Coarse Revealed Preference*

Gaoji Hu† Jiangtao Li‡ John K.-H. Quah§ Rui Tang¶

December 18, 2018

Abstract

We identify necessary and sufficient conditions under which a coarse data set can be coarsely rationalized by a linear order (or weak order). The conditions are easy to check, and efficient algorithms are provided. We apply our theory to investigate the observable restrictions of several economic models including (1) rational choice with imperfect observation; (2) multiple preferences; (3) monotone multiple preferences; and (4) minimax regret.

*We thank Mark Dean, David Dillenberger, Faruk Gul, Shaowei Ke, Barton Lipman, Jay Lu, Pietro Ortoleva, Wolfgang Pesendorfer, Satoru Takahashi, Can Urgun, Chen Zhao, and seminar participants at the National University of Singapore, Singapore Management University, Chinese University of Hong Kong, Columbia University, 2018 Summer School of Econometric Society, and Princeton Microeconomic Theory Student Lunch Seminar for helpful comments. Part of this paper was written while Li was visiting the Northwestern university, and he would like to thank the institution for hospitality and support.

†Nanyang Technological University, hu.gaoji@gmail.com

‡Singapore Management University, jtli@smu.edu.sg

§Johns Hopkins University and National University of Singapore, john.quah@jhu.edu

¶Princeton University, ruit@princeton.edu

1 Introduction

Pioneered by Samuelson (1938), revealed preference is one of the most influential ideas in economics and has been applied to a number of areas of economics, including consumer theory (Afriat (1967)), general equilibrium theory (Brown and Matzkin (1996)), and industrial organization (Carvajal et al. (2013)), among many others.

In a typical revealed preference exercise, it is assumed that there is an observer who records the choice behavior of the decision maker (DM). Alternative x is revealed to be preferred to alternative y if and only if x is chosen when y is also available. Thus, the standard revealed preference argument hinges on an implicit assumption that the observer could perfectly observe the DM's choice in different feasible sets.

Suppose that the observer does not have perfect observation of the DM's choice. Rather, she has only coarse observations of the form (A, B), where A is a feasible set from which the DM needs to make a choice, and B is a nonempty subset of A. The interpretation of (A, B) is that the observer knows that, when facing feasible set A, the DM chooses some alternative in B, but she does not know the DM's exact choice. Suppose that the observer has multiple coarse observations $\mathcal{O} = \{(A_i, B_i)\}_{i=1}^n$, termed a coarse data set, we are interesting in testing whether the observed choice behavior of the DM is consistent with the rational choice model where the DM has a strict preference $\succ$ over alternatives and chooses the $\succ$-maximal alternative for all feasible sets from which the DM needs to make a choice. A coarse data set $\mathcal{O} = \{(A_i, B_i)\}_{i=1}^n$ is said to be coarsely rationalizable by a strict preference if there exists a strict preference $\succ$ over alternatives such that the maximal element in A_i according to $\succ$ lies in B_i. Consider the following example.

Example 1 (A coarse data set that is not coarsely rationalizable by a strict preference)**.** *Consider a coarse data set $\mathcal{O}$ consisting of the following $n\,(n \geq 5)$ observations:*

1. $A_1 = \{x_1, x_2, x_3, x_4\}, B_1 = \{x_1, x_2\}$;
2. $A_2 = \{x_2, x_3, x_4, x_5\}, B_2 = \{x_2, x_3\}$;

   $\ldots$;

n. $A_n = \{x_n, x_1, x_2, x_3\}, B_4 = \{x_n, x_1\}$.

We claim that $\mathcal{O}$ is not coarsely rationalizable by a strict preference. Indeed, suppose that there exists a strict preference $\succ$ that coarsely rationalizes $\mathcal{O}$. Each observation i reveals that there exists some alternative x in B_i such that $x \succ^* y$ for all $y \in A_i \setminus \{x\}$. That is,

1. *(1a)* $x_1 \succ^* x_2$, $x_1 \succ^* x_3$, $x_1 \succ^* x_4$, *or (1b)* $x_2 \succ^* x_1$, $x_2 \succ^* x_3$, $x_2 \succ^* x_4$;
2. *(2a)* $x_2 \succ^* x_3$, $x_2 \succ^* x_4$, $x_2 \succ^* x_5$, *or (2b)* $x_3 \succ^* x_2$, $x_3 \succ^* x_4$, $x_3 \succ^* x_5$;

   $\ldots$;

n. *(na)* $x_n \succ^* x_1$, $x_n \succ^* x_2$, $x_n \succ^* x_3$, *or (nb)* $x_1 \succ^* x_n$, $x_1 \succ^* x_2$, $x_1 \succ^* x_3$.

One way to proceed is to consider all possible combinations. For example, suppose that (1a), (2a), …, and (na) hold simultaneously. Since (1a) requires that $x_1 \succ^* x_2$*, (2a) requires that* $x_2 \succ^* x_3$*, …, and (na) requires that* $x_n \succ^* x_1$*, we have a contradiction. By going through all combinations and deriving a contradiction for each case, one can conclude that* $\mathcal{O}$ *is not coarsely rationalizable by a strict preference. Having said that, proceeding in this way runs into computational issues as the number of observations increases.*

In this paper, we provide a systematic analysis of the revealed preference theory under coarse information. We identify necessary and sufficient conditions under which a coarse data set can be coarsely rationalized by a linear order (or weak order). For sake of clarity, we start by discussing the coarse rationalizability of a data set by a linear order. A coarse data set $\mathcal{O} = \{(A_i, B_i)\}_{i=1}^n$ is said to be coarsely rationalizable by a linear order if there exists a linear order P over alternatives such that the maximal element in A_i according to P lies in B_i.We identify a necessary and sufficient condition, termed Coarse SARP, for coarse rationalizability by a linear order. Coarse SARP requires that

Coarse SARP. For any $\emptyset \neq \mathcal{O}' \subset \mathcal{O}$, $\cup_{(A_i,B_i)\in\mathcal{O}'} A_i \setminus \cup_{(A_i,B_i)\in\mathcal{O}'} (A_i \setminus B_i) \neq \emptyset$.

That Coarse SARP is a necessary condition for coarse rationalizability by a linear order should come as no surprise. Suppose that a data set is coarsely rationalizable by a linear order. Consider any nonempty subcollection $\emptyset \neq \mathcal{O}' \subset \mathcal{O}$. Since for any $(A_i, B_i) \in \mathcal{O}'$ the maximal element in A_i is not contained in $A_i \setminus B_i$, the maximal element in $\cup_{(A_i,B_i)\in\mathcal{O}'} A_i$ is not contained in $\cup_{(A_i,B_i)\in\mathcal{O}'} (A_i \setminus B_i)$. We show that Coarse SARP is also sufficient for coarse rationalizability by a linear order.

Notably, it is easy to check whether the Coarse SARP condition holds. In the standard revealed preference exercise, the usual treatment is to derive preference relations between alternatives from each observation, and then check for acyclicity. Rather than deriving preference relations between alternatives from each observation, our approach offers for a more convenient way of testing the data set by working with sets of observations and sets of alternatives. We provide an efficient algorithm that drastically decreases the difficulty of checking whether the a coarse data set is coarsely rationalizable by a linear order. For an arbitrary coarse data set $\mathcal{O}$ with m alternatives and n data points, the number of steps required in the algorithm is at most $\min\{m, n\}$.

Our theory can be readily applied to test the observable restrictions of the rational choice model with imperfect observation. We note that there are several instances in which the imperfect observation of the DM's choice arises naturally. This occurs in situations where the observer sees an agent committing to choosing from some subset of a feasible set, without observing her ultimate choice. For example, consider a two-stage choice of the DM. The DM is assumed to know all restaurants/ menus that he might select. The DM chooses a meal eventually, but his initial choice is of a restaurant/ menu from which he will later chooses his meal. If the observer only has knowledge of the restaurant/ menu that the DM chooses, but not the DM's ultimate choice, this could be modeled as rational choice under imperfect observation. For another example, consider $X = X_1 \times X_2$, where X_1 is the set of alternatives available at date 1, and X_2 is the set of available alternatives at date 2. The agent chooses (x_1, x_2) from a nonempty feasible set $A \subseteq X$. If the observer only has knowledge that the choice of the DM at date 1, but not the choice at date 2, this again could be modeled as rational choice under imperfect observation. Such a framework would be suitable in a variety of contexts, such as intertemporal choice problems, where information about the choice(s) of the agent is revealed in stages.[1]

We hasten to emphasize that our framework works beyond the rational choice model with imperfect observation. We also motivate our theory from a theoretical perspective. We show that our theory can also be applied to investigate the observable restrictions of a number of behavioral models including the multiple preferences model, the monotone multiple preferences model, and the minimax regret model. While there is no imperfect observation of the DM's choice, imperfect information arises from these behavioral models themselves. To the best of our knowledge, our paper is the first to derive the revealed preference tests for these models. Our theory facilitates the study because we can transform the problem of checking rationalizability under these models into a problem of checking the coarse rationalizability of certain coarse data sets by a linear order.

[1]This has been briefly discussed in Nishimura et al. (2015) to point to the flexible nature of the revealed preference framework.

We then proceed to discuss the coarse rationalizability of a coarse data set by a weak order. A data point is defined to be a tuple (A, B, D) where A is a feasible set, B and D are disjoint subsets of A. The interpretation of (A, B, D) is that the observer knows that, when facing feasible set A, the exact choice of the DM is contained in B, and she also knows the alternatives in D are not optimal for the DM faced with A. Our analysis here parallels the analysis for the case of a linear order. A coarse data set $\mathcal{O} = \{(A_i, B_i, D_i)\}_{i=1}^n$ is said to be coarsely rationalizable by a weak order if there exists a weak order R over alternatives such that the set of maximal elements in A_i according to R has at least one common element with B_i and is disjoint with D_i. We identify a necessary and sufficient condition, termed Coarse GARP, for coarse rationalizability by a weak order. An efficient algorithm is provided to test the Coarse GARP condition.

Before we move on to talk about the model and the results, we wish to discuss two closely related papers. Fishburn (1976) studies when a choice function, which maps each set of alternatives in a domain of feasible sets into a non-empty subset of itself (called the choice set), can be representable by a linear order. A choice function is representable by a linear order if some linear order on the alternatives has its maximal element within each feasible set, which is contained in the choice set of the feasible set. Fishburn (1976) provides an axiom that he calls the partial congruence axiom to characterize representable choice functions, and the core logic is that in every nonempty collection of data points, there exists at least one alternative that is in the choice set for every feasible set that contains it. This axiom is essentially what we call Coarse SARP. Our analysis differs from Fishburn (1976) in several ways. First, we propose an algorithm to test the Coarse SARP condition when there are finitely many data points, and such a revealed preference analysis is absent in Fishburn (1976). Second, we show that our theory can be applied to investigate several models of non-classical decision making. Third, our formulation of rationalization by a weak order and the corresponding analysis is not covered in Fishburn (1976).

More recently, de Clippel and Rozen (2018) point out a methodological pitfall when testing choice theories with limited data in the the recent bounded rationality literature. They then present testable implications for several bounded rationality theories, and propose an enumeration procedure to test these implications. Both the enumeration procedure in their paper and the algorithm proposed here share the same core logic in Fishburn (1976). While de Clippel and Rozen (2018) address the question of incomplete data in the the recent bounded rationality literature, we explore related ideas in different settings. Among other things, we formulate an efficient test for coarse rationalizability by a weak order.

Section 2 presents the basics of the model. Section 3 identifies a necessary and sufficient condition for a coarse data set to be coarsely rationalizable by a linear order. Section 4 applies our theory to investigate the observable restrictions of several economic models including (1) rational choice with imperfect observation; (2) multiple preferences; (3) monotone multiple preferences; and (4) minimax regret. Section 5 studies coarse rationalizability by a weak order.

2 Preliminaries

We work with any arbitrarily fixed nonempty set X, which can be viewed as the universal set of alternatives. Let $\mathcal{X}$ be the collection of all nonempty subsets of X. A coarse data set $\mathcal{O}$ is a collection of data points $\{(A_i, B_i)\}_{i=1}^n$ where for each i, $A_i \in \mathcal{X}$ and B_i is a nonempty subset of A_i. In the general model, (A_i, B_i) need not have any intrinsic meanings. In each of the applications that we consider, additional structure will be imposed, and we shall specify the interpretation of (A_i, B_i). To simplify the statements below, we write C_i rather than $A_i \setminus B_i$. We also use the following notations:

$$A(\mathcal{O}') := \cup_{(A_i,B_i)\in\mathcal{O}'} A_i,$$
$$B(\mathcal{O}') := \cup_{(A_i,B_i)\in\mathcal{O}'} B_i,$$
$$C(\mathcal{O}') := \cup_{(A_i,B_i)\in\mathcal{O}'} C_i$$

for any $\emptyset \neq \mathcal{O}' \subseteq \mathcal{O}$. Throughout the rest of the paper, unless it leads to confusion, we abuse the notation by suppressing the set delimiters, e.g., writing x rather than $\{x\}$.

A linear order is a reflexive, complete, transitive, and antisymmetric binary relation, denoted by P. A strict preference is the asymmetric part of a linear order, denoted by $\succ$. We use $\max(A, P)$ (resp. $\max(A, \succ)$) to represent the maximal element in A according to P (resp. $\succ$).

We say that a coarse data set $\mathcal{O} = \{(A_i, B_i)\}_{i=1}^n$ is coarsely rationalized by some linear order P if

$$\max(A_i, P) \in B_i$$

for all i. If a coarse data set is coarsely rationalized by some linear order P, we say that the coarse data set is coarsely rationalizable by a linear order.

3 Theory

In this section, we propose a condition that we call Coarse SARP and show that Coarse SARP is both necessary and sufficient for coarse rationalizability by a linear order. Notably, the Coarse SARP condition is easy to check.

To fix ideas, let us first consider the familiar case that B_i is a singleton set for all i. Without loss of generality, we can rewrite $\mathcal{O}$ as $\{(A_i, x_i)\}_{i=1}^n$. We say that x_i is revealed to be ranked above y if $y \in A_i$ and $y \neq x_i$. It is well known that the data set is rationalizable by a linear order if and only if the data set obeys the strong axiom of revealed preference (SARP) condition, which says that there is no revealed cycle on the set of alternatives X. However, when B_i contains more than one alternative for some i, the revealed preference analysis becomes less straightforward, as illustrated by Example 1.

We now consider the general case that B_i need not be a singleton set for some i. If B_i is a not singleton set for some i, then (A_i, B_i) no longer reveals which element is the maximal element in A_i. Nevertheless, (A_i, B_i) reveals that the maximal element in A_i is not contained in C_i. For any nonempty subcollection $\mathcal{O}' = \{(A_{k_j}, B_{k_j})\}_{j=1}^m$ $(m \leq n)$ of $\mathcal{O}$, since the maximal element in A_{k_j} is not contained in C_{k_j} for all j, the maximal element in $A(\mathcal{O}')$ is necessarily not contained in $C(\mathcal{O}')$. This simple logic suggests the following necessary condition for coarse rationalizability by a linear order that we call Coarse SARP:

Coarse SARP. For any $\emptyset \neq \mathcal{O}' \subseteq \mathcal{O}$, $A(\mathcal{O}') \setminus C(\mathcal{O}') \neq \emptyset$.

We revisit Example 1 to illustrate how to use the Corase SARP condition to show that the data set is not coarsely rationalizable by a linear order.

Example 2 (Example 1 Revisited)**.** *The data set $\mathcal{O}$ is the same as in Example 1. It is easy to see that $A(\mathcal{O}) \setminus C(\mathcal{O}) = \emptyset$, which is a violation of the Coarse SARP condition. Thus, we can conclude that the data set $\mathcal{O}$ is not coarsely rationalizable by a linear order.*

We have argued above that Coarse SARP is a necessary condition for a data set to be coarsely rationalizable by a linear order. Theorem 1 below shows that Coarse SARP is also a sufficient condition for coarse rationalizability by a linear order.

Theorem 1. *A coarse data set $\mathcal{O} = \{(A_i, B_i)\}_{i=1}^n$ is coarsely rationalizable by a linear order if and only if it satisfies the Coarse SARP condition.*

The logic of the proof (the if-part) can be summarized as follows. Suppose that a coarse data set $\mathcal{O}$ satisfies the Coarse SARP condition. We shall explicitly construct a linear order that coarsely rationalizes $\mathcal{O}$. To do this, we first aggregate all the data points in $\mathcal{O}$. Note that the set of alternatives that might be the globally maximal elements are alternatives in $A(\mathcal{O}) \setminus C(\mathcal{O})$, which is nonempty by the Coarse SARP condition. We then construct an incomplete binary relation that ranks x above y if $x \in A(\mathcal{O}) \setminus C(\mathcal{O})$ and $y \in C(\mathcal{O})$. It is easy to see that any incomplete binary relation constructed in this way coarsely rationalizes a data point (A_i, B_i) if A_i contains some element in $A(\mathcal{O}) \setminus C(\mathcal{O})$. Thus, we can remove such data points from consideration, which leads to a simpler problem (a coarse data set with fewer data points). We then repeat the above process. In each step, we construct some incomplete binary relation that coarsely rationalizes some data points, and removing these data points leads to a simpler problem. Since $\mathcal{O}$ is finite, the process ends after finitely many steps. Finally, we can extend the incomplete binary relations to a linear order that coarsely rationalizes the data set $\mathcal{O}$. The proof below formalizes this idea.

Proof of Theorem 1. (The if-part) Suppose that the coarse data set $\mathcal{O}$ satisfies the Coarse SARP condition. In what follows, we shall explicitly construct a linear order that coarsely rationalizes $\mathcal{O}$. Without loss of generality, we assume that $A(\mathcal{O}) = X$.

We start with $\mathcal{O}_1 := \mathcal{O}$. Let $S_1 := A(\mathcal{O}_1) \setminus C(\mathcal{O}_1)$. Since $\mathcal{O}$ satisfies the Coarse SARP property, we know that $S_1 \neq \emptyset$. We proceed by induction. Suppose that we have constructed $\mathcal{O}_k$ and S_k for some $k \geq 1$ and $\mathcal{O}_k \neq \emptyset$, we construct $\mathcal{O}_{k+1}$ and S_{k+1} as follows:

$$\mathcal{O}_{k+1} := \{(A_i, B_i) \in \mathcal{O}_k : A_i \cap S_k = \emptyset\} \text{ and}$$
$$S_{k+1} := A(\mathcal{O}_{k+1}) \setminus C(\mathcal{O}_{k+1}).$$

We claim that for $\mathcal{O}_k \neq \emptyset$, $\mathcal{O}_{k+1}$ is a proper subset of $\mathcal{O}_k$. Since $\mathcal{O}$ satisfies the Coarse SARP property, if $\mathcal{O}_k \neq \emptyset$, we know that $S_k = A(\mathcal{O}_k) \setminus C(\mathcal{O}_k) \neq \emptyset$. Therefore, $A(\mathcal{O}_k) \cap S_k \neq \emptyset$, and there exists some $(A_i, B_i) \in \mathcal{O}_k$ such that $A_i \cap S_k \neq \emptyset$ and is eliminated when constructing $\mathcal{O}_{k+1}$ from $\mathcal{O}_k$. Now that we have established that for $\mathcal{O}_k \neq \emptyset$, $\mathcal{O}_{k+1}$ is a proper subset of $\mathcal{O}_k$, we can conclude that the construction stops after finitely many steps, when $\mathcal{O}_t \neq \emptyset$ and $\mathcal{O}_{t+1} = \emptyset$ for some t. Let us redefine

$$S_{t+1} := A(\mathcal{O}) \setminus \cup_{k=1}^t S_k.$$

We now argue that $\{S_k\}_{k=1}^{t+1}$ constitutes a partition of the set $A(\mathcal{O})$. To see this, consider S_j and $S_{j'}$ where $1 \leq j < j' \leq t$. Note that for any $(A_i, B_i) \in \mathcal{O}_{j'}$, it must be that $A_i \cap S_j = \emptyset$. Otherwise, (A_i, B_i) would have been eliminated in earlier steps. Since this is true for all $(A_i, B_i) \in \mathcal{O}_{j'}$, we know that $A(\mathcal{O}_{j'}) \cap S_j = \emptyset$. Since $S_{j'} = A(\mathcal{O}_{j'}) \setminus C(\mathcal{O}_{j'}) \subseteq A(\mathcal{O}_{j'})$, we have $S_j \cap S_{j'} = \emptyset$.

Next, we define a strict partial order that ranks x above y if $x \in S_j$, $y \in S_{j'}$, and $j' > j$. The strict partial order is well defined, since $\{S_k\}_{k=1}^{t+1}$ constitutes a partition of the set $A(\mathcal{O})$. It follows from the Szpilrajn's extension theorem (Szpilrajn (1930)) that we can extend the strict partial order to a linear order P. We now show that the linear order P constructed in this way necessarily satisfies that $\max(A_i, P) \in B_i$ for all i. For any $(A_i, B_i) \in \mathcal{O}$, $(A_i, B_i) \in \mathcal{O}_k \setminus \mathcal{O}_{k+1}$ for some k. Since $A_i \cap S_k \neq \emptyset$, and $A_i \cap S_{k'} = \emptyset$ whenever $k' < k$,

$$\max(A_i, P) \in A_i \cap S_k = (B_i \cap S_k) \cup (C_i \cap S_k) = B_i \cap S_k \subseteq B_i.$$

This completes the proof. $\square$

Remark 1. The Coarse SARP property reduces to the SARP property in the special case that B_i is a singleton set for all i. Without loss of generality, we rewrite $\mathcal{O}$ as $\{(A_i, x_i)\}_{i=1}^n$. (a) (SARP $\implies$ Coarse SARP) Suppose that the Coarse SARP property is violated, then there exists $\mathcal{O}' = \{(A_{k_j}, x_{k_j})\}_{j=1}^m$ such that $A(\mathcal{O}') \setminus C(\mathcal{O}') = \emptyset$. Therefore, for all x_{k_j}, we have $x_{k_j} \in C_{k_{j'}}$ for some j'. Therefore, $(A_{k_{j'}}, x_{k_{j'}})$ reveals that $x_{k_{j'}} P^* x_{k_j}$. Since this is true for all x_{k_j} and since $\mathcal{O}'$ is finite, P^* is cyclical and the SARP property is violated. (b) (Coarse SARP $\implies$ SARP) Suppose that the SARP property is violated, then we can find a sequence $\{x_{k_j}\}_{j=1}^m$ such that $x_{k_{j+1}} \in A_{k_j}$ with $l = 1, 2, \ldots, m-1$ and $x_{k_1} \in A_{k_m}$. Then for the subcollection of data points $\mathcal{O}' = \{(A_{k_j}, x_{k_j})\}_{j=1}^m$, we have $A(\mathcal{O}') \setminus C(\mathcal{O}') = \emptyset$. Therefore, the Coarse SARP property is violated.

We now propose a simple algorithm to check whether a coarse data set is coarsely rationalizable by a linear order (or equivalently, whether the Coarse SARP condition is satisfied).

ALGORITHM (linear order).

STEP 1. Set $k := 1$ and $\mathcal{O}' := \mathcal{O}$.

STEP 2. Define $\mathcal{O}_k := \mathcal{O}'$. If $\mathcal{O}_k = \emptyset$, stop and output *Rationalizable*; otherwise, proceed to STEP 3.

Step 3. Define $S_k := A(\mathcal{O}_k) \setminus C(\mathcal{O}_k)$. If $S_k = \emptyset$, stop and output *Not Rationalizable*. Otherwise, set $\mathcal{O}' := \{(A_i, B_i) \in \mathcal{O}_k : A_i \cap S_k = \emptyset\}$. Derive k' such that $k' = k + 1$. Set $k := k'$. Go to Step 2.

It follows from the proof of Theorem 1 that the Algorithm (linear order) can be used to determine whether a coarse data set is coarsely rationalizable by a linear order. Notably, the algorithm is efficient. Since the algorithm removes at least one data point and at least one alternative at each step, for an arbitrary coarse data set $\mathcal{O} = \{A_i, B_i\}_{i=1}^n$, the number of steps required in the algorithm is at most $\min\{|A(\mathcal{O})|, n\}$.[2] In particular, this implies that the algorithm works well when the number of alternatives is large but the number of data points is small.

When a coarse data set $\mathcal{O} = \{(A_i, B_i)\}_{i=1}^n$ is coarsely rationalizable by a linear order, the linear order that coarsely rationalizes $\mathcal{O}$ is not necessarily unique. Given the multiplicity of rationalizing linear orders, the readers might wonder, whether we can identify for a given pair of alternatives x and y, whether one alternative is ranked above the other for every linear order that coarsely rationalizes $\mathcal{O}$. We say that x is surely ranked above y, denoted by xP^sy, if for every linear order P that coarsely rationalizes $\mathcal{O}$, it holds that xPy. It is easy to see that P^s is transitive, since each linear order that coarsely rationalizes $\mathcal{O}$ is transitive.

We assume that the coarse data set $\mathcal{O}$ is coarsely rationalizable by a linear order. It is easy to to identify whether xP^sy if $(\{x, y\}, x) \in \mathcal{O}$ or $(\{x, y\}, y) \in \mathcal{O}$, as the former reveals that x is surely ranked above y and the latter reveals that y is surely ranked above x. In all other cases, we can apply the idea of Varian (1982) and insert one additional data point $(\{x, y\}, y)$ into the original coarse data set. We then check whether the new coarse data set $\mathcal{O}^* := \mathcal{O} \cup \{(\{x, y\}, y)\}$ is coarsely rationalizable by a linear order. If $\mathcal{O}^*$ is coarsely rationalizable by a linear order, then there exists a linear order that ranks y above x and coarsely rationalizes the original coarse data set $\mathcal{O}$. This rejects the hypothesis that xP^sy. If $\mathcal{O}^*$ is not coarsely rationalizable by a linear order, then it must be that every linear order that coarsely rationalizes $\mathcal{O}$ ranks x above y. In other words, xP^sy.

[2] For any set S, we denote by $|S|$ its cardinality.

4 Applications

In this section, we apply our theory in Section 3 to investigate the observable restrictions of several economic models. The theory immediately carries over to the rational choice model with imperfect observation. Interestingly, our theory can also be used to develop revealed preference tests for several behavioral models including the multiple preferences model and the minimax regret model. To the best of our knowledge, the revealed preference tests for these models are not yet known. We also introduce a variation of the multiple preference model that we call monotone multiple preference model, and show that our theory can be used to develop the revealed preference test for this model as well.

4.1 Rational Choice with Imperfect Observation

In this subsection, we investigate the observable restrictions of the rational choice model, with a twist that the analyst has imperfect observation of the choice made by the DM. As in the standard rational choice model, the DM has a strict preference $\succ$ over X, and she chooses $\max(A, \succ)$ from each feasible set $A \in \mathcal{X}$. Unlike in the standard rational choice model, the analyst does not observe the exact choice of the DM. Rather, she only observes that the DM chooses some alternative from a subset of the feasible set. We call this model rational choice with imperfect observation.

We represent the observed behavior of the DM by (Σ, f), where $\Sigma \subseteq \mathcal{X}$ and $f(A)$ is superset of the choice of the DM in $A \in \Sigma$. The interpretation is that, the observer does not observe the exact choice of the DM, but she knows that the exact choice happens in set $f(A)$ for each $A \in \Sigma$. We say that the data set (Σ, f) is rationalizable under the rational choice model with imperfect observation if there exists a strict preference $\succ$ such that

$$\max(A, \succ) \in f(A)$$

for all $A \in \Sigma$.

For a given data set (Σ, f), we construct a corresponding coarse data set $\mathcal{O} = \{(A_i, B_i)\}_{i=1}^n$ as follows: $(A, B) \in \mathcal{O}$ if and only if

$$A \in \Sigma \text{ and } f(A) = B.$$

Then (Σ, f) is rationalizable under the rational choice model with imperfect observation if and only if $\mathcal{O}$ is coarsely rationalizable by a linear order. The results in Section 3 immediately carry over to this setting, and the arguments are thus not repeated here.

4.2 Multiple Preferences

In this subsection, we investigate the observable restrictions of the multiple preferences model; see, for example, Salant and Rubinstein (2008). In contrast with the single preference model, the choice behavior of the DM may be a result of multiple rationales. Formally, the DM has a set $\triangleright$ of strict preferences, and she chooses

$$f_{\triangleright}(A) := \{x \in A : x = \max(A, \succ) \text{ for some } \succ \in \triangleright\}$$

from each feasible set A.

We represent the choice behavior of the DM by (Σ, f), where $\Sigma \subseteq \mathcal{X}$ and $f(A)$ is the set of all alternatives that the DM chooses in $A \in \Sigma$. We say that (Σ, f) is rationalizable by multiple preferences if there exists a set $\triangleright$ of strict preferences such that

$$f_{\triangleright}(A) = f(A)$$

for all $A \in \Sigma$. In what follows, we shall identify necessary and sufficient conditions under which (Σ, f) can be rationalizable by multiple preferences.

For a given data set (Σ, f), let us first consider the following coarse data set $\mathcal{O} = \{A, f(A)\}_{A\in\Sigma}$. If (Σ, f) is rationalizable by multiple preferences, then there must exist at least one preference $\succ$ such that $\max(A, \succ) \in f(A)$ for each $A \in \Sigma$. In other words, the coarse data set $\mathcal{O}$ must be rationalizable by a linear order, and thus the Coarse SARP property is a necessary condition for rationalizability by multiple preferences. The following example illustrates that the Coarse SARP property is not a sufficient condition.

Example 3 (A data set that is not rationalizable by multiple preferences)**.** *Let* $X = \{x, y, z\}$*. Consider a data set* (Σ, f) *consisting of the following three observations:*

1. $f(\{x, y\}) = x$;
2. $f(\{y, z\}) = y$;
3. $f(\{x, y, z\}) = \{x, z\}$.

The coarse data set $\mathcal{O} = \{(\{x,y\},x),(\{(y,z)\},y)\},\{(\{x,y,z\},\{x,z\})\}$ *satisfies the Coarse SARP property. However,* (Σ, f) *is not rationalizable by multiple preferences. Suppose that* (Σ, f) *is rationalizable by multiple preferences, by definition, there exists a set* $\triangleright$ *of strict preferences such that* $f_{\triangleright}(A) = f(A)$ *for each* $A \in \Sigma$*. Then* $f(\{x,y\}) = x$ *reveals that* $x \succ y$ *for all* $\succ \in \triangleright$*, and* $f(\{y,z\}) = y$ *reveals that* $y \succ z$ *for all* $\succ \in \triangleright$*. But then, by the transitivity of strict preference, it must be that* $x \succ z$ *for all* $\succ \in \triangleright$*, which contradicts the third observation that* $f(\{x,y,z\}) = \{x,z\}$*.*

For rationalizability by multiple preferences, we need to ensure that, for each $A \in \Sigma$ and each $x \in f(A)$, there is a strict preference such that x is the maximal element in A. Furthermore, the maximal element according to this strict preference lies in $f(A')$ for any other set $A' \in \Sigma$. This suggests the following divide-and-conquer approach. For each $A \in \Sigma$ and $x \in f(A)$, we construct a coarse data set $\mathcal{O}_{A,x}$ indexed by (A,x) as follows:

$$\mathcal{O}_{A,x} := \{(A', f(A'))\}_{A' \in \Sigma, A' \neq A} \cup (A, x).$$

That is, for each $A \in \Sigma$ and each $x \in f(A)$, the coarse data set $\mathcal{O}_{A,x}$ is derived from $\mathcal{O}$ by replacing $(A, f(A))$ with $(A, \{x\})$. Let

$$\mathfrak{D} := \{\mathcal{O}_{A,x}\}_{A \in \Sigma,\, x \in f(A)}.$$

Our logic at the beginning of this paragraph suggests that a necessary condition for the data set (Σ, f) to be rationalizable by multiple preferences is that each $\mathcal{O}_{A,x}$ constructed in this way is rationalizable by a linear order.

Theorem 2 below shows that (Σ, f) is rationalizable by multiple preferences if and only if each $\mathcal{O}_{A,x}$ in $\mathfrak{D}$ is rationalized by a linear order.

Theorem 2. (Σ, f) *is rationalizable by multiple preferences if and only if each* $\mathcal{O}_{A,x}$ *in* $\mathfrak{D}$ *is rationalizable by a linear order.*

Proof of Theorem 2. (The if-part) Suppose that for each $A \in \Sigma$ and $x \in A$, $\mathcal{O}_{A,x}$ is rationalizable by a linear order. Let $P_{A,x}$ denote a linear order that rationalizes $\mathcal{O}_{A,x}$ and we denote by $\succ_{A,x}$ the strict preference induced by $P_{A,x}$. We then have $\max(A, \succ_{A,x}) = x \in f(A)$ and $\max(A', \succ_{A,x}) \in f(A')$ for $A' \in \Sigma$ and $A' \neq A$. We claim that the set of strict preferences $\{\succ_{A,x}\}_{A \in \Sigma,\, x \in f(A)}$ rationalizes (Σ, f) under the multiple preferences model. It suffices to show that

$$f_{\{\succ_{A,x}\}_{A \in \Sigma,\, x \in f(A)}}(A') = f(A')$$

for each $A' \in \Sigma$. But this follows immediately from the construction of the set of strict preferences, since for each $A' \in \Sigma$, (1) each element x in $f(A')$ is the maximal element in A' according to the strict preference $\succ_{A',x}$; and (2) for any preference $\succ$ in $\{\succ_{A,x}\}_{A\in\Sigma,\, x\in f(A)}$, the maximal element according to $\succ$ lies in $f(A')$. $\square$

Recall from Section 3 that it is easy to test whether a coarse data set is rationalizable by a linear order. Thus, Theorem 2 provides a tractable way to test whether a given data set (Σ, f) is rationalizable by multiple preferences.

In what follows, we provide an axiomatization of the multiple preferences model. Building upon Theorem 1 and Theorem 2, we show that the multiple preferences model can be characterized by the Sen's α axiom (see Sen (1971)) and an additional axiom that we call betweenness. The Sen's α axiom says that if an element is not chosen in a set that contains it, then this element is not chosen in any superset of this set. The betweenness axiom says that if B is a superset of A and the chosen alternatives in B coincide with those in A, then the alternatives chosen in any set that contains A but is contained in B must also be the same alternatives chosen in A. Formally,

Sen's α Axiom. For any $A, B \in \mathcal{X}$, $f(A \cup B) \cap A \subseteq f(A)$.

Betweenness Axiom. For any $A, B \in \mathcal{X}$ with $A \subseteq B$, if $f(A) = f(B)$, then $f(C) = f(A)$ for any C such that $A \subseteq C \subseteq B$.

Theorem 3. *$(\mathcal{X}, f)$ is rationalizable by multiple preferences if and only if it satisfies the Sen's α axiom and the betweenness axiom.*

Proof of Theorem 3. (The only-if part) Suppose that $(\mathcal{X}, f)$ is rationalizable by multiple preferences. By definition, there exists a set $\triangleright$ of strict preferences such that $f_{\triangleright}(A) = f(A)$ for all $A \in \mathcal{X}$. To see that the Sen's α axiom holds, suppose that $x \in f(A \cup B) \cap A$, then there exists some $\succ \in \triangleright$ such that $x = \max(A \cup B, \succ)$. Therefore, $x = \max(A, \succ)$, and $x \in f(A)$. For the betweenness axiom, suppose that $A \subseteq C \subseteq B$ and $f(A) = f(B)$, it suffices to show that $f(A) \subseteq f(C) \subseteq f(B)$. We first show that $f(A) \subseteq f(C)$. For any $x \in f(A)$, since $f(A) = f(B)$, we have $x \in f(B)$ and there exists some $\succ \in \triangleright$ such that $x = \max(B, \succ)$. Since $x \in f(A) \subseteq A \subseteq C \subseteq B$, we have $x = \max(C, \succ)$ and thus $x \in f(C)$. We now show that $f(C) \subseteq f(B)$. For any $x \in f(C)$, there exists some $\succ \in \triangleright$ such that $x = \max(C, \succ)$. It must be that $x = \max(B, \succ)$. Otherwise, $\max(B, \succ) \in B \setminus C$ and $\max(B, \succ) \notin A$, which contradicts that $f(A) = f(B)$.

(The if-part) Suppose that $(\mathcal{X}, f)$ satisfies the Sen's α axiom and the betweenness axiom, we show that $(\mathcal{X}, f)$ is rationalizable by multiple preferences. For notational simplicity, we denote by $g(A) := A \setminus f(A)$ the collection of alternatives that are not chosen for each feasible set $A \in \mathcal{X}$.

Claim 1. *For any $A \in \mathcal{X}$, $f(f(A)) = f(A)$.*

Proof. For any $A \in \mathcal{X}$, we have $f(A) = f(f(A) \cup A) \cap f(A) \subseteq f(f(A))$, where the equality follows from the fact that $f(A) \subseteq A$, and the set inclusion relation follows from the Sen's α axiom. The claim then follows since $f(A) \in \mathcal{X}$ and $f(f(A)) \subseteq f(A)$.

Claim 2. *For any $A \in \mathcal{X}$, if $x \in g(A)$, then $f(f(A) \cup \{x\}) = f(A)$.*

Proof. Note that $f(A) \subseteq f(A) \cup \{x\} \subseteq A$. It then follows from Claim 1 and the betweenness axiom that $f(f(A) \cup \{x\}) = f(A)$.

Claim 3. *For any $A, B \in \mathcal{X}$, if $f(A) \subseteq B$, then $f(B) \cap g(A) = \emptyset$.*

Proof. Suppose to the contrary, there exists some $x \in f(B) \cap g(A)$. Since x is not chosen in A, by Claim 2, x is not chosen in $f(A) \cup \{x\}$. By the Sen's α axiom, x is not chosen in a superset of $f(A) \cup \{x\}$. Therefore, $x \notin f(B)$. We have a contradiction.

Claim 4. *For any $B \in \mathcal{X}$ and $\emptyset \neq \mathcal{X}' \subseteq \mathcal{X}$, if $\cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A) \subseteq B$, then $f(B) \cap \cup_{A\in\mathcal{X}'} g(A) = \emptyset$.*

Proof. Consider any $\emptyset \neq \mathcal{X}' \subseteq \mathcal{X}$. By the Sen's α axiom, $g(C) \subseteq g(\cup_{A\in\mathcal{X}'} A)$ for all $C \in \mathcal{X}'$. Therefore, $\cup_{A\in\mathcal{X}'} g(A) \subseteq g(\cup_{A\in\mathcal{X}'} A)$. We then have

$$\begin{aligned} f(\cup_{A\in\mathcal{X}'} A) &= \cup_{A\in\mathcal{X}'} A \setminus g(\cup_{A\in\mathcal{X}'} A) \\ &\subseteq \cup_{A\in\mathcal{X}'} A \setminus \cup_{A\in\mathcal{X}'} g(A) \\ &= \cup_{A\in\mathcal{X}'} (f(A) \cup g(A)) \setminus \cup_{A\in\mathcal{X}'} g(A) \\ &= \cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A). \end{aligned}$$

Thus, if $\cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A) \subseteq B$, we must have that $f(\cup_{A\in\mathcal{X}'} A) \subseteq B$. By Claim 3, $f(B) \cap g(\cup_{A\in\mathcal{X}'} A) = \emptyset$. The claim follows since $\cup_{A\in\mathcal{X}'} g(A) \subseteq g(\cup_{A\in\mathcal{X}'} A)$.

Claim 5. *The coarse data set $\mathcal{O} = \{A, f(A)\}_{A\in\mathcal{X}}$ satisfies the Coarse SARP property.*

Proof. Suppose to the contrary, there exists $\emptyset \neq \mathcal{X}' \subseteq \mathcal{X}$ such that $\cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A) = \emptyset$. Consider an arbitrary nonempty set $B \in \mathcal{X}'$, since $\cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A) = \emptyset \subseteq B$, by Claim 4, we have $f(B) \cap \cup_{A\in\mathcal{X}'} g(A) = \emptyset$. We have arrived at a contradiction, since $f(B) \subseteq \cup_{A\in\mathcal{X}'} f(A) \subseteq \cup_{A\in\mathcal{X}'} g(A)$, where the second set inclusion relation follows from the assumption that $\cup_{A\in\mathcal{X}'} f(A) \setminus \cup_{A\in\mathcal{X}'} g(A) = \emptyset$.

Claim 6. *For each $A \in \mathcal{X}$ and $x \in A$, the coarse data set $\mathcal{O}_{A,x}$ satisfies the Coarse SARP property.*

Proof. Suppose to the contrary, for some $A \in \mathcal{X}$ and $x \in f(A)$, the coarse data set $\mathcal{O}_{A,x}$ does not satisfy the Coarse SARP property. Then there exists $\emptyset \neq \mathcal{X}' \subseteq \mathcal{X}$ such that

$$\cup_{B\in\mathcal{X}'} \ f_{A,x}(B) \setminus \cup_{B\in\mathcal{X}'} g_{A,x}(B) = \emptyset. \tag{1}$$

We know from Claim 5 that $\mathcal{O} = \{A, f(A)\}_{A\in\mathcal{X}}$ satisfies the Coarse SARP property. Since $\mathcal{O}$ and $\mathcal{O}_{A,x}$ only differ when the feasible set is A, we can conclude that $A \in \mathcal{X}'$. Let us consider $\mathcal{X}'' := \mathcal{X}' \setminus \{A\}$. Since $\mathcal{O} = \{A, f(A)\}_{A\in\Sigma}$ satisfies the Coarse SARP property, we have

$$\cup_{B\in\mathcal{X}''} \ f_{A,x}(B) \setminus \cup_{B\in\mathcal{X}''} g_{A,x}(B) \neq \emptyset. \tag{2}$$

It follows from (1) and (2) that $\cup_{B\in\mathcal{X}''} f_{A,x}(B) \setminus \cup_{B\in\mathcal{X}''} g_{A,x}(B) \subseteq g_{A,x}(A) \subseteq A$, and $x \in \cup_{B\in\mathcal{X}''} g_{A,x}(B)$. Therefore, $x \in f(A) \cap \cup_{B\in\mathcal{X}''} g_{x,A}(B)$. But Claim 4 requires that $f(A) \cap \cup_{B\in\mathcal{X}''} g_{x,A}(B) = \emptyset$. Hence, we arrive at a contradiction.

By Theorem 1 and Theorem 2, $(\mathcal{X}, f)$ is rationalizable by multiple preferences. This completes the proof. □

4.3 Monotone Multiple Preferences

In this subsection, we introduce a new model of decision making that we call monotone multiple preferences. Consider a DM who makes choices in each period $t = 1, 2, \ldots, n$. In each period t, the DM has a set of rationales (strict preferences) $\triangleright_t$. As in the multiple preferences model an alternative is chosen from a feasible set if and only if it is maximal according to one of the rationales. Formally, if the DM has a set of strict preferences $\triangleright_t$, then she chooses

$$f_{\triangleright_t}(A) := \{x \in A : x = \max(A, \succ) \text{ for some } \succ \in \triangleright_t\}$$

from the feasible set A. The departure from the multiple preferences model analyzed in the last subsection is that we allow the set of rationales to shrink over time. Formally, we impose the following monotone structure in the sense of set inclusion on the sequence of sets of strict preferences:

$$t > t' \implies \rhd_t \subseteq \rhd_{t'}.$$

It is easy to see that the monotone multiple preferences model incorporates the multiple preferences model as a special case.

The monotone multiple preferences model thus has a simple interpretation: The DM initially has a large set of rationales. As time goes by, she is able to narrow down the list of rationales that are used to make choices. Let us now offer some justifications for this model. Consider the common motivation for the multiple preferences model that the DM's choices are made in the presence of framing effect; see, for example, Salant and Rubinstein (2008). Along this line of reasoning, our monotone multiple preferences model requires no more than that the DM can partly overcome the framing effect over time. Alternatively, one may think that the DM initially does not fully understand her own preference and is only certain that her preference lies in some set of preferences. Over time, the DM can rule out some preferences from the initial set of preferences as her true preference.

In what follows, we shall develop the revealed preference tests for the monotone multiple preferences model. Suppose that the analyst observes the DM's choices, and would like to test whether the DM's choices are consistent with the monotone multiple preferences model. Due to the monotone structure of the sets of strict preferences, it shouldn't come as a surprise that whether the analyst observes the timing of the DM's choices plays an important role. We distinguish the following two cases and propose revealed preference tests in each case. In the first case, the analyst observes the feasible sets, the choices of the DM, and the timing of the choices. In the second case, the analyst observes the feasible sets, the choices of the DM, but not the timing of the choices.

We first consider the case that the analyst observes the feasible sets, the choices of the DM, and the timing of the choices. The data set can be represented by the tuple $\{(A_t, B_t, t)\}_{t=1}^{n}$ where $t = 1, 2, \ldots, n$ indicates the timing of the DM's choices, A_t denotes the feasible set in period t, and $\emptyset \neq B_t \subset A_t$ denotes the DM's choices from A_t. We say that $\{(A_t, B_t, t)\}_{t=1}^{n}$ is rationalizable by monotone multiple preferences if there is a monotone sequence of sets of strict preferences $\{\triangleright_t\}_{t=1}^n$ such that

$$B_t = f_{\triangleright_t}(A_t)$$

for all $t = 1, 2, \ldots, n$.

This case is easier to handle. Suppose that the data set $\{(A_t, B_t, t)\}_{t=1}^n$ is rationalizable by monotone multiple preferences. Consider any observation (A_t, B_t, t) and fix $x \in B_t$. There necessarily exists a strict preference $\succ \in \triangleright_t$ that ranks x maximal. Due to the monotone structure of the sets of strict preferences, we also know that $\succ \in \triangleright_{t'}$ for any $t' < t$. In words, the rationale $\succ \in \triangleright_t$ that ranks x maximal in A_t must also be used to make choices in any earlier period. Subsequently, it must be the case that $\max(A_{t'}, \succ) \in B_{t'}$ for any $t' < t$. Put succinctly, for each (A_t, B_t, t) and each $x \in B_t$, there must exist a strict preference $\succ$ such that $\max(A_t, \succ) = x$ and $\max(A_{t'}, \succ) \in B_{t'}$ for all $t' < t$.

For each (A_t, B_t, t) and each $x \in B_t$, we construct a corresponding coarse data set as follows:

$$\mathcal{O}_{A_t,x} = \{(A_{t'}, B_{t'}) : t' < t\} \cup \{(A_t, \{x\})\}.$$

The analysis in the last paragraph suggests that $\mathcal{O}_{A_t,x}$ is rationalizable by a linear order. Let $\mathfrak{D} = \{\mathcal{O}_{A_t,x}\}_{A_t, x \in B_t}$ denote the family of such coarse data sets. Theorem 4 below shows that $\{(A_t, B_t, t)\}_{t=1}^n$ is rationalizable by monotone multiple preferences if and only if each $\mathcal{O}_{A_t,x}$ in $\mathfrak{D}$ is rationalizable by a linear order.

Theorem 4. *$\{(A_t, B_t, t)\}_{t=1}^n$ is rationalizable by monotone multiple preferences if and only if each $\mathcal{O}_{A_t,x}$ in $\mathfrak{D}$ is rationalizable by a linear order.*

Proof. (The if-part) Suppose that each $\mathcal{O}_{A_t,x}$ in $\mathfrak{D}$ is rationalizable by a linear order. Let $\triangleright_t$ denote the collection of strict preferences such that $\succ \in \triangleright_t$ if and only if $\succ$ rationalizes $\mathcal{O}_{A_t,x}$ for some $x \in B_t$. We show that $\{(A_t, B_t, t)\}_{t=1}^n$ is rationalizable under the monotone multiple preferences model by the sequence of sets of strict preferences $\{\triangleright_t\}_{t=1}^n$ constructed in this way. By construction, $B_t = f_{\triangleright_t}(A_t)$ for all $t = 1, 2, \ldots, n$. Thus, it suffices to show that $t > t'$ implies $\triangleright_t \subseteq \triangleright_{t'}$. To see this, note that if $t' < t$, $\succ \in \triangleright_t$ implies that $\succ$ also rationalizes $\mathcal{O}_{A_{t'},x}$ for some $x \in B_{t'}$. Thus, $\succ \in \triangleright_{t'}$. □

Remark 2. The analysis above should remind our readers of the analysis in the multiple preferences model. In the multiple preference model, the set of multiple preferences used by the DM do not vary as the DM faces different feasible sets. As such, a rationale that is used to rationalize a particular choice from a feasible set must be used in all the other choices. Whereas in this model, we allow the DM's set of rationales to shrink over time. Thus, the rationale that is used in period t is also used in any earlier period, and subsequently, the construction of the coarse data set is different from our construction in the multiple preferences model.

Next, we consider the case in which the analyst only observes the feasible sets, the DM's choices, but not the timing of the choices. While this introduces additional complexities beyond the case in which the analyst also observes the timing of the choices, we show that the logic of the analysis above carries over to this case.

Since the timing of the choices is not observed, the data set can be represented by $\{(A_i, B_i)\}_{i=1}^n$ where A_i denotes the feasible set and $\emptyset \neq B_i \subseteq A_i$ denotes the DM's choices from A_t. We should emphasize that i is simply an index of the data point and does not correspond to the timing of the choices. We say that $\{(A_i, B_i)\}_{i=1}^n$ is rationalized by monotone multiple preferences if there is a permutation $\rho : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and a monotone sequence of sets of preferences $\{\triangleright_t\}_{t=1}^n$ such that

$$B_i = f_{\triangleright_{\rho(i)}}(A_i)$$

for all $i = 1, 2, \ldots, n$. The permutation ρ represents the revealed timing of the choices.

To develop the revealed preferences tests for this case, it is useful to revisit the logic for the case in which the timing of the choices is observed. If the timing of the choices is observed, as Theorem 4 shows, it suffices to construct a family of coarse data sets and check whether each coarse data set is rationalizable by a linear order. The construction of the coarse data sets, however, requires the knowledge of the timing of the choices. When the timing of the choices is not observed, one naive way to proceed is to consider all permutations ρ and for each permutation we can invoke Theorem 4 to check for rationalizability. However, this is computationally infeasible if n is large. The problem can be circumvented by the following observation. Rather than going through all permutations ρ to determine the timing of the choices, we first study the problem of determining which choice occurs in the last period. Suppose that $\{(A_i, B_i)\}_{i=1}^n$ is rationalizable by monotone multiple preferences. Then if a choice, say (A_l, B_l), occurs in the last period, then for each x in B_j, the corresponding coarse data set $\{(A_i, B_i)\}_{i \neq j} \cup \{(A_j, x)\}$ is rationalizable by a linear order. Once we have determined which choice occurs in the last period, we can apply similar logic to the remaining data points and determine which choice occurs in the second last period. The logic continues and we have a tractable way of determining the timing of all the choices. Theorem 5 below states this formally.

Theorem 5. $\{(A_i, B_i)\}_{i=1}^n$ *is rationalizable by monotone multiple preferences if and only if for any subcollection* $\{(A_{k_j}, B_{k_j})\}_{j=1}^m$, *there is* $(A_l, B_l) \in \{(A_{k_j}, B_{k_j})\}_{j=1}^m$ *such that for each* $x \in B_l$, $\{(A_{k_j}, B_{k_j})\}_{j=1}^m \setminus \{(A_l, B_l)\} \cup \{(A_l, x)\}$ *is rationalizable by a linear order.*

Proof. (The if-part) The assumption ensures that for any collection of data points, there is a candidate for the choice that occurs last among the data points. Thus, one can simply starts with all the data points and determine one possible timing of all the choices. The result follows by applying Theorem 5 to that timing of all the choices. □

4.4 Minimax Regret

The minimax regret model is first proposed by Savage (1951) to model a DM who might anticipate regret and thus incorporate in their choice the desire to minimize the worst-case regret. In this subsection, we show that, by applying our results in Section 3, one can easily obtain the observable restrictions of the minimax regret model.

We shall adopt the framework of the state space that has been discussed by, for example, Kreps (1979) and Dekel et al. (2001). Let $u : X \to R$ be a utility function for the DM. Under utility function u, the regret of choosing x towards y is represented by $u(y) - u(x)$. Given a finite set of utility functions $\mathcal{U}$, the worst-case regret of choosing x from a feasible set $A \in \mathcal{X}$ is

$$\max_{y \in A} \max_{u \in \mathcal{U}} \{u(y) - u(x)\}.$$

The DM chooses an alternative that minimizes the worst-case regret. Formally, the DM has a finite set of utility functions $\mathcal{U}$ defined on X such that she chooses

$$\min_{x \in A} \left\{ \max_{y \in A} \max_{u \in \mathcal{U}} \{u(y) - u(x)\} \right\}$$

for all $A \in \mathcal{X}$.

We represent the choice behavior of the DM by (Σ, f), where $\Sigma \subseteq \mathcal{X}$ and $f(A)$ is the alternative that the DM chooses in $A \in \Sigma$. We assume that f is a choice function. We say that (Σ, f) is rationalizable under the minimax regret model if there exists a finite set of utility functions $\mathcal{U}$ such that

$$f(A) = \arg\min_{x \in A} \left\{ \max_{y \in A} \max_{u \in \mathcal{U}} \{u(y) - u(x)\} \right\}.$$

for each $A \in \Sigma$.

Suppose that (Σ, f) is rationalizable under the minimax regret model with a finite set of utility functions $\mathcal{U}$. We define the relative regret of x towards y as

$$\phi(x, y) := \max_{u \in \mathcal{U}} \{u(y) - u(x)\}.$$

Thus, the choice function could be rewritten as

$$f(A) = \arg\min_{x \in A} \left\{ \max_{y \in A} \phi(x, y) \right\}. \tag{3}$$

Given the flexibility to construct the set of utility functions to make the choice behavior of the DM consistent with the minimax regret model, the readers might doubt, whether there are any observable restrictions of the minimax regret model. The following example presents a data set (Σ, f) that is not rationalizable under the minimax regret model.

Example 4 (A data set that is not rationalizable under the minimax regret model)**.** *Let $X = \{x, y, z, w\}$ and $A = X$. Consider a data set (Σ, f) including the following three observations: $f(A) = x$, $f(A \setminus z) = y$, and $f(A \setminus w) = y$.*

Suppose that (Σ, f) is rationalizable under the minimax regret model. Since $f(A) = x$ and $f(A \setminus z) = y$, x generates a lower worst-case regret in set A than y does, but y generates a lower worst-case regret than x does if we remove the alternative z from the alternative set A. Therefore, the relative regret of y towards z is the worst-case regret of y in set A, and is higher than the worst-case regret of x in set A. But then, it cannot be that $f(A \setminus w) = y$, as x generates a lower worst-case regret in set $A \setminus w$ than the relative regret of y towards z, which is the worst-case regret of y in set $A \setminus w$. We arrive at a contradiction.

The next lemma shows that, without loss of generality, we can work with the following choice function:

$$f(A) = \arg\min_{x \in A} \left\{ \max_{y \in A \setminus x} \phi(x, y) \right\}. \tag{4}$$

This simplifies the analysis below.

Lemma 1. *There exists a set of utility function $\mathcal{U}$ such that (3) holds for each $A \in \Sigma$ if and only if (4) holds for each $A \in \Sigma$ under the same $\mathcal{U}$.*

By Lemma 1, the rationalizability of (Σ, f) under the minimax regret model could be transformed to the problem of finding a finite set of utility functions $\mathcal{U}$ such that

$$\max_{r \in A \setminus f(A)} \phi(f(A), r) < \max_{r \in A \setminus z} \phi(z, r)$$

for any $z \in A \setminus f(A)$.

Interestingly, the rationalizability of a data set under the minimax regret model is related to the rationalizability of a corresponding coarse data set by a linear order. Before we discuss the construction of the corresponding coarse data set, it is perhaps best to consider the following thought experiment, which illustrates the logic of the construction. Suppose that (Σ, f) includes the following observation $f(\{x, y, z\}) = x$. If (Σ, f) is rationalizable under the minimax regret model, then it must be the case that the worst-case regret of any alternative different from x in $\{x, y, z\}$ is higher than the worst-case regret of x in $\{x, y, z\}$. That is, $f(\{x, y, z\}) = x$ reveals that

$$\max\{\phi(y, x), \phi(y, z)\} > \max\{\phi(x, y), \phi(x, z)\}, \text{ and}$$
$$\max\{\phi(z, x), \phi(z, y)\} > \max\{\phi(x, y), \phi(x, z)\}.$$

For notational simplicity, in what follows, we shall use (x, y) to represent the maximal regret of x from y.

Following the logic in the last paragraph, for any (Σ, f), we are now ready to construct the corresponding coarse data set. Let $\bar{X} := \{(x, y) \in X \times X : x \neq y\}$. For any (Σ, f), the corresponding coarse data set $\bar{\mathcal{O}}$ can be constructed as follows: $(\bar{A}_i, \bar{B}_i) \in \bar{\mathcal{O}}$ if and only if there exists $y \in A \in \Sigma$ with $y \neq f(A)$ such that

$$\bar{A}_i = \{y \times (A \setminus y)\} \cup \{f(A) \times (A \setminus f(A))\}, \text{ and}$$
$$\bar{B}_i = y \times (A \setminus y).$$

The interpretation of the data point $(\bar{A}_i, \bar{B}_i)$ is then that alternative y which is not chosen in A generates a higher worst-case regret in A than $f(A)$ does. The following theorem shows that (Σ, f) is rationalizable under the minimax regret model if and only if the corresponding coarse data set $\bar{\mathcal{O}}$ is rationalizable by a linear order.

Theorem 6. (Σ, f) *is rationalizable under the minimax regret model if and only if the corresponding coarse data set $\bar{\mathcal{O}}$ is rationalizable by a linear order.*

Proof of Theorem 6. (The only if-part) Suppose that (Σ, f) is rationalizable under the minimax regret model, by definition, there exists a finite set of utility functions $\mathcal{U}$ such that

$$f(A) = \arg\min_{x \in A} \left\{ \max_{y \in A \setminus x} \phi(x, y) \right\}$$

holds for each $A \in \Sigma$. We claim that ϕ could represent a complete and transitive binary relation defined on $\bar{X}$. Formally, denote this binary relation as R^R which we call the regret relation. We have that $\phi(x, y) R^R \phi(z, w)$ if and only if $\phi(x, y) \geq \phi(z, w)$. For each $A \in \Sigma$,

$$f(A) = \arg\min(\Big(\max(x \times A \setminus z, R^R)\Big)_{x \in A}, R^R)$$

Thus, the relation R^R satisfies that for any $(\bar{A}, \bar{B}) \in \mathcal{O}$, there is some set $A \in \Sigma$ such that $\bar{A} = \{z\} \times A \backslash \{z\} \cup \{f(A)\} \times A \backslash \{f(A)\}$ with $z \neq f(A)$ and $\bar{B} = \{z\} \times A \backslash \{z\}$. As a result, it must be that there is some $y \neq z$ such that $(z, y) R^* (f(A), \hat{y})$ and not $(f(A), \hat{y}) R^* (z, y)$ for any $\hat{y} \in A \backslash \{f(A)\}$. Easy to see that if we simply assigns some tie-breaking rule to R^* and transfer it to R^{**} which is a linear order over $\bar{X}$, it still holds that there is some $y \neq z$ such that $(z, y) R^{**} (f(A), \hat{y})$ and not $(f(A), \hat{y}) R^{**} (z, y)$ for any $\hat{y} \in A \backslash \{f(A)\}$. As a result, the maximal element in each $\bar{A}$ must be contained in the set $\bar{B}$ according to the linear order R^{**}. We finish the sufficient part of the proof.

(The if-part) Suppose that $\bar{\mathcal{O}}$ is rationalizable by a linear order, by definition, there exists a linear order P defined on $\bar{X}$ such that for each $(\bar{A}, \bar{B}) \in \bar{\mathcal{O}}$, $\max(\bar{A}, P) \in \bar{B}$. Take the asymmetric part of P as $\triangleright$. It suffices to show that there exists a finite set of utility function $\mathcal{U}$ such that for any $(x, y), (z, w) \in \bar{X}$, $(x, y) \triangleright (z, w)$ if and only if $\phi(x, y) > \phi(z, w)$, where ϕ is constructed from $\mathcal{U}$.

As $\bar{X}$ is finite, we can find a function $\beta : \bar{X} \to (1, 2)$ such that $(x, y) \triangleright (z, w)$ if and only if $\beta(x, y) > \beta(z, w)$. Now consider a finite set of utility functions $\mathcal{U} := \{u_{x,y}\}_{(x,y) \in \bar{X}}$, where the utility functions in $\mathcal{U}$ are indexed by $(x, y) \in \bar{X}$. For each $(x, y) \in \bar{X}$, define $u_{x,y}$ as follows:

$$u_{x,y}(z) = \begin{cases} 0, & \text{if } z = x, \\ \beta(x, y), & \text{if } z = y, \\ \frac{\beta(z,w)}{2}, & \text{otherwise.} \end{cases}$$

Thus, $u_{x,y}(y) - u_{x,y}(x) = \beta(x, y) \in (1, 2)$. Furthermore, we claim that $u_{z,w}(y) - u_{z,w}(x) < 1$ if $(z, w) \neq (x, y)$. To see this, first consider the case in which $w \neq y$. By construction of the utility functions and the β function,

$$u_{z,w}(y) - u_{z,w}(x) \leq u_{z,w}(y) \leq \frac{\beta(z, w)}{2} \in (0, 1).$$

Now consider the case in which $w = y$. Since $(z, w) \neq (x, y)$, we have $z \neq x$. By construction of the utility functions, $u_{z,w}(x) = \frac{\beta(z,w)}{2}$ and $u_{z,w}(y) = \beta(z, w)$. Therefore, $u_{z,w}(y) - u_{z,w}(x) = \frac{\beta(z,w)}{2} \in (0, 1)$. This implies that

$$\phi(x, y) = \max_{u \in \mathcal{U}} [u(y) - u(x)] = u_{x,y}(y) - u_{x,y}(x) = \beta(x, y).$$

Therefore, we have explicitly constructed a finite set of utility functions $\mathcal{U}$ such that for any $(x, y), (z, w) \in \bar{X}$, $(x, y) \triangleright (z, w)$ if and only if $\phi(x, y) > \phi(z, w)$, where ϕ is constructed from $\mathcal{U}$. This completes the proof. □

We revisit Example 4 to illustrate how to use Theorem 6 to show that the data set is not rationalizable.

Example 5 (Example 4 Revisited)**.** *The data set is the same as in Example 4. We construct the corresponding coarse data set $\bar{\mathcal{O}}$ as follows:*

1. $\bar{A}_1 = \{(y, x), (y, z), (y, w), (x, y), (x, z), (x, w)\}, \bar{B}_1 = \{(y, x), (y, z), (y, w)\}$;
2. $\bar{A}_2 = \{(z, x), (z, y), (z, w), (x, y), (x, z), (x, w)\}, \bar{B}_2 = \{(z, x), (z, y), (z, w)\}$;
3. $\bar{A}_3 = \{(w, x), (w, y), (w, z), (x, y), (x, z), (x, w)\}, \bar{B}_3 = \{(w, x), (w, y), (w, z)\}$;
4. $\bar{A}_4 = \{(x, y), (x, w), (y, x), (y, w)\}, \bar{B}_4 = \{(x, y), (x, w)\}$;
5. $\bar{A}_5 = \{(w, x), (w, y), (y, x), (y, w)\}, \bar{B}_5 = \{(w, x), (w, y)\}$;
6. $\bar{A}_6 = \{(x, y), (x, z), (y, x), (y, z)\}, \bar{B}_6 = \{(x, y), (x, z)\}$;
7. $\bar{A}_7 = \{(z, x), (z, y), (y, x), (y, z)\}, \bar{B}_7 = \{(z, x), (z, y)\}$.

By Theorem 1, the coarse data set $\bar{\mathcal{O}}$ is not rationalizable by a linear order. In particular, for $\bar{\mathcal{O}}' = \{(\bar{A}_1, \bar{B}_1), (\bar{A}_4, \bar{B}_4), (\bar{A}_6, \bar{B}_6)\}$, $B(\bar{\mathcal{O}}') \setminus C(\bar{\mathcal{O}}') = \emptyset$. By Theorem 6, (Σ, f) is not rationalizable under the minimax regret model.

5 Weak order

Our analysis above centers around the coarse rationalizability of a coarse data set by a linear order. We shall now extend our analysis to the case of coarse rationalizability by a weak order. A weak order is a complete and transitive binary relation, denoted by R. We abuse the notation slightly and denote the asymmetric part of R by P. We write $\max(A, R)$ to denote the maximal alternatives in A according to R. If xRy, we say that x is ranked weakly above y. If xRy and not yRx, we say that x is ranked strictly above y.

One way to define the coarse rationalizability of a coarse data set by a weak order is as follows: A coarse data set $\mathcal{O} = \{(A_i, B_i)\}_{i=1}^n$ is rationalized by a weak order R if

$$\max(A_i,\ R) \cap B_i \neq \emptyset$$

for all i. Note that every coarse data set can be rationalizable by a weak order as the DM could be indifferent among all alternatives in X. In many settings, however, we may have some prior knowledge of the DM's preferences. Then, we could incorporate such prior knowledge of the DM's preference into the revealed preference exercise.[3]

Throughout the rest of this section, we shall model a data point to be a tuple (A, B, D) where A is a feasible set, B and D are disjoint subsets of A, and $B \cup D \neq \emptyset$. The interpretation of (A, B, D) is that when the DM is faced with the feasible set A, the analyst knows that the exact choice of the DM is contained in B. Furthermore, she also knows the alternatives in D are not maximal for the DM in A. A coarse data set $\mathcal{O}$ is a collection of data points $\{(A_i, B_i, D_i)\}_{i=1}^n$, where B_i and D_i are disjoint subsets of A_i, and $B_i \cup D_i \neq \emptyset$ for all i. We say that a coarse data set $\mathcal{O} = \{(A_i, B_i, D_i)\}_{i=1}^n$ is coarsely rationalizable by a weak order if there exists a weak order R defined on X such that

$$\max(A_i, R) \cap B_i \neq \emptyset \text{ and } \max(A_i, R) \cap D_i = \emptyset$$

for all i.

To fix ideas, let us first consider the familiar case that B_i is a singleton set for all i. Without loss of generality, we can rewrite $\mathcal{O}$ as $\{(A_i, x_i, D_i)\}_{i=1}^n$. We say that x_i is revealed

[3] For example, suppose that the analyst has prior knowledge that the DM ranks x strictly above y and also suppose that the DM chooses z from $\{x, y, z\}$. Then the choice reveals that the DM ranks z weakly over x and ranks z strictly above y.

to be ranked weakly above y if $y \in A_i$ and $y \neq x_i$ and that x_i is revealed to be ranked strictly above z if $z \in D_i$. It is well known that the data set is rationalizable by a weak order if and only if the data set obeys the generalized axiom of revealed preference (GARP) condition, which says that there is no strict revealed cycle on the set of alternatives X. However, when B_i contains more than one alternative for some i, the revealed preference analysis becomes less straightforward.

As in the case of linear order, we proceed to identify necessary and sufficient conditions for a coarse data set to be coarsely rationalized by a weak order. For ease of notation, we write

$$D(\mathcal{O}') = \cup_{(A_i,B_i,D_i)\in\mathcal{O}'} D_i.$$

A natural conjecture is that we can work with $B(\mathcal{O}') \setminus D(\mathcal{O}')$ and check whether it is empty for each subcollection $\emptyset \neq \mathcal{O}' \subseteq \mathcal{O}$. Indeed, if a coarse data set $\mathcal{O} = \{(A_i, B_i, D_i)\}_{i=1}^n$ is coarsely rationalizable by a weak order, by definition, for all (A_i, B_i, D_i), B_i contains an alternative that is strictly ranked above all alternatives in D_i. Thus, for each subcollection $\emptyset \neq \mathcal{O}' \subseteq \mathcal{O}$, $B(\mathcal{O}')$ contains an alternative that is strictly ranked above all alternatives in $D(\mathcal{O}')$, and $B(\mathcal{O}') \setminus D(\mathcal{O}') \neq \emptyset$. While this is necessary, this is not sufficient for the coarse rationalizability of a coarse data set by a weak order.

The reason that working with $B(\mathcal{O}') \setminus D(\mathcal{O}')$ for each subcollection $\emptyset \neq \mathcal{O}' \subseteq \mathcal{O}$ is not sufficient is as follows. This logic overlooks the information that for all (A_i, B_i, D_i), some alternative in B_i is ranked weakly above all alternatives in A_i. In what follows, we propose a procedure that incorporates such information, and identify a necessary and sufficient condition for coarse rationalizability of a coarse data set by a weak order.

Suppose that a coarse data set $\mathcal{O} = \{(A_i, B_i, D_i)\}_{i=1}^n$ is coarsely rationalizable by a weak order. By definition, for all i, B_i contains an alternative that is strictly ranked above all alternatives in D_i and weakly ranked above all alternatives in A_i. Consequently, for any $\emptyset \neq \mathcal{O}' \subseteq \mathcal{O}$, $B(\mathcal{O}')$ is directly revealed to contain an alternative that is strictly ranked above all alternatives in $D(\mathcal{O}')$. This logic is exactly the same as that in the case of coarse rationalizability by a linear order. However, this is not sufficient for rationalizability of a coarse data set by a weak order, because it does not capture that for all i, B_i contains an alternative that weakly ranked above all alternatives in A_i.

Suppose that $B_i \subseteq D(\mathcal{O}')$ for some $(A_i, B_i, D_i) \in \mathcal{O}'$. Since B_i contains an alternative that is weakly ranked above all alternatives in A_i, $D(\mathcal{O}')$ contains an alternative that is weakly ranked above all alternatives in A_i. By the transitivity of the weak order, $B(\mathcal{O}')$ contains an alternative that is strictly ranked above all alternatives in A_i. In other words, alternatives in A_i cannot be the maximal elements in $A(\mathcal{O}')$. Thus, the logic expands the set of alternatives that cannot be maximal in $A(\mathcal{O}')$.

Motivated by the logic in the previous paragraph, we define the following operator $\mathcal{E}^1$ such that

$$\mathcal{E}^1(\mathcal{O}', E) := (\cup_{(A_i,B_i,D_i)\in\mathcal{O}':B_i\subseteq E} A_i) \cup E \tag{5}$$

for all $\emptyset \neq \mathcal{O}' \subseteq \mathcal{O}$ and $E \in \mathcal{X}$. It follows that E contains an alternative that is weakly ranked above all alternatives in $\mathcal{E}^1(\mathcal{O}', E)$. Recursively, we can define

$$\mathcal{E}^n(\mathcal{O}',\, E) := \mathcal{E}^1(\mathcal{O}',\, \mathcal{E}^{n-1}(\mathcal{O}', E)).$$

By the same reasoning, for all n, $\mathcal{E}^{n-1}(\mathcal{O}',\, E)$ contains an alternative that is weakly ranked above all alternatives in $\mathcal{E}^n(\mathcal{O}', E)$. By the transitivity of the weak order, E contains at least one alternative that is weakly ranked above every alternative in $\mathcal{E}^n(\mathcal{O}', E)$ for all n.

By the construction of the operator $\mathcal{E}^1$, $\mathcal{E}^1(\mathcal{O}', E)$ is increasing in E in the sense that if $E \subseteq E'$ then $\mathcal{E}^1(\mathcal{O}', E) \subseteq \mathcal{E}^1(\mathcal{O}', E')$. Since $E \subseteq \mathcal{E}^1(\mathcal{O}', E)$, we have

$$\mathcal{E}^n(\mathcal{O}', E) \subseteq \mathcal{E}^{n+1}(\mathcal{O}', E)$$

for all n. Let N be the smallest integer such that $\mathcal{E}^{N+1}(\mathcal{O}', E) = \mathcal{E}^N(\mathcal{O}', E)$. Since X is finite, $\mathcal{E}^n(\mathcal{O}', E)$ necessarily converges to $\mathcal{E}^N(\mathcal{O}', E)$ in finitely many steps. Let $\mathcal{E}(\mathcal{O}', E) = \mathcal{E}^N(\mathcal{O}', E)$.

Since $D(\mathcal{O}')$ contains at least one alternative that is weakly ranked above every alternative in $\mathcal{E}(\mathcal{O}', D(\mathcal{O}'))$ and $B(\mathcal{O}')$ contains an alternative that is strictly ranked above all alternatives in $D(\mathcal{O}')$, by transitivity of weak order, $B(\mathcal{O}')$ necessarily contains an alternative that is strictly ranked above all alternatives in $\mathcal{E}(\mathcal{O}', D(\mathcal{O}'))$. As such, we have $B(\mathcal{O}') \setminus \mathcal{E}(\mathcal{O}', D(\mathcal{O}')) \neq \emptyset$. For notational simplicity, we write $\mathcal{E}(\mathcal{O}')$ rather than $\mathcal{E}(\mathcal{O}', D(\mathcal{O}'))$. Our analysis so far suggests the following necessary condition for coarse rationalizability by a weak order that we call Coarse GARP:

Coarse GARP. For any $\emptyset \neq \mathcal{O}' \subseteq \mathcal{O}$, $B(\mathcal{O}') \setminus \mathcal{E}(\mathcal{O}') \neq \emptyset$.

Theorem 7 below shows that Coarse GARP is also sufficient for coarse rationalizability of a coarse data set by a weak order.

Theorem 7. *A coarse data set* $\mathcal{O} = \{(A_i, B_i, D_i)\}_{i=1}^n$ *is coarsely rationalizable by a weak order if and only if it satisfies the Coarse GARP property.*

The necessity has been discussed in the previous paragraph. For sufficiency, we explicitly construct a weak order that coarsely rationalizes $\mathcal{O}$. To do this, we first decompose $\mathcal{O}$ by iteratively excluding the data points that involve the (revealed) highest-ranked alternatives. Simultaneously, X is partitioned by iteratively excluding the (revealed) highest-ranked alternatives. Next, we define a weak order on X according to the decomposition of X. Finally, we verify that such a weak order indeed rationalizes the data set $\mathcal{O}$.

Proof of Theorem 7. (The if-part) Suppose that $\mathcal{O}$ satisfies the Coarse GARP property. We show that $\mathcal{O}$ is coarsely rationalizable by a weak order by explicitly constructing a weak order that coarsely rationalizes $\mathcal{O}$.

Let $\mathcal{O}_1 := \mathcal{O}$ and $S_1 := B(\mathcal{O}_1) \setminus \mathcal{E}(\mathcal{O}_1)$. Since $\mathcal{O}$ satisfies the Coarse GARP property, we know that $S_1 \neq \emptyset$. We proceed by induction. Suppose that we have constructed $\mathcal{O}_k$ and S_k for some $k \geq 1$ and $\mathcal{O}_k \neq \emptyset$, we construct $\mathcal{O}_{k+1}$ and S_{k+1} as follows:

$$\mathcal{O}_{k+1} := \{(A_i, B_i, D_i) \in \mathcal{O}_k : B_i \cap P_k = \emptyset\};$$
$$S_{k+1} := B(\mathcal{O}_{k+1}) \setminus \mathcal{E}(\mathcal{O}_{k+1}).$$

We claim that for each $\mathcal{O}_k \neq \emptyset$, $\mathcal{O}_{k+1}$ is a proper subset of $\mathcal{O}_k$. Since $\mathcal{O}$ satisfies the Coarse GARP property, if $\mathcal{O}_k \neq \emptyset$, we know that $S_k = B(\mathcal{O}_k) \setminus \mathcal{E}(\mathcal{O}_k) \neq \emptyset$. Thus, $B(\mathcal{O}_k) \cap S_k \neq \emptyset$, and there exists some $(A_i, B_i, D_i) \in \mathcal{O}_k$ such that $B_i \cap S_k \neq \emptyset$ and is eliminated when constructing $\mathcal{O}_{k+1}$ from $\mathcal{O}_k$. Now that we have established that for $\mathcal{O}_k \neq \emptyset$, $\mathcal{O}_{k+1}$ is a proper subset of $\mathcal{O}_k$, we can conclude that the construction stops after finitely many steps, when $\mathcal{O}_t \neq \emptyset$ and $\mathcal{O}_{t+1} = \emptyset$ for some t. Let us redefine

$$S_{t+1} := X \setminus (\cup_{k=1}^t S_k).$$

We now argue that $\{S_k\}_{k=1}^{t+1}$ constitutes a partition of X.

Claim 7. $\mathcal{E}(\mathcal{O}_{k+1}) \subseteq \mathcal{E}(\mathcal{O}_k)$ *for* $k = 1, 2, ..., t-1$.

Proof. By the construction of the operator $\mathcal{E}^1$, $\mathcal{E}^1(\mathcal{O}', E)$ is increasing in both $\mathcal{O}'$ and E. Therefore,

$$\mathcal{O}_{k+1} \subseteq \mathcal{O}_k \implies \mathcal{E}^1(\mathcal{O}_{k+1}, D(\mathcal{O}_{k+1})) \subset \mathcal{E}^1(\mathcal{O}_k, D(\mathcal{O}_k)).$$

By induction, we know that $\mathcal{E}^n(\mathcal{O}_{k+1}, D(\mathcal{O}_{k+1})) \subseteq \mathcal{E}^n(\mathcal{O}_k, D(\mathcal{O}_k))$ for all n. Thus, $\mathcal{E}(\mathcal{O}_{k+1}) \subseteq \mathcal{E}(\mathcal{O}_k)$.

Claim 8. *$\{S_k\}_{k=1}^{t+1}$ constitutes a partition of X.*

Proof. By construction, $S_{t+1} = X \setminus (\cup_{k=1}^{t} S_k)$. Thus, to show that $\{S_k\}_{k=1}^{t+1}$ constitutes a partition of X, it suffices to show that $S_j \cap S_{j'} = \emptyset$ for $1 \leq j < j' \leq t$. This follows from

$$S_j = B(\mathcal{O}_j) \setminus \mathcal{E}(\mathcal{O}_j), \text{ and}$$
$$S_{j'} = B(\mathcal{O}_{j'}) \setminus \mathcal{E}(\mathcal{O}_{j'}) \subseteq B(\mathcal{O}_{j'}) \subseteq \mathcal{E}(\mathcal{O}_{j'-1}) \subseteq \mathcal{E}(\mathcal{O}_j),$$

where the first set inclusion is trivial and the third set inclusion follows from Claim 7. To see that the second set inclusion holds, let us consider an arbitrary $(A_i, B_i, D_i) \in \mathcal{O}_{j'}$. By the construction of $\mathcal{O}_{j'}$ and $S_{j'-1}$, it must be that

$$B_i \cap S_{j'-1} = B_i \cap (B(\mathcal{O}_{j'-1}) \setminus \mathcal{E}(\mathcal{O}_{j'-1})) = \emptyset. \tag{6}$$

Otherwise, (A_i, B_i, D_i) would have been eliminated in earlier steps. Since $(A_i, B_i, D_i) \in \mathcal{O}_{j'} \subseteq \mathcal{O}_{j'-1}$, we know that $B_i \subseteq B(\mathcal{O}_{j'-1})$. It then follows from (6) that $B_i \subseteq \mathcal{E}(\mathcal{O}_{j'-1})$. Since this holds for all $(A_i, B_i, D_i) \in \mathcal{O}_{j'}$, we can conclude that $B(\mathcal{O}_{j'}) \subseteq \mathcal{E}(\mathcal{O}_{j'-1})$.

We now construct a weak order R defined on X such that (1) xRy and yRx if and only if $x, y \in S_k$ for some k; and (2) xPy if and only if $x \in S_j$ and $y \in S_{j'}$ with $j < j'$. Since $\{S_k\}_{k=1}^{t+1}$ constitutes a partition of X, the weak order R is well defined.

We are left with the task to show that R coarsely rationalizes the coarse data set $\mathcal{O}$. In what follows, we verify that

$$\max(A_i, R) \cap B_i \neq \emptyset \text{ and } \max(A_i, R) \cap D_i = \emptyset$$

for all i. Fix i, and suppose that $(A_i, B_i, D_i) \in \mathcal{O}_k \setminus \mathcal{O}_{k+1}$ for some $1 \leq k \leq t$, we know that $B_i \cap S_k \neq \emptyset$. Moreover, $B_i \cap S_{k'} = \emptyset$ for all $k' < k$. Otherwise, (A_i, B_i, D_i) would have been eliminated in earlier steps. We show that $B_i \subseteq \mathcal{E}(\mathcal{O}_{k'})$ and $A_i \subseteq \mathcal{E}(\mathcal{O}_{k'})$ for all $k' < k$. Since $B_i \subseteq B(\mathcal{O}_k) \subseteq B(\mathcal{O}_{k'})$, the definition of $S_{k'}$, i.e., $P_{k'} = B(\mathcal{O}_{k'}) \setminus \mathcal{E}(\mathcal{O}_{k'})$, and $B_i \cap P_{k'} = \emptyset$ imply that $B_i \subseteq \mathcal{E}(\mathcal{O}_{k'})$. It follows from $B_i \subseteq \mathcal{E}(\mathcal{O}_{k'})$ that $A_i \subseteq \mathcal{E}(\mathcal{O}_{k'})$, because $A_i \subseteq \mathcal{E}^1(\mathcal{O}_{k'}, \mathcal{E}(\mathcal{O}_{k'})) = \mathcal{E}(\mathcal{O}_{k'})$, where the set inclusion follows from the definition of $\mathcal{E}^1(\cdot)$ and the equality from the definition of $\mathcal{E}(\cdot)$. Therefore, we have $A_i \cap S_{k'} = A_i \cap B(\mathcal{O}_{k'}) \setminus \mathcal{E}(\mathcal{O}_{k'}) = \emptyset$

for all $k' < k$, i.e., A_i contains no alternative that is higher ranked than those in S_k. Note that $B_i \cap S_k \neq \emptyset$ implies $A_i \cap P_k \neq \emptyset$. Hence,

$$\max(A_i, R) \cap B_i = A_i \cap P_k \cap B_i \neq \emptyset.$$

Furthermore, since $S_k \cap \mathcal{E}(\mathcal{O}_k) = \emptyset$ and $D_i \subseteq \mathcal{E}(\mathcal{O}_k)$, we have

$$\max(A, R) \cap D_i = A_i \cap S_k \cap D_i = \emptyset.$$

$\square$

Remark 3. The Coarse GARP property reduces to the GARP property in the special case that B_i is a singleton set for all i. Without loss of generality, we rewrite $\mathcal{O}$ as $\{(A_i, x_i, D_i)\}_{i=1}^n$. (a) (GARP $\implies$ Coarse GARP) Suppose that the Coarse GARP property is violated, then there exists $\mathcal{O}' = \{(A_{k_j}, x_{k_j}, D_{k_j})\}_{j=1}^m$ such that $B(\mathcal{O}') \setminus \mathcal{E}(\mathcal{O}') = \emptyset$. Therefore, for x_{k_j}, x_{k_j} is ranked strictly below another $x_{k_{j'}}$. Since this is true for all x_{k_j} and since $\mathcal{O}'$ is finite, we have a violation of the GARP property. (b) (Coarse GARP $\implies$ GARP) Suppose that the GARP property is violated, then we can find a sequence $\{x_{k_j}\}_{j=1}^m$ such that $x_{k_{j+1}} \in A_{k_j}$ with $l = 1, 2, \ldots, m-1$ and $x_{k_1} \in D_{k_m}$. Then for the subcollection of data points $\mathcal{O}' = \{(A_{k_j}, x_{k_j}, D_{k_j})\}_{j=1}^m$, we have $B(\mathcal{O}') \setminus \mathcal{E}(\mathcal{O}') = \emptyset$. Therefore, the Coarse SARP property is violated.

Next, we propose a simple algorithm to check whether a coarse data set is coarsely rationalizable by a weak order (or equivalently, whether the Coarse SARP condition is satisfied). The algorithm is a variant of the one we proposed in Section 3, where k and $\mathcal{O}'$ are variables.

Stratification Algorithm*.

Step 1. Set $k := 1$ and $\mathcal{O}' := \mathcal{O}$.

Step 2. Define $\mathcal{O}_k := \mathcal{O}'$. If $\mathcal{O}_k = \emptyset$, stop and output *Rationalizable*; otherwise, proceed to Step 3.

Step 3. Define $S_k := B(\mathcal{O}_k) \setminus \mathcal{E}(\mathcal{O}_k)$. If $S_k = \emptyset$, stop and output *Not Rationalizable*; otherwise, set $\mathcal{O}' := \{(A, B, D) \in \mathcal{O}_k : B \cap S_k = \emptyset\}$. Derive k' such that $k' = k + 1$. Set $k := k'$. Go to Step 2.

We now seek to identify for a given pair of alternatives x and y, whether one alternative is ranked above the other for every weak order that rationalizes $\mathcal{O}$. We say that x is surely weakly ranked above y, denoted by $xR^s y$, if for each weak order R that coarsely rationalizes $\mathcal{O}$, it holds that xRy. It is straightforward to see that R^s is transitive, since each weak order that coarsely rationalizes $\mathcal{O}$ is transitive. The definition for P^s is similar. We seek to identify the relation of $xR^s y$ and $xP^s y$. In what follows, we assume that the coarse data set $\mathcal{O}$ is coarsely rationalizable by a weak order.

To test $xR^s y$, we add the data point $(\{x, y\}, y, x)$ into the original coarse data set $\mathcal{O}$, and work with the new coarse data set $\mathcal{O}^* := \mathcal{O} \cup (\{x, y\}, y, x)$. If $\mathcal{O}^*$ is coarsely rationalizable by a weak order, then there exists a weak order that strictly ranks y above x and coarsely rationalizes the original coarse data set $\mathcal{O}$. As a result, the hypothesis that $xR^s y$ is rejected. If $\mathcal{O}^*$ is not coarsely rationalizable by a weak order, then every weak order R that coarsely rationalizes $\mathcal{O}$ necessarily satisfies that xRy. In other words, $xR^s y$. Similarly, to test whether $xP^s y$, we add the observation $(\{x, y\}, y, \emptyset)$ into the original coarse data set $\mathcal{O}$, and work with the new coarse data set $\mathcal{O} \cup \{(\{x, y\}, y, \emptyset)\}$.

References

Afriat, S. N. (1967): "The Construction of Utility Functions from Expenditure Data," *International Economic Review*, 8, 67–77.

Brown, D. J. and R. L. Matzkin (1996): "Testable Restrictions on the Equilibrium Manifold," *Econometrica*, 1249–1262.

Carvajal, A., R. Deb, J. Fenske, and J. K.-H. Quah (2013): "Revealed Preference Tests of the Cournot Model," *Econometrica*, 81, 2351–2379.

de Clippel, G. and K. Rozen (2018): "Bounded Rationality and Limited Datasets," Mimeo. Brown University.

Dekel, E., B. L. Lipman, and A. Rustichini (2001): "Representing Preferences with a Unique Subjective State Space," *Econometrica*, 69, 891–934.

Fishburn, P. (1976): "Representable Choice Functions," *Econometrica*, 44, 1033–1043.

Kreps, D. M. (1979): "A Representation Theorem for 'Preference for Flexibility'," *Econometrica*, 47, 565–577.

Nishimura, H., E. A. Ok, and J. Quah (2015): "A Comprehensive Approach to Revealed Preference Theory," Working paper version.

Salant, Y. and A. Rubinstein (2008): "(A,f): Choice with Frames," *Review of Economic Studies*, 75, 1287–1296.

Samuelson, P. (1938): "A Note on the Pure Theory of Consumer's Behaviour," *Economica*, 5, 61–71.

Savage, L. J. (1951): "The Theory of Statistical Decision," *Journal of the American Statistical Association*, 46, 55–67.

Sen, A. K. (1971): "Choice Functions and Revealed Preference," *Review of Economic Studies*, 38, 307–317.

Szpilrajn, E. (1930): "Sur l'extension de l'ordre partiel," *Fundamenta Mathematicae*, 16, 386–389.

Varian, H. R. (1982): "The Nonparametric Approach to Demand Analysis," *Econometrica*, 50, 945–973.