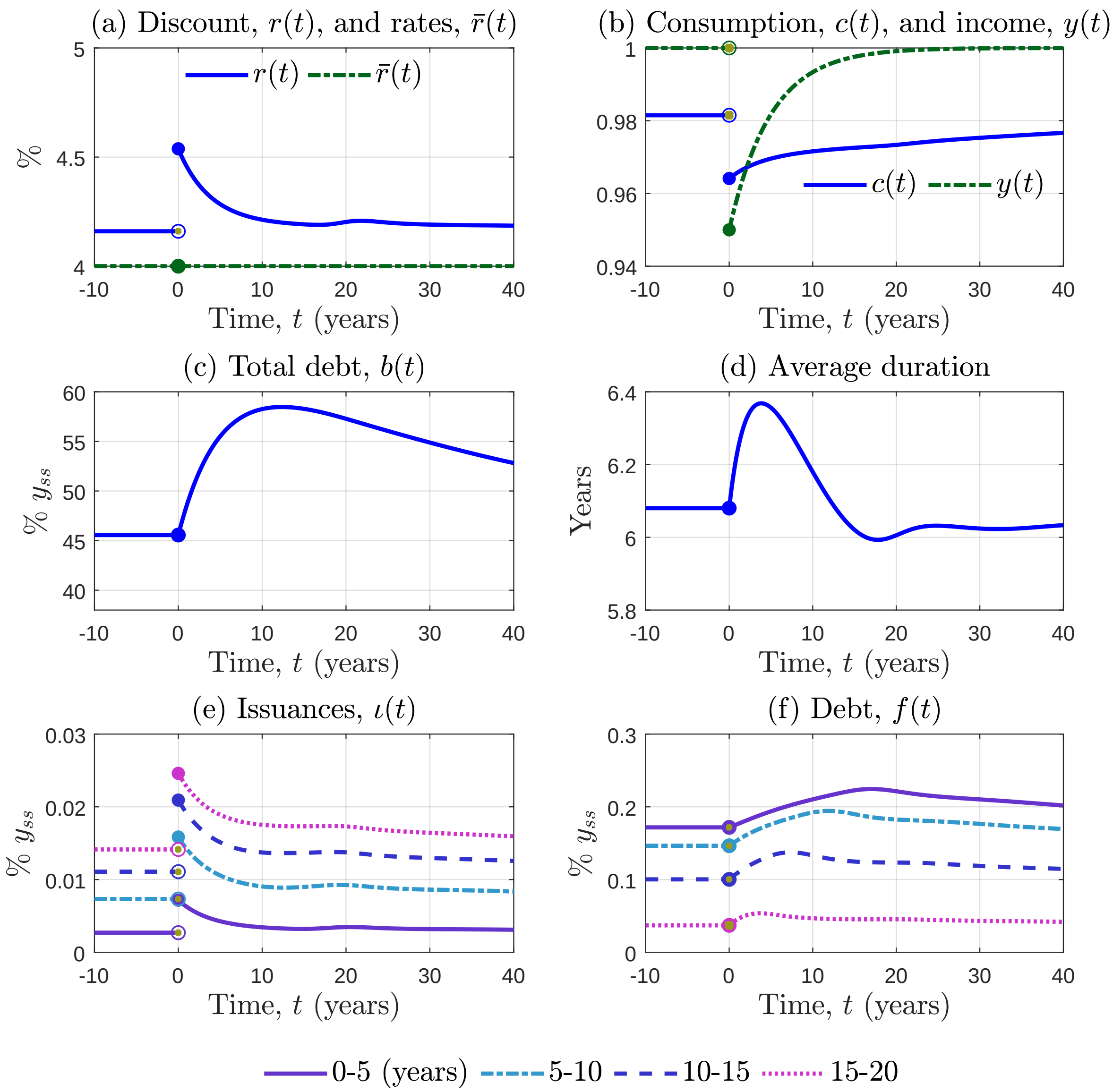

Figure 2.3: Response to an unexpected drop in income.

growth produces an increase in the domestic discount on impact which reverts down back to steady state. Since there is an increase in domestic discounting, domestic valuations decrease, which acts like a reduction in the perceived cost of debt. Given that bond prices are constant, the optimal issuance rule (2.12) dictates an increase in the issuances at all maturities, as displayed in panel (e) and hence in the outstanding amount of total debt (panels c and f), $b(t) \equiv \int_0^T f(\tau, t)\, d\tau$. One noticeable feature is that new issuances are not homogeneously distributed across maturities, but are more concentrated in longer maturities, as expected. This is because long-term domestic valuations are more sensitive to changes in the discount. This produces an *increase* in the average debt duration of the portfolio, computed as the average of the Macaulay duration of each individual bond.

The intuition for the pattern is that in response to a negative income shock, the Government attempts to smooth consumption by issuing more debt. Because of the liquidity costs, the Gov-

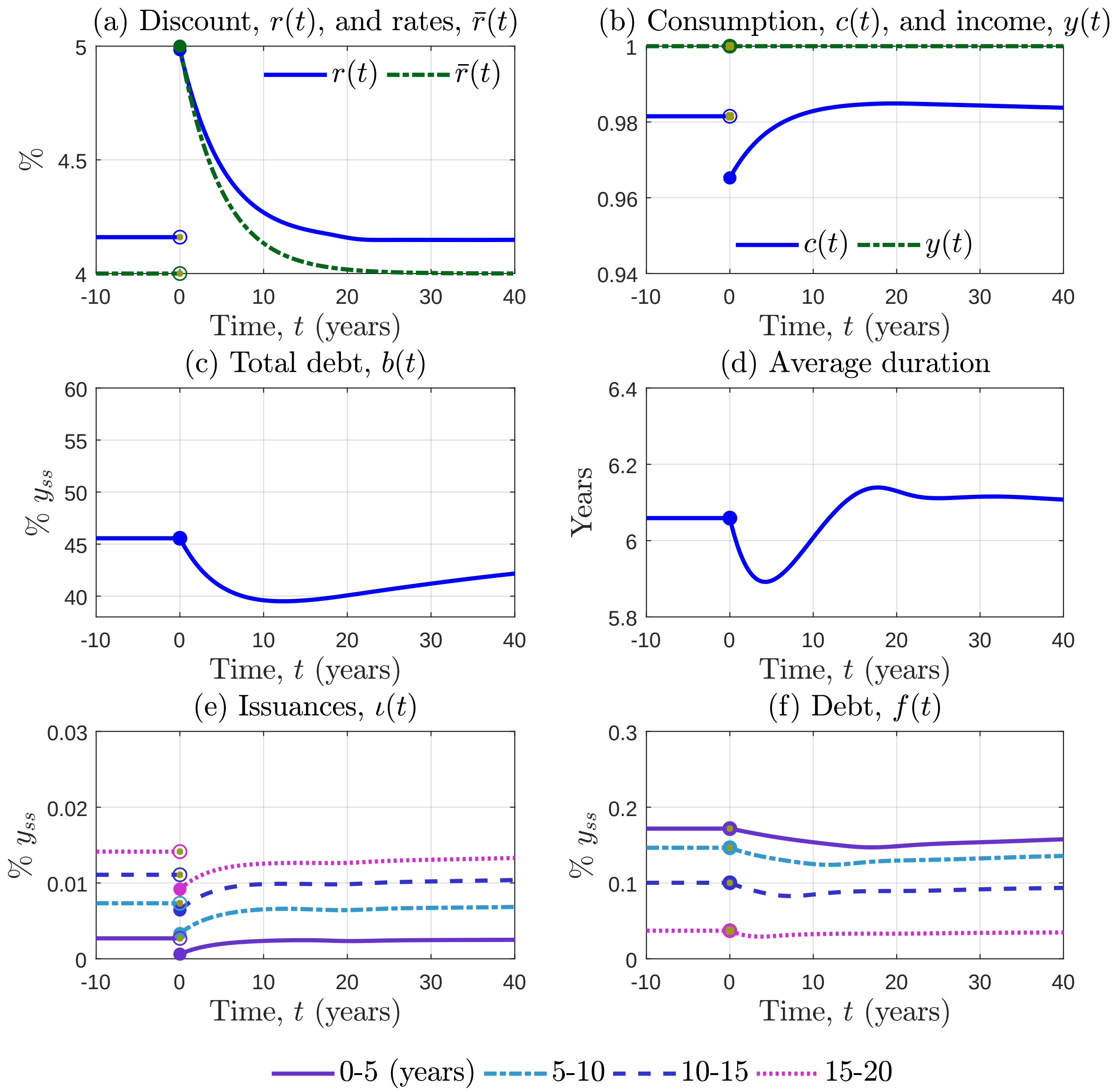

Figure 2.4: Response to an unexpected shock to interest rates.

ernment smooths consumption only partially, and this generates the pattern that lengthens the maturity.

Next, consider an unexpected shock to $r^*(t)$ as presented in Figure 2.4. We let $dr^*(t) = \alpha_r \left(r^*_{ss} - r^*(t)\right) dt$, with $\alpha_r = 0.2$ also taken from the Spanish data. On impact, the interest rate increases from 4 to 5 percent. The shock produces a consumption drop of the same magnitude as the income shock, which makes both shocks comparable. An interest shock turns on the yield management force, in addition to consumption smoothing.

We can interpret the transition as follows. On impact, the domestic discount also jumps to 5 percent, as shown in panel (a) of Figure 2.4. This narrows the valuation gap across all maturities. The initial effect of a narrower valuation gap is a decrease in all issuances (panel e). This captures the notion that upon an interest rates increase, the Government wants to sacrifice present consumption to mitigate a higher debt burden. A noticeable feature is that the interest rate shock produces an initial *reduction* in the average debt duration. This occurs even though the path of consumption resembles the one of the income shock that produces the opposite prediction regarding the debt maturity. The reason why maturity shrinks with the rate shock is that in this case, the valuation gaps decrease. The valuation gap is proxied by interest rate gap $r(t) - r^*(t)$, which becomes closer with interest shocks, but widens again as consumption recovers. This tells us that the yield management force dominates consumption smoothing.

In order to isolate the yield management channel, Figure F.1 found in the Appendix displays the transitions when $\sigma = 0$. The effects of Figure 2.4 are magnified because consumption smoothing operates in the opposite direction than yield management. In Figure F.1, once the shock arrives and bond prices decline the Government decides to directly repurchase debt, specially at long maturities, and to resell it when prices goes up.

3 Risk

In the previous section we analyze the optimal debt-management under perfect foresight. This section presents a characterization that allows for risk. The goal is to understand how the anticipation of shocks affects the shape of the optimal debt profile and the ability to insure.

3.1 The model with risk

Risk is modeled as a single jump at a random date. We introduce notation to distinguish pre- from post-jump variables: for any variable x we represent its value prior to the shock by a hat: $\hat{x}(t)$, and use $x(t)$ to express the value after the jump event.

The exogenous state $X(t)$, comprising income, $y(t)$, and the risk-free rate, $r^*(t)$, now features a jump at a random date t^o. Prior to the jump, the state is denoted as $\hat{X}(t)$. The random time is exponentially distributed with a parameter ϕ. Upon the jump, a new value of the state is drawn, $X(t^o) \sim G(\cdot|\hat{X}(t^o))$, where $G(\cdot)$ is the conditional distribution of the exogenous state given its value prior to the shock.[10] We let $\{y(t^o), r^*(t^o)\}$ take values in the compact space $[r_l^*, r_h^*] \times [y_l, y_h]$ and let $G(\cdot|\hat{X}(t^o))$ be a discrete or continuous distribution. We denote the conditional expectation operator under the distribution $G(\cdot; X(t^-))$ by $\mathbb{E}_t^X[\cdot]$.

Once the jump occurs, each element $x(t) \in X(t)$ follows a continuous mean-reverting process:

$$\dot{x}(t) = -\alpha^x (x(t) - x_{ss})$$

where α^x captures the speed of mean reversion to its steady-state value x_{ss}. Obviously, there

[10]Formally, $X(t)$ is right continuous. If the jump occurs at t^o, the left-limit $\hat{X}(t^o) \equiv \lim_{s \uparrow t^o} \hat{X}(s)$ jumps to some new $X(t^o) \sim G(\cdot; X(t^o))$.

is risk before the one-time jump, but after the jump the economy becomes the perfect-foresight one of the previous section.

With risk, the bond price satisfies a standard pricing equation with a single jump. The flow value of the bond, δ, plus the expected price appreciation should be equal to the value of the bond at the risk free rate $\hat{r}^*(t)\,\hat{\psi}(\tau,t)$. Thus, the market price is now given by:

$$\hat{r}^*(t)\,\hat{\psi}(\tau,t) = \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \phi\mathbb{E}_t^X\left[\psi(\tau,t,X_t) - \hat{\psi}(\tau,t)\right], \text{ if } t < t^o, \tag{3.1}$$

with the terminal condition $\hat{\psi}(0,t) = 1$. As before, the market price $\psi(\tau,t,X_t)$ denotes the perfect foresight price of the bond after the shock if the initial state is X_t, which still satisfies (2.4). The price that the Government receives is still given by (2.5).

Prior to the shock, the Government problem is:

$$\hat{V}\left[\hat{f}(\cdot,0), X(0)\right] = \max_{\{\hat{\iota}(\cdot)\}\in\mathcal{I}} \mathbb{E}_t\left[\int_0^{t^o} e^{-\rho t}U(\hat{c}(t))\,dt + e^{-\rho t^o}\mathbb{E}_{t^o}^X\left[V\left[\hat{f}(\cdot,t^o), X(t^o)\right]\right]\right], \tag{3.2}$$

where $V\left[\hat{f}(\cdot,t^o), X(t^o)\right]$ is the value given by (2.6). The maximization is subject to the law of motion of debt (2.1), the budget constraint (2.2), the initial condition $\hat{f}(\cdot,0) = f_0$, and bond prices are given by (3.1), just as in the perfect foresight environment.

3.2 Solution: Risk-adjusted valuations

To characterize the problem, we adapt the perfect-foresight solution to allow for risk. The proof of the solution to the problem with risk is in Appendix D.6. The approach is the same as the one used to prove Proposition 1: We construct a Lagragian and then apply infinite-dimensional optimization techniques to obtain the first-order conditions. Proposition 2 below is the analogue to Proposition 1.

Proposition 2. *(Necessary conditions) If a solution $\{c(t), \iota(\tau,t), f(\tau,t)\}_{t=0}^{\infty}$ to (3.2) exists, it satisfies the same conditions of the perfect-foresight solution (Proposition 1) where valuations prior to the shock now satisfy the PDE*

$$\hat{r}(t)\,\hat{v}(\tau,t) = \delta + \frac{\partial\hat{v}}{\partial t} - \frac{\partial\hat{v}}{\partial\tau} + \phi\mathbb{E}_t^X\left[v(\tau,t)\frac{U'(c(t))}{U'(\hat{c}(t))} - \hat{v}(\tau,t)\right],\ \tau\in(0,T], \tag{3.3}$$

with boundary condition $\hat{v}(0,t) = 1$.

Proposition 2 shows how the decentralization scheme of the perfect foresight problem carries over to the problem with risk: The optimal issuance rule still holds, but applies to the pre-shock valuations, which now are adjusted for risk. After the shock, the transition and asymptotic limits coincide with those of the perfect foresight problem that takes as initial condition the debt profile at the time of the shock.

This decentralization shows that the effect of risk is captured exclusively through a correction in domestic valuations. The PDE is similar to the one that corresponds to the riskless case, equation (2.10). The only difference is the last term on the right. This term is akin to the correction in any expected present-value formula that features a jump. In the particular case of domestic valuations, the expression captures that upon the shock arrival valuations jump from $\hat{v}(\tau, t)$ to $v(\tau, t)$, latter term being corrected by the ratio of marginal utilities. This ratio of marginal utilities captures an "exchange rate" between states. The domestic valuation after the shock, $v(\tau, t)$, is measured in goods after the arrival of the shock, but the valuation $\hat{v}(\tau, t)$ is expressed in goods prior to the shock. The ratio of marginal utilities tells us how goods are relatively valued in terms of utilities, before *via-à-vis* after the shock. If a shock produces a drop in consumption, the ratio of marginal utilities associated with that state is greater than one. This correction captures that payments associated with a bond are more costly in states where marginal utility jumps. This extra kick in the valuations affect bonds of different maturity differently, as we illustrate next.

Although Proposition 2 provides a characterization of the solution with risk, its computation involves solving a fixed point problem over a family of debt distributions. The reason for this complexity is that each instant prior to the shock is associated with a jump in consumption from $\hat{c}(t)$ to $c(t)$. The size of the jump is itself a function of the distribution $f(\cdot, t^o)$. The potential consumption jump affects the choice of issuances, through its influence on valuations. A solution to the transition then is a fixed point in a family of debt distributions, one for each instant prior to the shock, that must be consistent with a consumption path after the shock. This complexity renders the numerical solution of the problem unfeasible, and one is forced to employ only approximate solutions. If instead of a single jump, we were to consider multiple jumps, the complexity would escalate.[11]

Our approach to analyze the influence of risk is to study the risky steady state (RSS) of the problem. The approach is similar to the one introduced by Coeurdacier et al. (2011). In our context, the RSS is defined as the asymptotic limit of variables prior to the realization of the shock. In other words, the RSS is the state to which the solution converges when shocks are expected to arrive but have not yet materialized. In our setting this amounts to state that we study the solution as t^o converges to infinity. We obtain an explicit solution to the valuations in the RSS:

[11] An analogous challenge appears in models of incomplete markets which is why the literature uses the approximation in Krusell and Smith (1998).

$$\hat{v}_{rss}(\tau) = e^{-\int^{\tau}(\hat{r}_{rss}+\phi)du} + \int_0^{\tau} e^{-(\hat{r}_{rss}+\phi)(\tau-s)} \left(\delta + \phi \mathbb{E}^X_{rss}\left[\frac{U'(c(0))}{U'(\hat{c}_{rss})} v(\tau,0)\right]\right) ds, \text{ for } \tau \in (0,T]. \tag{3.4}$$

where $v(\tau,0)$ is the valuation of the perfect-foresight problem with initial condition f_{rss}.

The advantage of the RSS approach is that the RSS can be characterized as a fixed point in the space of consumption policies, an object that is feasible to compute. In fact, the method to calculate a solution adds only one more unknown to the perfect-foresight algorithm; now we must obtain a RSS consumption. Namely, the idea is to propose a guess of c_{rss} and the path $\{c(t)\}$ and then compute the valuations using (3.4) for the RSS and (2.10) for the valuation after the shock. We then obtain issuances from the simple issuance rule (2.12). Finally, given issuances we compute the debt profile in the RSS using the law of motion of debt (2.1). Thus budget constraint (2.2) yields a new c_{rss} and new path $\{c(t)\}$. Naturally, $\{c(t)\}$ is the solution to the perfect-foresight problem where $f(\tau,0) = \hat{f}_{rss}(\tau)$.

We can exploit a decomposition to explain how risk alters the debt distribution from a steady-state to the RSS.The main force that shapes the maturity profile under risk is *insurance*. Insurance is captured by the ratio of marginal utilities $U'(c(0))/U'(\hat{c}_{rss})$ in equation (3.3). Insurance is achieved in two ways, via *self-insurance* and *hedging*. By self insurance we refer to the fact that, in some cases, the Government may decide to reduce its RSS debt stock to minimize the fall in consumption after a negative shock. If the shock produces a change in the valuation of the portfolio, the Government might in principle, hedge the jump in consumption by building a portfolio that offsets the impact of the shock. Some examples without liquidity costs are useful to understand self-insurance and hedging in further detail.

3.3 No liquidity costs

We now consider the case without liquidity costs, $\bar{\lambda} = 0$. With positive liquidity costs, adjustments in portfolios are costly. By studying the problem at the limit where liquidity costs are zero, we can understand the extent to which the Government can obtain insurance given the set of bonds it has available. Thus, it clarifies the extent to which liquidity costs limit insurance.

Towards that goal, we note that the necessary conditions of the problem are the same with and without liquidity costs, including the issuance rule. If the issuance rule holds, issuances are bounded if and only if valuations and prices are equal, $v = \psi$ and $\hat{v} = \hat{\psi}$. If we substitute $v = \psi$ and $\hat{v} = \hat{\psi}$ in the PDE for valuations, equation (3.3), and subtract the bond PDE from both sides, equation (3.1), we obtain a premium condition that must hold for all bonds:[12]

$$\hat{r}(t) - \phi \mathbb{E}_t^X \left[\frac{U'(c(t))}{U'(\hat{c}(t))} \cdot \frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)} \right] = \hat{r}^*(t) - \phi \mathbb{E}_t^X \left[\frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)} \right]. \tag{3.5}$$

The analysis of the different solutions of equation (3.5) provides useful information about the role of hedging and self-insurance. We analyze each case in turn.

Perfect hedging: replicating the complete-markets allocation. Equation (3.5) replicates the complete-markets allocation when consumption follows a continuous path, i.e., when $\hat{c}(t^o) = c(t^o; X_{t^o})$ for any realization of the shock. This is because international investors are risk-neutral and the Government is risk averse. If consumption does not jump, condition (3.5) implies $\hat{r} = \hat{r}^*$. In a complete markets economy consumption growth satisfies $\frac{\dot{c}(t)}{c(t)} = \frac{r^*(t)-\rho}{\sigma}$, the same rule that it follows in a deterministic problem. Naturally, there is no RSS with positive consumption if $r^*(t) < \rho$, but consumption converges asymptotically towards zero.

Consumption does not jump when it is possible to form a perfect hedge, a debt profile that generates a capital gain that exactly offsets the shock. Any shock changes the net-present value of income. Given the path of rates, the optimal consumption rule and the initial post-shock consumption produce a net-present value of consumption. A perfect hedge thus produces the capital gains such the net present value of consumption minus income at the time of the shock (denoted by $\Delta B(t^o, X(t^o)))$ is covered to the point where pre- and post-shock consumption are equal: $c(t^o) = \hat{c}(t^o)$. This must be true for any shock. In the context of the model, a perfect hedge exists if the debt distribution satisfies at all times t

$$\Delta B(t, X(t)) = -\int_0^T \left(\psi(\tau, t; X(t)) - \hat{\psi}(\tau,t)\right) \hat{f}(\tau,t)\, d\tau \tag{3.6}$$

for any possible realization of $X(t)$. This family of equations is a generalization of the discrete-shock and discrete-bonds matrix conditions that guarantee market completion in Duffie and Huang (1985), Angeletos (2002) or Buera and Nicolini (2004).[13]

In our model, perfect hedging is available in the case of an interest rate shock taking N possible values. Then, there is continuum of solutions that satisfy equation (3.6). In this case we can use a range of maturities $[0, T]$ that is as short as we want to hedge. The shorter the range, the more extreme the positions we obtain. A second observation has to do with the direction of hedges. Consider the case of a single jump in interest rates ($N = 1$). To offset the reduction in the net-present value of income, the debt profile must generate an increase in

[12]This equation is recovered also by solving the problem with $\lambda = 0$ directly. The proof is available upon request.

[13]In the case of discrete shocks and discrete bonds, the existence of complete-markets solution requires the presence of at least $N + 1$ bonds for N shocks. In the case of a continuum of shocks, the condition requires the invertibility of a linear operator. Proving conditions on G that guarantee that family of solutions exceeds the scope of the paper.In this case, equation (3.6) is just a system of N linear integral equations, for every t, known as Fredholm equations of the first kind.

wealth. This requires an increase in short-term assets and long-term liabilities.

No hedging: only self insurance. The opposite to the complete-markets outcome is the case of income shocks. In this case bond prices do not change, $\psi = \hat{\psi} = 1$. Therefore, it is not possible to generate capital gains with a debt profile. Instead, the only solution to (3.5) is:

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \frac{\hat{r}^*(t) + \phi\left(\mathbb{E}_t^X\left[\frac{U'(c(t))}{U'(\hat{c}(t))}\right] - 1\right) - \rho}{\sigma}.$$

This is a situation in which no hedging is available, because the asset space does not allow any form of external insurance. Instead, the Government must self-insure. Self insurance is captured by the ratio of marginal utilities which effectively lowers $\hat{r}(t)$. To solve for consumption, this extreme case coincides with a single-bond economy without interest-rate risk. The jump in consumption is given by the jump in the net present value of income. The solution to $c(t)$ in this case is known and found, for example in Wang et al. (2016). Without presenting the solution to $c(t)$, we can already note that the ratio of marginal utilities increases as the level of assets falls. This means that, provided a sufficiently low level of debt, the economy reaches a RSS with positive consumption. The convergence in consumption is a manifestation of self-insurance.

General case. The general case with both income and interest rates shocks described by equation (3.5) features an intermediate point between the two extreme cases described above as both a partial hedging and self-insurance emerge.[14] Furthermore, as long as the support of the shocks has cardinality N the debt profile is indeterminate, as only N points of the debt distribution are pinned down.[15]

3.4 Quantitative analysis: the risky steady state

We now return to our calibration to analyze how quantitatively important are the forces that shape the debt profile presence of risk. To do so, we obtain a RSS and compute post-shock transitions. We compare these dynamics against the dynamics that begin at the deterministic steady state (DSS) and follow the responses of unexpected shocks. Naturally in both cases the economy converges to the DSS as time progresses. We calibrate the shock intensity ϕ to 0.02. The value for the intensity is obtained by computing the cross-country frequency of a yearly output decline greater or equal than 5 percent. We maintain the rest of the calibration.

The comparisons are reported in Figures 3.1 and 3.2 where solid lines denoted by 'DSS' are the dynamics of the perfect-foresight section and dashed lines denoted by 'RSS' describe the

[14] This case can be solved via dynamic programming using aggregate debt at market values $\hat{B}$ as a state variable. $\hat{B}$ is defined as in (2.16) with pre-shock prices and debts. Equation (3.5) holds for every maturity, so given $\hat{B}$, it represents a family of first-order conditions for $f(\tau, t)$. The debt profile then is associated with an insurance cost of $\dot{\hat{B}}$.

[15] The proof is similar follows the one in Appendix D.8.

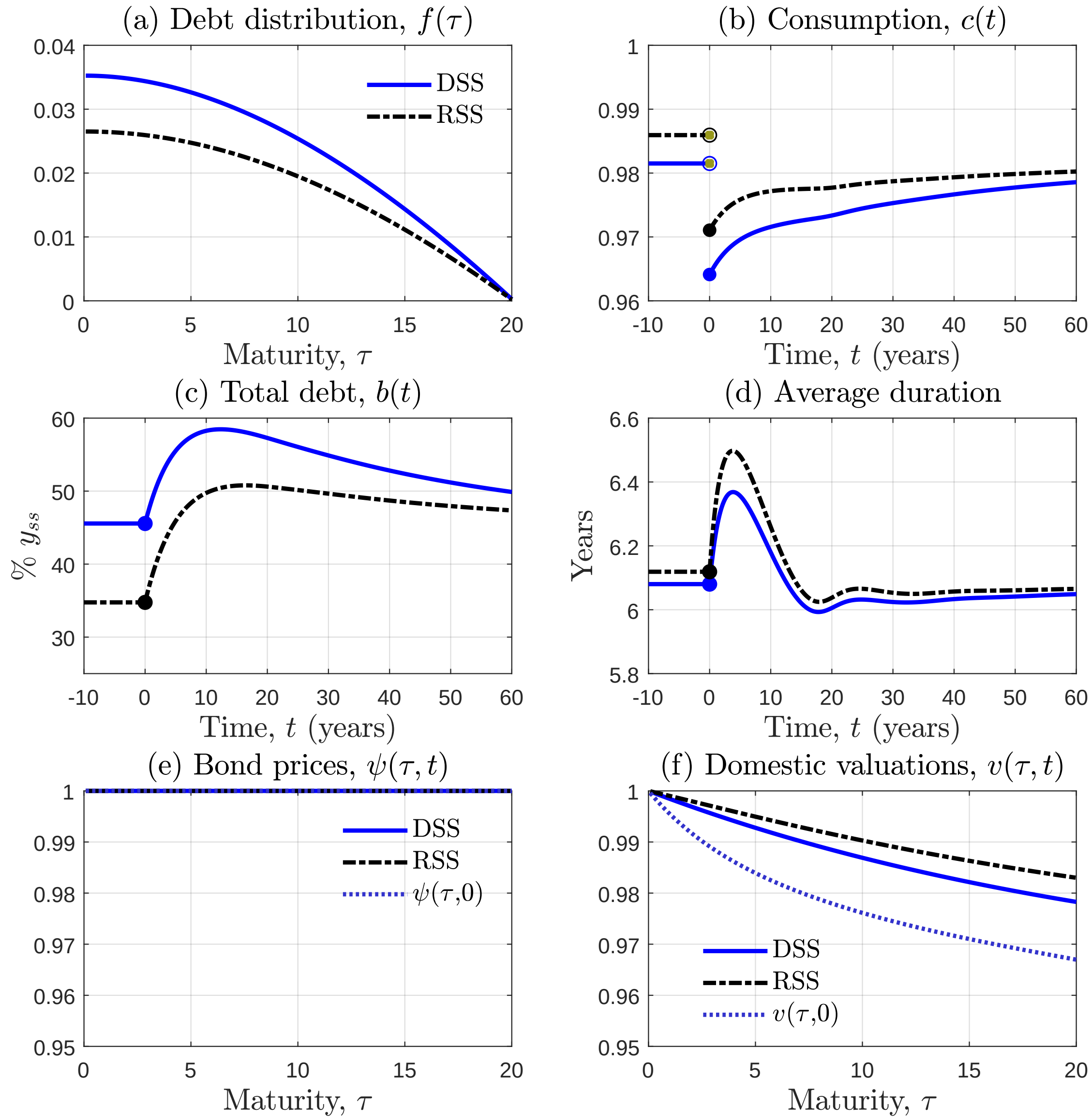

Figure 3.1: Response to a shock to income. RSS is model starting at the risky steady state and DSS at the deterministic steady state.

dynamics when the shock is expected.

Figure 3.1 reports responses to a shock that produces a 5 percent income drop. Compared to the DSS, in the RSS the total debt stock prior to the shock (panels a and c) is smaller, 35 per cent of GDP compared to 45 per cent. The reason for this decline is the self-insurance force. Since rates do not move, the hedging ex-ante and yield-management ex-post are shut down. Thus, the decline in the stock of debt is a race between two forces, consumption smoothing and self-insurance. These forces are encoded in the valuations. Recall from the RSS valuation formula (3.4) that two terms influence RSS valuations, the ratio of marginal utilities and post-shock valuations. Panel (f) displays the valuation immediately after the shock arrival, $v(\tau,0)$. This value is lower than the DSS valuations, $v_{ss}(\tau)$, reflecting that once the shock hits, $r(t)$ will increase. Despite that $v(\tau,0)$ is lower than $v_{ss}(\tau)$, the RSS valuation sare higher, $v_{rss}(\tau)$

greater than $v(\tau, 0)$, and this mechanically occurs because of the larger ratio of marginal utilities $\frac{U'(c(0))}{U'(\hat{c}_{rss})} = \left(\frac{0.986}{0.971}\right)^2 = 1.03$. This tells us that self-insurance and consumption smoothing operate in opposite directions.

Another feature is that duration is slightly higher in the RSS (panel d). The difference among the patterns can be entirely explained through the valuations at the RSS, $\hat{v}_{rss}(\tau)$, displayed in panel (f). Again, this pattern is the result of the two countervailing forces. The ratio of marginal utilities steers valuations towards a shorter duration because shorter duration bonds have a higher change avoiding a shock, which increases the valuation of its coupons. However, as we saw under perfect foresight, once the shock hits, consumption-smoothing extends the average duration to reduce rollovers. This is captured by the steepness of $v(\tau, 0)$. All in all, self-insurance dominates, producing a lower stock of debt and consumption smoothing tilts the distribution towards longer duration.

Next, consider the case of an interest rate shock. The stock of debt is 30 per cent of GDP in the RSS, a smaller value than the 45 per cent in the deterministic steady state, as displayed in panel (c) of Figure 3.2. The reduction is seen in all maturities (panel a). Again, we can dissect what forces drive these results by examining the effects on valuations and prices. As with income shock, consumption-smoothing operates in the opposite direction of self-insurance. Furthermore, the internal discount is higher in this example —we can see this in the lower $v(\tau, 0)$ valuations. The result is that valuations are very similar between the deterministic steady state and the risky steady state.

We can understand the role of the hedging by comparing the RSS solution to the RSS of a risk-neutral government, $\sigma = 0$. This comparison is reported in in Figure F.2 in the Appendix. The Figure reveals that the overall stock of debt is not modified substantially. However, we do see the presence of some hedging. As discussed in the case without liquidity costs, the correct hedge to an interest-rate shock is to hold more long-term debt on the margin and less short-term debt. We observe that pattern, but the effect is small. This suggests that liquidity costs introduce a cost to setup the hedge ex-ante, and a cost to unwind the position. In our calibration, this cost is large and mutes the hedging force.

Once we know that valuations do not change and that hedging is not effective, we can conclude that the reduction in debt follows from yield management. That is, from the reduction in the bond price in the RSS relative to the DSS (panel e). In the RSS, the price of bonds anticipates a probable rate increase in the future. This translates into an effect that is akin to bringing the rate closer to ρ. As a result, the Government avoids a high interest-rate expense and holds less debt.

In terms of the debt duration (panel d), we do not see that the anticipation of the interest shock changes the duration substantially. Most of the differences appear in ex-post, and they respond to the different influence of consumption smoothing.

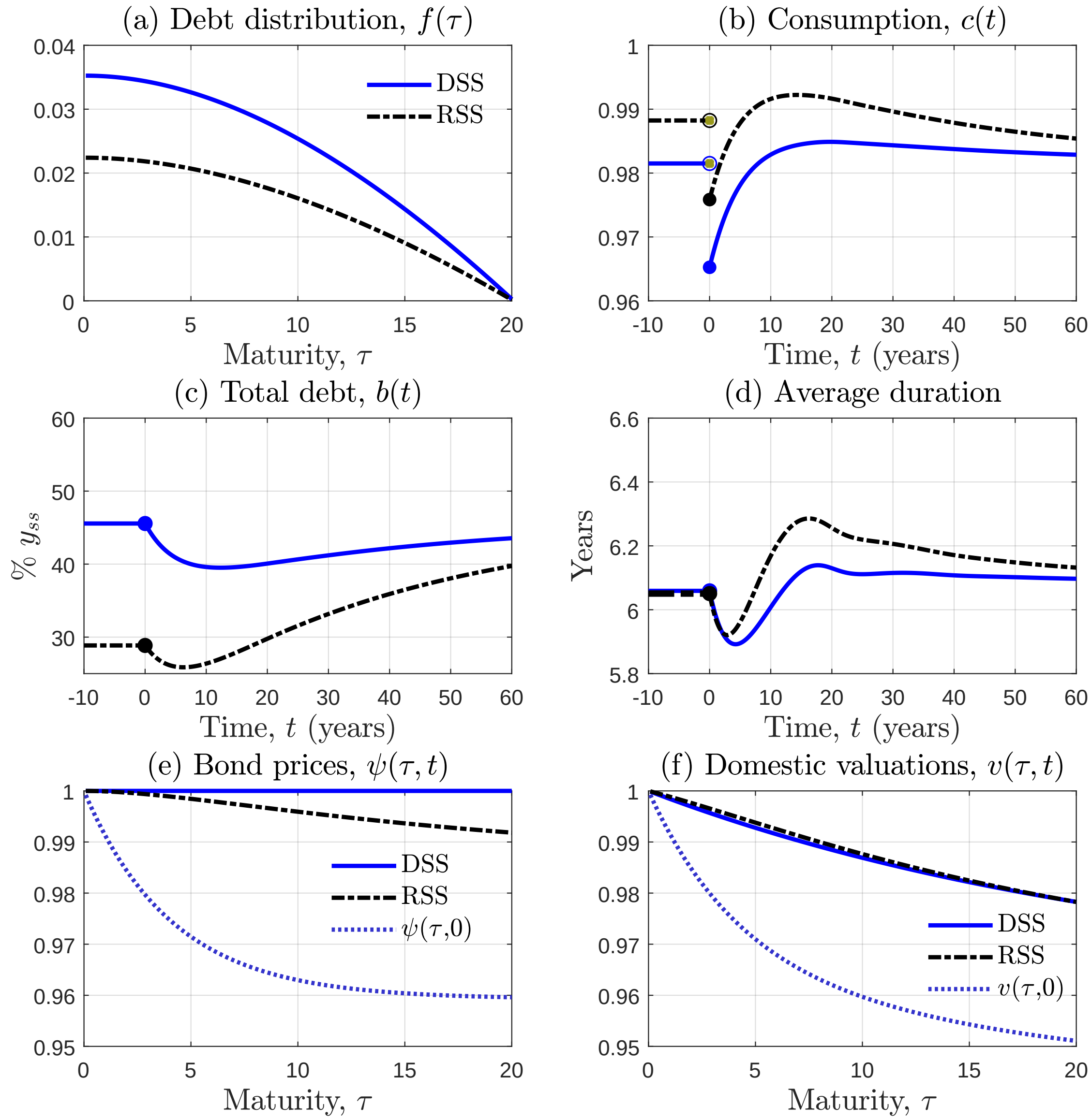

Figure 3.2: Response to a shock to interest rates. RSS is model starting at the risky steady state and DSS at the deterministic steady state.

4 Default

This section extends the model with risk to allow for the possibility of default. The nature of the problem changes because now bond prices depend on Government actions.

4.1 The option to default

Consider the model in Section 3.1 but now, upon the realization of the shock to $X(t)$, the Government has the option to default on all its debts. If the Government exercises the option, it defaults on all debts and its value is

$$V^D(t) = \int_t^\infty e^{-\rho(s-t)} U\left((1-\kappa)y(t)\right) ds + \varepsilon_t. \tag{4.1}$$

The value is the discounted utility of a fraction κ of the endowment plus a a zero-mean random variable, ε_t. The variable ε_t captures randomness around the decision to default. Thus, the post-default value is also a random variable, centered around the expected value of moving into perpetual autarky with an output loss κ. Denote by $\Theta(\cdot)$ and $\theta(\cdot)$ the cumulative distribution and the probability density function of $V^D(t)$, respectively. Note that the Government can hold foreign bonds ($f(\tau, t) < 0$). Default implies that these bonds are expropriated by foreign investors.

If the Government chooses not to default after the shock, the economy becomes the perfect-foresight economy that we have studied in Section 2. The Government strategically defaults when $V^D(t^o) > V\left[\hat{f}(\cdot, t), X(t)\right]$, that is, if the value after default is greater than the value obtained by continuing to service its debts. In the latter case, the post-shock value coincides with the perfect-foresight value of an initial debt profile $\hat{f}(\cdot, t^o)$. Therefore, at any given time the default probability is $1 - \Theta\left(V\left[\hat{f}(\cdot, t), X(t)\right]\right)$ and the probability of repayment is $\Theta\left(V\left[\hat{f}(\cdot, t), X(t)\right]\right)$.

Market prices must be adjusted for credit risk. The corresponding pricing equation is:

$$\hat{r}^*(t)\,\hat{\psi}(\tau, t) = \delta + \frac{\partial \hat{\psi}}{\partial t} - \frac{\partial \hat{\psi}}{\partial \tau} + \phi \mathbb{E}_t^X\left[\Theta\left(V\left[\hat{f}(\cdot, t), X(t)\right]\right)\psi(\tau, t, X(t)) - \hat{\psi}(\tau, t)\right], \text{ if } t < t^o. \tag{4.2}$$

After the shock, the pricing equation is again (2.4). Prior to the shock, the price equation is similar to the version with risk, but now the post-shock price is multiplied by the probability of repayment. The idea is that the expected change in the price after the shock is the perfect-foresight price times the repayment probability, $\Theta(V)\,\psi$. The option to default implies that the Government's actions affect bond prices, a new feature which alters the nature of the solution to the Government's problem.

With a default option, the Government's problem now becomes:

$$\max_{\{\hat{\imath}(\cdot)\}\in\mathcal{I}} \mathbb{E}_t\left[\int_0^{t^o} e^{-\rho t} U(\hat{c}(t))\,dt + e^{-\rho t^o}\mathbb{E}_{t^o}^X\left[\max\left\{V[f(\cdot, t^o), X(t^o)], V^D(t^o, X_{t^o})\right\}\right]\right] \tag{4.3}$$

subject to the law of motion of debt, (2.1), the budget constraint, (2.2), and the pricing equation (4.2).

In this problem, the Government *commits* at time zero to a debt program prior to realization of the shock. Notwithstanding, the Government cannot commit not to default. The assumption of commitment is different from most of the literature on sovereign default.[16] In particular, commitment eliminates the issue of "debt dilution" discussed, for instance, by Hatchondo et al. (2016). To our knowledge, the solution to this problem without commitment must rely on a numerical approximation. The solution with commitment can be interpreted as follows. Prior to the shock, there are no substantial changes in fundamentals that should trigger a default

[16] See, for instance Aguiar and Amador (2013) and references therein.

episode. If bond holdings are concentrated among a few international banks, then the Government can commit to a plan and be disciplined with an immediate sudden stop as in Dovis et al. (Forthcoming).

4.2 Default-adjusted valuations

We adapt the framework to allow for default. The main novelty is that the Government takes into account how its decisions affect market prices and the future decisions to default. To ease notation, we define $V(t+s) \equiv V\left[f\left(\cdot, t+s\right), X\left(t+s\right)\right]$.

Proposition 3. *(Necessary conditions) If a solution $\{c(t), \iota\left(\tau, t\right), f\left(\tau, t\right), \psi(\tau, t)\}_{t\in[0,\infty)}$ to (4.3) exists, it satisfies the same conditions of the perfect-foresight solution (Proposition 1) where valuations prior to the shock are replaced by valuations that satisfy the following PDE,*

$$\begin{aligned}
\hat{r}\left(t\right)\hat{v}\left(\tau, t\right) &= \delta + \frac{\partial \hat{v}}{\partial t} - \frac{\partial \hat{v}}{\partial \tau}\dots \\
&\quad + \phi\left(\mathbb{E}_s^X\left[\left(\Theta\left(V(t)\right) + \Omega(t)\right)\frac{U'\left(c\left(t\right)\right)}{U'\left(\hat{c}\left(t\right)\right)} v\left(\tau, t\right)\right] - \hat{v}\left(\tau, t\right)\right), \qquad (4.4)\\
\hat{v}\left(0, t\right) &= 1.
\end{aligned}$$

where

$$\Omega(t) = \theta\left(V(t)\right)U'\left(\hat{c}\left(t\right)\right)\dots$$
$$\int_0^T \int_{\max\{t+m-T,0\}}^t e^{-\int_z^t (r^*(u)-\hat{r}(u))du} \frac{\psi\left(m, t\right)}{2\bar{\lambda}}\left[1 - \left(\frac{\hat{v}\left(m+t-z, z\right)}{\hat{\psi}\left(m+t-z, z\right)}\right)^2\right] dzdm.$$

Proposition 3 shows that the option to default alters valuations, but that the principle of a simple issuance rule still holds. We can compare the expression for valuations with default, (4.4), to the expression for market prices (4.2). Without the option to default, prices and valuations only differ in the discount and the adjustment for risk. Default introduces two new features. The first feature is that default allows for a new form of insurance. This is captured by the repayment probability, Θ. Valuations after the shock are multiplied by the repayment probability in equation (4.4), since debt is worthless after a default. The same repayment probability appears in the market price (4.2), but unlike valuations, market prices do not include the change in marginal utilities. This captures an improvement in *insurance*. As we explained in the previous section, the increase in marginal utility is driven by a self-insurance motive. The option to default reduces this motive because the probability of repayment is low when a shock triggers an increase in the marginal utility ratio—put it simply, Θ offsets the effects of $\frac{U'(c(t))}{U'(\hat{c}(t))}$ in the valuations in particularly adverse states. This feature captures how default improves insurance under incomplete markets (an idea found for example in Zame, 1993).

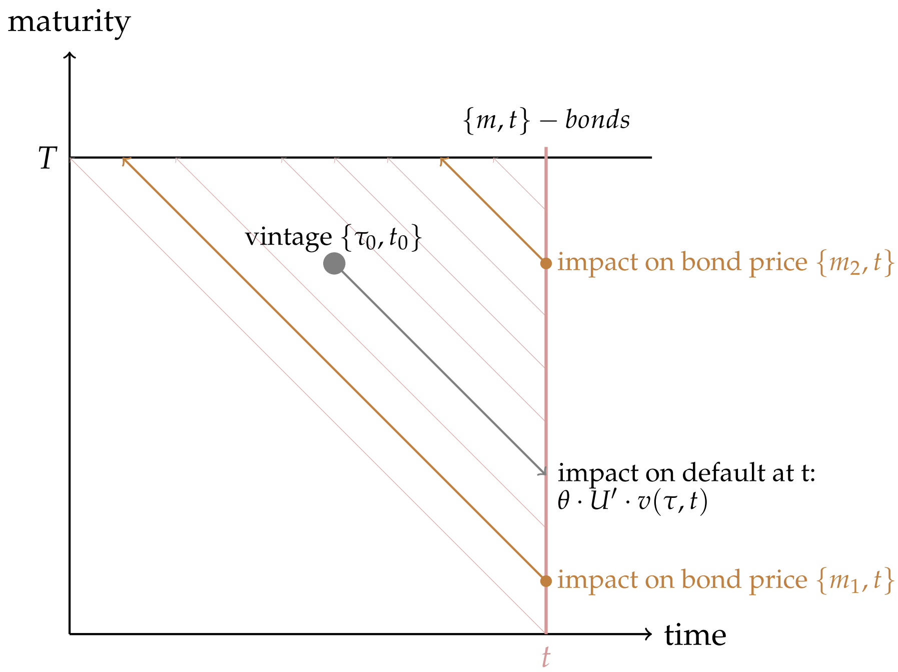

Figure 4.1: Illustration of the revenue echo effect (axes inverted).

The second feature reflects the role of *incentives*. Once the default option appears, the Government cannot commit to repay. The inability to honor debts in the future, impacts bonds prices. Hence, to evaluate how much to issue of a (τ, t)-bond, the Government should not only consider the current revenue and future repayment costs of that bond, but also consider that each marginal issuance depresses the issuance price of bonds of different maturity and issuance dates. The reduction in revenues from each issuance is the marginal cost of reducing the incentives to repay, and is captured by Ω. We refer to Ω as the *revenue-echo effect* for the reason we explain next.

We unpack the terms inside Ω to uncover an interpretation. We aid the explanation with Figure 4.1 whose axes represent time and maturity respectively. The revenue-echo Ω is the product of the marginal probability of default, $\theta U'$ times a double integral. To understand why, consider a small issuance of a $\{\tau_0, t_0\}$-bond—located at the gray dot in Figure 4.1. At each instant, the bond matures and time progress in the direction $\{-1, 1\}$, as depicted by the gray ray that starts at the issuance point. By time t, the bond has a maturity τ. At t, the issuance has a marginal impact on the repayment probability. If we multiply the term $\theta U'$ inside Ω by the term $\frac{U'(c(t))}{U'(\hat{c}(t))} v$ in the valuation we, obtain marginal effect on the repayment probability:

$$\underbrace{\theta \left(V \left[f \left(\cdot, t \right), X_t \right] \right) U' \left(c \left(t \right) \right)}_{\text{effect on repayment probability}} \cdot v \left(\tau, t \right).$$

Hence, the term $\theta U'$ is present to reflect a marginal effect on the repayment probability at t.

The double integral captures how a lower repayment probability at time t impacts the revenues generated by issuances in all moments prior to t obtained from all issuances. Notice how the range of integration covers all maturities $m \in [0, T]$ in the outer integral. The inner integral covers the relevant times prior to t, $z \in \max\{t + m - T, 0\}$. The marginal effect on repayment of the $\{\tau, t\}$-bond affects the repayment probability of all bonds at t, which are indexed by a maturity m—these are depicted in the vertical line at time t. Furthermore, the effect of default at time t impacts past prices in an "echo effect": Any bond price at a moment prior to t that is still alive at time t includes the discounted value of a price at t, $e^{-\int_z^t (\hat{r}^*(u))du} \psi(m, t)$ for a specific maturity m. For example, the price of the bond $\{m, t\}$, affects the price of all bonds $\{m + t - z, z\}$ indexed by $z \in \max\{t + m - T, 0\}$. Each ray that departs from the vertical line at t depicts one such family of bonds. Thus, if we multiply the change on the repayment probability at t by $e^{-\int_z^t (\hat{r}^*(u))du} \psi(m, t)$, we then obtain the reduction in the price of the $\{m + t - z, z\}$ bond. We can do the same for all bonds in $m \in [0, T]$, the outer integral, and past times z, the inner integral, to obtain the marginal effect that the $\{\tau, t\}$-bond has on all past prices.

This marginal impact on past prices affects past revenues. Thus, fix a maturity m and a time t. If we want to get the effect on revenues of a change in past prices, we must multiply the change in price by $\iota(1 - (\bar{\lambda}/2)\,\iota)$, the issuance amount net of the liquidity costs. If we use the optimal issuance rule (2.12), revenues are proportional to:

$$\frac{1}{2\bar{\lambda}}\left[1 - \left(\frac{\hat{v}(m + t - z, z)}{\hat{\psi}(m + t - z, z)}\right)^2\right].$$

Thus, when this terms is inside the double integral it captures the impact on past revenues that an increase in default probabilities has. Once we multiply these effects on past revenues by $e^{\int_z^t \hat{r}(u)du}$, we bring past reductions in revenues into the current period t. The echo effect is present at any instant prior to the maturity of the bond, which is why it appears as a flow in equation (4.4). When the Government considers the marginal issuance of the $\{\tau_0, t_0\}$-bond, its valuation is the present value of all the echo effects Ω that last throughout the life of the bonds. This reduction in revenues is part of the issuance consideration.

As in Section 3.2, the computation of the solution in proposition 3 involves solving for a fixed point in a a family of time-varying debt distributions. In this case the problem is even more complex, due to the interplay between debt management and bond prices that was absent in previous sections. However, as with the case of version with risk, we can obtain a solution in the risky steady state.

4.3 Default without liquidity costs

We return to the case without liquidity costs, $\bar{\lambda} = 0$, but allow for default. Without liquidity costs, we have again that a solution neccesarily features equality between valuations and prices,

$v = \psi$ and $\hat{v} = \hat{\psi}$. As a result, the condition that characterizes the solution without default, (3.5), is modified to:

$$\hat{r}(t) - \phi \mathbb{E}_t^X \left[\left(\Theta (V(t)) + \Omega(t) \right) \frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)} \cdot \frac{U'(c(t))}{U'(\hat{c}(t))} \right] = \hat{r}^*(t) - \phi \mathbb{E}_t^X \left[\Theta (V(t)) \frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)} \right]. \quad (4.5)$$

As in the case without default, we can explain how condition (4.5) characterizes the solution depending on the set of bonds and shocks.

On the impossibility of perfect hedging. The presence of default interrupts the ability to share risk. Efficient risk sharing requires a continuous consumption path along non-default states. To see how default interrupts risk-sharing, consider the case where interest-rate shocks allow complete asset spanning. Assume that $\hat{c}(t) = c(t)$ holds in non-default states, as in the version without default. In this case, condition (4.5) becomes

$$\hat{r}(t) - \phi \mathbb{E}_t^X \left[\Omega(t) \frac{\psi(\tau,t)}{\hat{\psi}(\tau,t)} \right] = \hat{r}^*(t).$$

This equation cannot if two maturities feature a different price jump. However, full asset spanning requires a different price jump at two maturities. This contradiction implies that even when the set of securities can provide insurance in non-default states, the Government's solution with commitment does not adopt a perfectly insuring scheme. The distortion follows because the echo-effect acts differently than the risk premium, it distorts valuations but not prices—we can see even under risk-neutrality.

Default allows some hedging. Consider the case of only income shocks. Without default, we noted that there was no hedging role for maturity but now we show that with default, there is a role. Default opens the possibility of a partial hedging because prior to the shock, different maturities are priced differently. Post-shock prices are alway $\psi(\tau, t^o) = 1$. This means that once a shock hits, the Government can exploit the change in the yield curve to obtain capital gains in its porfolio. The change in the risk premium is akin to the spanning effect of an interest-rate jump.

General case. The option to default interrupts insurance across non-default states, but allows price variation even without interest-rate risk. As long as the cardinality of shocks is discrete, the maturity profile is indeterminate—a formal proof is found in Appendix D.8. One extreme case of indeterminacy is that of a shock which does not produce a jump in income nor interests, but only grants a default option. Aguiar et al. (Forthcoming) studies that shock in a discrete-time model similar to ours but without commitment.

4.4 Quantitative analysis

We now continue with the quantitative analysis, but incorporate default. We calibrate the output loss, κ, to 1.5 per cent. This is a lower value than the 2 per cent used in, for example, Aguiar and Gopinath (2006). We calibrate a lower output loss because the literature usually assumes reinsertion to capital markets after some periods. Here autarky is absorbing. A lower κ is meant to produce a similar autarky value as if we allowed for reinsertion. The distribution of ε_t is a logistic with coefficient ς, as in common to discrete-choice models. We set ς to 100. This number produces a default in Spain in 32 per cent of the events when it is hit by an extreme shock. Coupled with the intensity of the extreme event, ϕ, the unconditional default probability is 0.6 percent per year, roughly a default every 157 years—according to Reinhart and Rogoff (2009) Spain experienced one default in during the years 1877-1982.

To gather a sense of the quantitative impact of default, figure 4.2 considers an income shock, reports the RSS and the post-shock transitions in the economy with default, in contrast to the case with risk but without default, the version studied in Section 3.4. We use the labels "Default", and "No default" correspondingly. Panel (e) and panel (f) show prices and valuations of the version with default at the RSS, the DSS and on impact of the shock.

Default leads to a reduction in the level and maturity of debt. In panel (c), we observe that default produces a reduction in total debt in the RSS, from 35 per cent to 30 per cent. Several forces are at play: the default premium makes domestic discounts closer to market rates and, thus, activates yield management which is a force towards less debt. Simultaneously, the revenue-echo effect increases internal valuations particularly for long-term bonds. In addition, default facilitates insurance because it reduces payments in high-marginal utility paths and because it enables hedging. One advantage of the framework is that we can unpack these effects, which we do next.

The reduction in the debt stock is dominated by incentives. We can see this by calculating the analogue of figure 4.2, but comparing the solution with default with a myopic solution with $\Omega = 0$. This comparison is reported in figure 4.3. Once we shut off the echo effect, the RSS debt stock increases from 30 to 40 per cent, a level even higher than without the default option. This pattern reflects the trade off between insurance and incentives. If we shut down the echo effect, default improves insurance. The incentives force is strong enough to reverse the pattern. Figure F.3 in Appendix F, compares the solution with default against the solution with default but for a risk neutral Government. The solution with risk neutrality is almost identical: the debt level, consumption, and the risk premium are the same. This occurs because the insurance and incentives channel offset each other. From the above discussion, we know that default provides insurance, so the echo-effect must be lower when $\sigma = 0$. This conclusion is intuitive because the echo effect is linked to the elasticity of inter-temporal substitution through the comparison of revenues across time.

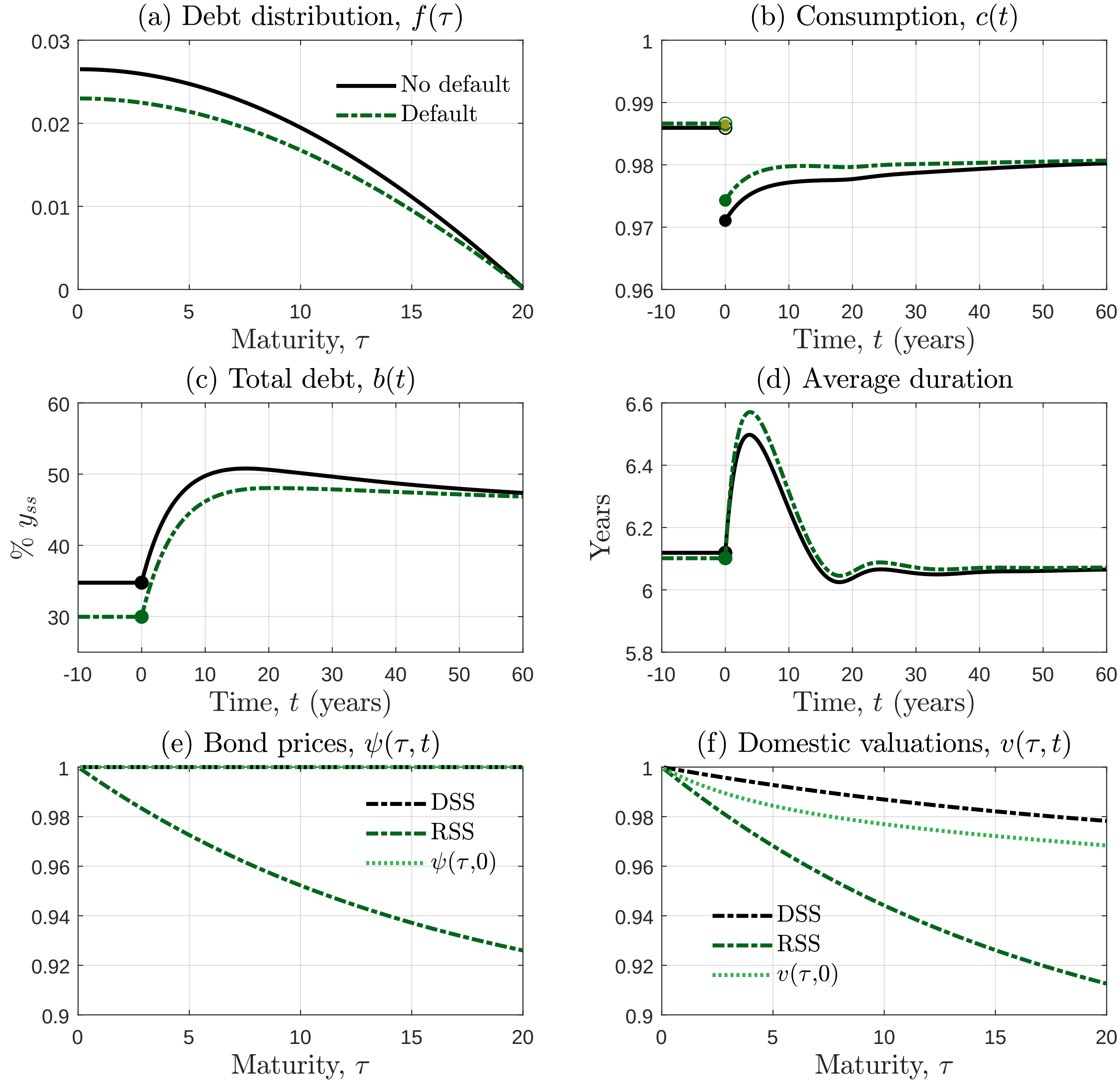

Figure 4.2: Response to a shock to income when the option to default is available. Panels (e) and (f) refer to the case with default.

The shortening of maturities can also be read off from figure 4.2. Earlier, we discussed that an increase in the risk premium has the effect of equalizing rates and that this activates yield management, a force towards shorter maturity. The increases in insurance offsets this effect because long-term debt is less expensive. Back in figure 4.3, we see that without the echo effect, maturity falls even further. At first, this result seems surprising because the echo effect penalizes longer bonds. However, the echo effect reduces the debt outstanding which lowers the default probability. Hence, the echo effect reduces the risk premium, which mitigates the yield management force that pushes towards shorter maturity.

We finalize the quantitative analysis with figure F.5 in Appendix F which we use to discuss the effect to a an interest-rate shock. In this case the RSS distribution is the same as the one without the option to default There is only a difference in the RSS consumption because prices

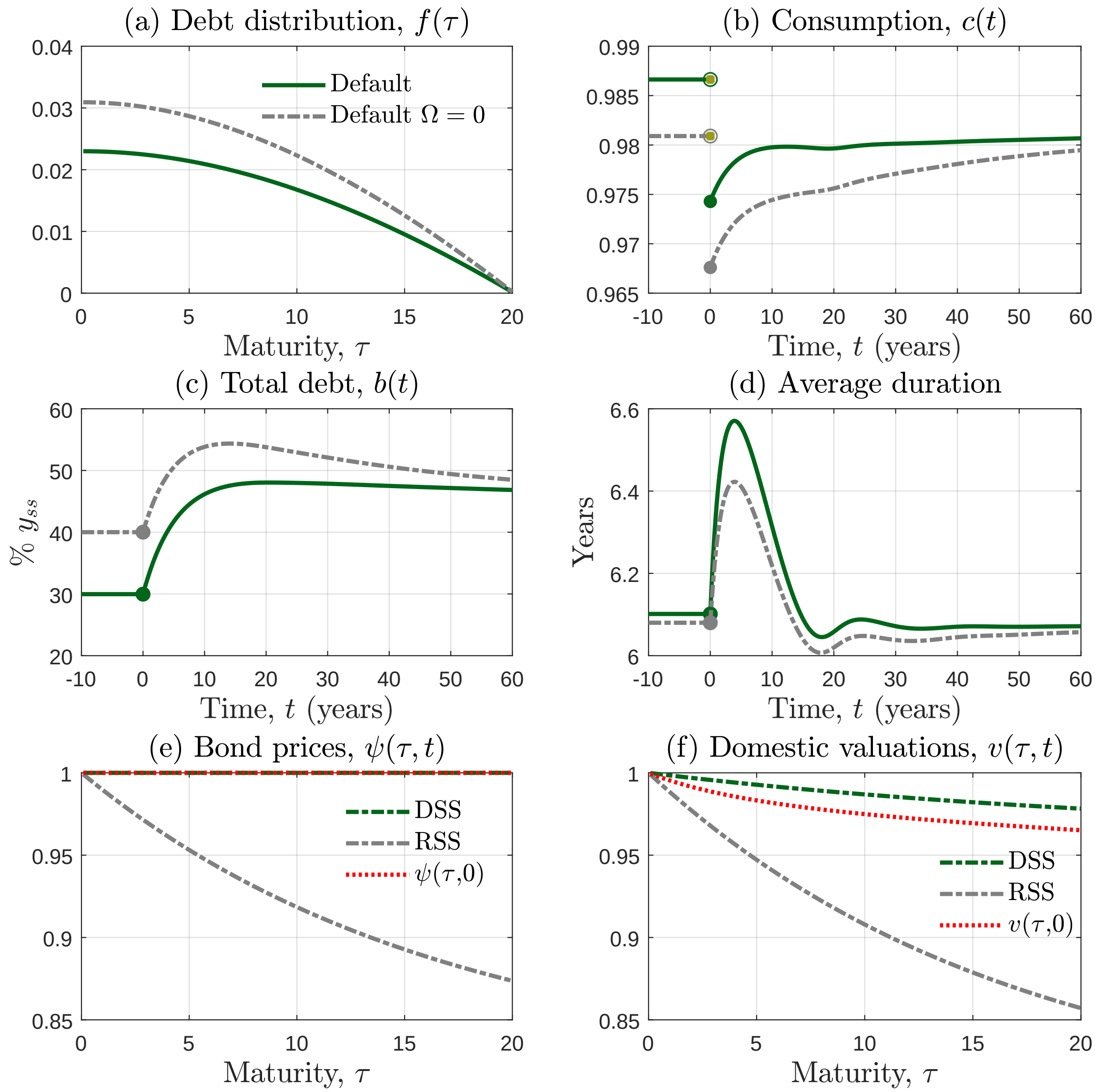

Figure 4.3: Response to a shock to income with $\Omega = 0$ when the option to default is available. Panels (e) and (f) refer to the case with $\Omega = 0$.

are lower with default. If we contrast the effect again with the case of risk-neutrality (figure F.4) we obtain only differences during the transition. Again, the yield management effect that is coupled with incentives counterbalances the improvement in insurance. Broner et al. (2013) or Arellano and Ramanarayanan (2012) present evidence that the debt maturity shrinks with default premia. The literature has emphasized the role of debt dilution where there is no commitment to a debt program. Here, we uncover that incentives is a force that shortens maturity in presence of income or interest-rate risk, even if debt dilution plays no role. Incentives is strong enough to overturn the the lengthening of maturity produced by self insurance.

5 Applications

The ultimate goal of the framework is to answer questions that are hard to answer with other approaches. Since the advantage of the framework is that we can work with a large number of bonds of arbitrary coupon structure, the most natural applications relate to questions about the number and type of bonds that a country should issue.

Application 1: exponential maturity vs. standard debt. As mentioned earlier, most quantitative applications employ bonds of exponential maturity (consols) whereas most governements issue bonds similar to the ones described in this paper. Without liquidity frictions and complete markets, the coupon structure does not matter. However, when we consider incomplete-markets, the option to default, or liquidity costs, the distinction matters because the valuations of consols and standard bonds differ. We can easily adapt the environment to compare the steady state and transitions; we use the "c" superscript to refer to the consols. The market price of a consol, ψ^c, satisfies:

$$\hat{r}^*(t)\psi^c(\tilde{\tau},t) = (\frac{1}{\tilde{\tau}}+\delta) - \frac{1}{\tilde{\tau}}\psi^c(\tilde{\tau},t) + \frac{\partial \psi^c(\tilde{\tau},t)}{\partial t}.$$

with $\lim_{\tilde{\tau}\to 0}\psi^c(\tilde{\tau},t) = 1$. The Government's problem is the same as before, but the budget constraint is modified to

$$c(t) = y(t) + \int_0^T \left[q^c(\tilde{\tau},t,\iota)\iota^c(\tilde{\tau},t) - \left(\frac{1}{\tilde{\tau}}+\delta\right) f^c(\tilde{\tau},t)\right] d\tilde{\tau},$$

where is again $q^c = \left(1-\frac{1}{2}\bar{\lambda}\iota\right)\psi^c$, and the law of motion of debt is:

$$\frac{\partial f^c(\tilde{\tau},t)}{\partial t} = \iota^c(\tilde{\tau},t) - \frac{1}{\tilde{\tau}} f^c(\tilde{\tau},t).$$

We can show that with consols, the issuance rule (2.12) remains the same, but internal valuations are:

$$r(t)\, v^c(\tilde{\tau},t) \;=\; (\frac{1}{\tilde{\tau}}+\delta) - \frac{1}{\tilde{\tau}} v^c(\tilde{\tau},t) + \frac{\partial v^c}{\partial t}, \text{ if } \tilde{\tau}\in(0,T],$$

where $\lim_{\tilde{\tau}\to 0} v(\tilde{\tau},t) = 1$. At a steady state with positive consumption, assuming that $\delta = r^*_{ss}$ we obtain that prices and valuations that satisfy $\psi^c_{ss}(\tilde{\tau}) = 1$ and $v^c_{ss}(\tilde{\tau}) = \left(r^*_{ss}+\frac{1}{\tilde{\tau}}\right)/\left(\rho+\frac{1}{\tilde{\tau}}\right)$. The steady-state debt profile is:

$$f^c_{ss}(\tilde{\tau}) = \frac{\tilde{\tau}\left(\rho - r^*_{ss}\right)}{\bar{\lambda}\left(\rho+\frac{1}{\tilde{\tau}}\right)}.$$

Opposite to the case with standard debt, the debt profile with consols increases with maturity $\partial f^c_{ss}(\tilde{\tau})/\partial\tilde{\tau} > 0$. This flipping of the maturity structure shows that liquidity costs make the choice between consols and standard debt not innocuous. For example, it can modify the ability to insure, or alter the decision to default. How much do consols bias results in default models merits further investigation. The framework can be used to delve deeper into that question.

Application 2: how many bonds should a country issue? In practice, countries issue at a discrete set of maturities. Next, we show how the model is adaptable to allow for issuances at a discrete set of points. One of the advantage of this extensions is that we can start to ask questions regarding the optimal number of bonds that a country should issue. We set the liquidity coefficient to be finite only for a discrete number of maturities $\{\tau_1, \tau_2, ..., \tau_N\}$.[17]

The rest of the solution is exactly as before, but the optimal issuance rule is:

$$\iota(\tau,t) = \begin{cases} \frac{1}{\bar{\lambda}} \cdot \frac{\psi(\tau,t) - v(\tau,t)}{\psi(\tau,t)}; & \tau \in [\tau_1, \tau_2, ..., \tau_N] \\ 0; & \tau \notin [\tau_1, \tau_2, ..., \tau_N] \end{cases}.$$

We obtain the following figure plots the solution for this case. We apply the discrete-issuance economy to study the effects of of the introduction an additional bond. We continue with the calibration but now allow for issuances of 5 bonds of 1, 6, 12, 60, and 240 months of maturity. To make comparisons, we adjust $\bar{\lambda}$ proportionally to the cardinality of bonds—we set $\bar{\lambda}(N) = 12 \cdot \bar{\lambda}\#N$ where $\#N$ is the number of bond markets. We compare the welfare at the RSS with and without the additional bond, for four scenarios: allowing and not allowing default, and with and without risk aversion. We then compare the welfare increase in Spain of adding a 10 year bond.

The welfare gains are presented in Figure . To transform welfare into a meaningful number within scenarios we do the following. Each scenario has its own RSS associated with the interest risk. For each RSS value functional, we obtain the constant annuity consumption that yields the same value for the Government. They y-axis reports how much of steady-state output the country would have to forego to obtain the certainty equivalent annuity.

Across scenarios, the annuity cost increases when we add the 10 year bond. This is because the level of debt increases in all scenarios, once we add the bond. An additional bond allows to spread out liquidity costs, even though the liquidity coefficient increases with the additional bond. It is also interesting output cost is higher with the option to default than without it even though, as we saw earlier, debt levels are lower at the RSS. This is because default is priced in by international investors, so the country is paying an insurance premium and a cost for not having the ability to commit. Adding an additional bond reduces the effect of default in all scenarios, because it provides an additional form of insurance. The overall order of magnitude of the effect is 0.1% of GDP which reflects the low cost of consumption fluctuations in macroeconomic

[17] Technically, issuances have to be modeled with Dirac delta functions at the issuance points, obtained as a limit where we shrink intervals. In terms of computations, this technical detail does not matter because the maturity space is discretized.

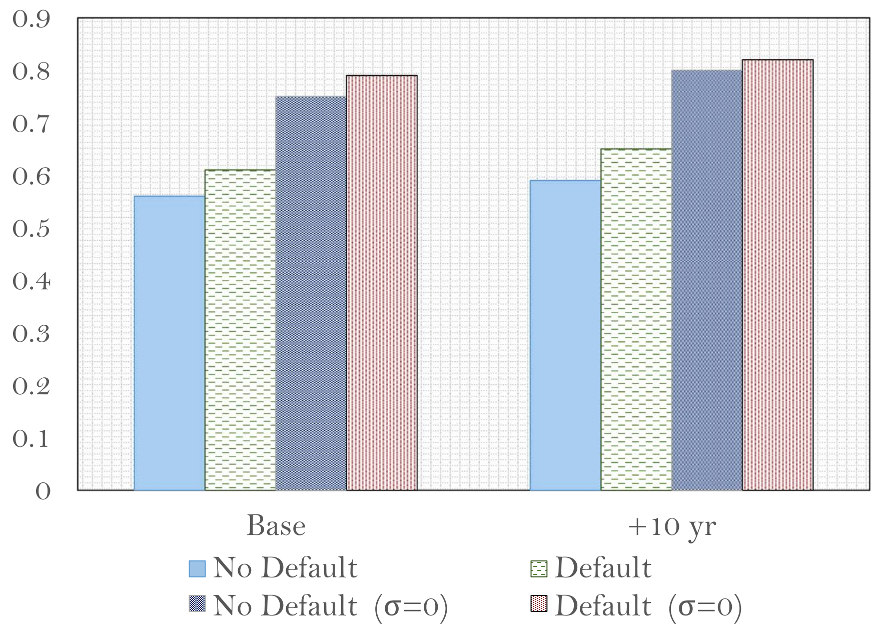

Figure 5.1: Benefits of Introducting a 10yr bond in Spain (consumption equivalent/GDP).

models.

Potential extensions to the current framework. A natural extension is to allow for multiple classes of bonds: for example, we could allow the Government to issue in multiple currencies as in Ottonello and Perez (Forthcoming) to study different ways of hedging. We can also endow the Government with an asset that can be used as collateral: this could allow certain types of debt to be secured. This is a way to introduce seniority and (as in Chatterjee and Eyigungor, 2015).

Another extension is to introduce liquidity costs that depend on the overall stock of debt rather than the issued amount, as in Vayanos and Vila (2009). In that case, valuations would be adjusted by a term similar to the revenue-echo effect, even without default. This is because the supply of bonds at a given maturity will affect future prices the market for each maturity, and this effect would be anticipated current market prices, just as the case with default.

We also imposed symmetry between liquidity costs for positive and negative issuances. A natural extension is to allow for a more general function of $\bar{\lambda}$, one that changes with with the sign of issuances and maturity. This extension can be used to understand the role of safe assets, as in (Bianchi et al., Forthcoming). Finally, we can investigate shocks to $\bar{\lambda}$ which can be interpreted as roll-over risk and even let this risk be a function of f, is with the case with default.

6 Conclusions

The paper develops techniques to study a consumption-savings problem where the state variable is a distribution over a set of assets. A central feature of the problem is the presence of liquidity costs that inhibit the ability to adjust the portfolio. A general principle emerges: to determine issuances a simple rule should compare domestic valuations with market prices. In presence of risk, valuations are adapted to capture jumps in marginal utility which encode an insurance force. When default is possible, valuations are modified to capture how default allows for risk-sharing and to introduce an echo-effect that internalizes the decline in revenues caused by the lack of commitment to repay.

The previous section discusses how the framework can be adapted to study various applications and it is not hard to see how the framework can be adapted to corporate or public finance settings. The framework faces two limitations. We believe that these cannot be tackled analytically, making the use of numerical methods necessary. First, when we study default, we analyze the problem with commitment to the issuance path. The solution without commitment requires the computation of how a current action today disciplines the future self. That term does not appear in our valuations because the planner has commitment. Second, our framework can characterize solutions when an unexpected shocks arrives only once. This already captures, up to a first order, expected impulse responses. However, an extension that admits recurrent shocks faces a challenge similar to the case in models with heterogeneous-agent models and aggregate shocks. Our hope is that the insights and techniques in this paper can serve as a stepping stone for that agenda.

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Appendix: A Framework for Debt-Maturity Management

Contents

section

A Equivalence between PDE and integral formulations

Valuations and prices are a continuous-time net present value formula. Their PDE representation is the analogue of the recursive representation in discrete time and the integral formulation is the equivalent of the sequence summations. The solutions to each PDE can be recovered easily via the method of characteristics or as an immediate application of the Feynman-Kac formula. All of the PDEs in this paper have an exact solution contained in table A.

Price (PF)	PDE	$r^*(t)\,\psi(\tau,t)=\delta+\frac{\partial\psi}{\partial t}-\frac{\partial\psi}{\partial\tau};\ \psi(0,t)=1$
	Integral	$e^{-\int_t^{t+\tau} r^*(u)du}+\delta\int_t^{t+\tau} e^{-\int_t^s r^*(u)du}ds$
Valuation (PF)	PDE	$r(t)\,v(\tau,t)=\delta+\frac{\partial v}{\partial t}-\frac{\partial v}{\partial\tau};\ v(0,t)=1$
	Integral	$e^{-\int_t^{t+\tau} r(u)du}+\delta\int_t^{t+\tau} e^{-\int_t^s r(u)du}ds$
Price (risk)	PDE	$\hat r^*(t)\,\hat\psi(\tau,t)=\delta+\frac{\partial\hat\psi}{\partial t}-\frac{\partial\hat\psi}{\partial\tau}+\phi\mathbb{E}_t^X\left[\psi(\tau,t,X_t)-\hat\psi(\tau,t)\right];\ \hat\psi(0,t)=1$
	Integral	$e^{-\int_t^{t+\tau}(\hat r^*(u)+\phi)du}+\int_t^{t+\tau}\left(\delta+\hat\psi(\tau-s,t+s)\right)e^{-\int_t^s(\hat r^*(u)+\phi)du}ds$
Valuation (risk)	PDE	$\hat r^*(t)\,\hat v(\tau,t)=\delta+\frac{\partial\hat v}{\partial t}-\frac{\partial\hat v}{\partial\tau}+\phi\mathbb{E}_t^X\left[v(\tau,t,X_t)-\hat v(\tau,t)\right];\ \hat v(0,t)=1$
	Integral	$e^{-\int_t^{t+\tau}(\hat r(u)+\phi)du}+\int_t^{t+\tau}\left(\delta+v(\tau-s,t+s)\frac{U'(c(t+s))}{U'(\hat c(t+s))}\right)e^{-\int_t^s(\hat r(u)+\phi)du}ds$
Price (default)	PDE	$\hat r^*(t)\,\hat\psi(\tau,t)=\delta+\frac{\partial\hat\psi}{\partial t}-\frac{\partial\hat\psi}{\partial\tau}+\phi\mathbb{E}_t^X\left[\Phi\left(V\left[\hat f(\cdot,t)\right]\right)\psi(\tau,t,X_t)-\hat\psi(\tau,t)\right];\ v(0,t)=1$
	Integral	$e^{-\int_t^{t+\tau}(\hat r^*(u)+\phi)du}+\int_t^{t+\tau}e^{-\int_t^s(\hat r^*(u)+\phi)du}\left(\delta+\phi\mathbb{E}_s^X\left[\Theta\left(V\left[\hat f(\cdot,t)\right]\right)\psi(\tau-s,t+s)\right]\right)ds$
Valuation (default)	PDE	$\hat r^*(t)\,\hat v(\tau,t)=\delta+\frac{\partial\hat v}{\partial t}-\frac{\partial\hat v}{\partial\tau}+\phi\left(\mathbb{E}_s^X\left[\left(\Theta(V(t))+\Omega(t)\right)\frac{U'(c(t))}{U'(\hat c(t))}v(\tau,t)\right]-\hat v(\tau,t)\right);\ \hat v(0,t)=1$
	Integral	$e^{-\int_t^{t+\tau}(\hat r^*(s)+\phi)ds}+\int_t^{t+\tau}e^{-\int_t^s(\hat r^*(u)+\phi)du}\left(\delta+\phi\mathbb{E}_s^X\left[\left(\Omega(t+s)+\Theta\left(V\left[\hat f(\cdot,t+s)\right]\right)\right)\frac{U'(c(t+s))}{U'(\hat c(t+s))}v(\tau-s,t+s)\right]\right)ds$
Debt Profile	PDE	$\frac{\partial f}{\partial t}=\iota(\tau,t)+\frac{\partial f}{\partial\tau}$
	Integral	$f(\tau,t)=\int_\tau^{min\{T,\tau+t\}}\iota(s,t+\tau-s)ds+\mathbb{I}[T>t+\tau]\cdot f(0,\tau+t)$

Table 1: Equivalence Between PDE and Integral Formulations

B Micro Model of Liquidity Costs

B.1 Environment

Here we describe the micro model of the liquidity costs in more detail. The model is a wholesale retail model of the secondary market of sovereign bonds.[18] Without loss of generality we focus on the bond issuance of $\iota(\tau, t)$ bonds at time t with maturity τ. We define s as the amount of time passed since the auction. The outstanding amount of bonds in hands of an atomistic banker, after a period of time s has passed after the issuance of the bond is:

$$I\left(s; \iota(\tau, t)\right) = \max(\iota(\tau, t) - \mu y_{ss} \cdot s, 0).$$

This implies that the bond inventory is exhausted by time:

$$\bar{s} = \frac{\iota(\tau, t)}{\mu y_{ss}}.$$

We consider that individual orders arrive randomly according to a Poisson distribution. The intensity at which bonds are sold per unit of time is given by:

$$\begin{aligned}\gamma^{(t,\tau)}\left(s\right) &= \frac{\mu y_{ss}}{I\left(s; \iota(\tau, t)\right)} \\ &= \frac{1}{\bar{s} - s} \text{ for } s \in \left[0, \min\{\tau, \bar{s}\}\right).\end{aligned}$$

This intensity $\gamma^{(t,\tau)}\left(s\right)$ is defined only between $[0, \min\{\tau, \bar{s}\})$, because after the bond matures or after the stock is exhausted there is no further selling.

B.2 Valuations

Investor's valuation. At time $t + s$, after a period of time s passed since the auction, the time to maturity is $\tau' = \tau - s$. The valuation of the bond by investors is:

$$\psi^{(t,\tau)}(\tau', s) = \psi(\tau - s, t + s).$$

They are risk neutral and discount future payoffs and the international risk free rate. Hence, the price equation satisfies the PDE (2.4):

$$r^{*}(t + s)\psi^{(t,\tau)}(\tau', s) = \delta - \frac{\partial \psi^{(t,\tau)}}{\partial \tau'} + \frac{\partial \psi^{(t,\tau)}}{\partial t},$$

with the terminal condition of $\psi^{(t,\tau)}(0, s) = 1$.

Banker's valuation. Now consider the valuation of the cash-flows of the bond from the perspective of the banker $q^{(t,\tau)}(\tau', s)$. Bankers are risk neutral but have a higher cost of capital. At each moment $t + s$ bankers meet

[18]See Foucault et al. (2013) for a textbook treatment of liquidity costs and Stoll (2003), Madhavan (2000) and Vayanos and Wang (2013a) for reviews. The three main economic forces explored by this literature are: (i) inventory management of financial intermediaries, (ii) adverse selection and, (iii) order processing costs. Seminal inventory problem papers are Ho and Stoll (1983), Huang and Stoll (1997) and Weill (2007). In the case of adverse selection, see Glosten and Milgrom (1985), Kyle (1985) or Easley and O'hara (1987). A general reference of the implications of order processing costs is Foucault et al. (2013). A large empirical literature has documented the presence of price impact. For example, see Madhavan and Smidt (1991), Madhavan and Sofianos (1998), Hendershott and Seasholes (2007) or Naik and Yadav (2003a,b).

investors and sell at a price $\psi^{(t,\tau)}(\tau',s)$. The valuation of the bankers, $q^{(t,\tau)}(\tau',s)$ satisfies:

$$(r^*(t+s)+\eta)q^{(t,\tau)}(\tau',s) = \delta - \frac{\partial q^{(t,\tau)}}{\partial \tau'} + \frac{\partial q^{(t,\tau)}}{\partial t} + \gamma^{(t,\tau)}(s)\left(\psi^{(t,\tau)}(\tau',s) - q^{(t,\tau)}(\tau',s)\right). \tag{B.1}$$

This expression takes this form because the banker extracts surplus when it matches. Before a match, bankers earn the flow utility, but upon a match, it's value jumps to $\psi^{(t,\tau)} - q^{(t,\tau)}$. This jump arrives with endogenous intensity $\gamma(s)$. The complication with this PDE is its terminal condition. If $\bar{s} \leq \tau$, its terminal condition is given by $q^{(t,\tau)}(\tau',\bar{s}) = \psi^{(t,\tau)}(\tau',\bar{s})$. If $\bar{s} > \tau$, the corresponding terminal condition is $q^{(t,\tau)}(0,s) = 1$, since by the expiration date, it is paid the principal equal to 1.

Competitive Auction Price. The date of the auction $s=0$, $\tau'=\tau$, the banker pays its expected valuation and hence the bond price demand faced by the Government is:

$$q(\iota,\tau,t) \equiv q^{(t,\tau)}(\tau,0).$$

This is because, there is free entry by investment banks into the auction. In equation (2.3) we expressed $q(\iota,\tau,t)$ as $q(\iota,\tau,t) = \psi(\tau,t) - \lambda(\iota,\tau,t)$. Thus,

$$\lambda(\iota,\tau,t) = \psi(\tau,t) - q(\iota,\tau,t),$$

is the object we are trying to find.

B.3 Solution

We now provide a first order linear approximation for the price at the auction, $q(\iota,\tau,t)$, for small issuances. The result is given by the following proposition:

Proposition 4. *A first-order Taylor expansion around $\iota = 0$ yields a linear auction price:*

$$q(\iota,\tau,t) \approx \psi(\tau,t) - \frac{1}{2}\frac{\eta}{\mu y_{ss}}\psi(\tau,t)\iota(\tau,t). \tag{B.2}$$

Thus, the approximate liquidity cost function is $\lambda(\tau,t,\iota) \approx \frac{1}{2}\bar{\lambda}\psi(\tau,t)\iota(\tau,t)$ where price impact is given by $\bar{\lambda}=\frac{\eta}{\mu y_{ss}}$.

Proof. Step 1. Exact solutions. The solution to $q(\iota,\tau,t)$ falls in one of two cases. Case 1. If $\bar{s} \leq \tau$, then:

$$q(\iota,\tau,t) = \frac{\int_0^{\bar{s}} e^{-\int_0^v (r^*(t+u)+\eta)du}\left(\delta(\bar{s}-v) + \psi(\tau-v,t+v)\right)dv}{\bar{s}}. \tag{B.3}$$

Case 2. If $\bar{s} > \tau$, then:

$$\begin{aligned} q(\iota,\tau,t) \quad &= \quad \int_0^{\tau} e^{-\int_0^s (r^*(t+u)+\eta)du}\left(\frac{\delta(\bar{s}-v)+\psi(\tau-v,t+v)}{\bar{s}}\right)dv \\ &\quad + e^{-\int_0^{\tau}(r^*(t+u)+\eta)du}\frac{(\bar{s}-\tau)}{\bar{s}}. \end{aligned} \tag{B.4}$$

We solve the PDE for q depending on the corresponding terminal conditions; respectively, $q^{(t,\tau)}(\tau',\bar{s}) = \psi^{(t,\tau)}(\tau',\bar{s})$ and $q^{(t,\tau)}(0,s) = 1$.

Case 1. Consider the first case. The general solution to the PDE equation for $q^{(t,\tau)}(\tau',s)$ is,

$$\begin{aligned} &\int_0^{\bar{s}-s} e^{-\int_0^v (r^*(t+u)+\eta+\gamma(u))du}\left(\delta + \gamma(s+v)\psi(\tau-v,t+v)\right)dv \quad + \\ &\qquad e^{-\int_0^{\bar{s}-s}(r^*(t+u)+\eta+\gamma(u))du}\psi(\tau'-(\bar{s}-s),t+(\bar{s}-s)). \end{aligned} \tag{B.5}$$

This can be checked by taking partial derivatives with respect to time and maturity and applying Leibniz's rule.[19] Consider the exponentials that appear in both terms of equation (B.5). These can be decomposed into $e^{-\int_0^v [r^*(t+u)+\eta]du} e^{-\int_0^v \gamma(u)du}$. Then, by definition of γ we have:

$$\begin{aligned} e^{-\int_0^v \gamma(u)du} &= e^{-\int_0^v \frac{1}{\bar{s}-u}du} \\ &= \frac{(\bar{s}-v)}{\bar{s}}. \end{aligned} \tag{B.6}$$

Thus, using (B.6) in (B.5) we can re-express it as:

$$\begin{aligned} q^{(t,\tau)}(\tau', s) &= \int_0^{\bar{s}-s} e^{-\int_0^v (r^*(t+u)+\eta)du} \frac{(\bar{s}-v)}{\bar{s}} \left(\delta + \gamma(s+v)\psi(\tau - v, t+v)\right) dv \\ &+ e^{-\int_0^{\bar{s}-s}(r^*(t+u)+\eta)du} \frac{s}{\bar{s}} \psi(\tau' - (\bar{s}-s), t+\bar{s}). \end{aligned}$$

When we evaluate this expression at $s = 0$, $\tau' = \tau$, and we replace $\gamma(v) = \frac{1}{\bar{s}-v}$, we arrive to:

$$\begin{aligned} q(\iota, \tau, t) &\equiv q^{(t,\tau)}(\tau, 0) \\ &= \int_0^{\bar{s}} e^{-\int_0^v (r^*(t+u)+\eta)du} \left(\frac{(\bar{s}-v)}{\bar{s}}\delta + \frac{\psi(\tau - v, t+v)}{\bar{s}} \right) dv. \end{aligned}$$

Case 2. The proof in the second case runs parallel. The general solution to the PDE equation in this case is:

$$\begin{aligned} q^{(t,\tau)}(\tau', s) &= \int_0^{\tau'} e^{-\int_0^v (r^*(t+u)+\eta+\gamma(u))du} \left(\delta + \gamma(s+v)\psi(\tau - v, t+v)\right) dv \\ &+ e^{-\int_0^v (r^*(t+u)+\eta+\gamma(u))du}. \end{aligned}$$

When we evaluate this expression at $s = 0$, $\tau' = \tau$:

$$\begin{aligned} q(\iota, \tau, t) &= \int_0^{\tau} e^{-\int_0^s (r^*(t+u)+\eta)du} \frac{(\bar{s}-v)}{\bar{s}} \left(\delta + \frac{\psi(\tau - v, t+v)}{(\bar{s}-v)} \right) dv \\ &+ e^{-\int_0^{\tau} (r^*(t+u)+\eta)du} \frac{(\bar{s}-\tau)}{\bar{s}}. \end{aligned}$$

Step. 2. Limit Behavior of $q(\iota, \tau, t)$. Price with Zero Issuances. Consider the limit $\iota(\tau, t) \longrightarrow 0$ for any $\tau > 0$, which implies that $\bar{s} \longrightarrow 0$. For both Case 1 and Case 2, equations (B.3) and (B.4),[20] it holds that:

$$\lim_{\iota(\tau,t)\longrightarrow 0} q(\iota, \tau, t) = \lim_{\bar{s}\longrightarrow 0} \frac{\int_0^{\bar{s}} e^{-\int_0^s (r^*(t+u)+\eta)du} \left(\delta(\bar{s}-s) + \psi(\tau - s, t+s)\right) ds}{\bar{s}}.$$

Now, both the numerator and the denominator, converge to zero as we take the limits. Hence, by L'Hôpital's rule, the limit of the price is the limit of the ratio of derivatives. The derivative of the numerator is obtained via Leibniz's

[19]Notice that we have directly replaced the value $\psi^{(t,\tau)}(\tau', s) = \psi(\tau - s, t+s)$.

[20]For for every $\tau < \bar{s}$, i.e. in case 2, it will be analogous since we are taking the limit when $\bar{s}$ converges to zero.

rule and thus,

$$\begin{aligned}
\lim_{\iota(\tau,t)\longrightarrow 0} q(\iota,\tau,t) &= \lim_{\bar{s}\longrightarrow 0} \frac{\left[e^{-\int_0^s (r^*(t+u)+\eta)du}(\delta(\bar{s}-s)+\psi(\tau-s,t+s))\right]|_{s=\bar{s}}}{1} \\
&= \lim_{\bar{s}\longrightarrow 0} e^{-\int_0^{\bar{s}} (r^*(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s}) \\
&= \psi(\tau,t).
\end{aligned}$$

Step. 3. Linear approximation of $q(\iota,\tau,t)$. The first order approximation of the function $q(\iota,\tau,t)$, the price at the auction, around $\iota = 0$ is given by:

$$q(\iota,\tau,t) \simeq q(\iota,\tau,t)\ |_{\iota=0} + \frac{\partial q(\iota,\tau,t)}{\partial \iota}\ |_{\iota=0}\ \iota(\tau,t).$$

We computed the first term in step 2. It is given by $\psi(\tau,t)$. Thus, our objective will be to obtain $\frac{\partial q(\iota,\tau,t)}{\partial \iota}\ |_{\iota=0}$. Observe that by definition of $\bar{s}$, it holds that:

$$\begin{aligned}
\frac{\partial q(\iota,\tau,t)}{\partial \iota} &= \frac{\partial \bar{s}}{\partial \iota}\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}} \\
&= \frac{1}{\mu y_{ss}}\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}},
\end{aligned}$$

where we used that $\bar{s} = \frac{\iota(\tau,t)}{\mu y_{ss}}$. For further reference note that

$$\frac{\partial q(\iota,\tau,t)}{\partial \iota}\ |_{\iota=0} = \lim_{\bar{s}\to 0}\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}}\frac{1}{\mu y_{ss}}. \tag{B.7}$$

Step 3.1. Derivative $\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}}$. Consider the price function corresponding to Case 1. The derivative of the price function with respect to $\bar{s}$ is given by:

$$\begin{aligned}
\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}} &= \frac{\partial}{\partial \bar{s}}\left(\frac{\int_0^{\bar{s}} e^{-\int_0^s (r^*(t+u)+\eta)du}\,(\delta(\bar{s}-s)+\psi(\tau-s,t+s))\,ds}{\bar{s}}\right) \\
&= \frac{e^{-\int_0^{\bar{s}} (r^*(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s}) + \int_0^{\bar{s}} \delta e^{-\int_0^s (r(t+u)+\eta)du}ds}{\bar{s}} \\
&\quad - \frac{\int_0^{\bar{s}} e^{-\int_0^s (r^*(t+u)+\eta)du}\,(\delta(\bar{s}-s)+\psi(\tau-s,t+s))\,ds}{\bar{s}^2} \\
&= \frac{e^{-\int_0^{\bar{s}} (r^*(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s}) + \int_0^{\bar{s}} \delta e^{-\int_0^s (r^*(t+u)+\eta)du}ds - q(\iota,\tau,t)}{\bar{s}}.
\end{aligned} \tag{B.8}$$

Note that in the last line we used the definition of $q(\iota,\tau,t)$ for Case 1.

Step 3.2. Re-writing the limit of $\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}}$. To obtain $\frac{\partial q(\iota,\tau,t)}{\partial \iota}\ |_{\iota=0}$ we compute $\lim_{\bar{s}\to 0}\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}}$ using equation (B.8). In equation (B.8) both the numerator and denominator converge to zero as $\bar{s} \longrightarrow 0$.[21] Thus we employ L'Hôpital's

[21] This because the limit of the three terms of the numerator of equation (B.8) are respectively:

$$\begin{aligned}
\lim_{\bar{s}\longrightarrow 0}\int_0^{\bar{s}} e^{-\int_0^s (r(t+u)+\eta)du}ds &= 0 \\
\lim_{\bar{s}\longrightarrow 0} e^{-\int_0^{\bar{s}} (r(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s})) &= \psi(\tau,t) \\
\lim_{\bar{s}\longrightarrow 0} q(\iota,\tau,t) &= \psi(\tau,t).
\end{aligned}$$

rule to obtain the derivative of interest. The derivative of the denominator is 1. Thus, the limit of (B.8), is now given by:

$$\lim_{\bar{s}\longrightarrow 0}\frac{\partial q(\iota,\tau,t)}{\partial \bar{s}}=\lim_{\bar{s}\longrightarrow 0}\frac{\partial}{\partial \bar{s}}\left[e^{-\int_0^{\bar{s}}(r^*(t+u)+\eta)du}\psi(\tau-\bar{s},t+\bar{s})+\int_0^{\bar{s}}\delta e^{-\int_0^{s}(r^*(t+u)+\eta)du}ds-q(\iota,\tau,t)\right]. \tag{B.9}$$

Step 3.3. Consider the first two terms of (B.9). Applying Leibniz's rule:

$$\lim_{\bar{s}\longrightarrow 0}\left\{\left(-\frac{\partial}{\partial\tau}\psi(\tau-\bar{s},t+\bar{s})+\frac{\partial}{\partial t}\psi(\tau-\bar{s},t+\bar{s})-(r^*(t+\bar{s})+\eta)\psi(\tau-\bar{s},t+\bar{s})\right)e^{-\int_0^{\bar{s}}(r^*(t+u)+\eta)du}\right.$$
$$\left.+\delta e^{-\int_0^{\bar{s}}(r^*(t+u)+\eta)du}\right\}.$$

The previous limit is given by:

$$-\frac{\partial}{\partial\tau}\psi(\tau,t)+\frac{\partial}{\partial t}\psi(\tau,t)-(r^*(t)+\eta)\psi(\tau,t)+\delta.$$

Using the valuation of the international investors, we can rewrite the previous equation as:

$$\begin{aligned}-\frac{\partial}{\partial\tau}\psi(\tau,t)+\frac{\partial}{\partial t}\psi(\tau,t)-(r^*(t)+\eta)\psi(\tau,t)+\delta&=r^*(t)\psi(\tau,t)-(r^*(t)+\eta)\psi(\tau,t)\\&=-\eta\psi(\tau,t).\end{aligned} \tag{B.10}$$

This the first two terms of the limit of $\frac{\partial q(\iota,\tau,t)}{\partial\bar{s}}$ are equal to $-\eta\psi(\tau,t)$. Computing the limit of $\frac{\partial q(\iota,\tau,t)}{\partial\bar{s}}$: last term. The last term of (B.9) is given by

$$\begin{aligned}-\lim_{\bar{s}\longrightarrow 0}\frac{\partial q(\iota,\tau,t)}{\partial\bar{s}}&=-\lim_{\bar{s}\longrightarrow 0}\frac{\partial q(\iota,\tau,t)}{\partial\iota}\frac{\partial\iota}{\partial\bar{s}}\\&=-\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0}\mu y_{ss},\end{aligned} \tag{B.11}$$

where we used (B.7). Thus, from (B.10) and (B.11), the derivative (B.8) is given by:

$$\lim_{\bar{s}\longrightarrow 0}\frac{\partial q(\iota,\tau,t)}{\partial\bar{s}}=-\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0}\mu y_{ss}-\eta\psi(\tau,t). \tag{B.12}$$

Plugging (B.12) in (B.7) we obtain that:

$$\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0}=\left(-\mu y_{ss}\frac{\partial q(\iota,\tau,t)}{\partial\bar{\iota}}\mid_{\iota=0}-\eta\psi(\tau,t)\right)\frac{1}{\mu y_{ss}}.$$

Rearranging terms, we conclude that:

$$\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0}=-\frac{\eta\psi(\tau,t)}{2\mu y_{ss}}. \tag{B.13}$$

Step 4. Taylor Expansion. A first-order Taylor expansion around zero emissions yields:

$$\begin{aligned}q(\iota,\tau,t)&\simeq q(\iota,\tau,t)\mid_{\iota=0}+\frac{\partial q(\iota,\tau,t)}{\partial\iota}\mid_{\iota=0}\iota(\tau,t),\\&=\psi(\tau,t)-\frac{\eta\psi(\tau,t)}{2\mu y_{ss}}\iota(\tau,t).\end{aligned}$$

where we used (B.13). We can define price impact as $\bar{\lambda}=\frac{\eta}{\mu y_{ss}}$. This concludes the proof. □

C Calibration Notes

In this Appendix, we describe the sources of the data and the calibration procedure for the parameters values and shocks used in the numerical exercises in subsection 2.4, subsection 3.4, and subsection 4.4.

Income Process: ρ_y

We obtain the series for the Spanish Gross Domestic Product for the period Q1 1995 to Q2 2018 from FRED economic data `https://fred.stlouisfed.org/series/CLVMNACSCAB1GQES`. We estimate an AR(1) process for the de-trended seasonally adjusted output. The de-trending uses a Hodrick-Prescott filter with a parameter of 1600. The estimated model is $\log y_t = \rho_y \log y_{t-1} + \sigma_y \epsilon^y_{t-1}$. The estimated persistence of quarterly income (ρ_y) is 0.95 with a standard deviation (σ_y) of 0.375. This corresponds to a value of $\alpha_y = (1-\rho_y)/(3\times\Delta t) = 0.2$, where we fix $\Delta t = 1/12$ for all our numerical exercises. The details of the numerical procedure to solve the model are described in Appendix E.

Interest Rates Process: ρ_r

We obtain from the Bundesbank monthly data for the 1 month Euribor nominal rate. The period is Q11999 to Q22018. The source of the data is: `https://www.bundesbank.de/action/en/744770/bbkstatisticsearch?query=euribor`. We then obtain the annualized average (geometric mean) quarterly rate, i^q_t, as $\left[\Pi^3_{m=1}(1+i^m_t)\right]^{\frac{1}{3}} = (1+i^q_t)$. Using the quarterly inflation rate for the Eurozone (overall index not seasonally adjusted), obtained from the ECB Statistical Data Warehouse, the source of the data is `http://sdw.ecb.europa.eu/`, we compute the annualized real rate at a quarterly frequency as $\frac{1+i^q_t}{1+\pi^q_t} = 1 + r^q_t$. We then fit an AR(1) process for the level of the real interest rate $r_t = \mu_r + \rho_r r_{t-1} + \sigma_r \epsilon^r_{t-1}$. The estimated persistence of the quarterly real rate (ρ_r) is 0.95, which translates into $\alpha_r = (1-\rho_r)/(3\times\Delta t) = 0.2$, where $\Delta t = 1/12$. The standard deviation (σ_r) is equal to 0.410.

Dealers Cost of Capital: η

We approximate the cost of capital of dealers, η, as follows. For each one of the five largest US banks by Assets (as of August 2018), we obtain the current (as of August 2018) credit rating from Fitch, Standard and Poor's and Moody's. At the same time we obtain AA, A, BBB daily option-adjusted spreads (OAS) for US corporate bonds from FRED Economic Data for the period January 1st of 1997 to August 27th of 2018. The source of the data is: `https://fred.stlouisfed.org/search?st=ICE+BofAML+US+Corporate+AA+Option-Adjusted+Spread`. Options adjusted spreads measure the spread of a family of US Corporate issuers of the same credit rating adjusting for any embedded option.[22] We denote $OAS_t(\text{Rating})$ as a function that maps a rating and a moment in time to the option-adjusted spread. For each bank i we fix the current credit rating, $\text{Rating}_{i,t}$, for the entire sample of daily observations at the final value, $\text{Rating}_{i,T}$. Then, we obtain a time series for the spread of each bank as $\eta_{i,t} = OAS_t(\text{Rating}_{i,T})$ for $t = 1....T$. We weight the spreads of each bank by their relative assets (as of 2018). Finally, we obtain an estimated value of the spread of intermediaries in our model as $\hat{\eta} = \frac{\sum^T_{t=1}\sum^5_{i=1}\omega_{i,T}\eta_{i,t}}{T}$, equal to 149 basis points.

[22] Bank of America Merrill Lynch computes the index. More precisely, the option-adjusted spread for bonds is "the number of basis points that the fair value government spot curve is shifted to match the present value of discounted cash flows to the bond's price. For securities with embedded options, such as call, sink or put features, a log-normal short interest rate model is used to evaluate the present value of the securities potential cash flows." Thus, the index fits a term structure model for different securities, to separate the value of the security from any embedded option by subtracting or adding the value of the option.

Parameter	Description	Value
γ	Sovereign's risk aversion	2
δ	Coupon rate	0.01
ρ_y	Persistence of output	0.952
ρ_r	Persistence of short rate	0.949
σ_y	Standard deviation of output (pct/quarter)	0.375
σ_r	Standard deviation of short rate (pct/quarter)	0.410
η	Cost of Capital for Intermediaries (pct/py)	1.49
ϕ	Poisson Int. of a Large Shock (pct/py)	2.00
Δy	Drop in output (pct)	5.00
Δr	Increase in rates (pct)	1.00
ς	Logistic p.d.f. scale parameter	100

Table 2: Summary Baseline Calibration

Spanish Debt Profile: Figure 2.1

For the maturity debt profile of Spain featured in Figure 2.1, we use data from the Spanish Treasury. The data can be accessed at `www.tesoro.es/en/deuda-publica/estadísticas-mensuales` and corresponds to the debt profile as of July the 31st of 2018. We use the total debt.

Large Shock Intensity: ϕ

We calibrate the intensity of the shock based on the data from Barro and Ursua (2010). This data-set is obtained from `https://scholar.harvard.edu/barro/publications/barro-ursua-macroeconomic-data`. Out of 1600 year-country observations for OECD countries, 34 of them correspond to an output drop of more than 5 percent. This amounts to an estimated frequency equal to 2.13 percent per year; one large shock every 50 years. We calibrate the arrival of shocks in the risky steady state, ϕ, to a value equal to 0.02 (2.00 percent per year).

Large Shock Size: $\Delta y(0), \Delta r(0)$

We will fix 5 percent as the size of the shock to output, and we denote it as $\Delta y(0)$. For interest rates, the shock to the short rate is the one that implies the same drop in consumption than with the shock to output, given the persistence of rates (as well as the other parameters). This is an increase in rates from 4 to 5 percent, and we denote it as $\Delta r(0)$.

Scale Logistic Distribution: ς

To calibrate ς we match the unconditional default probability of Spain during the period 1877-1982. We proceed as follows. According to Barro and Ursua (2010), for the period 1945 to 2009 the most significant year to year drop in income for Spain was 4.8 percent. Thus, we will fix the size of the shock output to 5.0 percent and the intensity to $\phi = 0.02$, as in Section 3. The preference shocks are distributed according to a logistic distribution with a probability density function given by:

$$f(\varepsilon) = \frac{\varsigma e^{-\varsigma\varepsilon}}{(1 + e^{-\varsigma\varepsilon})^2}.$$

We set the scale parameter, ς, equal to 100. As we mentioned in the main text, for our calibration, this value of the parameter produces a default in 32 percent of the events when an extreme shock hits Spain's output. Given the intensity of the extreme shock, $\phi = 0.02$, this implies an unconditional default probability equal to 0.6 percent per year, roughly a default every 157 years. This is in line with the findings of Reinhart and Rogoff (2009) in which Spain experienced one default during the period 1877-1982.

D Proofs

D.1 Proof of Proposition 1

Proof. First we construct a Lagrangian on the space of functions g such that are Lebesgue integrable, $\left\| e^{-\rho t/2} g\left(\tau, t\right) \right\|^2 < \infty$. The Lagrangian, after replacing $c(t)$ from the budget constraint, is:

$$\begin{aligned}\mathcal{L}\left[\iota, f\right] &= \int_0^\infty e^{-\rho t} U\left(y\left(t\right) - f\left(0,t\right) + \int_0^T \left[q\left(\tau, t, \iota\right) \iota\left(\tau, t\right) - \delta f\left(\tau, t\right)\right] d\tau \right) dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t} j\left(\tau, t\right) \left(-\frac{\partial f}{\partial t} + \iota\left(\tau, t\right) + \frac{\partial f}{\partial \tau} \right) d\tau dt,\end{aligned}$$

where $j\left(\tau, t\right)$ is the Lagrange multiplier associated to the law of motion of debt.

We consider a perturbation $h\left(\tau, t\right), e^{-\rho t} h \in L^2\left(\left[0, T\right] \times \left[0, \infty\right)\right)$, around the optimal solution. Since the initial distribution f_0 is given, any feasible perturbation must satisfy $h\left(\tau, 0\right) = 0$. In addition, we know that $f\left(T, t\right) = 0$ because $f\left(T^+, t\right) = 0$ (by construction) and issuances are infinitesimal. Thus, any admissible variation must also feature $h\left(T, t\right) = 0$. At an optimal solution f, the Lagrangian must satisfy$\mathcal{L}\left[\iota, f\right] \geq \mathcal{L}\left[\iota, f + \alpha h\right]$ for any perturbation $h(\tau, t)$.

Taking the derivative with respect to α —i.e., computing the Gâteaux derivative, for any suitable $h\left(\tau, t\right)$ we obtain:

$$\begin{aligned}\left.\frac{\partial}{\partial \alpha} \mathcal{L}\left[\iota, f + \alpha h\right]\right|_{\alpha=0} &= \int_0^\infty e^{-\rho t} U'\left(c(t)\right) \left[-h\left(0, t\right) - \int_0^T \delta h\left(\tau, t\right) d\tau \right] dt \\ &\quad - \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j\left(\tau, t\right) d\tau dt \\ &\quad + \int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial \tau} j\left(\tau, t\right) d\tau dt.\end{aligned}$$

We employ integration by parts to show that:

$$\begin{aligned}\int_0^\infty \int_0^T e^{-\rho t} \frac{\partial h}{\partial t} j\left(\tau, t\right) d\tau dt &= \int_0^T \int_0^\infty e^{-\rho t} \frac{\partial h}{\partial t} j\left(\tau, t\right) dt d\tau \\ &= \int_0^T \left(\lim_{s \to \infty} e^{-\rho s} h\left(\tau, s\right) j\left(\tau, s\right)\right] - h\left(\tau, 0\right) j\left(\tau, 0\right) \right) d\tau \\ &\quad - \int_0^T \int_0^\infty e^{-\rho t} \left(\frac{\partial j\left(\tau, t\right)}{\partial t} - \rho j\left(\tau, t\right) \right) h(\tau, t) dt d\tau,\end{aligned}$$

and

$$\int_0^\infty e^{-\rho t} \int_0^T \frac{\partial h}{\partial \tau} j\left(\tau, t\right) d\tau dt = \int_0^\infty e^{-\rho t} \left[h\left(T, t\right) j\left(T, t\right) - h\left(0, t\right) j\left(0, t\right) - \int_0^T h(\tau, t) \frac{\partial j}{\partial \tau} d\tau \right] dt.$$

Replacing these calculations in the Lagrangian, and equating it to zero, yields:

$$\begin{aligned}0 &= \int_0^\infty e^{-\rho t}U'(c(t))\left[-h(0,t)-\int_0^T \delta h(\tau,t)\,d\tau\right]dt \\ &+\int_0^\infty\int_0^T e^{-\rho t}\left(-\rho j-\frac{\partial j}{\partial \tau}+\frac{\partial j}{\partial t}\right)h(\tau,t)\,d\tau dt \\ &+\int_0^\infty e^{-\rho t}\left(h(T,t)\,j(T,t)-h(0,t)\,j(0,t)\right)dt \\ &-\int_0^\infty \lim_{s\to\infty} e^{-\rho s}h(\tau,s)\,j(\tau,s)\,d\tau+h(\tau,0)\,j(\tau,0)\,.\end{aligned}$$

We rearrange terms to obtain:

$$\begin{aligned}0 &= -\int_0^\infty e^{-\rho t}\left[U'(c(t))-j(0,t)\right]h(0,t)\,dt \\ &+\int_0^\infty\int_0^T e^{-\rho t}\left(-\rho j-U'(c)\,\delta-\frac{\partial j}{\partial \tau}+\frac{\partial j}{\partial t}\right)h(\tau,t)\,d\tau dt \\ &-\int_0^\infty e^{-\rho t}\left(h(T,t)\,j(T,t)\right)dt \\ &-\int_0^\infty \lim_{s\to\infty} e^{-\rho s}h(\tau,s)\,j(\tau,s)\,d\tau+h(\tau,0)\,j(\tau,0)\,.\end{aligned} \tag{D.1}$$

Since $h(T,t)=h(\tau,0)=0$ is a condition for any admissible variation, then, both the third line in equation (D.1) and the second term in the fourth line are equal to zero. Furthermore, because (D.1) needs to hold for any feasible variation $h(\tau,t)$, all the terms that multiply $h(\tau,t)$ should equal zero. The latter, yields a system of necessary conditions for the Lagrange multipliers:

$$\begin{aligned}\rho j(\tau,t) &= -\delta U'(c(t))+\frac{\partial j}{\partial t}-\frac{\partial j}{\partial \tau},\text{ if }\tau\in(0,T], \\ j(0,t) &= -U'(c(t))\,,\ \text{ if }\tau=0, \\ \lim_{t\to\infty} e^{-\rho t}j(\tau,t) &= 0,\text{ if }\tau\in(0,T].\end{aligned} \tag{D.2}$$

Next, we perturb the control. We proceed in a similar fashion:

$$\begin{aligned}\tfrac{\partial}{\partial\alpha}\mathcal{L}\left[\iota+\alpha h,,f\right]\Big|_{\alpha=0} = &\int_0^\infty e^{-\rho t}U'(c(t))\left[\int_0^T\left(\frac{\partial q}{\partial \iota}\iota(\tau,t)+q(\tau,t,\iota)\right)h(\tau,t)\,d\tau\right]dt \\ &+\int_0^\infty\int_0^T e^{-\rho t}h(\tau,t)\,j(\tau,t)\,d\tau dt.\end{aligned}$$

Collecting terms and setting the Lagrangian to zero, we obtain:

$$\int_0^\infty\int_0^T e^{-\rho t}\left[j(\tau,t)+U'(c(t))\left(\frac{\partial q}{\partial \iota}\iota(\tau,t)+q(\tau,t,\iota)\right)\right]h(\tau,t)\,d\tau dt=0.$$

Thus, setting the term in parenthesis to zero, amounts to setting:

$$U'(c(t))\left(\frac{\partial q}{\partial \iota}\iota(\tau,t)+q(\tau,t,\iota)\right)=-j(\tau,t)\,. \tag{D.3}$$

Next, we define the Lagrange multiplier in terms of goods:

$$v(\tau,t) = -j(\tau,t)/U'(c(t)). \tag{D.4}$$

Taking the derivative of $v(\tau,t)$ with respect to t and τ we can express the necessary conditions, (D.2) in terms of v. In particular, we transform the PDE in (D.2) into the summary equations in the Proposition. That is:

$$\begin{aligned}
\left(\rho - \frac{U''(c(t))\,c(t)}{U'(c(t))}\frac{\dot{c}(t)}{c(t)}\right) v(\tau,t) &= \delta + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \tau}, \text{ if } \tau \in (0,T], \\
v(0,t) &= 1, \text{ if } \tau = 0, \\
\lim_{t\to\infty} e^{-\rho t} v(\tau,t) &= 0, \text{ if } \tau \in (0,T];
\end{aligned}$$

and the first-order condition, (D.3), is now given by:

$$\frac{\partial q}{\partial \iota}\iota(\tau,t) + q(\tau,t,\iota) = v(\tau,t)$$

as we intended to show. □

D.2 Duality

Given a path of resources $y(t)$, the primal problem, the one solved in Section 2, is given by:

$$V[f(\cdot,0)] = \max_{\{\iota(\tau,t),c(t)\}_{t\in[0,\infty),\tau\in[0,T]}} \int_t^\infty e^{-\rho(s-t)} u(c(s))ds \text{ s.t.}$$

$$c(t) = y(t) - f(0,t) + \int_0^T [q(\tau,t,\iota)\,\iota(\tau,t) - \delta f(\tau,t)]\,d\tau$$

$$\frac{\partial f}{\partial t} = \iota(\tau,t) + \frac{\partial f}{\partial \tau}; f(\tau,0) = f_0(\tau).$$

Here we show that this problem has a dual formulation. This dual formulation, minimizes the resources needed to sustain a given path of consumption $c(t)$:

$$D[f(\cdot,0)] = \min_{\{\iota(\tau,t)\}_{t\in[0,\infty),\tau\in[0,T]}} \int_0^\infty e^{-\int_0^t r(s)ds} y(t)dt \text{ s.t.}$$

$$c(t) = y(t) - f(0,t) + \int_0^T [q(\tau,t,\iota)\iota(\tau,t) - \delta f(\tau,t)]\,d\tau$$

$$\frac{\partial f}{\partial t} = \iota(\tau,t) + \frac{\partial f}{\partial \tau}; f(\tau,0) = f_0(\tau)$$

$$r(t) = \rho - \frac{U''(c(t))\,c(t)}{U'(c(t))}\frac{\dot{c}(t)}{c(t)}.$$

Proposition 5. *Consider the solution $\{c^*(t),\iota^*(\tau,t),f^*(\tau,t)\}_{t\in[0,\infty),\tau\in(0,T]}^{\infty}$ to the Primal Problem given f_0. Then, given the path of consumption $c^*(t)$, $\{y^*(t),\iota^*(\tau,t),f^*(\tau,t)\}_{t\in[0,\infty),\tau\in(0,T]}$ solves the Dual Problem where:*

$$y^*(t) = c^*(t) + f^*(0,t) + \int_0^T [q(\tau,t,\iota^*)\iota^*(\tau,t) - \delta f^*(\tau,t)]\,d\tau.$$

Proof. Step 1. We start following the steps of Proposition 1. We construct the Lagrangian for the Dual Problem in the space $\left\| e^{-\rho t/2} g\left(\tau, t\right) \right\|^2 < \infty$. After replacing the resources $y(t)$ needed to support a path of consumption $c(t)$ the budget constraint, is:

$$
\begin{aligned}
\mathcal{L}\left[\iota, f\right] &= \int_0^\infty e^{-\int_0^t r(t)ds} \left(c\left(t\right) + f\left(0, t\right) - \int_0^T \left[q\left(\tau, t, \iota\right) \iota\left(\tau, t\right) - \delta f\left(\tau, t\right)\right] d\tau \right) dt \\
&+ \int_0^\infty \int_0^T e^{-\int_0^t r(t)ds} v\left(\tau, t\right) \left(-\frac{\partial f}{\partial t} + \iota\left(\tau, t\right) + \frac{\partial f}{\partial \tau} \right) d\tau dt,
\end{aligned}
$$

where $v\left(\tau, t\right)$ is the Lagrange multiplier associated to the law of motion of debt. We again consider a perturbation $h\left(\tau, t\right), e^{-\rho t} h \in L^2\left(\left[0, T\right] \times \left[0, \infty\right)\right)$, around the optimal solution. Recall that because f_0 is given, and $f\left(T, t\right) = 0$, any feasible perturbation needs to meet: $h\left(\tau, 0\right) = 0$ and $h\left(T, t\right) = 0$. At an optimal solution f, it must be the case that $\mathcal{L}\left[\iota, f\right] \geq \mathcal{L}\left[\iota, f + \alpha h\right]$ for any feasible perturbation $h(\tau, t)$. This implies that

$$
\begin{aligned}
\left.\frac{\partial}{\partial \alpha} \mathcal{L}\left[\iota, f + \alpha h\right]\right|_{\alpha=0} &= \int_0^\infty e^{-\int_0^t r(t)ds} \left[h\left(0, t\right) + \int_0^T \delta h\left(\tau, t\right) d\tau \right] dt \\
&- \int_0^\infty \int_0^T e^{-\int_0^t r(t)ds} \frac{\partial h}{\partial t} v\left(\tau, t\right) d\tau dt \\
&+ \int_0^\infty \int_0^T e^{-\int_0^t r(t)ds} \frac{\partial h}{\partial \tau} v\left(\tau, t\right) d\tau dt.
\end{aligned}
$$

We again employ integration by parts to show that:

$$
\begin{aligned}
\int_0^\infty \int_0^T e^{-\int_0^t r(t)ds} \frac{\partial h}{\partial t} v\left(\tau, t\right) d\tau dt &= \int_0^T \int_0^\infty e^{-\int_0^t r(t)ds} v\left(\tau, t\right) \frac{\partial h}{\partial t} dt d\tau \\
&= \int_0^T \left(\lim_{s \to \infty} e^{-\int_0^t r(t)ds} h\left(\tau, s\right) v\left(\tau, s\right)\right] - h\left(\tau, 0\right) v\left(\tau, 0\right) \right) d\tau \\
&- \int_0^T \int_0^\infty e^{-\int_0^t r(t)ds} \left(\frac{\partial v\left(\tau, t\right)}{\partial t} - r(t) v\left(\tau, t\right) \right) h(\tau, t) dt d\tau \\
&= \int_0^T \left(\lim_{s \to \infty} e^{-\int_0^t r(t)ds} h\left(\tau, s\right) v\left(\tau, s\right) - h\left(\tau, 0\right) v\left(\tau, 0\right) \right) d\tau - \\
&\int_0^\infty e^{-\int_0^s r(u)du} \int_0^T \left(\frac{\partial v\left(\tau, t\right)}{\partial t} - r(t) v\left(\tau, t\right) \right) h(\tau, t) d\tau dt,
\end{aligned}
$$

and

$$
\int_0^\infty e^{-\int_0^t r(t)ds} \int_0^T \frac{\partial h}{\partial \tau} v\left(\tau, t\right) d\tau dt = \int_0^\infty e^{-\int_0^t r(t)ds} \left[h\left(T, t\right) v\left(T, t\right) - h\left(0, t\right) v\left(0, t\right) - \int_0^T h(\tau, t) \frac{\partial v}{\partial \tau} d\tau \right] dt.
$$

Replacing these calculations in the Lagrangian, and equating it to zero, yields:

$$
\begin{aligned}
0 \;=\; & \int_0^\infty e^{-\int_0^t r(t)ds}\left[h\left(0,t\right)+\int_0^T \delta h\left(\tau,t\right)d\tau\right]dtt \\
&+\int_0^\infty \int_0^T e^{-\int_0^t r(t)ds}\left(-r(t)v-\frac{\partial v}{\partial \tau}+\frac{\partial v}{\partial t}\right)h\left(\tau,t\right)d\tau dt \\
&+\int_0^\infty e^{-\int_0^t r(t)ds}\left(h\left(T,t\right)v\left(T,t\right)-h\left(0,t\right)v\left(0,t\right)\right)dt \\
&-\int_0^\infty \lim_{s\to\infty} e^{-\int_0^s r(u)du}h\left(\tau,s\right)v\left(\tau,s\right)d\tau.
\end{aligned}
$$

Again, the previous equation needs to hold for any feasible variation $h(\tau,t)$, all the terms that multiply $h(\tau,t)$ should be equal to zero. The latter, yields a system of necessary conditions for the Lagrange multipliers, and substituting for the value of $r(t)$:

$$
\begin{aligned}
\left(\rho-\frac{U''\left(c\left(t\right)\right)c\left(t\right)}{U'\left(c\left(t\right)\right)}\frac{\dot{c}(t)}{c\left(t\right)}\right)v\left(\tau,t\right) &= \delta+\frac{\partial v}{\partial t}-\frac{\partial v}{\partial \tau},\text{ if }\tau\in(0,T], \\
v\left(0,t\right) &= 1,\text{ if }\tau=0, \\
\lim_{t\to\infty} e^{-\rho t}v\left(\tau,t\right) &= 0,\text{ if }\tau\in(0,T].
\end{aligned}
\tag{D.5}
$$

By proceeding in a similar fashion with the control we arrive to:

$$
\left(\frac{\partial q}{\partial \iota}\iota\left(\tau,t\right)+q\left(\tau,t,\iota\right)\right)=-v\left(\tau,t\right). \tag{D.6}
$$

Note that system of equation (D.5) to (D.6) plus the budget constraint, the law of motion of debt, and initial debt f_0, are precisely the conditions that characterize the solution of the primal problem. □

D.3 Asymptotic Behavior

Here we formally prove the limit conditions that we discussed after Proposition 1. In particular, the provide a complete asymptotic characterization. The following Proposition provides a summary.

Proposition 6. *Assume that $\rho > r^*_{ss}$, there exists a steady state if and only if $\bar{\lambda} > \bar{\lambda}_o$ for some $\bar{\lambda}_o$. If instead, $\bar{\lambda} \leq \bar{\lambda}_o$, there is no steady state but consumption converges asymptotically to zero. In particular, the asymptotic behavior is:*

Case 1 (High Liquidity Costs). *For liquidity costs above the threshold value $\bar{\lambda} > \bar{\lambda}_o$, variables converge to a steady state characterized by the following system:*

$$
\frac{\dot{c}_{ss}}{c_{ss}} = 0
$$

$$
r_{ss} = 0 \tag{D.7}
$$

$$
\iota_{ss}(\tau) = \frac{\psi_{ss}(\tau)-v_{ss}(\tau)}{\bar{\lambda}\psi_{ss}\left(\tau\right)}, \tag{D.8}
$$

$$
v_{ss}(\tau) = \frac{\delta}{\rho}(1-e^{-\rho\tau})+e^{-\rho\tau} \tag{D.9}
$$

$$
f_{ss}(\tau) = \int_\tau^T \iota_{ss}(s)ds \tag{D.10}
$$

$$
c_{ss} = y_{ss}-f_{ss}(0)+\int_0^T\left[\psi_{ss}(\tau)\iota_{ss}(\tau)-\frac{\bar{\lambda}\psi_{ss}\left(\tau\right)}{2}\iota_{ss}(\tau)^2-\delta f_{ss}(\tau)\right]. \tag{D.11}
$$

Case 2 (Low Liquidity Costs). *For liquidity costs below the threshold value $0 < \bar{\lambda} \leq \bar{\lambda}_o$, variables converge asymptotically to:*

$$\lim_{s\to\infty}\frac{c(s)}{c(t)} = e^{-\frac{(\rho - r_\infty(\bar{\lambda}))(s-t)}{\sigma}}$$
$$v_\infty(\tau, r_\infty(\bar{\lambda})) = \frac{\delta}{r_\infty(\bar{\lambda})}(1 - e^{-r_\infty(\bar{\lambda})\tau}) + e^{-r_\infty(\bar{\lambda})\tau}$$
$$\iota_\infty(\tau, r_\infty(\bar{\lambda})) = \frac{\psi_{ss}(\tau) - v_\infty(\tau, r_\infty(\bar{\lambda}))}{\bar{\lambda}\psi_{ss}(\tau)}$$
$$f_\infty(\tau, r_\infty(\bar{\lambda})) = \int_\tau^T \iota_\infty(s, r_\infty(\bar{\lambda}))ds$$

where $r_\infty(\bar{\lambda})$ satisfies $r^*_{ss} \leq r_\infty(\bar{\lambda}) < \rho$ and solves:

$$\begin{aligned} c_\infty &= 0 \\ &= y_{ss} - f_\infty(0, r_\infty(\bar{\lambda})) + \int_0^T \left[\iota_\infty(\tau, r_\infty(\bar{\lambda}))\psi(\tau) - \frac{\bar{\lambda}\psi_{ss}(\tau,t)}{2}\iota_\infty(\tau, r_\infty(\bar{\lambda}))^2 - \delta f_\infty(\tau, r_\infty(\bar{\lambda}))\right] d\tau. \end{aligned}$$

Threshold. *The threshold $\bar{\lambda}_o$ solves $|c_{ss}|_{\bar{\lambda}=\bar{\lambda}_o} = 0$ in (D.11) and $\lim_{\bar{\lambda}\to\bar{\lambda}_o} r_\infty(\bar{\lambda}) = \rho$.*

Proof. Step 1. First observe that as $\bar{\lambda} \to \infty$, the optimal issuance policy (2.12) approaches $\iota(\tau, t) = 0$. Thus, for that limit, $c_{ss} = y > 0$ and $f_{ss}(\tau) = 0$.

Step 2. Next, consider the system in *Case 1* of Proposition 6 as a guess of a solution. Note that equations (D.8) to (D.11) meet the necessary conditions of Proposition 1 as long as $r(t) = \rho$. This because: $\iota_{ss}(\tau)$ meets the first order condition with respect to the control; $v_{ss}(\tau)$ solves the PDE for valuations; given $\iota_{ss}(\tau)$ and $v_{ss}(\tau)$ the stock of debt solves the KFE, thus, is given by $\int_\tau^T \iota_{ss}(s)ds$; and consumption is pinned down by the budget constraint. In addition, by construction, consumption determined in (D.11) does not depend on time; i.e. $\dot{c}(t) = 0$ and this implies that

$$r_{ss} \equiv r(t) = \rho.$$

Thus, the only thing we need to check is that there exists some $\bar{\lambda}$ finite such that consumption is positive.

Step 3. The system in equations (D.8) to (D.11) is continuous in $\bar{\lambda}$. Therefore, because $c_{ss} = y > 0$ for $\bar{\lambda} \to \infty$, there exists a value of $\bar{\lambda}$ such that the implied consumption by equations (D.8) to (D.11) is positive.

Step 4. We now prove that there is an interval where this solution holds. In particular, we will show that c_{ss} decreases as $\bar{\lambda}$ increases. Observe that, steady state internal valuations $v_{ss}(\tau)$ in (D.9) and bond prices $\psi(\tau)$ are independent of $\bar{\lambda}$. Steady-state debt issuance's $\iota_{ss}(\tau)$ in (D.8) are a monotonously decreasing function of $\bar{\lambda}$, because

$$\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}}\iota_{ss}(\tau) < 0,$$

and therefore the total amount of debt at each maturity $f_{ss}(\tau)$ in (D.10) is also decreasing with $\bar{\lambda}$, because

$$\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}} = -\frac{1}{\bar{\lambda}}f_{ss}(\tau) < 0.$$

If we take derivatives with respect to $\bar{\lambda}$ in the budget constraint (D.11) we obtain:

$$\begin{aligned}\frac{\partial c_{ss}}{\partial \bar{\lambda}} &= -\frac{\partial f_{ss}(0)}{\partial \bar{\lambda}} + \int_0^T \left[\psi_{ss}(\tau)\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \frac{\psi_{ss}(\tau)}{2}\iota_{ss}(\tau)^2 - \bar{\lambda}\psi_{ss}(\tau)\,\iota_{ss}(\tau)\frac{\partial \iota_{ss}(\tau)}{\partial \bar{\lambda}} - \delta\frac{\partial f_{ss}(\tau)}{\partial \bar{\lambda}}\right] d\tau \\ &= \frac{1}{\bar{\lambda}} f_{ss}(0) - \frac{1}{\bar{\lambda}}\int_0^T \left[\psi_{ss}(\tau)\iota_{ss}(\tau) + \bar{\lambda}\frac{\psi_{ss}(\tau)}{2}\iota_{ss}(\tau)^2 - \bar{\lambda}\psi_{ss}(\tau)\,\iota_{ss}(\tau)^2 - \delta f_{ss}(\tau)\right] d\tau \\ &= -\frac{1}{\bar{\lambda}} c_{ss} < 0.\end{aligned}$$

Observe that $\iota_{ss}(\tau)$ can be made arbitrarily small by increasing $\bar{\lambda}$. Thus, there exist a value of $\bar{\lambda} \geq 0$ such that $c_{ss} = 0$ in the system above. We denote this value by $\bar{\lambda}_0$.

Step 5. For $\bar{\lambda} \leq \bar{\lambda}_o$, if a steady state exists, it would imply $c_{ss} < 0$, outside of the range of admissible values. Therefore, there is no steady state in this case. Assume that the economy grows asymptotically at rate $g_\infty(\bar{\lambda}) \equiv \lim_{t\to\infty}\frac{1}{c(t)}\frac{dc}{dt}$. If $g_\infty(\bar{\lambda}) > 0$ then consumption would grow to infinity, which violates the budget constraint. Thus, if there exists an asymptotic the growth rate, it is negative: $g_\infty(\bar{\lambda}) < 0$. If we define $r_\infty(\bar{\lambda})$ as

$$r_\infty(\bar{\lambda}) \equiv (\rho + \sigma g(\bar{\lambda})) < \rho,$$

the growth rate of the economy can be expressed as

$$g_\infty(\bar{\lambda}) = -\frac{(\rho - r_\infty(\bar{\lambda}))}{\sigma}.$$

When this is the case, the asymptotic valuation is

$$v_\infty(\tau, r_\infty(\bar{\lambda})) = \frac{\delta\left(1 - e^{-r_\infty(\bar{\lambda})\tau}\right)}{r_\infty(\bar{\lambda})} + e^{-r_\infty(\bar{\lambda})\tau}.$$

To obtain the discount factor bounds, observe that if $v_\infty(\tau, r_\infty(\bar{\lambda})) \leq \psi_{ss}(\tau)$ the optimal issuance is non-negative. Otherwise issuances would be negative at all maturities and the country would be an asymptotic net asset holder. This cannot an optimal solution as this implies that consumption can be increased just by reducing the amount of foreign assets. Therefore, $r_\infty(\bar{\lambda}) \geq r^*$. Finally, by definition $r_\infty(\bar{\lambda}) < \rho$. $\square$

D.4 No liquidity costs: $\bar{\lambda} = 0$

Proposition 7. *(Optimal Policy with Liquid Debt) Assume that $\lambda(\tau, t, \iota) = 0$. If a solution exists, then consumption satisfies equation (2.11) with $r^*(t) = r(t)$ and the initial condition $B(0) = \int_0^\infty \exp\left(-\int_0^s r^*(u)du\right)(c(s) - y(s))\,ds$. Given the optimal path of consumption, any solution $\iota(\tau, t)$ consistent with (2.1), (2.16) and*

$$\dot{B}(t) = r^*(t)B(t) + c(t) - y(t), \text{ for } t > 0, \tag{D.12}$$

is an optimal solution.

Proof. Step 1. The first part of the proof is just a direct consequence of the first-order condition $v(\tau, t) = \psi(\tau, t)$ for bond issuance. Bond prices are given by (2.4) while the Government valuations are given by (2.10). Since both equations must be equal in a bounded solution, we conclude that

$$r^*(t) = r(t) = \rho - \frac{U''(c(t))}{U'(c(t))}\frac{dc}{dt},$$

must describe the dynamics of consumption.

Step 2. The second part of the proof derives the law of motion of $B(t)$. First we take the derivative with respect to time at both sides of definition (2.16), that we repeat for completion:

$$B(t) = \int_0^T \psi(\tau,t) f(\tau,t)\, d\tau.$$

Recall that, from the law of motion of debt, equation (2.1), it holds that:

$$\iota(\tau,t) = -\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \tau}.$$

To express the budget constraint in terms of f, we substitute $\iota(\tau,t)$ into the budget constraint:

$$c(t) = y(t) - f(0,t) + \int_0^T \left[\psi(\tau,t)\left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial \tau}\right) - \delta f(\tau,t)\right] d\tau. \tag{D.13}$$

We would like to rewrite equation (D.13). Therefore, first, we apply integration by parts to the following expression:

$$\int_0^T \psi(\tau,t)\frac{\partial f}{\partial \tau} d\tau = \psi(T,t) f(T,t) - \psi(0,t) f(0,t) - \int_0^T \frac{\partial \psi}{\partial \tau} f(\tau,t)\, d\tau.$$

As long as the solution is smooth, it holds that $f(T,t) = 0$. Further, recall that by construction $\psi(0,t) = 1$. Hence:

$$\int_0^T \psi(\tau,t)\frac{\partial f}{\partial \tau} d\tau = -f(0,t) - \int_0^T \frac{\partial \psi}{\partial \tau} f(\tau,t)\, d\tau. \tag{D.14}$$

Second, from the pricing equation of international investors, we now that

$$\frac{\partial \psi}{\partial \tau} = -r^*(t)\,\psi(\tau,t) + \delta + \frac{\partial \psi}{\partial t}.$$

Then, we obtain:

$$\int_0^T \psi(\tau,t)\frac{\partial f}{\partial \tau} d\tau = -f(0,t) - \int_0^T [\delta + \psi_t(\tau,t) - r(t)\psi(\tau,t)]\, f(\tau,t)\, d\tau. \tag{D.15}$$

We substitute (D.14) and (D.15) this expression into (D.13), and thus:

$$\begin{aligned}
c(t) =& y(t) - f(0,t) + \int_0^T \left[\psi(\tau,t)\frac{\partial f}{\partial t} - \delta f(\tau,t)\right] d\tau ...\\
&- \left\{ -f(0,t) - \int_0^T [\delta + \psi_t(\tau,t) - r(t)\psi(\tau,t)]\, f(\tau,t)\, d\tau \right\}\\
=& y(t) + \int_0^T [\psi(\tau,t) f_t(\tau,t) + \psi_t(\tau,t) f(\tau,t)]\, d\tau - \int_0^T r^*(t)\psi(\tau,t) f(\tau,t)\,.
\end{aligned}$$

Rearranging terms and employing the definitions above, we obtain:

$$\dot{B}(t) = c(t) - y(t) + r^*(t) B(t),$$

as desired. $\square$

D.5 Limiting Distribution: $\bar{\lambda} \to 0$

Proposition 8. *(Limiting distribution) In the limit as liquidity costs vanish, $\bar{\lambda} \to 0$, the asymptotic optimal issuance is given by*

$$\iota_{\infty}^{\bar{\lambda}\to 0}(\tau) = \lim_{\bar{\lambda}\to 0} \iota_{\infty}(\tau) = \frac{1 + \left[-1 + (r^*/\delta - 1)\, r_{ss}^* \tau\right] e^{-r_{ss}^*\tau}}{1 + \left[-1 + (r^*/\delta - 1)\, r_{ss}^* T\right] e^{-r_{ss}^* T}} \frac{\psi_{ss}(T)}{\psi_{ss}(\tau)} \kappa, \tag{D.16}$$

where constant $\kappa > 0$ is such that $y_{ss} - f_{\infty}^{\bar{\lambda}\to 0}(0) + \int_0^T \left[\iota_{\infty}^{\bar{\lambda}\to 0}(\tau)\, \psi_{ss}(\tau) - \delta f_{\infty}^{\bar{\lambda}\to 0}(\tau)\right] d\tau = 0$, and $f_{\infty}^{\bar{\lambda}\to 0}(\tau) = \int_{\tau}^{T} \iota_{\infty}^{\bar{\lambda}\to 0}(s)\, ds$.

Proof. Consider the following limit:

$$\begin{aligned}
\iota_{\infty}^{\bar{\lambda}\to 0}(\tau) &= \lim_{\bar{\lambda}\to 0} \iota_{\infty}(\tau, r_{\infty}(\bar{\lambda})) \\
&= \lim_{\bar{\lambda}\to 0} \frac{\psi_{ss}(\tau) - v_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{\bar{\lambda}\psi_{ss}(\tau)} \\
&= \lim_{\bar{\lambda}\to 0} \frac{1}{\bar{\lambda}\psi(\tau)} \left[\frac{\delta\left(1 - e^{-r_{ss}^*\tau}\right)}{r_{ss}^*} - \frac{\delta\left(1 - e^{-r_{\infty}(\bar{\lambda})\tau}\right)}{r_{\infty}(\bar{\lambda})} + e^{-r_{ss}^*\tau} - e^{-r_{\infty}(\bar{\lambda})\tau} \right].
\end{aligned}$$

This is a limit of the form $\frac{0}{0}$ as $\lim_{\bar{\lambda}\to 0} r_{\infty}(\bar{\lambda}) = r^*$.[23] We do not have an expression for $r_{\infty}(\bar{\lambda})$, so we cannot apply L'Hôpital's rule directly. Instead, we compute the following limit:

$$\lim_{\bar{\lambda}\to 0} \frac{\iota_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{\iota_{\infty}(T, r_{\infty}(\bar{\lambda}))} = \lim_{r_{\infty}(\bar{\lambda})\to r^*} \frac{\frac{\delta\left(1-e^{-r^*\tau}\right)}{r^*} - \frac{\delta\left(1-e^{-r_{\infty}(\bar{\lambda})\tau}\right)}{r} + e^{-r^*\tau} - e^{-r_{\infty}(\bar{\lambda})\tau}}{\frac{\delta\left(1-e^{-r^*T}\right)}{r^*} - \frac{\delta\left(1-e^{-r_{\infty}(\bar{\lambda})T}\right)}{r} + e^{-r^*T} - e^{-r_{\infty}(\bar{\lambda})T}} \frac{\psi(T)}{\psi(\tau)},$$

which also has a limit of the form $\frac{0}{0}$. Now we can apply L'Hôpital's. We obtain:

$$\begin{aligned}
\lim_{\bar{\lambda}\to 0} \frac{\iota_{\infty}(\tau, r_{\infty}(\bar{\lambda}))}{\iota_{\infty}(T, r_{\infty}(\bar{\lambda}))} &= \frac{\frac{-\delta r^* \tau e^{-r^*\tau} + \delta\left(1-e^{-r^*\tau}\right)}{r^{*2}} + \tau e^{-r^*\tau}}{\frac{-\delta r^* T e^{-r^*T} + \delta\left(1-e^{-r^*T}\right)}{r^{*2}} + T e^{-rT}} \frac{\psi(T)}{\psi(\tau)} \\
&= \frac{1 + \left[-1 + (r^*/\delta - 1)\, r^*\tau\right] e^{-r^*\tau}}{1 + \left[-1 + (r^*/\delta - 1)\, r^* T\right] e^{-r^*T}} \frac{\psi(T)}{\psi(\tau)}.
\end{aligned}$$

If we define

$$\kappa \equiv \lim_{\bar{\lambda}\to 0} \iota_{\infty}(T, r_{\infty}(\bar{\lambda}))$$

then

$$\lim_{\bar{\lambda}\to 0} \iota_{\infty}(\tau, r_{\infty}(\bar{\lambda})) = \frac{1 + \left[-1 + (r^*/\delta - 1)\, r^*\tau\right] e^{-r^*\tau}}{1 + \left[-1 + (r^*/\delta - 1)\, r^* T\right] e^{-r^*T}} \frac{\psi(T)}{\psi(\tau)} \kappa.$$

The value of κ then must be consistent with zero consumption:

$$y_{ss} - f_{\infty}^{\bar{\lambda}\to 0}(0) + \int_0^T \left[\iota_{\infty}^{\bar{\lambda}\to 0}(\tau)\, \psi_{ss}(\tau) - \delta f_{\infty}^{\bar{\lambda}\to 0}(\tau)\right] d\tau = 0,$$

for $f_{\infty}^{\bar{\lambda}\to 0}(\tau) = \int_{\tau}^{T} \iota_{\infty}^{\bar{\lambda}\to 0}(s)\, ds$. ∎

[23] We drop the sub-index ss to ease the notation.

D.6 Proof of Proposition 2

We need first the following lemma:

Lemma 1. *Given a fix t, the Gâteaux derivative of the post-shock value functional $V[f(\cdot,t)]$, defined by equation (2.6), with respect to the debt distribution $f(\cdot,t)$ is the valuation $j(\tau,t) = -U'(c(t))\, v(\tau,t)$* satisfying equation (2.8):

$$\frac{\partial}{\partial\alpha} V\left[f(\tau,t) + \alpha h(\tau,t)\right]\Big|_{\alpha=0} = \int_0^T j(\tau,t)\, h(\tau,t)\, d\tau.$$

Proof. To simplify notation assume, without loss of generality, that $t = 0$. To avoid confusions, we denote by (ι^*, f^*) the optimal issuance policy and debt distributions. First, note that

$$V\left[f(\cdot,0)\right] = \mathcal{L}\left[\iota^*, f^*\right].$$

This follows from the fact that

$$\begin{aligned}
V\left[f(\cdot,0)\right] &= \int_0^\infty e^{-\rho t} U\left(y(t) - f^*(0,t) + \int_0^T \left[q(\tau,t,\iota^*)\,\iota^*(\tau,t) - \delta f^*(\tau,t)\right] d\tau\right) dt \\
&= \int_0^\infty e^{-\rho t} U\left(y(t) - f^*(0,t) + \int_0^T \left[q(\tau,t,\iota^*)\,\iota^*(\tau,t) - \delta f^*(\tau,t)\right] d\tau\right) dt \\
&+ \int_0^\infty \int_0^T e^{-\rho t} j^*(\tau,t)\left(-\frac{\partial f^*}{\partial t} + \iota^*(\tau,t) + \frac{\partial f^*}{\partial\tau}\right) d\tau dt, \\
&= \mathcal{L}\left[\iota^*, f^*\right].
\end{aligned}$$

where the first line is the definition of $V\left[f(\cdot,0)\right]$, the second line is the fact that

$$-\frac{\partial f^*}{\partial t} + \iota^*(\tau,t) + \frac{\partial f^*}{\partial\tau} = 0$$

for for every τ, t and the last line is the definition of the Lagrangian. Next we compute $\frac{\partial}{\partial\alpha}\mathcal{L}\left[\iota^*, f^* + \alpha h(\tau,0)\right]\Big|_{\alpha=0}$. The derivative with respect to a general variation $h(\tau,t)$ is given by:

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\mathcal{L}\left[\iota, f + \alpha h\right]\Big|_{\alpha=0} &= \int_0^\infty e^{-\rho t} U'(c(t))\left[-h(0,t) - \int_0^T \delta h(\tau,t)\, d\tau\right] dt \\
&\quad - \int_0^\infty \int_0^T e^{-\rho t}\frac{\partial h}{\partial t} j(\tau,t)\, d\tau dt \\
&\quad + \int_0^\infty \int_0^T e^{-\rho t}\frac{\partial h}{\partial\tau} j(\tau,t)\, d\tau dt.
\end{aligned} \tag{D.17}$$

We employ integration by parts to show that:

$$\begin{aligned}
\int_0^\infty \int_0^T e^{-\rho t}\frac{\partial h}{\partial t} j(\tau,t)\, d\tau dt &= \int_0^T \left(\lim_{s\to\infty} e^{-\rho s}\left[h(\tau,s)\, j(\tau,s)\right] - h(\tau,0)\, j(\tau,0)\right) d\tau \\
&\quad - \int_0^T \int_0^\infty e^{-\rho t}\left(\frac{\partial j(\tau,t)}{\partial t} - \rho j(\tau,t)\right) h(\tau,t) dt d\tau
\end{aligned}$$

and

$$\int_0^\infty e^{-\rho t}\int_0^T \frac{\partial h}{\partial\tau} j(\tau,t)\, d\tau dt = \int_0^\infty e^{-\rho t}\left[h(T,t)\, j(T,t) - h(0,t)\, j(0,t) - \int_0^T h(\tau,t)\frac{\partial j}{\partial\tau} d\tau\right] dt.$$

Note that we have not yet used optimality. Using the particular case of interest, i.e.:

$$h(\tau,0) = h(\tau,t)\delta(t),$$

where $\delta(t)$ is the Dirac delta, and plugging it in equation (D.17):

$$\begin{aligned}
\left.\frac{\partial}{\partial\alpha}\mathcal{L}\left[\iota^*, f^* + \alpha h(\cdot,0)\right]\right|_{\alpha=0} = {} & U'\left(c^*(0)\right)\left[-h(0,0) - \int_0^T \delta h\left(\tau,0\right) d\tau\right] \\
& + \int_0^T h\left(\tau,0\right) j\left(\tau,0\right) d\tau \\
& + \int_0^T \left(\frac{\partial j\left(\tau,0\right)}{\partial t} - \rho j\left(\tau,0\right)\right) h(\tau,0) d\tau, \\
& - h\left(0,0\right) j\left(0,0\right) - \int_0^T h(\tau,0)\frac{\partial j}{\partial\tau} d\tau dt
\end{aligned}$$

Because (ι^*, f^*) is an optimum, we know that for all $\tau \in (0, T]$ the following holds

$$\frac{\partial j\left(\tau,0\right)}{\partial t} - \rho j\left(\tau,0\right) - \frac{\partial j\left(\tau,0\right)}{\partial\tau} - \delta = 0$$

and for $\tau = 0$ it also holds that $U'\left(c^*(0)\right) = -j\left(0,0\right)$. This implies that

$$\left.\frac{\partial}{\partial\alpha}\mathcal{L}\left[\iota^*, f^* + \alpha u(\cdot,0)\right]\right|_{\alpha=0} = \int_0^T h\left(\tau,0\right) j\left(\tau,0\right) d\tau.$$

Thus,

$$\begin{aligned}
\left.\frac{\partial}{\partial\alpha}\mathcal{L}\left[\iota^*, f^* + \alpha h(\cdot,0)\right]\right|_{\alpha=0} &= \int_0^T h\left(\tau,0\right) j\left(\tau,0\right) d\tau \\
&= \left.\frac{\partial}{\partial\alpha} V\left[f(\cdot,0) + \alpha h(\cdot,0)\right].\right|
\end{aligned}$$

□

Proof. Proposition 2. Then we can proceed with the proof of Proposition 2. The Lagrangian is:

$$\begin{aligned}
\mathcal{L}\left[\hat{\iota}, \hat{f}\right] = \mathbb{E}^{t^o}\Bigg[& \int_0^{t^o} e^{-\rho s} U\left(\hat{c}(s)\right) ds + e^{-\rho t^o}\mathbb{E}^X_{t^o}\left\{V\left[\hat{f}\left(\cdot,t^o\right), X\left(t^o\right)\right]\right\} \\
& + \int_0^{t^o}\int_0^T e^{-\rho s}\hat{\jmath}\left(\tau,s\right)\left(-\frac{\partial\hat{f}}{\partial s} + \hat{\iota}\left(\tau,s\right) + \frac{\partial\hat{f}}{\partial\tau}\right) d\tau ds\Bigg]
\end{aligned}$$

where $\mathbb{E}^{t^o}$ denotes the expectation with respect to the random time t^o. In this case $\hat{\jmath}\left(\tau,s\right)$ is the Lagrange multiplier associated to the law of motion of debt, before the shock.

Step 1. Re-writing the Lagrangian. Proceeding as in the proof of the risk-less case, as an intermediate step we integrate by parts the terms that involve time or maturity derivatives of $\hat{f}$. The Lagrangian $\mathcal{L}\left[\hat{\iota}, \hat{f}\right]$ can thus be expressed as:

$$
\begin{aligned}
\mathcal{L}\left[\hat{\imath},\hat{f}\right] = \mathbb{E}^{t^o}&\left[\int_0^{t^o} e^{-\rho s} U\left(\hat{c}(s)\right) ds + e^{-\rho t^o} V\left[\hat{f}\left(\cdot,t^o\right), X\left(t^o\right)\right]\right. \\
&- \int_0^T e^{-\rho t^o} \hat{f}\left(\tau,t^o\right) \hat{\jmath}\left(\tau,t^o\right) d\tau + \int_0^T \hat{f}\left(\tau,0\right) \hat{\jmath}\left(\tau,0\right) d\tau \\
&+ \int_0^{t^o}\int_0^T e^{-\rho s} \hat{f}\left(\tau,s\right)\left(\frac{\partial \hat{\jmath}}{\partial s} - \rho \hat{\jmath}\left(\tau,s\right)\right) d\tau ds \\
&+ \int_0^{t^o} e^{-\rho s} \hat{f}\left(T,s\right) \hat{\jmath}\left(T,s\right) ds - \int_0^{t^o} e^{-\rho s} \hat{f}\left(0,s\right) \hat{\jmath}\left(0,s\right) ds \\
&- \int_0^{t^o}\int_0^T e^{-\rho s} \hat{f}\left(\tau,s\right) \frac{\partial \hat{\jmath}}{\partial \tau} d\tau ds \\
&+ \int_0^{t^o}\int_0^T e^{-\rho s} \hat{\jmath}\left(\tau,s\right) \hat{\imath}\left(\tau,s\right) d\tau ds.
\end{aligned}
$$

If we group terms, substitute the terminal conditions $\hat{f}\left(T,s\right) = 0$ and compute the expected value with respect to t^o, we can express the Lagrangian $\mathcal{L}\left[\hat{\imath},\hat{f}\right]$ as:

$$
\begin{aligned}
\mathcal{L}\left[\hat{\imath},\hat{f}\right] &= \int_0^\infty e^{-(\rho+\phi)s} U\left(\hat{c}(s)\right) ds \\
&+ \int_0^\infty e^{-(\rho+\phi)s} \phi V\left[\hat{f}\left(\cdot,s\right)\right] ds \\
&- \int_0^\infty \int_0^T e^{-(\rho+\phi)s} \phi \hat{f}\left(\tau,s\right) \hat{\jmath}\left(\tau,s\right) d\tau ds \\
&+ \int_0^T \hat{f}\left(\tau,0\right) \hat{\jmath}\left(\tau,0\right) d\tau \\
&+ \int_0^\infty \int_0^T e^{-(\rho+\phi)s} \hat{f}\left(\tau,s\right)\left(\frac{\partial \hat{\jmath}}{\partial s} - \rho \hat{\jmath}\left(\tau,s\right)\right) d\tau ds \\
&- \int_0^\infty e^{-(\rho+\phi)s} \hat{f}\left(0,s\right) \hat{\jmath}\left(0,s\right) ds \\
&- \int_0^\infty \int_0^T e^{-(\rho+\phi)s} \hat{f}\left(\tau,s\right) \frac{\partial \hat{\jmath}}{\partial \tau} d\tau ds \\
&+ \int_0^{t^o}\int_0^T e^{-(\rho+\phi)s} \hat{\jmath}\left(\tau,s\right) \hat{\imath}\left(\tau,s\right) d\tau ds.
\end{aligned}
$$

Step 2: Gâteaux Derivatives. Next, we compute the Gâteaux derivatives with respect to each of the two arguments of the Lagrangian at a time. *Step 2.1: Gateaux derivative with respect to issuances.* We consider a perturbation around optimal issuances. Equalizing the Gâteaux derivative with respect to issuances to zero, i.e. $\frac{\partial}{\partial \alpha} \mathcal{L}\left[\hat{\imath} + \alpha h, \hat{f}\right]\Big|_{\alpha=0} = 0$, the result is identical to the riskless case:

$$
U'\left(\hat{c}\left(t\right)\right)\left(\frac{\partial q}{\partial \hat{\imath}} \hat{\imath}\left(\tau,t\right) + q\left(\tau,t,\hat{\imath}\right)\right) = -\hat{\jmath}\left(\tau,t\right).
$$

Step 2.2: Gateaux derivative of V with respect to the debt density. The Gâteaux derivative of the continuation value with respect to the debt density is

$$
\frac{d}{d\alpha} V\left[\hat{f}\left(\cdot,s\right) + \alpha h\left(\cdot,s\right), X\left(s\right)\right]\Big|_{\alpha=0} = \mathbb{E}_s^X\left\{\int_0^T j\left(\tau,s\right) h\left(\tau,s\right) d\tau\right\}.
$$

where we have applied Lemma 1. *Step 2.3: Gateaux derivative of the Lagrangian with respect to the debt density.* Since the distribution at the beginning $\hat{f}(\tau,0)$ is given, any feasible perturbation must feature $h(\tau,0)=0$ for any $\tau\in(0,T]$. In addition, we know that $h(T,t)=0$, because $\hat{f}(T,t)=0$. The Gâteaux derivative of the Lagrangian with respect to the debt density is:

$$\begin{aligned}
\frac{\partial}{\partial\alpha}\mathcal{L}\left[\hat{\imath},\hat{f}+\alpha h\right]\Big|_{\alpha=0} = & \int_0^\infty e^{-(\rho+\phi)s}U'(\hat{c}(s))\left[-h(0,s)+\int_0^T(-\delta)h(\tau,s)\,d\tau\right]ds \\
& + \int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\mathbb{E}_s^X\left[j(\tau,s)h(\tau,s)\right]d\tau ds \\
& - \int_0^\infty\int_0^T e^{-(\rho+\phi)s}\theta h(\tau,s)\hat{\jmath}(\tau,s)\,d\tau ds \\
& + \int_0^T h(\tau,0)\hat{\jmath}(\tau,0)\,d\tau \\
& + \int_0^\infty\int_0^T e^{-(\rho+\phi)s}h(\tau,s)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}(\tau,s)\right)ds d\tau \\
& - \int_0^\infty e^{-(\rho+\phi)s}h(0,s)\hat{\jmath}(0,s)\,ds \\
& - \int_0^\infty\int_0^T e^{-(\rho+\phi)s}h(\tau,s)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds.
\end{aligned}$$

The value of the Gâteaux derivative of the Lagrangian for any perturbation must be zero. Thus, again a necessary condition is to have all terms that multiply any entry of $h(\tau,s)$ add up to zero. We summarize the necessary conditions into:

$$\rho\hat{\jmath}(\tau,s)=-\delta U'(\hat{c}(s))+\frac{\partial\hat{\jmath}}{\partial s}-\frac{\partial\hat{\jmath}}{\partial\tau}+\phi\left[\mathbb{E}_s^X j(\tau,s)-\hat{\jmath}(\tau,s)\right], \tag{D.18}$$

$$\hat{\jmath}(0,s)=-U'(\hat{c}(s)). \tag{D.19}$$

Step 3: From Lagrange multipliers to valuations. We now employ the definitions of $\hat{v}(\tau,s)=-\hat{\jmath}(\tau,s)/U'(\hat{c}(s))$ and $v(\tau,s)=-j(\tau,s)/U'(c(s))$. Thus, we can express equations (D.18)-(D.19) as:

$$\begin{aligned}
\hat{r}(s)\hat{v}(\tau,s) &= \delta+\frac{\partial\hat{v}}{\partial s}-\frac{\partial\hat{v}}{\partial\tau}+\phi\left[\mathbb{E}_s^X\frac{U'(c(s))}{U'(\hat{c}(s))}v(\tau,s)-\hat{v}(\tau,s)\right] \\
\hat{v}(0,s) &= 1.
\end{aligned}$$

Therefore valuations can be expressed as:

$$\begin{aligned}
\hat{v}(\tau,t) \quad = \quad & e^{-\int_t^{t+\tau}(\hat{r}(s)+\phi)ds} \\
& +\theta\int_t^{t+\tau}e^{-\int_t^s(\hat{r}(u)+\phi)du}\left(\delta+\mathbb{E}_s^X\left[\frac{U'(c(t+s))}{U'(\hat{c}(t+s))}v(\tau-s,t+s)\right]\right)ds,
\end{aligned}$$

as we wanted to show. □

D.7 Proof of Proposition 3

Proof. Step 1. Setting the Lagrangian. Let $\mathbb{V}\left[\hat{f}\left(\cdot,t^{o}\right),X\left(t^{o}\right)\right]$ denote the expected value of the Government, at the instant t^{o} where the option to default is available, but prior to the decision of default. This value equals:

$$\mathbb{E}_{t^{o}}^{X}\left[\underbrace{\Gamma\left(V\left[\hat{f}\left(\cdot,t^{o}\right)\right],X\left(t^{o}\right)\right)}_{\text{Default}}+\underbrace{\Theta\left(V\left[\hat{f}\left(\cdot,t^{o}\right),X\left(t^{o}\right)\right]\right)V\left[\hat{f}\left(\cdot,t^{o}\right),X\left(t^{o}\right)\right]}_{\text{No default}}\right],$$

where the first term in the expectation is the expected utility conditional on default given by $\Gamma\left(x\right)\equiv\int_{x}^{\infty}zd\Theta(z)$. The second term is the probability of no default time the perfect-foresight value. The Lagrangian is:

$$\begin{aligned}\mathcal{L}\left[\hat{\imath},\hat{f},\hat{\psi}\right] &= \mathbb{E}^{t^{o}}\left[\int_{0}^{t^{o}}e^{-\rho s}U\left(\hat{c}(s)\right)ds+e^{-\rho t^{o}}\mathbb{V}\left[\hat{f}\left(\cdot,t^{o}\right),X\left(t^{o}\right)\right]\right.\\ &\quad+\int_{0}^{t^{o}}\int_{0}^{T}e^{-\rho s}\hat{\jmath}\left(\tau,s\right)\left(-\frac{\partial\hat{f}}{\partial s}+\hat{\imath}\left(\tau,s\right)+\frac{\partial\hat{f}}{\partial\tau}\right)d\tau ds\\ &\quad\left.+\int_{0}^{t^{o}}\int_{0}^{T}e^{-\rho s}\hat{\mu}\left(\tau,s\right)\left(-\hat{r}^{*}\left(s\right)\hat{\psi}(\tau,s)+\delta+\frac{\partial\hat{\psi}}{\partial s}-\frac{\partial\hat{\psi}}{\partial\tau}\right)d\tau ds\right].\end{aligned}$$

In the Lagrangian, $\mathbb{E}^{t^{o}}$ denotes the conditional expectation with respect to the random time t^{o}. Here $\hat{\jmath}\left(\tau,s\right)$ and $\hat{\mu}\left(\tau,s\right)$ are the Lagrange multipliers. The first set of multipliers, $\hat{\jmath}\left(\tau,s\right)$, are associated with the law of motion of debt and appears also in previous sections. The second set of multipliers, $\hat{\mu}\left(\tau,s\right)$, are associated with the law of motion of bond prices. This terms appears because the Government acknowledges that its influence on the maturity profile influences the incentives to default, and hence affects bond prices. This happens through the terminal condition:

$$\begin{aligned}\hat{\psi}(\tau,t^{o}) &= \mathbb{E}_{t^{o}}^{X}\left\{\Theta\left(V\left[\hat{f}\left(\cdot,t^{o}\right)\right]\right)\psi(\tau,t^{o})\right\},\\ \hat{\psi}(0,t) &= 1.\end{aligned}$$

The terminal condition reflects that at date t^{o}, the bond price is zero if default occurs or the perfect-foresight price, $\psi(\tau,t^{o})$, if default does not occur.

Step 1.2. Re-writing the Lagrangian. Proceeding as in the proof of the riskless case, as an intermediate step we integrate by parts the terms that involve time or maturity derivatives of $\hat{f}$ and $\hat{\psi}$. The Lagrangian $\mathcal{L}\left[\hat{\imath},\hat{f},\hat{\psi}\right]$ can thus be expressed as:

$$
\begin{aligned}
&\mathbb{E}^{t^o}\left[\int_0^{t^o} e^{-\rho s} U\left(\hat{c}(s)\right) ds + e^{-\rho t^o} \mathbb{V}\left[\hat{f}\left(\cdot, t^o\right), X\left(t^o\right)\right] - \int_0^T e^{-\rho t^o} \hat{f}\left(\tau, t^o\right) \hat{\jmath}\left(\tau, t^o\right) d\tau \right.\\
&+ \int_0^T \hat{f}\left(\tau, 0\right) \hat{\jmath}\left(\tau, 0\right) d\tau + \int_0^{t^o} \int_0^T e^{-\rho s} \hat{f}\left(\tau, s\right) \left(\frac{\partial \hat{\jmath}}{\partial s} - \rho \hat{\jmath}\left(\tau, s\right)\right) ds d\tau \\
&+ \int_0^{t^o} e^{-\rho s} \hat{f}\left(T, s\right) \hat{\jmath}\left(T, s\right) ds - \int_0^{t^o} e^{-\rho s} \hat{f}\left(0, s\right) \hat{\jmath}\left(0, s\right) ds \\
&- \int_0^{t^o} \int_0^T e^{-\rho s} \hat{f}\left(\tau, s\right) \frac{\partial \hat{\jmath}}{\partial \tau} d\tau ds + \int_0^{t^o} \int_0^T e^{-\rho s} \hat{\jmath}\left(\tau, s\right) \hat{\imath}\left(\tau, s\right) d\tau ds \\
&+ \int_0^{t^o} \int_0^T e^{-\rho s} \hat{\mu}\left(\tau, s\right) \left(-\hat{r}^*\left(s\right) \hat{\psi}(\tau, s) + \delta\right) d\tau ds, \\
&+ \int_0^T \left[e^{-\rho t^o} \hat{\mu}\left(\tau, t^o\right) \hat{\psi}\left(\tau, t^o\right) - \hat{\mu}\left(\tau, 0\right) \hat{\psi}\left(\tau, 0\right)\right] d\tau \\
&- \int_0^{t^o} \int_0^T e^{-\rho s} \hat{\psi}(\tau, s) \left(\frac{\partial \hat{\mu}}{\partial s} - \rho \hat{\mu}(\tau, s)\right) d\tau ds \\
&- \int_0^{t^o} e^{-\rho s} \left[\hat{\mu}\left(T, s\right) \hat{\psi}\left(T, s\right) - e^{-\rho s} \hat{\mu}\left(0, s\right) \hat{\psi}\left(0, s\right)\right] ds \\
&\left. + \int_0^{t^o} \int_0^T e^{-\rho s} \hat{\psi}\left(\tau, s\right) \frac{\partial \hat{\mu}}{\partial \tau} d\tau ds \right].
\end{aligned}
$$

Step 1.3. Computing expectations. If we group terms, substitute the terminal conditions $f\left(T, s\right) = 0$ and $\hat{\psi}(\tau, t^o) = \Theta\left(V\left[\hat{f}\left(\cdot, t^o\right)\right]\right) \psi(\tau, t^o)$ and compute the expected value with respect to t^o, we can express the Lagrangian $\mathcal{L}\left[\hat{\imath},\hat{f},\hat{\psi}\right]$ as:

$$\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}U\left(\hat{c}(s)\right)ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}\hat{f}\left(0,s\right)\hat{\jmath}\left(0,s\right)ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}\left[\hat{\mu}\left(T,s\right)\hat{\psi}\left(T,s\right)-\mu\left(0,s\right)\hat{\psi}\left(0,s\right)\right]ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{f}\left(\tau,s\right)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}\left(\tau,s\right)\right)dsd\tau\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{f}\left(\tau,s\right)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\jmath}\left(\tau,s\right)\hat{\imath}\left(\tau,s\right)d\tau ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\mu}\left(\tau,s\right)\left(-\hat{r}^*\left(s\right)\hat{\psi}(\tau,s)+\delta\right)d\tau ds\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\psi}(\tau,s)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\hat{\mu}(\tau,s)\right)d\tau ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\hat{\psi}\left(\tau,s\right)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds\\
&+\int_0^T f\left(\tau,0\right)\hat{\jmath}\left(\tau,0\right)d\tau-\int_0^T\hat{\mu}\left(\tau,0\right)\hat{\psi}\left(\tau,0\right)d\tau\\
&+\int_0^\infty e^{-(\rho+\phi)s}\phi\mathbb{V}\left[\hat{f}\left(\cdot,s\right)\right]ds\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\hat{f}\left(\tau,s\right)\hat{\jmath}\left(\tau,s\right)d\tau ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\hat{\mu}\left(\tau,s\right)\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}\left(\cdot,s\right)\right]\right)\psi(\tau,s)\right\}d\tau.
\end{aligned}$$

Next, we compute the Gâteaux derivatives with respect to each of the three arguments of the value function at a time.

Step 2. Computing the derivatives. Step 2.1. Gâteaux derivative with respect to the issuances. If we consider a perturbation around issuances and equalize it to zero, $\frac{\partial}{\partial\alpha}\mathcal{L}\left[\hat{\imath}+\alpha h,\hat{f},\hat{\psi}\right]\Big|_{\alpha=0}=0$, the result is identical to the risk-less case:

$$U'\left(\hat{c}\left(t\right)\right)\left(\frac{\partial q}{\partial\iota}\hat{\imath}\left(\tau,t\right)+q\left(t,\tau,\hat{\imath}\right)\right)=-\hat{\jmath}\left(\tau,t\right).$$

Step 2.2. Gâteaux derivative with respect to the debt density. Since the distribution at the beginning $f\left(\tau,0\right)$ is given, any feasible perturbation must feature $h(\tau,0)=0$ for any $\tau\in(0,T]$. In addition, we know that $h(T,t)=0$, because $f(T,t)=0$. The Gâteaux derivative of the continuation value with respect to the debt density is:

$$\frac{d}{d\alpha}\mathbb{V}\left[\hat{f}\left(\cdot,s\right)+\alpha h\left(\cdot,s\right),X\left(s\right)\right]\Big|_{\alpha=0}=\mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)\int_0^T j\left(\tau,s\right)h\left(\tau,s\right)d\tau\right\},$$

where we have taken into account the fact that $\frac{d}{dx}\left(\Gamma\left(x\right)+\Theta\left(x\right)x\right)=\Theta\left(x\right)$ and −from the perfect foresight problem− :

$$\frac{d}{d\alpha}V\left[\hat{f}\left(\cdot,s\right)+\alpha h\left(\cdot,s\right)\right]\Big|_{\alpha=0}=\int_0^T j\left(\tau,s\right)h\left(\tau,s\right)d\tau.$$

Similarly, the Gâteaux derivative of the terminal bond price with respect to the debt density is

$$\frac{d}{d\alpha}\ \mathbb{E}_s^X \left\{\Theta\left(V\left[\hat{f}\left(\cdot,s\right)+\alpha h\left(\cdot,s\right)\right]\right)\psi(\tau,s)\right\}\Big|_{\alpha=0} = \dots$$

$$\mathbb{E}_s^X \left\{\theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)\psi(\tau,s)\int_0^T j\left(\tau',s\right)h\left(\tau',s\right)d\tau'\right\},$$

where $\theta(x)\equiv\frac{d}{dx}\Theta\left(x\right)$ is the probability density. The Gâteaux derivative of the Lagrangian with respect to the debt density, $\frac{\partial}{\partial\alpha}\ \mathcal{L}\left[\hat{\imath},\hat{f}+\alpha h,\hat{\psi}\right]\Big|_{\alpha=0}$, is thus:

$$\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}U'\left(\hat{c}\left(s\right)\right)\left[-h\left(0,s\right)+\int_0^T\left(-\delta\right)h\left(\tau,s\right)d\tau\right]ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}h\left(0,s\right)\hat{\jmath}\left(0,s\right)ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}h\left(\tau,s\right)\left(\frac{\partial\hat{\jmath}}{\partial s}-\rho\hat{\jmath}\left(\tau,s\right)\right)dsd\tau\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}h\left(\tau,s\right)\frac{\partial\hat{\jmath}}{\partial\tau}d\tau ds\\
&+\int_0^T h\left(\tau,0\right)\hat{\jmath}\left(\tau,0\right)d\tau\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\theta\mathbb{E}_s^X\left\{\Theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)j\left(\tau,s\right)\right\}h\left(\tau,s\right)d\tau ds\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi h\left(\tau,s\right)\hat{\jmath}\left(\tau,s\right)d\tau ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\phi\mu\left(m,s\right)\mathbb{E}_s^X\left\{\theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)\psi(m,s)\int_0^T j\left(\tau,s\right)h\left(\tau,s\right)d\tau\right\}dm.
\end{aligned}$$

The value of the Gâteaux derivative of the Lagrangian for any perturbation, must be zero, i.e. $\frac{\partial}{\partial\alpha}\ \mathcal{L}\left[\hat{\imath},\hat{f}+\alpha h,\hat{\psi}\right]\Big|_{\alpha=0}=0$. Thus, a necessary condition that all terms that multiply any entry of $h\left(\tau,s\right)$ add up to zero. We summarize the necessary conditions into:

$$\begin{aligned}
\rho\hat{\jmath}\left(\tau,s\right) &= \left(-\delta\right)U'\left(\hat{c}\left(s\right)\right)+\frac{\partial\hat{\jmath}}{\partial s}-\frac{\partial\hat{\jmath}}{\partial\tau} &\text{(D.20)}\\
&+\phi\mathbb{E}_s^X\left\{\left[\Theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)+\theta\left(V\left[\hat{f}\left(\cdot,s\right),X\left(s\right)\right]\right)\int_0^T\hat{\mu}\left(m,s\right)\psi(m,s)dm\right]j\left(\tau,s\right)-\hat{\jmath}\left(\tau,s\right)\right\}, &\text{(D.21)}\\
\hat{\jmath}\left(0,s\right) &= -U'\left(\hat{c}\left(s\right)\right).
\end{aligned}$$

Step 2.3. Gâteaux derivative with respect to the bond price. In the case of the Gâteaux derivatives with respect to the evolution of the price $\hat{\psi}$, $\frac{\partial}{\partial\alpha}\ \mathcal{L}\left[\hat{\imath},f,\hat{\psi}+\alpha h\right]\big|_{\alpha=0}$, we need to work first with the Lagragian before expectations have been computed. The reason is the following: only bonds that mature after default can be affected by the Governement's policies and hence the variations have to be zero for those bonds that mature before default, $\tilde{h}(\tau,t)=0$, if $\tau+t<t^o$. To incorporate this, we assume that admissible perturbations are of the form $\tilde{h}(\tau,t)=h(\tau,t)1_{\{\tau+t\geq t^o\}}$,

where $h(\tau,t)$ is unrestricted. The Gâteaux derivative is then

$$
\begin{aligned}
&\mathbb{E}^{t^o}\left[\int_0^{t^o} e^{-\rho s}U'\left(\hat{c}\left(s\right)\right)\left(\int_0^T \hat{\imath}\left(\tau,s\right)\frac{\partial q}{\partial\hat{\psi}}1_{\{\tau+s\geq t^o\}}h\left(\tau,s\right)d\tau\right)ds\right.\\
&+\int_0^{t^o}\int_0^T e^{-\rho s}\hat{\mu}\left(\tau,s\right)\left(-\hat{r}^*\left(s\right)1_{\{\tau+s\geq t^o\}}h(\tau,s)\right)d\tau ds,\\
&-\int_0^T \hat{\mu}\left(\tau,0\right)1_{\{\tau\geq t^o\}}h\left(\tau,0\right)d\tau\\
&-\int_0^{t^o}\int_0^T e^{-\rho s}1_{\{\tau+s\geq t^o\}}h(\tau,s)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\hat{\mu}(\tau,s)\right)d\tau ds\\
&-\int_0^{t^o} e^{-\rho s}\left[1_{\{T+s\geq t^o\}}h\left(T,s\right)\hat{\psi}\left(T,s\right)-e^{-\rho s}1_{\{s\geq t^o\}}h\left(0,s\right)\hat{\psi}\left(0,s\right)\right]ds\\
&\left.+\int_0^{t^o}\int_0^T e^{-\rho s}1_{\{\tau+s\geq t^o\}}h\left(\tau,s\right)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds\right].
\end{aligned}
$$

Note that the perturbation is only around $\hat{\psi}\left(\tau,s\right)$ and not $\psi\left(\tau,s\right)$, the terminal price after default, which is given. Since at maturity, bonds have a value of 1, $h\left(0,s\right)=0$, because no perturbation can affect that price. If we compute the expectation with respect to the random arrival time t^o, we get:

$$
\begin{aligned}
&\int_0^\infty e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)U'\left(\hat{c}\left(s\right)\right)\left[\int_0^T \hat{\imath}\left(\tau,s\right)\frac{\partial q}{\partial\hat{\psi}}h\left(\tau,s\right)d\tau\right]ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)\hat{\mu}\left(\tau,s\right)\left(-\hat{r}^*\left(s\right)h(\tau,s)\right)d\tau ds\\
&-\int_0^T\left(1-e^{-\phi\tau}\right)\mu\left(\tau,0\right)h\left(\tau,0\right)d\tau\\
&-\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)h(\tau,s)\left(\frac{\partial\hat{\mu}}{\partial s}-\rho\mu(\tau,s)\right)d\tau ds\\
&-\int_0^\infty e^{-(\rho+\phi)s}\left(1-e^{-\phi T}\right)\mu\left(T,s\right)h\left(T,s\right)ds\\
&+\int_0^\infty\int_0^T e^{-(\rho+\phi)s}\left(1-e^{-\phi\tau}\right)\hat{\psi}\left(\tau,s\right)\frac{\partial\hat{\mu}}{\partial\tau}d\tau ds,
\end{aligned}
$$

where we use

$$\mathbb{E}^{t^o}\left[1_{\{\tau+s>t^o>s\}}\right]=e^{-\phi s}\left(1-e^{-\phi\tau}\right).$$

Again, as the Gâteaux derivative should be zero for any suitable $h(\tau,s)$, the optimality condition is

$$
\begin{aligned}
\left(\hat{r}^*\left(s\right)-\rho\right)\hat{\mu}(\tau,s) &= U'\left(\hat{c}\left(s\right)\right)\hat{\imath}\left(\tau,s\right)\frac{\partial q}{\partial\hat{\psi}}-\frac{\partial\hat{\mu}}{\partial s}+\frac{\partial\hat{\mu}}{\partial\tau},\\
\hat{\mu}\left(T,s\right) &= 0,\\
\hat{\mu}\left(\tau,0\right) &= 0.
\end{aligned}
$$

The solution to this PDE is

$$\hat{\mu}\left(\tau,s\right)=\int_{\max\{s+\tau-T,0\}}^s e^{-\int_z^s(\hat{r}^*(u)-\rho)du}U'\left(\hat{c}\left(z\right)\right)\hat{\imath}\left(\tau+s-z,z\right)\frac{\partial q}{\partial\hat{\psi}}\left(\tau+s-z,z\right)dz.$$

If we integrate the discount factor of the Government with respect to time, we obtain the following identity:

$$\int_z^s \hat{r}(u)\,du = \int_z^s \rho du - \int_z^s \frac{U''(\hat{c}(u))}{U'(\hat{c}(u))}\hat{c}(u)\frac{\dot{\hat{c}}(u)}{\hat{c}(u)}du.$$

Therefore, we have that:

$$\int_z^s \hat{r}(u)\,du - \int_z^s \rho du = -\,log\left(U'(\hat{c}(u))\right)\big|_z^s.$$

Therefore, we have the following identity:

$$e^{\int_z^s \rho du} = e^{\int_z^s \hat{r}(u)du}\frac{U'(\hat{c}(s))}{U'(\hat{c}(z))}.$$

Thus, the PDE for $\hat{\mu}(\tau,s)$ can be written as:

$$\hat{\mu}(\tau,s) = U'(\hat{c}(s))\int_{\max\{s+\tau-T,0\}}^s e^{-\int_z^s(\hat{r}^*(u)-\hat{r}(u))du}\hat{\iota}(\tau+s-z,z)\frac{\partial q}{\partial\hat{\psi}}(\tau+s-z,z)\,dz.$$

Thus, the bond also satisfies the PDE

Also note that

$$\begin{aligned}\hat{\iota}\frac{\partial q}{\partial\hat{\psi}} &= \left(\hat{\iota}-\frac{\bar{\lambda}}{2}(\hat{\iota})^2\right)\\ &= \frac{1}{2\bar{\lambda}}\left(\frac{\hat{\psi}^2-\hat{v}^2}{\hat{\psi}^2}\right)\\ &= \frac{1}{2\bar{\lambda}}\left(1-\frac{\hat{v}^2}{\hat{\psi}^2}\right).\end{aligned}$$

where the second line uses the optimal issuance rule. Hence, we can write the price multiplier as:

$$\hat{\mu}(\tau,s) = U'(\hat{c}(s))\int_{\max\{s+\tau-T,0\}}^s e^{-\int_z^s(\hat{r}^*(u)-\hat{r}(u))du}\frac{1}{2\bar{\lambda}}\left(1-\left(\frac{\hat{v}(\tau+s-z,z)}{\hat{\psi}(\tau+s-z,z)}\right)^2\right)dz.$$

We employ this solution in the main text.

Step 3: From Lagrange multipliers to valuations. We now employ the definitions of $\hat{v}(\tau,s) = -\hat{\jmath}(\tau,s)/U'(\hat{c}(s))$ and $v(\tau,s) = -j(\tau,s)/U'(c(s))$, we can express equations (D.20)-(D.21) as

$$\begin{aligned}\hat{r}(s)\hat{v}(\tau,s) &= \delta+\frac{\partial\hat{v}}{\partial s}-\frac{\partial\hat{v}}{\partial\tau}\\ &\quad+\phi\mathbb{E}_s^X\left\{\left[\Theta(V(s))+\theta(V(s))\int_0^T\hat{\mu}(m,s)\psi(m,s)dm\right]\frac{U'(c(s))}{U'(\hat{c}(s))}v(\tau,s)-\hat{v}(\tau,s)\right\},\\ \hat{v}(0,s) &= 1,\end{aligned}$$

where we use the notation $\Theta(V(s)) \equiv \Theta(V[f(\cdot,s),X(s)])$ and $\theta(V(s)) \equiv \theta(V[f(\cdot,s),X(s)])$. Therefore valuations can be expressed as

$$\begin{aligned}\hat{v}(\tau,t) &= e^{-\int_t^{t+\tau}(\hat{r}(s)+\phi)ds}\\ &\quad+\phi\int_t^{t+\tau}e^{-\int_t^s(\hat{r}(u)+\phi)du}\left(\delta+\mathbb{E}_s^X\left[(\Theta(V(t+s))+\Omega(t+s))\frac{U'(c(t+s))}{U'(\hat{c}(t+s))}v(\tau-s,t+s)\right]\right)ds,\end{aligned}$$

where

$$\Omega(t) = \theta\left(V\left(t\right)\right) \int_0^T \hat{\mu}\left(m,t\right)\psi(m,t)dm.$$

$\square$

D.8 The case with default without liquidity costs: $\bar{\lambda} = 0$

We show here that the maturity structure is indetermined in the case without liquidity costs and a finite support of G. In proposition 9 below we show how, if distribution $\hat{f}^*$ is a solutions of Problem (4.3), then another distribution $\hat{f}'$ is also a solution provided that

$$\int_0^T \left(\psi\left(\tau,t,X(t)\right) - \hat{\psi}\left(\tau,t\right)\right)\left(\hat{f}^*(\tau,t) - \hat{f}'(\tau,t)\right)d\tau = 0, \tag{D.22}$$

$$\int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V\left[\hat{f}^*\left(\cdot,t\right),X(t)\right]\right)\psi\left(\tau,t,X(t)\right)\right] - \hat{\psi}\left(\tau,t\right)\right)\left(\hat{f}^*(\tau,t) - \hat{f}'(\tau,t)\right)d\tau = 0. \tag{D.23}$$

Consider first the case of an income shock. Here $X(t)$ does not jump after the shock arrives. If the Government decides to default then the maturity profile at the moment of defaut is irrelevant. If the government decides instead to repay, the post-shock yield curve will be $\psi\left(\tau,t^o\right)$, which differs from $\hat{\psi}\left(\tau,t^o\right)$ as the post-shock default premium is zero. The maturity structure is indeterminate because conditions (D.22) and (D.23) are two integral equation with a continuum of unknowns, $\hat{f}'\left(\tau,t^o\right)$.

Consider next the case of an interest rate shock, in which $X(t)$ jumps with the option to default. Condition (D.22) is a system of integral equations, indexed by $X(t^o)$, where $\hat{f}'\left(\cdot,t^o\right)$ is the unknown. Provided that $X(t^o)$ may take N possible values, then we have at most N equations that need to be satisfied by the debt distribution. In addition we have equation (D.23) and the condition that the market debt should coincide. Notice that the number can be less than $N+2$ as in some states the Government may default and then condition D.22 is trivially satisfied for any debt profile that replicates the total debt at market prices before the shock arrival. In any case, the maturity structure is indeterminate.

The indeterminacy of the debt distribution in our model complements previous results in the literature. In particular, Aguiar et al. (Forthcoming) study a model of sovereign default similar to the one presented with the key difference that in their model the Government cannot commit to future debt issuances whereas in our paper it ca, conditional on repayment. Aguiar et al. (Forthcoming) find how in that case the Government only operates in the short end of the curve, making payments and retiring long-term bonds as they mature but never actively issuing or buying back such bonds. This is because short term bonds cannot be diluted. The authors also conjecture that the maturity structure would be indeterminate if the Government had full commitment over its issuance path. This is precisely the case we study here, confirming their conjecture.

Proposition 9. *Let* $\{\hat{\iota}^*(\tau,t),\hat{f}^*(\tau,t),\hat{c}^*(t)\}_{t\in[0,t^o]}$ *and* $\{\iota^*(\tau,t),f^*(\tau,t),c^*(t)\}_{t\in(t^o,\infty)}$ *be the solution of Problem (4.3) when* $\lambda\left(\iota,\tau,t\right) = 0$. *Let* $\{\hat{\iota}'(\tau,t),\hat{f}'(\tau,t),\hat{c}'(t)\}_{t\in[0,t^o]}$ *and* $\{\iota^*(\tau,t),f^*(\tau,t),c^*(t)\}_{t\in(t^o,\infty)}$ *be such that, for every* $t \le t^o$ *and every value of* X_t,

$$\hat{B}^*(t) = \hat{B}'(t) \tag{D.24}$$

$$\int_0^T \left(\psi\left(\tau,t,X_t\right) - \hat{\psi}\left(\tau,t\right)\right)\left(\hat{f}^*(\tau,t) - \hat{f}'(\tau,t)\right)d\tau = 0, \tag{D.25}$$

$$\int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V\left[\hat{f}^*\left(\cdot,t\right),X_t\right]\right)\psi\left(\tau,t,X_t\right)\right] - \hat{\psi}\left(\tau,t\right)\right)\left(\hat{f}^*(\tau,t) - \hat{f}'(\tau,t)\right)d\tau = 0, \tag{D.26}$$

and $B^*(t) = B'(t)$ *for every* $t > t^o$. *Then,* $\hat{c}'(t) = \hat{c}^*(t)$ *and* $c'(t) = c^*(t)$. *Thus,* $\{\hat{\iota}'(\tau,t), \hat{f}'(\tau,t), \hat{c}'(t)\}_{t\in[0,t^o]}$ *and* $\{\iota^*(\tau,t), f^*(\tau,t), c^*(t)\}_{t\in(t^o,\infty)}$ *is also optimal.*

Proof. Step 0. Default values. The value functional of a policy given an initial debt $f(\cdot,0)$ is given by:

$$\hat{V}[f(\cdot,0)] = \mathbb{E}_0\left[\int_0^{t^o} e^{-\rho t}U(\hat{c}(t))\,dt + \mathbb{E}_{V^D,X_{t^o}}\left[e^{-\rho t^o}V^O\left[V^D(t^o), f(\cdot,t^o)\right]\right]\right]$$

where the post-default value $V^O\left[V^D(t^o), f(\cdot,t^o), X_{t^o}\right] \equiv \max\left\{V^D(t^o), V[f(\cdot,t^o), X_{t^o}]\right\}$ and $V[f(\cdot,t^o), X_{t^o}]$ is the value of the perfect foresight solution. Note that from the solution of the problem with perfect foresight $V[f(\cdot,t^o), X_{t^o}] = V(B(t^o,X_{t^o}), X_{t^o})$ where $B(t^o,X_{t^o})$ is defined as

$$B(t^o, X_{t^o}) \equiv \int_0^T \psi(\tau, t^o, X_{t^o})\, f(\tau, t^o)d\tau.$$

Therefore the post-default value

$$V^O\left[V^D(t^o), f(\cdot,t^o), X_{t^o}\right] = V^O\left(V^D(t^o), B(t^o,X_{t^o}), X_{t^o}\right),$$

only depends on $B(t^o,X_{t^o})$. Because $B'(t^o,X_{t^o}) = B^*(t^o,X_{t^o})$ for every realization of X_{t^o} the default decision depends only on the market value of debt when the country receives the opportunity to default and not on the debt-maturity profile. Thus, continuation values are equal and it is enough to show that $\hat{c}^*(t) = \hat{c}'(t)$ for $t \le t^o$ to prove that the two policies yield the same utility. *Step 1. Pre-shock prices are equal.* Pre-shock prices solve

$$\begin{aligned}
\hat{r}^*(t)\,\hat{\psi}(\tau,t) &= \delta + \frac{\partial\hat{\psi}}{\partial t} - \frac{\partial\hat{\psi}}{\partial\tau} + \phi\mathbb{E}^X_t\left[\Theta\left(V[f(\cdot,t),X_t]\right)\psi(\tau,t,X_t) - \hat{\psi}(\tau,t)\right], \text{ if } t < t^o\\
\hat{\psi}(\tau,t^o) &= \mathbb{E}^X_{t^o}\left\{\Theta\left(V\left[\hat{f}(\cdot,t^o),X_{t^o}\right]\right)\psi(\tau,t^o)\right\}\\
\hat{\psi}(0,t) &= 1.
\end{aligned}$$

It holds that

$$\Theta\left(V[f^*(\cdot,t^o),X_{t^o}]\right) = \Theta\left(V(B^*(t^o,X_{t^o}),X_{t^o})\right) = \Theta\left(V\left(B'(t^o,X_{t^o}),X_{t^o}\right)\right) = \Theta\left(V\left[f'(\cdot,t^o),X_{t^o}\right]\right).$$

This is a consequence of the fact that

$$V\left(B^*(t^o,X_{t^o}),X_{t^o}\right) = V\left(B'(t^o,X_{t^o}),X_{t^o}\right).$$

Thus, pre-shock prices are equal for both policies. *Step 2. Law of of motion of debt before the shock arrival.* By definition $\hat{B}(t) = \int_0^T \hat{\psi}(\tau,t)\hat{f}(\tau,t)d\tau$. The dynamics of $\hat{B}(t)$ for $t < t^o$ are:

$$d\hat{B}(t) = \left(\int_0^T \left(\hat{\psi}_t(\tau,t)\hat{f}(\tau,t) + \hat{\psi}(\tau,t)\hat{f}_t(\tau,t)\right)d\tau\right)dt,$$

which, with similar derivations as in 7, yields to

$$d\hat{B}(t) = \left(\hat{c}(t) - y(t) + \hat{r}^*(t)\hat{B}(t) + \phi\int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V\left(B(t,X_t),X_t\right)\right)\psi(\tau,t,X_t)\right] - \hat{\psi}(\tau,t)\right)\hat{f}(\tau,t)d\tau\right)dt$$

Step 3. The expected jump. Note that (D.26) for all X_t implies that:

$$\phi \int_0^T \left(\mathbb{E}_{X_t}\left[\Theta\left(V\left(B^*(t,X_t),X_t\right)\right)\psi\left(\tau,t,X_t\right)\right]-\hat{\psi}\left(\tau,t\right)\right)\left(\hat{f}(\tau,t)-\hat{f}'(\tau,t)\right)d\tau=0, \tag{D.27}$$

Combining this equation with the law of motion of debt we get that before the shock arrival, $t < t^o$,

$$d\hat{B}^*(t) = d\hat{B}'(t). \tag{D.28}$$

Step 4. The actual jump. Condition (D.25) guarantees that the jump is the same for any X_t if the country does not default. If it defaults, condition (D.25) is trivially satisfied as $\hat{B}^*(t) = \hat{B}'(t)$ and the jump is also the same as market debt is then zero. Hence $\hat{B}^*(t^o) = \hat{B}'(t^o)$. Finally, taking all these results together we conclude that: $\hat{c}^*(t) = \hat{c}'(t)$ for all $t \leq t^o$. As the policy $\{\hat{\iota}'(\tau,t),\hat{f}'(\tau,t),\hat{c}'(t)\}_{t\in[0,t^o]}$ and $\{\iota^*(\tau,t),f^*(\tau,t),c^*(t)\}_{t\in(t^o,\infty)}$ achieves the same consumption path as the optimal, it is thus optimal. $\square$

E Computational Method

We describe the numerical algorithm used to jointly solve for the equilibrium value function, $v(\tau,t)$, bond price, $q(t,\tau,\iota)$, consumption $c(t)$, issuance $\iota(\tau,t)$ and density $f(\tau,t)$. The initial distribution is $f(\tau,0) = f_0(\tau)$. The algorithm proceeds in 3 steps. We describe each step in turn.

Step 1: Solution to the domestic value The steady state equation (2.10) is solved using an *upwind finite difference* scheme similar to Achdou et al. (2017). We approximate the value function $v_{ss}(\tau)$ on a finite grid with step $\Delta\tau : \tau \in \{\tau_1, ..., \tau_I\}$, where $\tau_i = \tau_{i-1} + \Delta\tau = \tau_1 + (i-1)\Delta\tau$ for $2 \le i \le I$. The bounds are $\tau_1 = \Delta\tau$ and $\tau_I = T$, such that $\Delta\tau = T/I$. We use the notation $v_i := v_{ss}(\tau_i)$, and similarly for the policy function $\iota_{i,}$. Notice first that the HJB equation involves first derivatives of the value function. At each point of the grid, the first derivative can be approximated with a forward or a backward approximation. In an upwind scheme, the choice of forward or backward derivative depends on the sign of the *drift function* for the state variable. As in our case, the drift is always negative, we employ a backward approximation in state:

$$\frac{\partial v(\tau_i)}{\partial\tau} \approx \frac{v_i - v_{i-1}}{\Delta\tau}. \tag{E.1}$$

The equation is approximated by the following upwind scheme,

$$\rho v_i = \delta + \frac{v_{i-1}}{\Delta\tau} - \frac{v_i}{\Delta\tau},$$

with terminal condition $v_0 = v(0) = 1$. This can be written in matrix notation as

$$\rho\mathbf{v} = \mathbf{u} + \mathbf{A}\mathbf{v},$$

where

$$\mathbf{A} = \frac{1}{\Delta\tau}\begin{bmatrix} -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{I-1} \\ v_I \end{bmatrix}, \mathbf{u} = \begin{bmatrix} \delta - 1/\Delta\tau \\ \delta \\ \delta \\ \vdots \\ \delta \\ \delta \end{bmatrix}. \tag{E.2}$$

The solution is given by

$$\mathbf{v} = (\rho\mathbf{I} - \mathbf{A})^{-1}\mathbf{u}. \tag{E.3}$$

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as $\mathbf{A}$.

To analyze the transitional dynamics, define $t^{\max}$ as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and time is discretized as $t_n = t_{n-1} + \Delta t$, in intervals of length

$$\Delta t = \frac{t^{\max}}{N-1},$$

where N is a constant. We use now the notation $v_i^n := v(\tau_i, t_n)$. The value function at $t^{\max}$ is the stationary solution computed in (E.3) that we denote as $\mathbf{v}^N$. We choose a forward approximation in time. The dynamic value equation (2.10) can thus be expressed

$$r^n\mathbf{v}^n = \mathbf{u} + \mathbf{A}\mathbf{v}^n + \frac{\left(\mathbf{v}^{n+1} - \mathbf{v}^n\right)}{\Delta t},$$

where $r^n := r(t_n)$. By defining $\mathbf{B}^n = \left(\frac{1}{\Delta t} + r^n\right)\mathbf{I} - \mathbf{A}$ and $\mathbf{d}^{n+1} = \mathbf{u} + \frac{\mathbf{v}^{n+1}}{\Delta t}$, we have

$$\mathbf{v}^n = \left(\mathbf{B}^n\right)^{-1} \mathbf{d}^{n+1}, \tag{E.4}$$

which can be solved backwards from $n = N-1$ until $n = 1$.

The optimal issuance is given by

$$\iota_i^n = \frac{1}{\bar{\lambda}} \frac{\left(\psi_i^n - v_i^n\right)}{\psi_i^n},$$

where ψ_i^n is computed in an analogous form to v_i^n.

Step 2: Solution to the Kolmogorov Forward equation Analogously, the KFE equation (2.1) can be approximated as

$$\frac{f_i^n - f_i^{n-1}}{\Delta t} = \iota_i^n + \frac{f_{i+1}^n - f_i^n}{\Delta \tau},$$

where we have employed the notation $f_i^n := f(\tau_i, t_n)$. This can be written in matrix notation as:

$$\frac{\mathbf{f}^n - \mathbf{f}^{n-1}}{\Delta t} = \mathbf{i}^n + \mathbf{A}^{\mathbf{T}}\mathbf{f}^n, \tag{E.5}$$

where $\mathbf{A}^{\mathbf{T}}$ is the transpose of $\mathbf{A}$ and

$$\mathbf{f}^n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{I-1}^n \\ f_I^n \end{bmatrix}, \mathbf{i}^n = \begin{bmatrix} \iota_1^n \\ \iota_2^n \\ \vdots \\ \iota_{I-1}^n \\ \iota_I^n \end{bmatrix}.$$

Given $\mathbf{f}_0$, the discretized approximation to the initial distribution $f_0(\tau)$, we can solve the KF equation forward as

$$\mathbf{f}_n = \left(\mathbf{I} - \Delta t \mathbf{A}^{\mathbf{T}}\right)^{-1} \left(\mathbf{i}^n \Delta t + \mathbf{f}_{n-1}\right), \quad n = 1, .., N. \tag{E.6}$$

Step 3: Computation of consumption The discretized budget constraint (2.2) can be expressed as

$$c^n = \bar{y}^n - f_1^{n-1} + \sum_{i=1}^{I} \left[\left(1_n - \frac{1}{2}\bar{\lambda}\iota_i^n\right)\iota_i^n \psi_i^n - \delta f_i^n\right]\Delta\tau, \quad n = 1, .., N.$$

Compute

$$r^n = \rho + \frac{\sigma}{c^n}\frac{c^{n+1} - c^n}{\Delta t}, n = 1, .., N-1.$$

Complete algorithm The algorithm proceeds as follows. First guess an initial path for consumption, for example $c^n = \bar{y}^n$, for $n = 1, .., N$. Set $k = 1$;

Step 1: Issuances. Given c_{k-1} solve step 1 and obtain ι.

Step 2: **KF.** Given ι solve the KF equation with initial distribution f_0 and obtain the distribution f.

Step 3: **Consumption**. Given ι and f compute consumption c. If $\|c - c_{k-1}\| = \sum_{n=1}^{N} \left|c^n - c_{k-1}^n\right| < \varepsilon$ then stop. Otherwise compute

$$c_k = \omega c + (1-\omega)\, c_{k-1}, \ \lambda \in (0,1),$$

set $k := k+1$ and return to step 1.

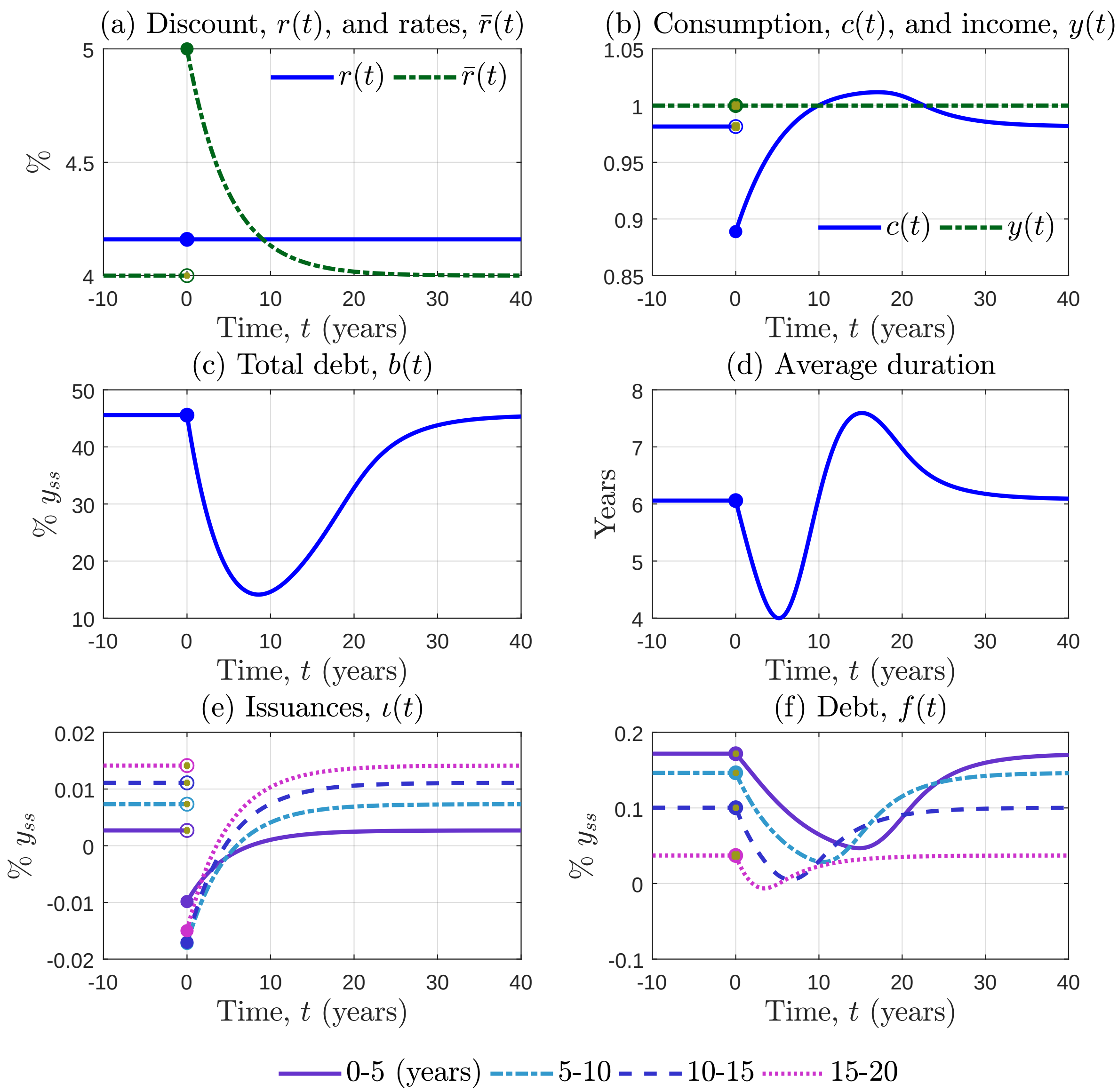

Figure F.1: Response to an unexpected shock to interest rates with $\sigma = 0$.

F Additional Figures

In this section we introduce the figures corresponding to the different exercises performed in the case or a risk-neutral Government, turning of the revenue-echo effect ($\Omega = 0$) or discrete issuances.

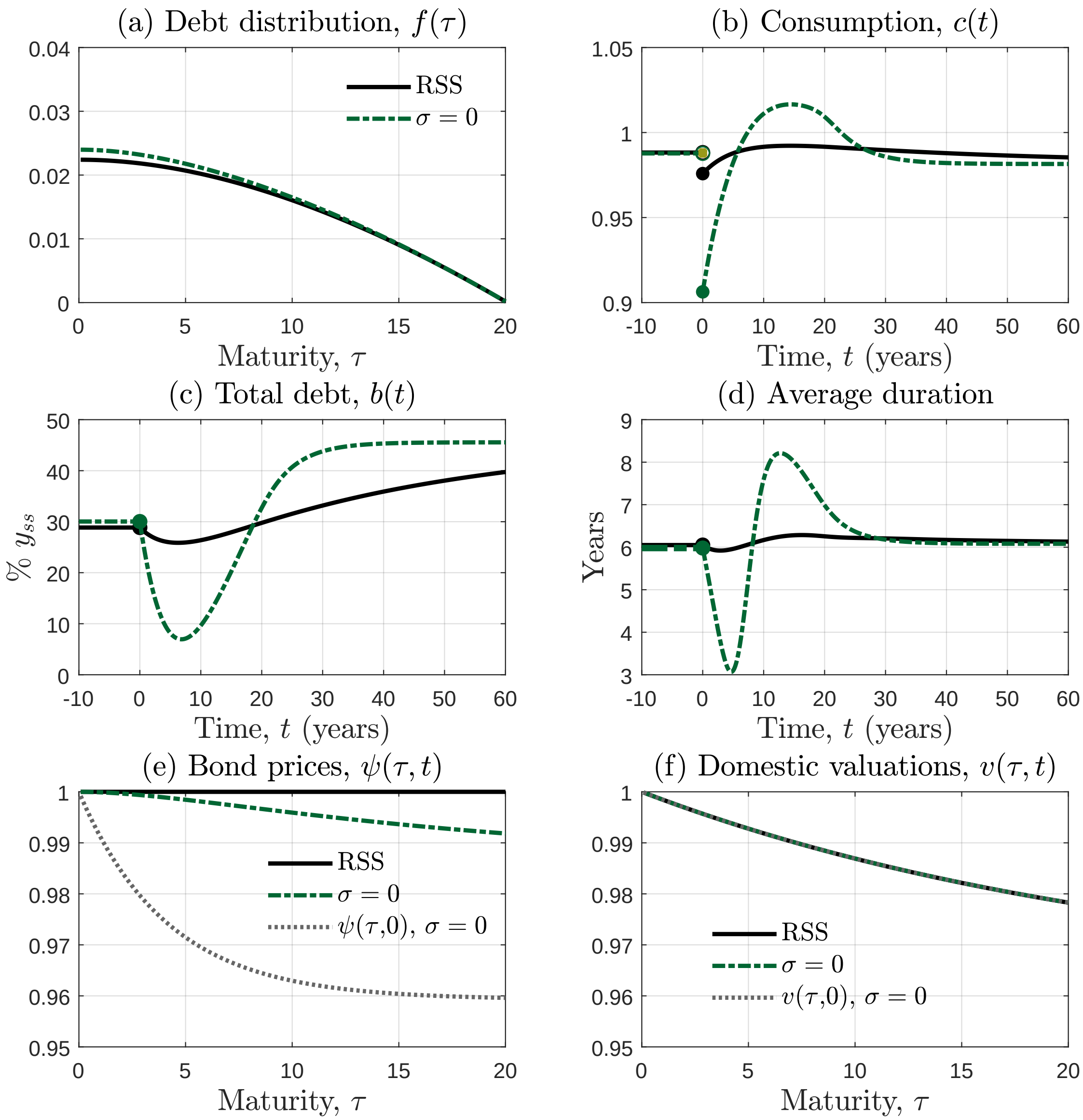

Figure F.2: Response to a shock to interest rates with $\sigma = 0$. RSS stands for the model starting at the "risky steady state" and DSS for "deterministic steady state".

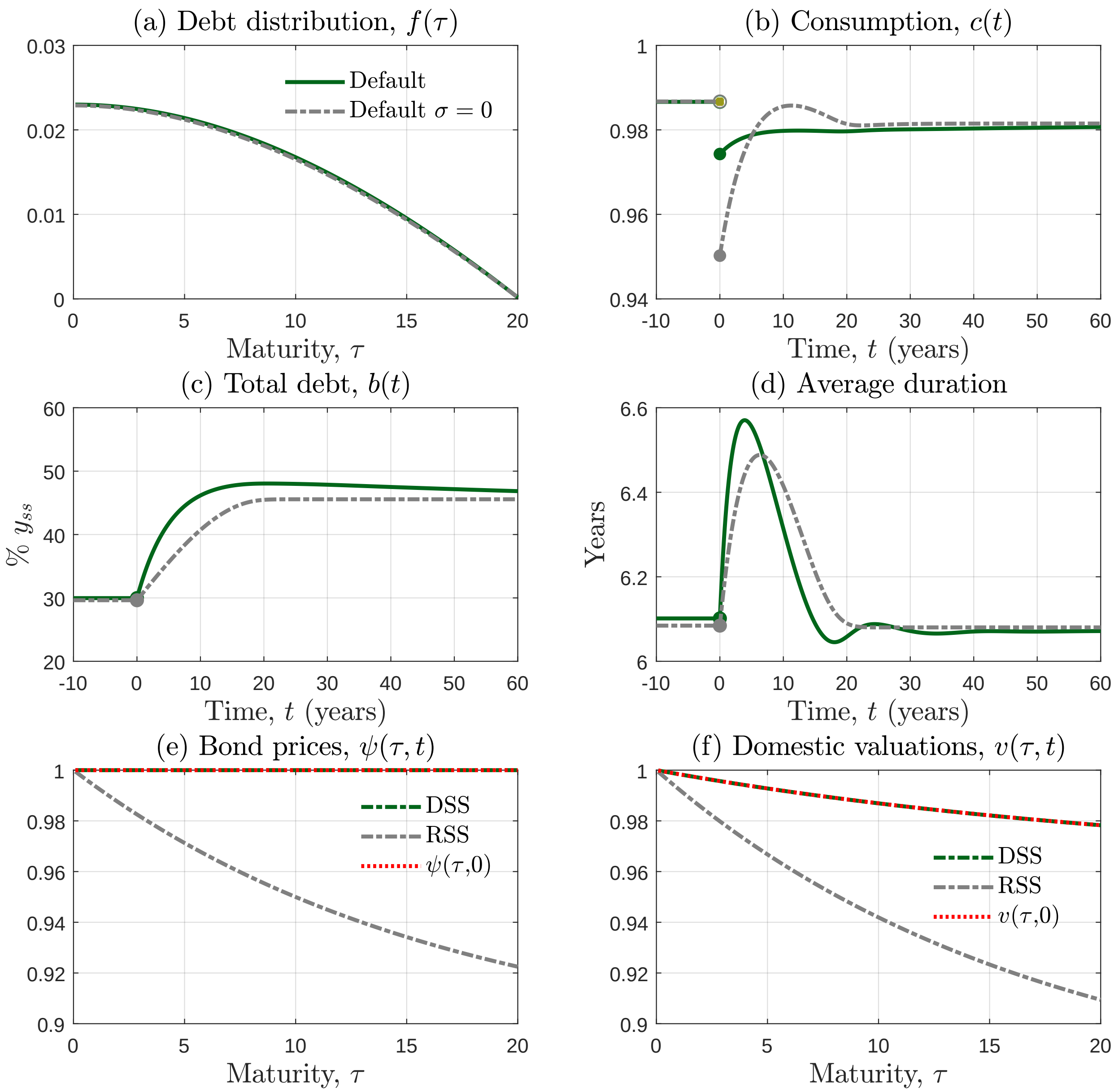

Figure F.3: Response to a shock to income with $\sigma = 0$ when the option to default is available. Panels (e) and (f) refer to the case with $\sigma = 0$.

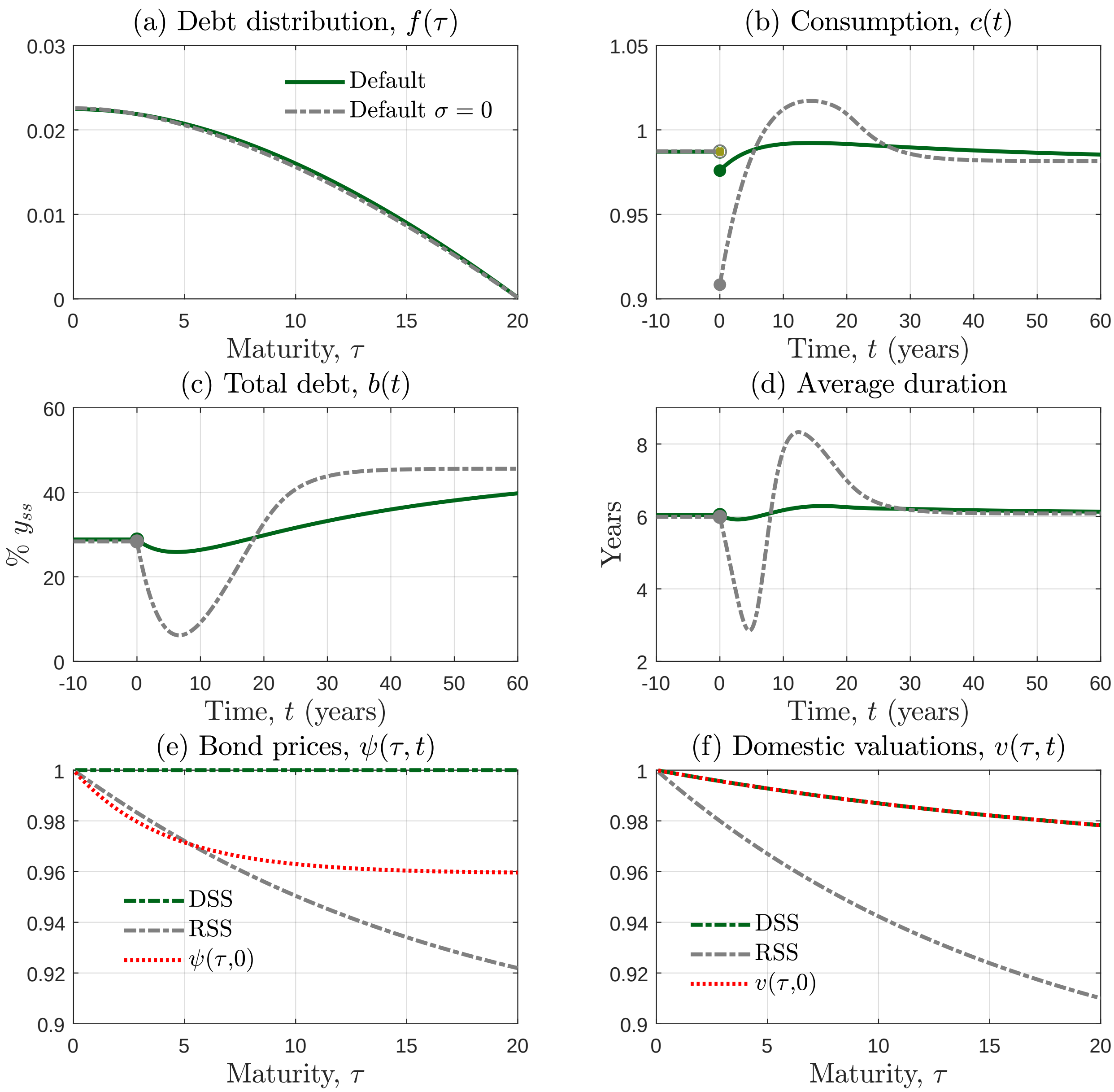

Figure F.4: Response to a shock to interest rates with $\sigma = 0$ when the option to default is available. Panels (e) and (f) refer to the case with $\sigma = 0$.

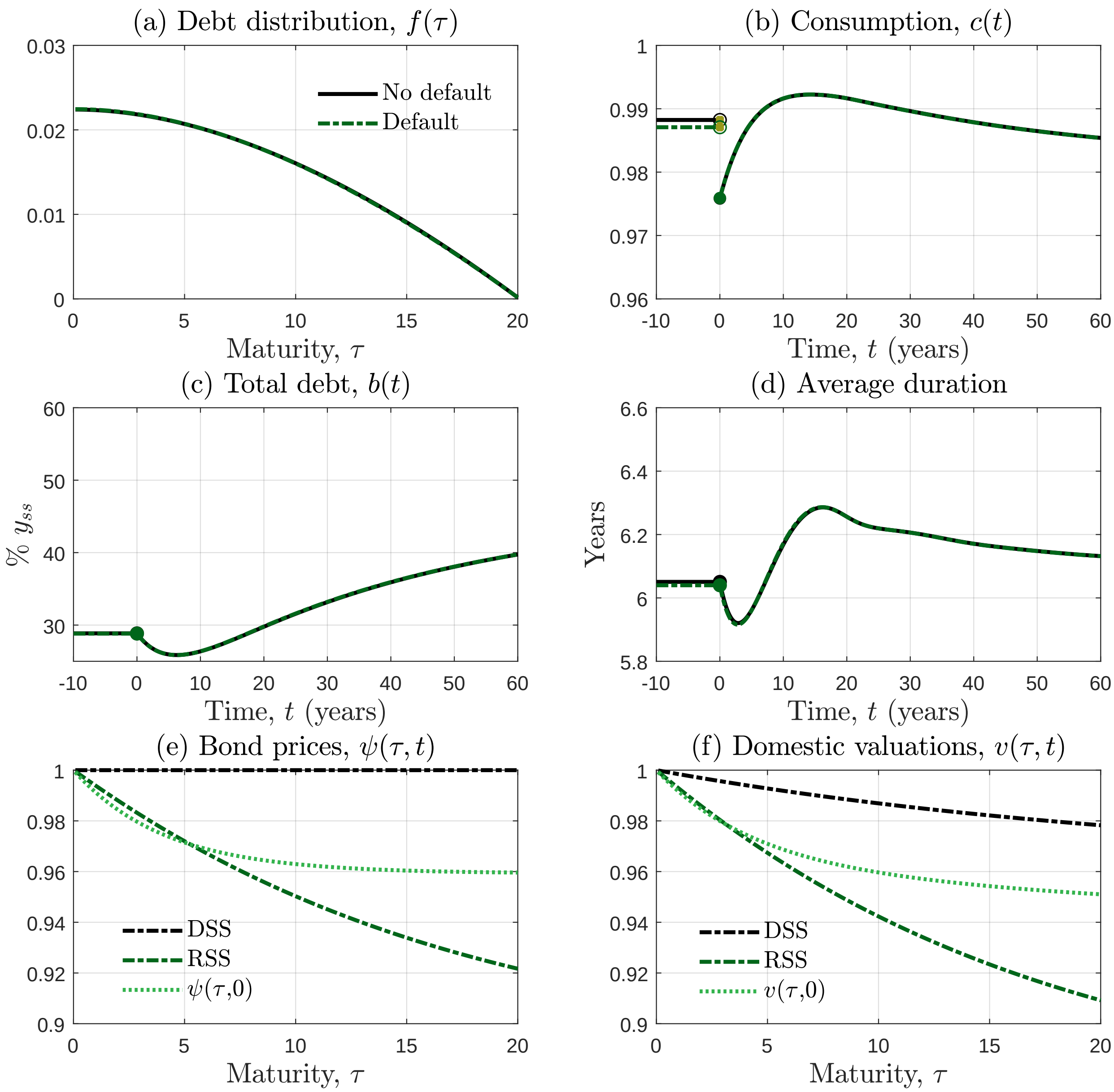

Figure F.5: Response to a shock to interest rates when the option to default is available. Panels (e) and (f) refer to the case with default.