Uncertain Technology

Xiaohan Ma[*] Roberto Samaniego[†]

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Abstract

We develop a general equilibrium business cycle model with imperfectly observed neutral and investment-specific technology shocks. Agents are uncertain about the level of each, and learn from noisy signals. Estimated to match US macroeconomic dynamics, the model implies that neutral technology shocks generate more volatile responses than in an environment with perfect information, whereas investment-specific shocks have more persistent and less volatile impact. Noise in the signals also alters agents' behavior persistently, even when the underlying fundamentals are unchanged and the noise itself is not persistent. Implications for business cycle analysis are explored.

Keywords: Uncertainty, productivity shocks, investment specific technical change, Bayesian learning, noise, business cycles.

[*]Texas Tech University, Department Of Economics, P.O. Box 41014, Lubbock, TX 79409. E-mail: xiaohan.ma@ttu.edu. Declarations of interest: none.

[†]Department of Economics, The George Washington University, 2115 G Street, NW Monroe Hall 340, Washington, DC 20052. E-mail: roberto@gwu.edu. Declarations of interest: none.

1 Introduction

Business cycle research tends to assume that economic fundamentals evolve according to stochastic processes, but that their current values are correctly observed. The objective of this paper is to explore business cycle dynamics in an environment where the values of current fundamentals are observed *only with noise*.

In particular, business cycle research finds that, in addition to the neutral productivity shocks introduced in Kydland and Prescott (1982), *investment specific* productivity shocks are important drivers of macroeconomic dynamics.[1] This work generally assumes that the value of investment-specific productivity at any given date is known with certainty. However, there are reasons to believe that the presence of investment specific technological change (ISTC) introduces *uncertainty about economic fundamentals* into the economic environment. The ISTC-uncertainty link has not been explicitly addressed in the literature, yet the link is evident: consider the disagreements about how to measure the rate of ISTC or about how to measure the contribution of ISTC to growth among key references such as Hulten (1992), Greenwood et al (1997), Cummins and Violante (2002) and Whelan (2003).

If there is uncertainty about the extent of ISTC at any given point in time, agents will have to *learn* about the quality of the capital they use. For example, when the author editing this paragraph bought the Lenovo T560 on which these words were first typed, he had a rough idea of the quality of the machine and its software, based on product descriptions and reviews, but could only begin to have a more precise idea through *use*. If there are *other* imperfectly observed fundamentals as well, observed output will not be a sufficient statistic for current ISTC. Thus, the idea of uncertain ISTC raises a more general question about business cycle models: can uncertainty about *current economic fundamentals* (such as ISTC) affect the response of the economy to changes in those fundamentals?

To answer this question, this paper introduces the notion that ISTC is linked to uncertainty about current fundamentals, and studies the implications of this uncertainty in an otherwise standard business cycle model. We find that technological uncertainty of this kind has important implications for macroeconomic dynamics. In addition, we find that introducing technological uncertainty can resolve the well-known "curse" of Barro and King (1984).

To begin, we provide suggestive evidence of the link between uncertainty and ISTC by exploiting the fact that ISTC is well known to vary across industries.[2] We empirically test whether popular measures of time-varying aggregate uncertainty from the macroeconomic literature are related to measures of uncertainty *particularly* in industries with high rates of ISTC. We measure industry uncertainty using the realized forecast errors of analysts predicting firm-level earnings-to-share ratios twelve months ahead, as reported in the I/B/E/S database. Using a difference in differences specification, we indeed find that

[1] See for example Greenwood, Hercowitz and Krusell (2000), Fisher (2006) and Justiniano, Primiceri and Tambalotti (2010).

[2] See Cummins and Violante (2002) and Samaniego (2010) inter alia.

high macroeconomic uncertainty is related to analysts making disproportionately larger forecast errors for firms in industries with high rate of ISTC compared to other industries, implying that there is uncertainty particularly associated with ISTC.

We then develop a model where economic agents never perfectly observe the values of the fundamentals of the economy, which are investment specific productivity shocks and neutral productivity shocks. We use this specification for several reasons. First, these shocks are known to be important fundamentals that drive the business cycle, and as shown by the empirical evidence, investment specific technological change is related to uncertainty. Second, there must be at least *two* uncertain fundamentals for agents to be unable to infer their values based on observed output. If there were only neutral technology shocks, for example, GDP would be a sufficient statistic for the unseen level of productivity. On the other hand, when there is uncertainty regarding current and past values of *investment specific* productivity, we show that the capital stock measured in efficiency units is also unknown, which implies that GDP is no longer a sufficient statistic for either of the fundamentals. To our knowledge, this is the first paper to observe that uncertainty about the nature of technology can lead to uncertainty about the capital stock, and the first paper to explore the implications of such uncertainty for economic dynamics. In addition, the fact that the level of ISTC at any given date is learned about slowly introduces a trade-off between learning more about the current capital stock and investing in new capital of uncertain quality – a dynamic consideration that can only be studied in a model with fundamental uncertainty of this kind.

Since agents do not observe the values of economic fundamentals at any given point in time, their behavior is guided instead by their *beliefs* regarding fundamentals, and these beliefs change over time based on noisy signals the agents receive. One such signal is the observed value of GDP. The other is a noisy signal of the value of investment specific technological change (ISTC), which is revealed when agents operate the aggregate production technology to produce GDP. This signal is a function of the actual value of ISTC and a stochastic *noise* term which itself may lead to fluctuations in investment and output even if fundamentals remain unchanged. Since this noise represents a signal that fundamentals have changed when in fact they have not, we call it a noise shock or a "fake news" shock.

The model is difficult to solve. Agents' beliefs about fundamentals are continuous functions, and these beliefs are state variables of the economy.[3] Moreover, these beliefs will not all have analytically tractable expressions. However, we show that agent beliefs approximately follow a multivariate Gauss-Markov process after linearization. In addition, the discrepancy between agents' expectations of the unobservables and their actual values follows a reduced VAR system. Based on this VAR system we demonstrate that the extent of uncertainty (the variance of agent beliefs about fundamentals) settles down to a constant. Also, in the absence of noise shocks, beliefs converge to the correct values of the uncertain economic fundamentals

[3]The learning rule in our paper is similar to that in Sargent and Williams (2005) where Bayesian updating through the Kalman filter is studied. The process of signal extraction is similar to that introduced in Edge, Laubach, and Williams (2007) where agents are confused between shocks to the level or to the growth rate of the technology.

asymptotically.

Using this approximation strategy, we deliver several analytical results. First, a shock to any of the fundamentals or a noise shock generates uncertainty about *all* fundamentals in subsequent periods. This is because the impact of any change in information or any change in output cannot be attributed with certainty to any particular source. Second, if there is uncertainty about the value of any particular fundamental at a given point in time, in subsequent periods uncertainty will spread to *all* fundamentals, for the same reason: agents cannot attribute observed macroeconomic outcomes to any particular fundamental. Third, a "fake news" or noise shock has a persistent impact on beliefs, and hence on behavior, even if the shock itself is not persistent. Thus, the presence of noise in the economic environment is a potential source of additional persistence in macroeconomic variables.

We then estimate the critical structural and shock parameters that characterize economic dynamics and the learning process in the theoretical model, allowing us to quantify the macroeconomic impact of ISTC uncertainty and of learning. We use Bayesian techniques to match the time series of model-generated output and consumption to their counterparts in US data. We then use the estimated model to explore *quantitatively* the role beliefs play in the transmission mechanism of technology and noise shocks. We find that the economy with uncertain fundamentals responds more strongly to changes in neutral productivity (TFP) than a model without fundamental uncertainty. In contrast, the response to an ISTC shock is less volatile but more persistent. We also find that a "fake news" shock changes agents' behavior persistently, even though the underlying fundamentals governing the economy remain unchanged. This is because it takes time for fake beliefs to converge to the true values even if there is no further noise, and because noise changes the beliefs about *all* fundamentals (not just ISTC), since imperfect information implies that agents do not know whether observed outcomes are due to noise or to an actual change in one or other fundamental. Overall, we find that "fake news" shocks are a significant source of variation, and that their impact on beliefs is persistent even though these shocks decay rapidly. Based on the estimated persistence of signals, the half-life of noise is about 1.5 quarters (4.5 months). However, because of its persistent impact on *beliefs*, the half-life of the change in GDP induced by a noise shock is over 8 quarters (2 years). The quantity of noise is non-trivial: in the estimated economy, the noise shock accounts for 10% of variation in output and 41% of variation in investment. This finding echoes Blanchard et al. (2013), which empirically shows that noise shock explains a sizable portion of US business cycles – although in their case there is no ISTC. A key puzzle in the business cycle has long been that these models lack strong internal propagation mechanisms, relying on highly persistent shocks to generate reasonable macroeconomic dynamics,[4] such as productivity shocks with quarterly persitsence well over 0.9. In contrast, the propagation of shocks through persistent beliefs is so powerful that the model economy matches the dynamics of output and consumption with a persistence of only about 0.7.

[4]See for example Cogley and Nason (1995).

A key application of our model is to provide a solution to the Barro-King (1984) "co-movement" puzzle, the result that consumption and investment tend to have low or negative comovement in RBC models when shocks other than TFP shocks (in our model, the ISTC shock) are important driving forces of fluctuations. This is inconsistent with the highly *positive* correlation observed in US data. Our model shows that the introduction of *uncertainty* about the forms technological progress that the model already contains can solve this issue, without introducing additional complex model features as in the related literature.[5] This works through the mechanism that, under uncertainty, economic agents do not know for sure whether or not an ISTC shock has hit at any given moment in time, and always put some weight on the possibility that any unexpected changes in their observed signals could be due to changes in neutral productivity as well, or even simply just due to noise. The correlation between consumption and investment in US data is 0.74, whereas in the model economy it is not far off, at 0.62. When we remove technological uncertainty from the model, the correlation drops to 0.26 if we re-estimate the parameters of the model, or to -0.48 if we do not.

The idea that expectations about the state of the economy could drive economic fluctuations dates back at least to Pigou's (1927) theory of "errors of undue optimism or undue pessimism," regaining popularity in part after the 2000 tech-driven stock market decline. Jermann and Quadrini (2007) relate this event to beliefs about the advent of a "new economy" of higher productivity growth. Our paper contains a notion of unwarranted optimism or pessimism regarding the current value of *investment-specific productivity*, an ISTC noise or "fake news" shock, that fluctuates and may be present at any time. Lorenzoni (2009) and Angeletos and La'O (2013) develop models of uncertainty about fundamentals based on private information about productivity shock. However, those models do not have ISTC, nor do they provide independent evidence of the form of uncertainty they consider. These papers also hinge on agent heterogeneity as the source of uncertainty, whereas there is no heterogeneity in our model, which is close to standard business cycle models except for the dynamics of learning.[6] Saijo (2017) assumes informational frictions about the depreciation rate of capital and about ISTC, so that economic agents use the observed capital stock and investment to estimate the unobserved shocks. In our paper, uncertainty is qualitatively different as it originates from the unobservability of different types of productivity shock: as is well known, under full information, the optimal response of agents to neutral and to investment-specific shocks is very different. In addition, Saijo (2017) requires nominal rigidities and countercyclical markups as used in other papers to obtain a reasonable comovement of macroeconomic variables. In contrast, our model shows the confusion about technologies itself can generate positive comovement, without resort to other frictions.

[5] To resolve the comovement puzzle, Ascari et al (2016) employ roundabout production and trend growth in neutral and investment technology; Basu and Bundick (2017) introduce countercyclical markups and sticky prices; Chen and Liao (2018) extend the standard sticky-price model to a two-sector model with durable goods.

[6] Also in Lorenzoni (2009) an infinite train of beliefs is the state variable of the aggregate economy, so that solving the model requires truncation of histories at some point. Our model is more complex in that multiple fundamentals are not observed, yet simpler in that the model possesses the Markov property: a history of only one period is required.

Our model features a signal about future ISTC, with noise. The presence of noise bears superficial similarity to models that emphasize non-fundamental business cycles, e.g. due to animal spirits or self-fulfilling expectations as in Farmer and Guo (1994): however, in our environment there is a unique equilibrium, and business cycles that are not driven by fundamentals instead occur because of information frictions and the possibility of error. On the other hand, the fact that there is a signal about future ISTC – albeit a noisy one – is related to the literature on "news" – information about future fundamentals – such as Beaudry and Portier (2004) or Schmitt-Grohe and Uribe (2012). In those papers, however, the current state of the world is known. In our environment, as likely in reality,[7] the current values of fundamentals are not known: rather, they are estimated by economic actors under conditions of imperfect information. It is not known to what extent business cycle dynamics are sensitive to the presence of uncertainty about current fundamentals – nor whether shocks to the noise inherent to these signals could generate economic fluctuations – and our paper provides answers to these questions.

Finally, our paper is related to the extensive literature on the importance of TFP and ISTC for business cycle dynamics, such as Greenwood et al (2000), Fisher (2006) and Justiniano et al (2010). Unlike those papers, our focus is on how *uncertainty* about these fundamentals affects economic dynamics. Jaimovich and Rebelo (2007) and Görtz and Tsoukalas (2013) consider a two-state Markov switching process for the evolution of the investment-specific technology, however in those papers the current value of the technology is always known. Our paper, instead, assumes a continuum of possible states of investment technology, and the uncertainty is about the current value of ISTC itself, as well as TFP, which implies in addition that the value of the *capital stock* is not known. Beliefs are continuous, and the estimation process tells us how rapid and quantitatively important the impact of learning about fundamentals such as ISTC is in the economic environment. We also provide empirical evidence that industries where ISTC is important are subject to greater uncertainty about current fundamentals, and thus ISTC can serve as the source or conduit of uncertainty

Section 2 reports evidence that there is more uncertainty about conditions in industries where ISTC is high. Section 3 describes the model and environment. Section 4 develops the solution strategy and properties of beliefs. Section 5 presents the data and estimation methodology and the quantitative results. Section 6 concludes with a discussion of suggestive evidence, and suggests direction for future research.

[7] For example, it is a commonplace to mention in introductory macroeconomics textbooks that one problem with stabilization policy is the inability of the fiscal or monetary authority to know the exact position of "potential GDP" as opposed to current GDP. Baumol and Blinder (2011) liken stabilization policy to "a poor rifleman shooting through dense fog at an erratically moving target with an inaccurate gun and slow-moving bullets." This paper is about fog and moving targets. In a companion paper, Ma (2016) studies the rifleman (in the form of monetary policy).

2 Motivating Evidence

Our paper explores an economic environment in which there is uncertainty about the value of current fundamentals, modeled as neutral- and investment-specific technology shocks. It is the inability to observe these two independently that leads agents to be uncertain as to the value of each productivity. On the other hand, if ISTC were not relevant for production, then signals such as observed output would be a sufficient statistic for current fundamentals.

If there is variation in the economy in production functions regarding the importance of ISTC, we should expect to observe measures of uncertainty being particularly pertinent for firms in spheres of activity where ISTC is important. Cummins and Violante (2002) find that the rate of ISTC varies significantly across *industries*, based on industry variation in the types of capital used. Thus, we investigate whether there is support for a link between ISTC and uncertainty by exploring whether uncertainty is particularly related to the rate of ISTC across industries.

Specifically, we develop firm level measures of uncertainty, and ask whether the firm level uncertainty varies disproportionately for firms in high-ISTC industries compared to other industries when overall economic uncertainty is high. The correct specification should (1) condition on all date-related factors that might affect firm uncertainty, including aggregate uncertainty itself; (2) condition on all firm characteristics that might affect uncertainty, including ISTC itself; and (3) allow for autocorrelated errors, which are highly likely to be present in a high-frequency panel.

Our specification is:

$$FirmUncertainty_{it} = \alpha_i + \delta_t + \beta \times ISTC_{j(i)} \times Uncertainty_t + \varepsilon_{it} \tag{1}$$

where α_i is a firm-level fixed effect, t denotes a month, and $j\left(i\right)$ is the industry j where firm i operates. $FirmUncertainty_{it}$ is a measure of firm level uncertainty. The firm fixed effect α_i accounts for all firm-level factors (including industry-level factors) that might affect the firm level uncertainty. The time fixed effect δ_t captures all time-varying factors that might affect the firm level uncertainty, including cyclical factors such as aggregate uncertainty itself. The coefficient β then captures the interaction between ISTC as measured in the industry in which firm i operates, and time-varying aggregate uncertainty $Uncertainty_t$. Given the concern that the errors ε_{it} might be autocorrelated, we use the method of Baltagi and Wu (1999) that allows for AR(1) autocorrelation in the errors, as well as an unbalanced panel.

To measure time-varying aggregate uncertainty, we draw on several recent papers[8]:

1. Bachmann et al (2013) measure uncertainty using the extent of dispersion on up/down forecasts about future business conditions, drawn from the Federal Reserve Bank of Philadelphia's Business

[8] All of these measures are available at monthly frequency.

Outlook Survey. We call this measure $Dispersion_t$.

2. Jurado et al (2015) construct an uncertainty measure using the deviation (conditional variance) of forecasts from the sample mean of a large number of macroeconomic time series. We call this $Unpredict_t$.

3. Bachmann et al (2013) also construct a measure of uncertainty using the number of articles in Google News mentioning "uncertainty" divided by the number of articles containing the word "today". We call this measure $Google_t$.

4. Bloom (2009) measures uncertainty using the volatility of the stock market within a month. We call this measure $Stock_t$.

To measure firm level uncertainty, we draw on the Institutional Brokers' Estimate System or I/B/E/S, available through a WRDS subscription and managed by Thomson Reuters. I/B/E/S contains the forecasts by thousands of analysts about financial data for a large number of publicly traded firms. The most widely reported forecast in I/B/E/S is the 1-year ahead earnings per share (EPS) forecast, which is available from 1981 to 2017. We use EPS forecasts because they are the most widely available in our database, and also because EPS ratios are a basic indicator of the profitability of a share, and are thus widely understood and followed both by financial analysts and their clients. I/B/E/S also reports realizations of the forecast data, collected from a variety of public data sources. Companies are included in the database as long as at least one analyst provides a forecast for that company. Forecasts are collected each day as they are released by analysts.[9]

We define the uncertainty at the firm i as the absolute value of the forecast error, i.e. the difference between the forecasted and the realized EPS value one year later. We use this measure because a large ex-post observed forecast error indicates that the forecast was made under conditions of imperfect information. Of course forecasts about any particular firm will always contain errors, but what we are looking for are systematic differences in forecast errors by industry, using specification 1. These forecasted and realized EPS data are available at daily frequency (although it is not the case that forecasts about any particular firm are necessarily made on a particular day). To generate the monthly measure of firm level uncertainty ($FirmUncertainty_{it}$), we use the last observed value in each month for each firm, on the assumption that the latest forecast (within a month) was based on the most updated information set about the firm for that month. We drop the top 5 percent of values of $FirmUncertainty_{it}$ to avoid the influence of outliers. We also exclude financial institutions defined as firms with a 2-digit NAICS code of 52. We exclude financials for two reasons. First, forecasts of financial firms may not relate as much to real conditions (such as their

[9]For further details, see https://wrdsweb.wharton.upenn.edu/wrds/support /Data/_001Manuals%20and%20Overviews/_003I-B-E-S/Release%20Notes/

Table 1: Panel Regression Results

The table reports estimates from the following specification:

$$FirmUncertainty_{it} = \alpha_i + \delta_t + \beta \times ISTC_{j(i)} \times Uncertainty_t + \varepsilon_{it.}$$

Here $FirmUncertainty_{it}$ is the absolute forecast error made at date t about firm i; α_i is a firm fixed effect; δ_t is a time fixed effect; $ISTC_j$ is the rate of ISTC as measured in industry j, drawn from Cummins and Violante (2002); $j(i)$ is the industry where firm i operates; and $Uncertainty_t$ is a measure of uncertainty at date t. The errors ε_{it} may be autocorrelated. Two and three asterisks represent statistical significance at the five and one percent levels respectively. Sources: I/B/E/S database, authors' calculations, Bloom (2009), Bachmann et al (2013) and Jurado et al (2015).

	Uncertainty measure			
Statistic	$Dispersion_t$	$Unpredict_t$	$Google_t$	$Stock_t$
Coefficient β	6.52***	15.9***	.0122***	.0808***
S.e.	(.258)	(.561)	(.00057)	(.00433)
Obs	446,621	523,849	427,640	446,621
Groups	8,173	9,176	8,044	8,173

rate of ISTC) as opposed to their operations that ultimately depend on claims on firms in other industries inherent in their asset holdings. Second, Ma and Samaniego (2018) find that uncertainty measured in the financial sector appears to lead uncertainty measured elsewhere. Since financials have high ISTC, we do not wish our results concerning ISTC to be contaminated by the special properties of the financial sector.

To measure ISTC, we employ the industry-level estimates of ISTC constructed from Cummins and Violante (2002). The capital flow tables available at the time allow us to distinguish between 62 private industries.[10] We use Compustat to assign an NAICS industry code to each firm in I/B/E/S, and construct a crosswalk between NAICS codes and the 62 industries in CV. We define $ISTC_j$ as the rate of ISTC reported for industry j.[11]

The estimation results are displayed in Table 1. All the coefficient estimates are positive and very highly significant, regardless of which aggregate uncertainty measure we use. Based on these results, we conclude that overall economic uncertainty, as illustrated by the standard measures of aggregate uncertainty available in the literature, is particularly related to uncertainty in high-ISTC industries. This empirical finding motivates the notion of uncertainty regarding ISTC in our theoretical model.

We also employ a different empirical approach to demonstrate our motivation. If uncertainty about

[10] Cummins and Violante (2002) measure the decline in the relative price of capital for 28 types of equipment and machinery over the period $1970-2000$. Then, they use the capital flow tables from the BEA to construct an industry-specific measure based on the share of investment in each type of capital good. They report 63 industries, one of which is "Federal Reserve Banks."

[11] Compustat reports NAICS codes for each firm at a level of aggregation varying between $2-$ and $6-$digit. If we were unable to assign a firm to one of the 63 CV industries we discarded that firm. We end up with data for 57 industries, since we exclude financials.

ISTC is a significant source of macroeconomic fluctuations, then uncertainty measured in high-ISTC industries should have a more significant impact on aggregates. We use the approach of Ma and Samaniego (2018) to develop two measures of uncertainty: one for firms in high-ISTC industries (defined as those with a reported rate of ISTC above the median) and one for firms in low-ISTC industries. In a standard recursively identified vector autoregression (VAR) estimation, we indeed find that high-ISTC uncertainty has a more significant impact on aggregates than uncertainty measured among low-ISTC firms. The details of this exercise are shown in Appendix 1.

3 Economic Environment

In the description of the economic environment we will focus on the social planner's problem. This way there are no inefficiencies nor informational problems arising from the choice of decentralization strategy, only from the information structure of the economy. The social planner suffers from the same informational frictions as the agents.

3.1 Preferences and Technology

The social planner maximizes the discounted expected utility of a representative agent. Output $\tilde{Y}_t$ is produced using an aggregate technology

$$\tilde{Y}_t = A_t e^{z_t} \tilde{K}_t^{\alpha} n_t^{1-\alpha}$$

where $\tilde{K}_t$ is capital, $n_t \in [0,1]$ is labor and $A_t e^{z_t}$ is neutral productivity. As discussed below, A_t captures the trend in neutral productivity whereas z_t captures temporary innovations.

Output may be consumed (C_t) or invested (I_t), so that $\tilde{Y}_t \geq C_t + I_t$. Agents earn utility $U(C_t, 1-n_t)$ from consumption and work.

The planner's problem is

$$
\begin{aligned}
& \max E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, n_t) \\
& s.t. \\
C_t + I_t \quad & \leq \quad A_t e^{z_t} \tilde{K}_t^{\alpha} n_t^{1-\alpha} \\
\tilde{K}_{t+1} \quad & = \quad B_t e^{q_t} \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t + (1-\delta)\tilde{K}_t \\
A_t \quad & = \quad e^{\gamma t} \qquad B_t = e^{\gamma_q t} \\
z_{t+1} \quad & = \quad \psi z_t + \varepsilon_{t+1} \qquad q_{t+1} = \rho q_t + w_{t+1} \\
& \varepsilon_{t+1} \tilde{} N(0, \sigma_\varepsilon), \epsilon_{t+1} \tilde{} N(0, \sigma_w), |\psi| < 1, |\rho| < 1.
\end{aligned}
$$

There is a cost associated with the adjustment of investment, captured by $S(\cdot)$, with $S(0) = 0$, $S'(\cdot) \geq 0$, $S''(\cdot) > 0$.

Notice there are two stochastic terms, z_t and q_t. The evolution of the capital stock is affected by investment specific technological change (ISTC), q_t. Variables A_t and B_t are respectively the deterministic trend of TFP and of ISTC. Variables z_t and q_t are respectively the TFP shock and ISTC shock, both of which follow persistent AR(1) processes. Variables ε_t and w_t are independent disturbances to the TFP and ISTC terms respectively. The capital depreciation rate is δ.

In what follows, we assume that:

$$
U(C_t, n_t) = \log(C_t) - \xi n_t.
$$

We reformulate the variables of the economy in terms of deviations from a balanced growth path. Specifically, we divide $\tilde{K}_t$ by its growth factor e^{g_k} to obtain the detrended capital stock $k_t = \frac{\tilde{K}_t}{e^{g_k t}}$. We also divide C_t I_t and $\tilde{Y}_t$ by their common growth factor e^g to get detrended series $c_t = \frac{C_t}{e^{gt}}, i_t = \frac{I_t}{e^{gt}}, y_t = \frac{\tilde{Y}_t}{e^{gt}}$, where $g = \frac{\gamma + \alpha\gamma_q}{1-\alpha}$ and $g_k = \frac{\gamma + \gamma_q}{1-\alpha}$ – see Greenwood et al (1997) and Fisher (2006).

Assumption 0 $\gamma_q > 0$ and $\gamma \geq -\alpha\gamma_q$.

Assumption 0 is a condition that is necessary and sufficient for there to be both economic growth and investment specific technological progress in this model $\forall \alpha \in (0,1)$.

After detrending and rewriting the utility function in terms of detrended variables, the agent's problem becomes:

$$
\max E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, n_t)
$$

subject to the feasibility constraint

$$c_t + i_t = e^{z_t} k_t^{\alpha} n_t^{1-\alpha}$$

and the capital accumulation equation

$$k_{t+1} e^{g_k} = e^{q_t} \left[1 - S\left(e^g \frac{i_t}{i_{t-1}} \right) \right] i_t + (1-\delta) k_t \tag{2}$$

where $S\left(e^g \frac{i_t}{i_{t-1}}\right) \equiv \frac{\varphi}{2}(e^g \frac{i_t}{i_{t-1}} - e^g)^2$ as in the literature. $\varphi \geq 0$ is the investment adjustment cost parameter.

The stochastic processes become

$$z_{t+1} = \psi z_t + \varepsilon_{t+1} \qquad q_{t+1} = \rho q_t + w_{t+1}.$$

Remark Notice that an implication of this change of variables is that the cyclical behavior of the model will not depend on the values of the productivity trends γ and γ_q, regardless of whether or not there is uncertainty regarding the current value of the cyclical terms q_t or z_t.

3.2 Imperfect Information

In our paper, there is imperfect information about the values of z_t and q_t. We capture this as follows. The true data generating processes for z_t and q_t are exogenous and known to the agents. However, neither agents nor the social planner observe the realized values of q_t or z_t. As we shall see, this implies that they do not know the capital stock either since in equation (2) the evolution of the capital stock depends on the value of q_t, which is unknown. Notice that this is not an assumption: it is a *consequence* of not observing past values of q_t. Instead, agents observe noisy signals regarding these shocks, which contain information about the unobserved fundamentals. The planner's decisions thus depend on expectations of future productivity and on *beliefs* about their current values. Agents develop posterior beliefs about the values of these fundamentals based on their prior beliefs, and on signals they receive, according to Bayes' rule.

We consider the case of fully Bayesian learning in the sense that the agents update their beliefs about the distributions of TFP, ISTC and the capital stock as new information arrives, and make current decisions knowing that their beliefs may be further updated in the future. In other words, the agents and the social planner take into account how future realizations of signals may alter their beliefs. This is implemented by allowing the beliefs to be a state variable of the economy, and by allowing the evolution of the state to be common knowledge, so agents understand how future signals will affect their beliefs.

The timing of events is as follows. At the beginning of each period t, TFP and ISTC shocks are realized, but are not observed by the social planner. Once labor input is chosen, labor and capital are introduced into the production technology, and output is realized and observed. Thus, one signal is the realized output y_t itself, which is a function of the realized but unobserved TFP level z_t, the unobserved capital stock measured in efficiency units k_t, and labor input n_t:

$$y_t = e^{z_t} k_t^{\alpha} n_t^{1-\alpha} \tag{3}$$

Since neither z_t nor k_t are perfectly observed, y_t is not a sufficient statistic for either.

The other signal is a noisy signal ϕ_t regarding the value of the current ISTC term q_t. It is a combination of the past signal ϕ_{t-1}, the unobserved realization of ISTC q_t and an iid disturbance v_t :

$$\phi_t = \pi\phi_{t-1} + (1-\pi)q_t + v_t, v_t\tilde{\ } N\left(0, \sigma_v\right), \pi \in [0, 1). \tag{4}$$

Under this formulation the signal ϕ_t is a combination of information and noise. As long as $\pi \in [0, 1)$, the signal contains some information regarding the current value of q_t. However, as long as $\sigma_v > 0$, the signal also contains noise. We refer to the disturbance v_t as a *noise shock*. In so doing, we use the term "noise" as "random or irregular fluctuations or disturbances which are not part of a signal ... or which interfere with or obscure a signal. (Oxford English Dictionary). However, since it also represents an element of a signal that is misleading or false, and to distinguish it from other notions of noise in the literature, we also refer to it as a "fake news" shock. The noise (or fake news) shock v_t is uncorrelated with the technological errors ε_t and ϵ_t.

The potential persistence in the signal is very important. On the one hand, the case in which $\pi = 0$ (no persistence) is interesting for its theoretical implications, because any propagation of noise shocks v_t in an environment with $\pi = 0$ will occur purely through the dynamics of the learning process. On the other hand, if we impose that $\pi = 0$, then in any quantitative implementation of the model the role of uncertainty will be closely tied to the length of a period. In addition, the extent to which the signal is informative $(1-\pi)$ is ultimately an empirical matter of independent interest that the calibration or estimation of the model can speak to. Thus we allow for the general case $\pi \in [0, 1)$, so information may be revealed gradually.

One interpretation of the disturbance v_t is that it is related to market sentiment. For example, if agents are overoptimistic about ISTC because $v_t > 0$, they would believe to some extent that it is the increase of q_t that leads to the increase of ϕ_t, even when this is not the case. As a result, agents might react by increasing investment, which could affect dynamics as agents gradually learn that q_t had not in fact changed. Such a fluctuation in economic activity would not involve any change in fundamentals. However, we refrain from using the term "sentiment shock" (as in Angeletos and La'O (2013), for example) because the nature of uncertainty in this model is different. In particular, in our paper the *value of the current ISTC shock* is imperfectly observed. The shock is about a perceived (but non-existent) change in the value of a fundamental, in this case specifically ISTC. This is why we refer to it as a "fake news" shock.

3.3 Beliefs

At the beginning of each period t, prior beliefs regarding the distribution of TFP, ISTC and the capital stock, denoted respectively as h_t^Z, h_t^Q and h_t^K, are taken as given. For the rest of the paper, $m(\cdot)$ denotes posterior beliefs, whereas $h(\cdot)$ denotes prior beliefs. Upper case variables such as Q, Z, K and Y denote random variables, whereas lower case variables such as q, z, k, y denote the realized values of the corresponding random variables. Values of z_t and q_t are drawn, but are not observed by the planner nor the agents. The capital stock in period t is a given quantity k_t, determined by investment and past realizations of ISTC, however agents do not know this quantity either.

We assume that the choice of labor is made in period t before observing any signals. This "time to build" assumption on labor is necessary due to the nature of the uncertainty in this economy: output itself is a signal of the value of fundamentals, and output cannot be observed until after the labor input is used in production.

After observing the two signals, the agents and the planner update their beliefs about z_t, q_t and k_t, following Bayes' rule, given their prior beliefs:

$$m_t^{Z,Q,K}(z, q, k|y_t, \phi_t) \propto h(y_t, \phi_t|z, q, k)h_t^{Z,Q,K}(z, q, k)$$

where $m_t^{Z,Q,K}$ is the joint posterior belief about z_t, q_t and k_t, and $h_t^{Z,Q,K}$ is the joint prior belief about z_t, q_t and k_t. Function $h(y_t, \phi_t|z, q, k)$ is the likelihood of y_t and ϕ_t conditional on z_t, q_t and k_t.

In addition, using the known stochastic process for TFP and ISTC, the planner can derive a prior belief h_{t+1}^Z about TFP for period $t+1$, and a prior belief h_{t+1}^Q about ISTC for period $t+1$. Knowing the true evolution process of the capital stock, the planner can also derive a prior belief for capital in period $t+1$, h_{t+1}^K, given the choice of i_t, which is (ex-ante) optimally chosen based on expectations of how these choices affect output in period $t+1$. The expectation of output y_{t+1},(the likelihood of a given value of output conditional on prior beliefs about the unobserved variables and on the choice variables for period $t+1$ $h(y_{t+1}|z, q, k)$) are generated according to the known production function (3). Similarly, prior beliefs about ϕ_{t+1} are derived from posterior beliefs about ϕ_t and the known stochastic processes for ϕ, q and v. The calculation of the updating of beliefs is shown in Appendix 3.

3.3.1 Social planner's problem

To summarize, in each period t the planner's problem is to:

1. Choose n_t before observing signals y_t and ϕ_t. We find it convenient to characterize this behavior with the use of two value functions, $V\left(h_t^Z, h_t^Q, h_t^K\right)$ and $W\left(m_t^Z, m_t^Q, m_t^K|\phi_t, y_t; n_t\right)$. V is the agent's expected discounted utility in period t before observing signals, whereas W is the value function after observing signals and updating beliefs accordingly. Then,

$$V\left(h_t^Z, h_t^Q, h_t^K\right) = \max_{n_t} \int\int W\left(m_t^Z, m_t^Q, m_t^K|\phi_t, y_t; n_t\right) h_t^{Y,Q}(y, \phi|h_t^Z, h_t^Q, h_t^K, n_t)dyd\phi \tag{5}$$

2. Choose consumption and investment to maximize utility after observing y_t and ϕ_t. This decision can be specified as a dynamic programming problem.

$$\begin{aligned} W\left(m_t^Z, m_t^Q, m_t^K|\phi_t, y_t; n_t\right) &= \max_{c_t, i_t} \left\{U(c_t, n_t) + \beta E_t V\left(h_{t+1}^Z, h_{t+1}^Q, h_{t+1}^K\right)\right\} \\ & s.t. \\ c_t + i_t &= y_t \\ h_{t+1}^K(k') &= \frac{e^{g_k}}{(1-\delta)} \int_{q=0}^{\infty} m_t^K \left(\frac{e^{g_k}k' - e^q\left[1 - S\left(e^g \frac{i_t}{i_{t-1}}\right)\right] i_t}{(1-\delta)}\right) m_t^Q dq \end{aligned} \tag{6}$$

and where $h_{t+1}^Z(z')$ and $h_{t+1}^Q(q')$ follow the conjugate calculation detailed in Appendix 3.

In equation (6), functional $E_t(\cdot)$ is the rational expectation based on prior beliefs pre-determined at time t. The first constraint is the budget constraint. The second captures the evolutions of beliefs. Note that agents understand how information revealed in this period will be taken into account in the future and, given the structure of the value function, they also take into account how information revealed next period will be taken into account subsequently.

3.3.2 Optimization conditions for V

The first order condition for n_t is:

$$\begin{aligned} & \int W_y\left(m_t^Z, m_t^Q, m_t^K|\phi_t, y_t; n_t\right) \frac{\delta h_t^Y(y, \phi|h_t^Z, h_t^K, n_t)}{\delta n_t} dy + \\ & \int \frac{\delta W\left(m_t^Z, m_t^Q, m_t^K|\phi_t, y_t; n_t\right)}{\delta n_t} h_t^Y(y, \phi|h_t^Z, h_t^K, n_t)dy \\ = & \int U_c(c_t, n_t) \frac{\delta h_t^Y}{\delta n_t} dy + \frac{\delta W\left(m_t^Z, m_t^Q, m_t^K|\phi_t, y_t; n_t\right)}{\delta n_t} = 0 \end{aligned}$$

The Envelope condition for n_t is

$$\frac{\delta W\left(m_t^Z, m_t^Q, m_t^K|\phi_t, y_t; n_t\right)}{\delta n_t} = U_n(c_t, n_t)$$

Combining them, we obtain the optimization condition for n_t

$$\int [U_n(c_t,n_t) + \beta U_c(c_t,n_t)\frac{\delta h_t^Y}{\delta n_t}]dy = 0$$

Here the planner chooses labor input given her beliefs, knowing that the marginal utility of labor will depend on the choice of consumption, which will depend on how much output is realized as well as on any other signals.

Next, the first order condition for c_t is

$$-U_c(c_t,n_t) + \frac{\delta \beta E_t V\left(h_{t+1}^Z, h_{t+1}^Q, h_{t+1}^K\right)}{\delta c_t} = 0$$

which balances the marginal utility of consumption now against the expected value of investing in new capital, given beliefs. The Envelope condition for c_t is then

$$\begin{aligned}
\frac{\delta V\left(h_t^Z, h_t^Q, h_t^K, n_{t-1}\right)}{\delta c_{t-1}} &= \int W_y\left(m_t^Z, m_t^Q, m_t^K|\phi_t, y_t; n_t\right)\frac{\delta h_t^Y(y_t|h_t^Z, h_t^Q, h_t^K, n_{t-1})}{\delta h_t^K}\frac{\delta h_t^K}{\delta c_{t-1}}dy \\
&+ \int \frac{\delta W\left(m_t^Z, m_t^Q, m_t^K|\phi_t, y_t; n_t\right)}{\delta m_t^K} h_t^Y(y_t|h_t^Z, h_t^Q, h_t^K, n_{t-1})\frac{\delta m_t^K}{\delta c_{t-1}}dy \\
&= \int [U_c(c_t,n_t)\frac{\delta h_t^Y}{\delta h_t^K}\frac{\delta h_t^K}{\delta c_{t-1}} + U_c(c_t,n_t)\frac{\delta c_t}{\delta m_t^K}\frac{\delta m_t^K}{\partial c_{t-1}}h_t^Y]dy
\end{aligned}$$

Combining them, we get the Euler equation for c_t

$$-U_c(c_t,n_t) + \beta\int U_c(c_{t+1},n_{t+1})[\frac{\delta h_{t+1}^Y}{\delta h_{t+1}^K}\frac{\delta h_{t+1}^K}{\delta c_t} + \frac{\delta c_{t+1}}{\delta m_{t+1}^K}\frac{\delta m_{t+1}^K}{\delta c_t}h_t^Y]dy = 0$$

where $\delta()$ denotes a functional derivative and $\partial()$ denotes a univariate derivative. The marginal cost of investment in period t in terms of consumption goods equals to the expected marginal benefit of capital in period $t+1$, due to an increase in expected future output and in the expected future capital stock.

Finally, given the adjustment cost function, the first order condition for investment i_t is

$$\int \mu_t\left[1 - S\left(e^g\frac{i_t}{i_{t-1}}\right) - S'\left(e^g\frac{i_t}{i_{t-1}}\right)e^g\frac{i_t}{i_{t-1}}\right]m_t^Q dq + \beta\int\frac{U_c(c_{t+1},n_{t+1})}{U_c(c_t,n_t)}\mu_{t+1}S'\left(e^g\frac{i_{t+1}}{i_t}\right)\left(e^g\frac{i_{t+1}}{i_t}\right)^2 h_{t+1}^Q dq = 1$$

where μ_t can be interpreted as the value of installed investment in terms of its cost, and is given by

$$\mu_t = \beta\int\frac{U_c(c_{t+1},n_{t+1})}{U_c(c_t,n_t)}\left((1-\delta)\mu_{t+1} + \frac{\delta h_{t+1}^Y}{\delta h_{t+1}^K}\right)h_{t+1}^Y dy$$

3.4 Implications of uncertainty

To shed light on the impact of uncertainty on economic behavior, we can compare the optimal conditions derived above for our model with those of a standard real business cycle model without uncertainty – i.e. a model where $v_t = 0$, $\pi = 0$ and the values of z_t and q_t are known before the choice of n_t.

For example, the optimality conditions for consumption in a standard RBC framework are

$$-U_c(c_t, n_t) + \beta \int U_c(c_{t+1}, n_{t+1})[\frac{\partial Y_{t+1}}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial c_t} + \frac{\partial c_{t+1}}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial c_t}]dy = 0$$

In contrast, under the setup with uncertainty, we have

$$-U_c\left(c_t, n_t\right) + \beta \int U_c\left(c_{t+1}, n_{t+1}\right)[\frac{\delta h^Y_{t+1}}{\delta h^K_{t+1}} \frac{\delta h^K_{t+1}}{\delta c_t} + \frac{\delta c_{t+1}}{\delta m^K_{t+1}} \frac{\delta m^K_{t+1}}{\delta c_t} h^Y_t]dy = 0$$

When there is no uncertainty, an increase in investment (a decrease in consumption) in the current period increases the capital stock unambiguously, which in turn increases future output. However, when there is uncertainty regarding the values of the technology shocks and of the capital stock, the effect of an increase in current investment in the current period is uncertain, as there are non-degenerate beliefs about fundamentals. Thus agents will be less likely to act on a given piece of information about z_t or q_t.

This would seem to suggest that macroeconomic variables are likely to be *less* volatile in an environment with uncertainty of our kind. There are countervailing effects however. First, we have an additional source of variation, the noise shock. Second, when a shock to a fundamental occurs, agents' beliefs are unlikely to have the correct value of fundamentals in expectation due to past shocks. Thus, for example, if agents believe productivity is high whereas it is actually low, a transition back to the steady state could involve beliefs first converging to the correct value before returning to the steady state, possibly leading to a boom-bust cycle as a response to a single shock. Thus it is not a priori clear whether volatility will be enhanced or ameliorated by the introduction of uncertainty. Indeed, Collard et al (2009) argue that uncertainty has an ambiguous effect on macroeconomic volatility in environments such as ours where agents solve a signal extraction problem. In fact, below we show that the impulse responses of the model suggest the presence of more volatility when agent beliefs are very different from the correct value at the time of shock impact. At the same time, agents' caution in the face of uncertainty leads the model to display *less* volatility on average.

4 Solution method

In our model, agents' prior beliefs about the values of z_t, q_t and k_t are state variables of the economy. In general these beliefs are continuous functions and, as is well known in the literature, it is generally difficult to solve problems where state variables include continuous functions.

In our case, calculating the evolution of beliefs following Bayesian learning requires a nonlinear filter. If the prior and the likelihood function are of the exponential class, then the prior and posterior might be conjugates with the appropriate choice of distributions, and there will be an analytical solution to the posterior resulting from Bayesian learning. However, for the updating of the distribution of capital to be explicitly solvable, the prior and posterior distributions would both have to be of the exponential class. For example, if we assume the prior distribution is exponential, since the posterior investment technology has to be log-normal given the shock process is normal, the belief about the sum of depreciated past capital and new investment using the uncertain investment technology does not belong to the exponential class anymore. The same argument applies to other conjugate pairs: to our knowledge there is no known analytical class of probability distributions that spans the positive reals and is robust to summation of a non-negative random variable from the same or another distribution, certainly not among the distributions thought to relevant in this context. In addition, since the impact of the agents' or the planner's decision regarding investment is multiplied by an unknown q and added to an unknown k, and since agents (or the planner) must take this into account in their decision making, it implies that the solution of the model requires the computation of the derivative of one function, such as the utility function, with respect to the distribution of capital, another continuous function. Functional derivatives would be required to solve the model, leading to an intractable solution given that beliefs about the capital stock themselves do not have an analytical solution.

We adopt a methodology to overcome these complications while keeping the main structure of the model. The method is based on a log-linear approximation using a Taylor expansion, which allows us to consider the evolution of beliefs about linearized variables rather than that of the original variables. Through the log-linearization, we are able to transform the Bayesian learning problem into a linear Gauss-Markov process under certain initial conditions, and implement the computation of the learning process using the Kalman filter. In this way, the evolution of beliefs can be parameterized such that *means and variances* enter in the decision-making, rather than the entire distribution.

We later show that the conditional variances of the unobservables (prediction errors) in the approximation converge to positive constants, and thereafter the evolution of the variables in the model depend only on these constants and on the mean beliefs about the capital stock, TFP and ISTC. We will focus on the economic implications of the model when beliefs about these conditional variances are stable.

4.1 Transformation of beliefs

First, some notation:

1. If X_t is the true value of X at period t, then $X_{t|t-1}$ and $X_{t|t}$ denote respectively the prior and posterior means of X at period t

2. $\Sigma_t^K, \Sigma_t^Z, \Sigma_t^Q$ are respectively the prior covariances of capital, TFP and ISTC at period t, while $\sigma_t^K, \sigma_t^Z, \sigma_t^Q$ are respectively their posterior variances.

3. For any variable X_t, define $X_t \equiv \bar{X} e^{\tilde{X}_t}$, so $\tilde{X}_t = \log X_t - \log \bar{X}$, and $\bar{X}$ is the steady state value of variable X. Similarly, $\tilde{X}_{t|t-1} = \log X_{t|t-1} - \log \bar{X}$ and $\tilde{X}_{t|t} = \log X_{t|t} - \log \bar{X}$

Then, we can rewrite the production function (3) and the signal process (4), and capital accumulation equation (2) as a linear system:

$$\begin{aligned}
\tilde{y}_t &= z_t + \alpha \tilde{k}_t + (1-\alpha)\tilde{n}_t \\
\tilde{\phi}_t &= \pi \tilde{\phi}_{t-1} + (1-\pi) q_t + \upsilon_t \\
\tilde{k}_t &= \frac{(1-\delta)}{e^{g_k}} \tilde{k}_{t-1} + \frac{\bar{\imath}}{\bar{k} e^{g_k}} \tilde{\imath}_{t-1} + \frac{\bar{\imath}(1+\beta) e^{2g} \varphi}{\bar{k} e^{g_k}} q_{t-1}
\end{aligned}$$

Together with the stochastic processes for TFP and ISTC:

$$\begin{aligned}
z_{t+1} &= \psi z_t + \varepsilon_t, \varepsilon_t \sim N(\mu_\varepsilon, \sigma_\varepsilon^2) \\
q_{t+1} &= \rho q_t + w_t, w_t \sim N(\mu_w, \sigma_w^2)
\end{aligned}$$

We can write a discrete time, linear and time varying state space system state space system of the original system, where the *measurement equations* are:

$$\begin{pmatrix} \tilde{y}_t - (1-\alpha)\tilde{n}_t \\ \tilde{\phi}_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1-\pi & 0 \end{pmatrix} \cdot \begin{pmatrix} z_t \\ q_t \\ \tilde{k}_t \end{pmatrix} + \begin{pmatrix} \pi & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\phi}_{t-1} \\ \tilde{\imath}_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \upsilon_t \end{pmatrix}$$

The *state equations* are:

$$\begin{pmatrix} z_t \\ q_t \\ \tilde{k}_t \end{pmatrix} = \begin{pmatrix} \psi & 0 & 0 \\ 0 & \rho & 0 \\ 0 & \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}} & \frac{(1-\delta)}{e^{g_k}} \end{pmatrix} \cdot \begin{pmatrix} z_{t-1} \\ q_{t-1} \\ \tilde{k}_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{\bar{\imath}}{\bar{k}e^{g_k}} \end{pmatrix} \cdot \begin{pmatrix} \tilde{\phi}_{t-1} \\ \tilde{\imath}_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ w_t \\ 0 \end{pmatrix}$$

or in matrix notation

$$\begin{aligned}
\mathbf{s}_t &= \mathbf{H}\mathbf{x}_t + \mathbf{B}\boldsymbol{y}_t + \boldsymbol{\eta}_t \\
\mathbf{x}_{t+1} &= \mathbf{F}\mathbf{x}_t + \mathbf{G}\mathbf{y}_t + \boldsymbol{\omega}_t
\end{aligned}$$

where $\mathbf{s}_t = \left\{ \tilde{y}_t - (1-\alpha)\tilde{n}_t, \tilde{\phi}_t \right\}$ is termed the observation vector, $\mathbf{x}_t = \left\{ z_t, q_t, \tilde{k}_t \right\}$ is the state vector, $\mathbf{y}_t = \left\{ \tilde{\phi}_{t-1}, \tilde{\imath}_{t-1} \right\}$ includes the observed variables, $\boldsymbol{\eta}_t = \{0, v_t\}$ is the shock vector on the signals, which follows $\mathbf{N}(\mathbf{0}, \mathbf{R})$; and $\boldsymbol{\omega}_t = \{\varepsilon_t, w_t, 0\}$ is the shock vector on the state variables, which follows $\mathbf{N}(\mathbf{0}, \mathbf{Q})$. $\mathbf{R}$ and $\mathbf{Q}$ are by assumption constant over time, as are matrices $\mathbf{H}$, $\mathbf{B}$, $\mathbf{F}$ and $\mathbf{G}$.

In order to obtain a Gauss-Markov process, we make assumptions on the initial values of the random variables:

Condition 1 *The initial TFP shock z_{-1} is a random Gaussian variable, independent of the noise processes, with $z_{-1} \sim N(z_{0|-1}, \Sigma_0^Z)$*

Condition 2 *The initial ISTC shock q_{-1} is a random Gaussian variable, independent of the noise processes, with $q_{-1} \sim N(q_{0|-1}, \Sigma_0^Q)$*

Condition 3 *The initial capital $\tilde{k}_{-1}$ is a random Gaussian variable, independent of the noise processes, with $\tilde{k}_{-1} \sim N(\tilde{k}_{0|-1}, \Sigma_0^{\tilde{K}})$*

The first two conditions are not restrictive from the perspective of the literature. The shock processes for neutral and investment specific productivity are generally modeled as being log normal. The third is the key assumption for enabling the approximation procedure implemented in what follows of the paper. Given these conditions and the assumption for the shock processes, it is straightforward to show that the process of beliefs $\{z_t, q_t, \tilde{k}_t\}$ and the process of signals $\{\tilde{y}_t - (1-\alpha)\tilde{n}_t, \tilde{\phi}_t\}$ are (jointly) Gaussian as well. The proof is in the Appendix 3. In addition, given the assumption that $\{v_t\}$ is white noise and independent of initial values of z_{-1}, q_{-1} and $\tilde{k}_{-1}$, the vector $\{z_t, q_t, \tilde{k}_t\}$ becomes a Markov process, which can be characterized using the Kalman filter. Thus, means and covariances are sufficient for the characterization of the linearized model.

4.2 Evolution of beliefs

Consider period t, after the signals y_t and $\tilde{\phi}_t$ regarding z_t, q_t, and $\tilde{k}_t$ are observed. Combined with the prior beliefs regarding these fundamentals, the social planner can derive posterior beliefs about z_t, q_t and $\tilde{k}_t$ using Bayes' rule. Then, given posterior beliefs about z_t, q_t and $\tilde{k}_t$, the social planner can derive prior beliefs for z_{t+1}, q_{t+1} and $\tilde{k}_{t+1}$, using the state updating process. This Bayesian learning process is implemented using a discrete time Kalman filter, which is the optimal minimum mean square error (MMSE) state estimator of the uncertain state variables, under Conditions 1-3 and given the log-linear approximation[12].

The evolution of the means of posterior beliefs about TFP, ISTC and the capital stock, and their joint posterior covariances are described by[13]:

[12] See, for example, Chen (2003).

[13] The detailed derivations and calculations of prior and posterior beliefs are shown in Appendix 2.

$$
\begin{aligned}
\mathbf{x}_{t|t} &= \mathbf{x}_{t|t-1} + \boldsymbol{\Sigma}_t \mathbf{H}^T (\mathbf{H}\boldsymbol{\Sigma}_t \mathbf{H}^T + \mathbf{R})^{-1} (\mathbf{s}_t - \mathbf{B}\boldsymbol{y}_t - \mathbf{H}\mathbf{x}_{t|t-1}) && (7) \\
\boldsymbol{\sigma}_t &= \boldsymbol{\Sigma}_t - \boldsymbol{\Sigma}_t \mathbf{H}^T (\mathbf{H}\boldsymbol{\Sigma}_t \mathbf{H}^T)^{-1} \mathbf{H}\boldsymbol{\Sigma}_t && (8)
\end{aligned}
$$

In addition, the updating process for prior beliefs and covariances are:

$$
\begin{aligned}
\mathbf{x}_{t+1|t} &= \mathbf{F}\mathbf{x}_{t|t} + \mathbf{G}\mathbf{y}_t \\
\boldsymbol{\Sigma}_{t+1} &= \mathbf{F}\boldsymbol{\sigma}_t \mathbf{F}^T + \mathbf{Q}
\end{aligned}
$$

First, the evolution of covariances, represented by $\boldsymbol{\Sigma}_t$ and $\boldsymbol{\sigma}_t$, is dependent on the covariance matrices of the shocks ($\mathbf{Q}$ and $\mathbf{R}$). In this sense, $\mathbf{Q}$ and $\mathbf{R}$ not only represent the volatility of the stochastic shocks, but also affect the Bayesian updating process and therefore the decisions of economic agents. Thus, even in a first-order linearized economy, the size of the shock variances still matter directly for the dynamics of the model with uncertainty through its impact on the learning process. Second, notice that the prior and posterior covariances of the unobservables $\boldsymbol{\Sigma}_t$ and $\boldsymbol{\sigma}_t$ are stationary processes, provided that the coefficients of the learning matrices ($\mathbf{F}$ and $\mathbf{H}$) and the covariances of the shocks ($\mathbf{Q}$ and $\mathbf{R}$) are not time-varying. In order words, $\boldsymbol{\Sigma}_t$ and $\boldsymbol{\sigma}_t$ do not depend on the realized values of the signals. They can therefore be computed in advance using the algebraic Riccati Equation, given the system matrices and the shock covariances.

Proposition 1 *$\forall \Sigma_0$, $\exists! \boldsymbol{\Sigma} : \lim_{\mathbf{t}\to\infty} \boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}$ where $-\infty < \|\boldsymbol{\Sigma}\| < \infty$.*

Proof. Theorem 13.2 in Hamilton (1994) implies the result provided the eigenvalues of $\mathbf{F}$ are inside the unit circle. This reduces to the condition that $|\psi| < 1$, $|\rho| < 1$, $\left|\frac{(1-\delta)}{e^{g_k}}\right| < 1$. The first two are satisfied by assumption and the third is because $\delta \in (0,1)$ and $g_k > 0$, as a consequence of Assumption 0. ■

However, the evolution of beliefs about means, represented by the unobserved state variables $\mathbf{x}_t$, *is* dependent on realized signals as well as on the term $\mathbf{P}_t \equiv \boldsymbol{\Sigma}_t \mathbf{H}^T (\mathbf{H}\boldsymbol{\Sigma}_t \mathbf{H}^T + \mathbf{R})^{-1}$ which is known as the optimal Kalman gain. The optimal Kalman gain assigns a measure of uncertainty to the estimation of the current state. If the gain is high, the learning process places more weight on the observations, and thus follows them more closely. With a low gain, the learning process follows the model predictions (the prior beliefs) more closely, smoothing out noise but decreasing the responsiveness to signals. For example, if the volatility of noise captured by $\mathbf{R}$ is large, the information embodied in the signal $\tilde{\phi}_t$ would receive less weight in the learning process and be followed less closely by agents. In our model, the stationary value of the Kalman gain matrix $\mathbf{P} \equiv \boldsymbol{\Sigma}\mathbf{H}^T (\mathbf{H}\boldsymbol{\Sigma}\mathbf{H}^T + \mathbf{R})^{-1}$ depends on the structural parameters of the economy, such as α, π and δ, the covariances of the shocks $\mathbf{Q}$ and $\mathbf{R}$, and the constant prior covariances $\boldsymbol{\Sigma}$. Hence, the steady state Kalman gain is also a constant and can be computed based on the values of the model parameters and the shock covariances: the realizations of the state variables do not matter.

4.3 Properties of beliefs

We now study the role of expectations in economic fluctuations in the context of the model. We can write out the simplified evolution of posterior beliefs in equation (7) as:

$$
\begin{aligned}
z_{t|t} &= z_{t|t-1} + P_{1,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{1,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1}) \\
q_{t|t} &= q_{t|t-1} + P_{2,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{2,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1}) \\
\tilde{k}_{t|t} &= \tilde{k}_{t|t-1} + P_{3,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{3,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})
\end{aligned}
$$

Here $P_{i,j}$ is the row i and column j of the Kalman gain once $\mathbf{\Sigma}_t \to \mathbf{\Sigma}$.

In order to demonstrate the discrepancy between the beliefs and the actual values of the uncertain fundamentals, we compute the difference between the posterior beliefs and the actual values for TFP, ISTC and capital respectively. Starting at a steady state, the discrepancy between the beliefs and the actual values can be summarized by a reduced VAR(1) system, given by[14]:

$$\mathbf{X}_t = \boldsymbol{\mu}_t + \mathbf{\Xi}\mathbf{X}_{t-1} + \mathbf{\Omega}\mathbf{u}_t,\ \mathbf{u}_t \sim N(\mathbf{u}, \mathbf{\Theta}). \tag{9}$$

where $\mathbf{X}_t = \begin{pmatrix} z_{t|t} - z_t \\ q_{t|t} - q_t \\ \tilde{k}_{t|t} - \tilde{k}_t \end{pmatrix}$ is the distance between posterior beliefs and the actual values; $\boldsymbol{\mu}_t = \begin{pmatrix} -\pi P_{1,2} \\ -\pi P_{3,2} \\ -\pi P_{2,2} \end{pmatrix} \tilde{\phi}_{t-1}$ is the weighted exogenous variable (signal about ISTC); $\mathbf{u}_t = \begin{pmatrix} \varepsilon_t \\ w_t \\ v_t \end{pmatrix}$ is the reduced shock, with $\mathbf{u}$ as the mean vector and $\mathbf{\Theta}$ as their variance-covariance matrix; $\mathbf{\Xi}$ is a linear function of the Kalman gain matrix $\mathbf{P}$ and the parameters of the model, including the persistence parameters and other structural parameters:

$$
\mathbf{\Xi} = \begin{pmatrix}
(1-P_{1,1})\psi & -(P_{1,2}\rho + P_{1,1}\alpha\frac{\bar{\imath}}{\bar{k}e^{g_k}}) & -P_{1,1}\alpha\frac{(1-\delta)}{e^{g_k}} \\
-P_{2,1}\psi & (1-P_{2,2})\rho - P_{2,1}\alpha\frac{\bar{\imath}}{\bar{k}e^{g_k}} & -P_{2,1}\alpha\frac{(1-\delta)}{e^{g_k}} \\
-P_{3,1}\psi & (1-P_{3,1}\alpha)\frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}} - P_{3,2}\rho & (1-P_{3,1}\alpha)\frac{(1-\delta)}{e^{g_k}}
\end{pmatrix}
$$

and $\mathbf{\Omega}$ is the coefficient for the shock processes

[14] Appendix 3 shows the details of the calculation.

$$\mathbf{\Omega} = \begin{pmatrix} (P_{1,1}-1) & P_{1,2} & P_{1,2} \\ P_{2,1} & (P_{2,2}-1) & P_{2,2} \\ P_{3,1} & P_{3,2} & -P_{3,2} \end{pmatrix}$$

We explore the properties of the learning process and its implications for the aggregate dynamics of the model. For these Propositions we assume:

Assumption 1 $\mathbf{\Sigma}_t = \mathbf{\Sigma}$: the variance-covariance matrix has converged to its long run value (steady state value).

Assumption 1 is not required for the following results: however we adopt Assumption 1 because it simplifies notation, and because we assume $\mathbf{\Sigma}_t = \mathbf{\Sigma}$ in the quantitative work that follows. This assumption is not as strong as it appears. In fact, an empirical estimation of a state space model as the one shown above with unobserved IST and TFP shocks shows that the variance-covariance matrix of the unobservables converges to constant steady state values quite fast, within a few quarters. It is nonetheless interesting to investigate the implications of the model with time-varying variances, which is solvable using our proposed solution methodology. This is left for future work.

Lemma 1 *All the elements of $\mathbf{P}_t$ are generically non-zero unless there is no uncertainty (i.e. unless z_t and q_t are perfectly observed and there are no unanticipated shocks to either variable.)*

Proof. As shown before, $\mathbf{P} \equiv \mathbf{\Sigma H}^T(\mathbf{H\Sigma H}^T + \mathbf{R})^{-1}$, where $\mathbf{\Sigma} = \mathbf{F}\boldsymbol{\sigma}\mathbf{F}^T + \mathbf{Q}$. Since both $\mathbf{F}\boldsymbol{\sigma}\mathbf{F}^T$ and $\mathbf{Q}$ are positive definite, $\mathbf{\Sigma} = \mathbf{0}$ if and only if $\mathbf{F}\boldsymbol{\sigma}\mathbf{F}^T = \mathbf{0}$ and $\mathbf{Q} = \mathbf{0}$. If $\mathbf{P} = \mathbf{0}$, i.e., $P_{1,1} = P_{1,2} = P_{2,1} = P_{2,2} = P_{3,1} = P_{3,2} = 0$, then it must be that $\boldsymbol{\sigma} = \mathbf{0}$ and $\mathbf{Q} = \mathbf{0}$. On the other hand, since $\boldsymbol{\sigma} = E(\mathbf{X}_t - \mathbf{X}_{t|t})(\mathbf{X}_t - \mathbf{X}_{t|t})'$, it equals to zero only the expected value equals to the actual value, implying the TFP and ISTC are observed, and since $\mathbf{Q}$ is the variance covariance matrix of the unobserved shocks then $\mathbf{Q} = 0$ also means no disturbances occur. This is contradicts the assumption of the framework. Therefore in general all the elements of $\mathbf{P}$ are not zero unless there is no uncertainty. ■

Lemma 2 *Lemma 1 is true even when the signal is not persistent, i.e., $\pi = 0$.*

Proof. Again, so long as there is uncertainty, i.e., $\mathbf{Q} \neq \mathbf{0}$ and $\boldsymbol{\sigma} \neq \mathbf{0}$, $\mathbf{\Sigma H}^T$ and $\mathbf{H\Sigma H}^T + \mathbf{R}$ will not equal $\mathbf{0}$ even when $\pi = 0$.. As a result $\mathbf{P} \neq \mathbf{0}$. ■

Proposition 2 *A shock to either of ε_t, w_t or v_t generates a discrepancy between the actual and expected values of z_t, q_t and k_t, even if agents' expectations were previously correct.*

Proof. Without initial uncertainty, in equation (9) we have $\boldsymbol{\Xi}\mathbf{X}_{t-1} = \mathbf{0}$. Since we assume the economy starts at a steady state without previous shocks, $\boldsymbol{\mu}_t = \mathbf{0}$ as well. The change in the expectation is reduced to $\mathbf{X}_t = \boldsymbol{\Omega}\mathbf{u}_t$. Lemma 1 proves that the matrix in front of $\mathbf{u}_t = \{\varepsilon_t, w_t, v_t\}$ is not diagonal in that none of the Kalman gain factors is zero generically. Therefore if any element of $\mathbf{u}_t \neq \mathbf{0}$ there will be a discrepancy between the expected and true states for *all* the unobservable variables next period. ■

This deviation between expectations and actual values can serve as a transmission mechanism for technology shocks. For example, when there is a shock to TFP, not only will it influence TFP and the economy in a way similar as in the standard RBC model, it also influences the beliefs about all the three unobservables. Of course, whether this learning process propagates or weakens fluctuations depends on how agents understand and filter the information contained in various signals. For example, if agents observe an increase in the signal $\tilde{\phi}_t$, they may tend to behave conservatively due to the fact that the signal could also rise due to a noise shock. As a result, movements in investment and output could be less volatile than under certainty.

Proposition 3 *Under* Assumption 1*, a discrepancy between actual and expected values for any of z, q, and k generates uncertainty for all of z, q and k in the subsequent periods, even when there are no shocks ($\varepsilon_t = w_t = v_t = 0$).*

Proof. Starting with the stationary Kalman gain, no signals initially, and $\varepsilon_t = w_t = v_t = 0$, beliefs evolve according to $\mathbf{X}_t = \boldsymbol{\Xi}\mathbf{X}_{t-1}$. $\boldsymbol{\Xi}$ is not diagonal as shown earlier: thus any uncertainty in the initial state would generate deviations in other variables and thus fluctuations in expectations. ■

Take the TFP shock z as an example. Suppose initially, the expectation of q and $\tilde{k}$ coincide with their actual values, so that $q_{t-1|t-1} - q_{t-1}$ and $\tilde{k}_{t-1|t-1} - \tilde{k}_{t-1}$ are zero, whereas $z_{t-1|t-1} - z_{t-1} \neq 0$. This initial difference not only leads to a deviation of expectation over z for period t, but also influences the expectation of q_t and $\tilde{k}_t$, making all the expectations deviate from their actual values thereafter, which in turn would influence the decisions of economic agents. As a result, initial uncertainty could still influence economic dynamics even with no subsequent noise.

Proposition 4 *Assuming there is no further noise, after a period with shocks, the discrepancy between the expectations and the actual values of $\tilde{z}$, $\tilde{q}$ and $\tilde{k}$ converges to zero provided $|\sum_{i=0}^{\infty} \boldsymbol{\Xi}^i| < \infty$.*[15]

Proof. The unconditional mean of the discrepancy of $\mathbf{X}_t$ using backward substitution yields

$$\lim_{t\to\infty} E\mathbf{X}_t = E[\sum_{i=0}^{\infty} \boldsymbol{\Xi}^i \mathbf{u_{t-i}}] = \sum_{i=0}^{\infty} \boldsymbol{\Xi}^i E[\mathbf{u_{t-i}}] = \mathbf{0}.$$

■

[15] Although we cannot prove this property in general, we find that this assumption holds for empirically relevant parameters.

Starting from arbitrary initial values of the discrepancy, economic agents are able to learn from the observed signals and update their beliefs gradually. Their expectations will eventually be consistent with the actual values of these unobserved fundamentals. Thus, only the repeated introduction of noise allows uncertainty to affect macroeconomic dynamics indefinitely.

Using the above, we now prove an important result. Since any shock affects beliefs regarding all state variables, and since these beliefs wear away only gradually, a non-fundamental or noise shock can have a persistent effect on all variables. This is true even when noise itself is not persistent.

Proposition 5 *Suppose that $\pi = 0$, so that noise shocks are not persistent. A noise shock $v_t \neq 0$ leads the expected values of z, q and k to deviate from their actual values in all subsequent periods.*

Proof. Starting with no expectation errors, when there is a noise shock in period t, we have $\mathbf{X}_t = \Xi\mathbf{X}_{t-1} + \boldsymbol{\Omega}\mathbf{v}_t$. By Lemma 2, the expected values of z, q and k are all influenced by the shock v and therefore deviate from the actual values at $t, \mathbf{X}_t \neq \mathbf{0}$. For the next period $t+1$, we have $\mathbf{X}_{t+1} = \boldsymbol{\mu}_{t+1} + \Xi\mathbf{X}_t$. No matter whether the signal is persistent or not (no matter whether $\boldsymbol{\mu}_{t+1} = \mathbf{0}$ or not in the subsequent periods), since $\mathbf{X}_t \neq \mathbf{0}$, $\mathbf{X}_{t+1} \neq \mathbf{0}$ before beliefs converge to their actual values. ■

5 Quantitative Analysis

We now analyze numerically the behavior of the model economy. It is worth noting that, unlike a standard model where it is assumed that the current values of fundamentals are observed, the basis for approximation is not the deterministic steady state. Instead, in the baseline agents are uncertain about the values of fundamentals. The extent of uncertainty (variance) has converged to a positive constant, following Assumption 1.

5.1 Estimation

To proceed with the numerical analysis, we must select a functional form for the utility function U. We follow Hansen (1985) and Rogerson (1988) in assuming that

$$U(c_t, n_t) = \log(c_t) - \xi n_t$$

which is equivalent to assuming an environment with a more general utility function and indivisible labor.

Then, following Kydland and Prescott (1982), we assign values to as many model parameters as possible from exogenous sources. Most of these values are standard. As in the related study of Fisher (2006) the discount factor is $\beta = 0.99$. The capital share is set to $\alpha = 0.33$, which is a standard value in the literature. The depreciation rate is $\delta = 0.025$ as in Hansen (1985), which is in the middle of the quarterly depreciation rates for equipment and structures estimated in Greenwood et al (1997). The detrending parameter for the capital stock at a quarterly frequency is $e^{g_k} = 1.0077$ and the detrending parameter for investment is $e^g = 1.0124$. These are consistent with the rate of secular decline in the relative price of capital and the rate of investment growth in Greenwood et al (1997). We set the disutility of labor $\xi = 0.84$ so that employment is 0.91 in the steady state, which is the ratio of employment to working age population in post-war US data.

The rest of the parameters characterize the evolution of beliefs and productivity shocks. They are the investment adjustment cost parameter φ, and parameters representing the conditional covariances of the prediction errors and the stochastic shock processes $\psi, \rho, \pi, \sigma_v, \sigma_w, \sigma_\varepsilon$. These parameters not only directly characterize the dynamics of the linearized economy through the standard shock processes, but also indirectly influence the behavior of our model with uncertainty and learning, as the values of informational gain in the learning process are calculated based on the estimates of the persistence of the shocks, as well as the size of variance of the shocks.

To jointly estimate these parameters, we use the Bayesian estimation technique to match the data of quarterly US time series to the model-simulated time series of corresponding variables. Specifically, the correspondence between the data on the left hand side and the variables from the model on the right hand side is:

$$\begin{bmatrix} \log GDP_t - \log GDP \\ \log CONSUMP_t - \log CONSUMP \end{bmatrix} = \begin{bmatrix} \tilde{y}_t \\ \tilde{c}_t \end{bmatrix}$$

where GDP_t and $CONSUMP_t$ are respectively the data for real GDP and real consumption, and GDP and $CONSUMP$ are their corresponding trends generated by one-sided HP filter. Data are extracted from National Income and Product Accounts (NIPA) dataset, between 1968Q3 and 2015Q3. Real GDP is obtained by dividing nominal GDP by the chain-weighted deflator for the consumption of nondurables and services. We define consumption as consumer expenditures on non-durables and services, and construct the real series for consumption by dividing nominal consumption expenditure by the chain-weighted consumption deflator[16]. We choose the prior distribution of the parameters as in Smets and Wouters (2007), such that the prior distribution provides least information for the estimation.

Given the prior distribution and observation variables, we estimate the posterior distribution and the mode of the structural and shock parameters in the model by maximizing the log posterior likelihood and then using the Metropolis-Hastings Makov Chain Monte Carlo technique to obtain a complete posterior distribution of the parameters. The estimates of the posterior mode and standard deviation of the parameters obtained from this procedure are shown in Table 2.

[16] For robustness, we have also conducted an alternative Bayesian estimation by using the data of real GDP, real consumption, and hours worked. The simulated dynamics of the economy and the statistics of the macroeconomic variables of interest based on this estimation strategy is similar with those reported in the paper. Results are available on request.

Table 2: Calibrated parameters in the model economy

See text for detailed discussions of how these parameters are matched or estimated.

Param	Description	Prior			Posterior			
		Density	Mean	S.d.	Mode	Mean	Std.	[5th,95th]
φ	Invest adjustment cost	InvG	5	0.5	3.35	3.35	0.21	$[3.10, 3.54]$
ψ	TFP persistence	Beta	0.5	0.2	0.69	0.63	0.12	$[0.40, 0.82]$
ρ	ISTC persistence	Beta	0.5	0.2	0.79	0.80	0.02	$[0.78, 0.83]$
π	signal persistence	Beta	0.5	0.2	0.59	0.60	0.11	$[0.35, 0.72]$
σ_ε	TFP s.d.	InvG	0.1	2	0.31	0.31	0.08	$[0.22, 0.44]$
σ_w	ISTC s.d.	InvG	0.1	2	1.83	1.73	0.11	$[1.57, 1.92]$
σ_v	noise s.d.	InvG	0.1	2	1.73	1.89	0.20	$[1.62, 2.18]$
$P_{1,1}$	Kalman gain of z from y	N/A			0.50			
$P_{1,2}$	Kalman gain of z from ϕ	N/A			-0.02			
$P_{2,1}$	Kalman gain of q from y	N/A			1.61			
$P_{2,2}$	Kalman gain of q from ϕ	N/A			0.56			
$P_{3,1}$	Kalman gain of k from y	N/A			1.52			
$P_{3,2}$	Kalman gain of k from ϕ	N/A			0.07			

5.2 Information in the estimated economy

The persistence of TFP is only 0.69, smaller than the common value of 0.95. This difference is likely due to the fact that in our model, the capital used in the production function is assumed to be the unobserved effective capital, different from the standard RBC model without uncertainty where the capital is interpreted as the physical capital stock. The ISTC shock is more persistent ($\rho = 0.79$). On the other hand, the signal process of our kind has never been estimated before. We find that the persistence parameter of signal π equals 0.59, so that the half life of a noise shock is about one and a half quarters. The standard deviation of the ISTC shock is the largest, so the ISTC shock is more volatile than the TFP shock. The standard deviation of the noise shock is smaller than that of ISTC but bigger than that of TFP, indicating that the noise shock is a significant potential source of volatility at business cycle frequencies.

The parameters of the stationary Kalman gain matrix are also of interest as they measure how the learning process places weight on the two signals and on prior beliefs. $P_{1,1}$ is the Kalman gain factor regarding information from output when updating z. The value of 0.50 indicates that the more important information source for updating beliefs about z is the observation of output. $P_{1,2}$ is the information from the observation of ϕ for the learning, and a value of -0.02 means that an observation of a higher ISTC signal ϕ leads to a downward revision of expectations of z. Intuitively, a higher value of ϕ implies that q may be higher than was initially expected, and therefore z might *not* be as high as initially expected to account for a given observation of GDP. $P_{2,1}$ and $P_{2,2}$ are Kalman gain factors regarding information when updating q. The values are positive, indicating that a higher output and ϕ will increase the expected value of q, and the information from output is more important than information from ϕ when updating beliefs about q. $P_{3,1}$ and $P_{3,2}$ are the Kalman gain factors regarding information from observed output and from the signal ϕ respectively when updating beliefs about k. As for q, the values indicate that a larger portion of the update on k comes from output than the signal ϕ.This is because economic agents know that the dynamics of signal is affected by both ISTC shock and noise.

Based on the estimation, it is also interesting to investigate how the precision of signal ϕ is dependent on the noise, measured by σ_v. A related notion is the "signal-to-noise" ratio (SNR) used in science and engineering that compares the level of a desired signal to the level of background noise. We can define the precision of the signal ϕ as the ratio of meaningful information to noise (an "unwanted signal"). Since the signal has a zero mean, the ratio can be expressed as a conditional relative volatility of the signal to the noise:

$$SNR = \frac{\sigma_{\phi}^2}{\sigma_v^2} = \frac{(1-\pi)^2\sigma_w^2 + \sigma_v^2}{\sigma_v^2} = 1 + (1-\pi)^2\frac{\sigma_w^2}{\sigma_v^2}$$

For the baseline estimation, given $\pi = 0.59, \sigma_w = 1.83$ and $\sigma_v = 1.73$, we obtain an SNR of 1.19. The ratio is bigger than unity, which means the signal has more information than noise. This is the reason why beliefs converge to the underlying fundamentals eventually. Another observation is that, when σ_v increases, indicating more noise, the signal-to-noise ratio will decrease and therefore the signal will become less precise.

Finally, a unique feature of our model is that the capital stock is not known with certainty at any date. The estimated economy yields a measure of the extent of uncertainty about the value of the capital stock, assuming Assumption 1 holds (convergence of the variance-covariance matrix of beliefs). We find that the standard deviation of the posterior belief distribution of log capital (relative to its steady state value) is 1.40 percent. Thus uncertainty about the *posterior* regarding the current value of $\log k_t$ is non-trivial. In addition, the standard deviation of the *prior* of log capital is 2 percent: the signals that agents receive do little to assuage their uncertainty about the capital stock, as it has accumulated along with the capital stock itself. In what follows, we explore the implications of this uncertainty for business cycle dynamics.

5.3 Business cycle facts

In order to understand the implications of uncertainty for the business cycle, we compare the business cycle facts implied by two related models. First, we look at the benchmark model with uncertainty and Bayesian learning, denoted Model U. We also look at the model without uncertainty and learning (a standard RBC model with investment adjustment costs), denoted as Model NU, in which there is no uncertainty about current values and choices of consumption and labor are made after observing these values. In order to obtain the business cycle statistics implied by Model NU, we estimate parameter values of Model NU with

Table 3: Business Cycle Statistics in different model economies.

Model U is the model in the paper with uncertainty and learning. Model NU is the standard RBC model with no uncertainty or learning.

$x =$	I	C	L
	Data		
σ_x/σ_Y	3.60	0.82	0.90
$corr(x, Y)$	0.91	0.87	0.89
	Model U		
σ_x/σ_Y	2.87	0.88	1.11
$corr(x, Y)$	0.85	0.93	0.96
	Model NU		
σ_x/σ_Y	3.19	0.57	2.24
$corr(x, Y)$	0.91	0.63	-0.44

the same types of shocks and using the same dataset as for the benchmark Model U. Table 3 shows the cyclical statistics calculated from data, and the simulated statistics of the two models with the parameter values separately estimated based on Model U and Model NU, as described above[17].

By comparing the simulated statistics of these two models, we learn what generates differences in behavior between our model with uncertainty and the model without uncertainty and learning. The main difference is that the volatility of investment relative to that of GDP is smaller in Model U than Model NU, whereas that of consumption is larger in Model U than Model NU. This is because the inability to clearly identify TFP and ISTC shocks leads to more conservative response of investment to perceived changes in ISTC. At the same time, agents may incorrectly identify ISTC shocks as TFP shocks, and therefore increase consumption more in response. Overall, our estimated model with uncertainty and learning fits well with the data[18]. Furthermore, the log data density generated from the Bayesian estimation of the two models suggests that Model U has a marginal data density of -384.49, larger than that of Model NU, -401.67, which implies that Model U overall fits data better compared to Model NU.

5.4 Unobserved shocks

By combining US data with Bayenesian techniques to estimate the model with unobserved TFP, ISTC and noise shocks, we are able to extract estimates of the structural shocks themselves over the time period of our dataset. These estimates are derived from the Kalman smoother at the posterior mode of the estimated parameters that the model needs to match the data.

[17] The statistics results from a simulation with 1100 periods, dropping the first 100 observations.

[18] This also suggests that we would likely obtain same conclusions about the dynamics of the economy if we instead choose to estimate our model using the generalized methods of moments methodology to match key macroeconomic variables directly (as opposed to our entire series matching procedure).

Figure 1 plots the time series of the extracted shocks between 1968Q3 and 2015Q3. First, consistent with the statistics shown in Table 2, the ISTC and noise shocks are generally more volatile compared to TFP shocks over the sample period. In addition, all the shocks seem to vary more towards the beginning of the time period, compared to more recent data. TFP shocks and noise shocks appear to settle down early in the mid-1980s, around the time of the Great Moderation, with ISTC shocks settling down a little later. Third, two recent periods intriguingly stand out.

One period is around the year 2000, when the US economy experienced a boom-bust cycle related to the information technology sector. In the literature, this recession has been attributed to an incorrect perception about the rate of progress of the investment technology, which is strongly related to information and communication technologies[19]. The boom started with an over optimistic expectation about the unobserved progress of ISTC, and ended with downward revision of the expectations and of investment and output[20]. The dynamics of the extracted shocks of our model provide evidence in support of this hypothesis. Around the year 2000, the noise shock regarding the ISTC increased to its highest levels since the 1990s, leading to an increase of the signal received by agents about ISTC. In fact, according to these estimates, the level of ISTC did not improve, as shown by the absence of any unusually large ISTC shocks around that time. Observing a high ISTC signal at that time, yet being unable to distinguish the true values of ISTC shocks and noise shocks, economic agents at that time would wrongly attribute the increasing ISTC signals to a higher ISTC shocks to some extent, and invest more than they would have if they had been able to observe the shocks.

The other period is around 2008, when the US economy experienced the Great Recession, with the turmoil originated in the housing and financial market and spreaded to the real sectors. Although most studies attribute this recession to the financial or uncertainty shocks, our evidence suggests the downturn of ISTC could also have contributed, as ISTC shocks at the time were at their lowest levels since the 1980s.

[19] See, Jaimovich and Rebelo (2009), and Ben Zeev (2018), among others.

[20] The stock market performace then is indicative of this episode. During the 2000s recession, the volatility of the NASDAQ stock index, which measures the expected performance of high-tech and innovation firms. is higher than the Willshire index, which measures the expected performance of the top 5000 big companies covering all sectors.

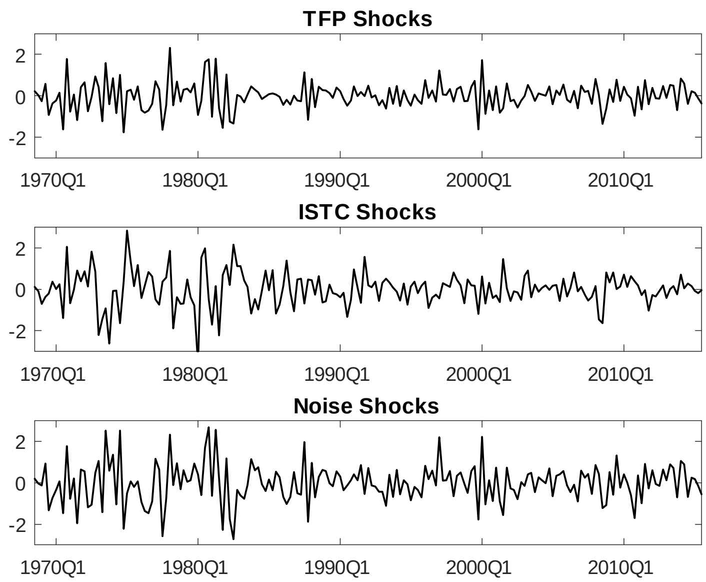

Figure 1 Time Series of Various Extracted Shocks

5.5 Dynamics of the Economy

In order to illustrate the influence of different shocks in the "uncertain" economy, we perform quantitative experiments by calculating the impulse responses of variables to each individual shock. To do this, we employ Assumption 1, and also:

Assumption 2 $z_{0|0} - z_0 = q_{0|0} - q_0 = \tilde{k}_{0|0} - \tilde{k}_0 = 0.$

In the initial period, the economy is assumed to be in a "correct" steady state where all variables are in the steady state value of zero and the believed output, capital stock and productivity values are accurate and identical to the actual output, capital stock and productivity values. When doing this analysis, we as *modelers* know that agents' beliefs of economic variables coincide with their true values. The economic agents, however, do not know this. They continue to expect shocks and also noise, so that the variance of their beliefs does not trend to zero, rather it converges to the stationary values discussed in the previous Section, as per Assumption 1. The responses following the shocks are measured using the percentage deviation from the steady state values. In the following impulse response figures, we record the behavior of macroeconomic variables responding to current shocks after the observation of output and signals about ISTC. The responses of *beliefs* about productivity, however, reflect changes in the posterior beliefs about the current shocks i.e. the prior beliefs for the following period.

5.5.1 Total factor productivity shock

In Figure 2, we plot the influence of a one standard deviation shock to TFP. The solid line depicts the mean impulse response effects under benchmark framework with imperfect information, Model U, and the dash-dot line depicts Model NU with no uncertainty. We also show the 10 and 90 percent confidence intervals using shaded area, and the 0 line using dashed line. On impact, the responses of output, investment, consumption, and hours worked are all stronger under uncertainty compared to the standard model, as if both positive supply and demand shocks occur in the economy. This is because, in this environment with uncertainty, both neutral and investment specific productivity are expected to rise, even if the only shock hitting the economy is the TFP shock. In addition, despite a lower expected value of TFP shock compared to the realized TFP shock (0.1 versus 0.3 on impact), since the dynamics of the variables are dominantly contributed by the ISTC shock, the additional expansionary effects due to the higher expected level of ISTC shock relative to the true investment specific technology (which is zero) outweighs the contractionary effects due to the lower expected level of TFP shock relative to the true neutral productivity. As a result, the responses of macroeconomic variables are more significant,

The speed of the convergence between agents' expected values and the actual values is rapid after a TFP shock. Based on the simulation, the initial difference between $z_{t|t}$ and z_t is 67% of its standard deviation and drops to barely 6% at the fourth quarter. Similarly, the difference between $q_{t|t}$ and q_t is 28% of its standard deviation, and only 3% at the fourth quarter.

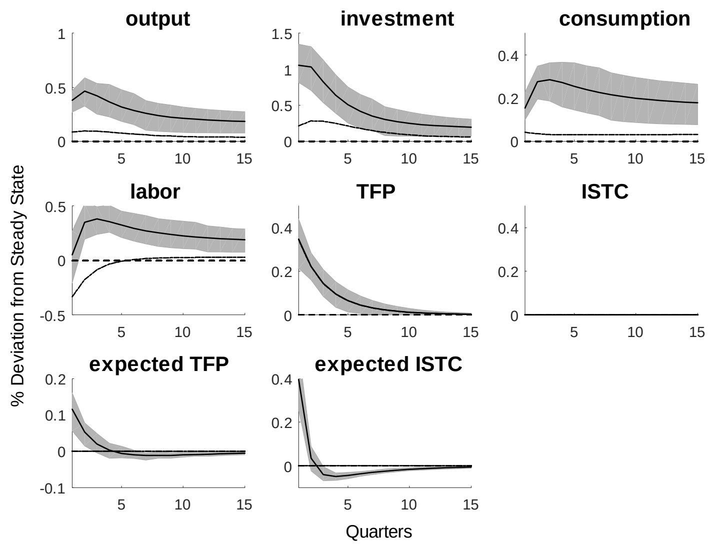

Figure 2 – Effects of a TFP shock.
The solid line is the mean impulse response in the benchmark model with uncertain technology; the shaded area is the 10 percent and 90 percent posterior intervals; the dash-dot line is the impulse response in the model with no uncertainty; the dotted line is the zero line.

5.5.2 Investment specific technology shock

The dynamics of an ISTC shock are richer. In Figure 3, the solid line depicts the mean impulse responses of economic variables to a one standard deviation shock to ISTC. The evolution of economic variables is more persistent in Model U than in Model NU. First, even though there is no change in the actual value, the expected TFP rises gradually and persistently after a slight decrease initially, because subsequent increases in output are attributed at least partly to a possible increase in TFP. In addition, the expected value of ISTC according to agents' beliefs does not even return to the steady state within 15 quarters. Beliefs about ISTC are different from the actual values due to the fact that agents can only observe a noisy signal for ISTC. As shown, on impact, expected ISTC is smaller than actual ISTC. Consequently, investment under uncertainty is smaller than without uncertainty, since agents' decisions are influenced by expected productivity values rather than the realized values. In the subsequent periods, expected ISTC is persistently lower than its actual value, because when agents observe the increased signal ϕ, they interpret it as a combination of changes in the disturbance and in the ISTC shock. As a result, agents tend to behave conservatively, leading to less investment and lower output compared to Model NU.

The convergence of agents' beliefs to the actual values of the fundamentals after an ISTC shock is much slower than when there is an TFP shock as shown in Figure 2. The simulation results suggest that the initial difference between $q_{t|t}$ and q_t is 78% of its standard deviation, dropping to 23% by the fourth quarter and to 7% by the eighth quarter. The difference between $z_{t|t}$ and z_t is also persistent: it increases to the peak of 23% at the sixth quarter and only declines to 9% by the fifteenth quarter. This explains why the response of output is also quite persistent when ISTC shocks occur.

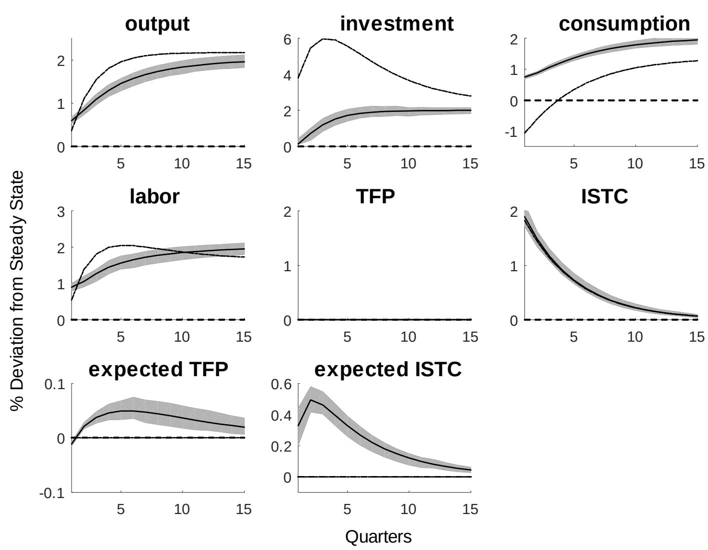

Figure 3 – Effects of an ISTC shock.

The solid line is the mean impulse response in the benchmark model with uncertain technology; the shaded area is the 10 percent and 90 percent posterior intervals; the dash-dot line is the impulse response in the model with no uncertainty; the dotted line is the zero line.

5.5.3 Noise shock

We now study the impact of noise shocks (or "fake news" shocks) on aggregate dynamics. Figure 4 shows the effects on economic variables of a one standard deviation (positive) noise shock. On impact, the signal increases following the increase in noise. Agents know that the observed change in the signal could be due to noise or to an actual increase in ISTC, and therefore put some weight on both possibilities. However they do not know that their initial beliefs about technologies are correct and consistent with the actual values of ISTC and TFP (which have not changed). As a result, they mistakenly update their beliefs about ISTC upwards, which leads to an increase in investment and a very slight decrease in consumption. Interestingly, agents also update their beliefs about TFP downwards initially, even though actual TFP has not changed. In the following periods, the signal about ISTC is still persistently higher than its steady state value given the estimated value of $\pi = 0.60$, even if the noise shock fades. The misperception of TFP and ISTC lasts longer, leading to higher level of macroeconomic variables than their steady state values during the simulated period. This result shows that the noise shock is able to generate economic persistence that is higher than suggested by similar models with noise shocks in the literature, such as Blanchard et al. (2013), through the mechanism that a one time noise shock can lead to persistent misperception about both technologies in this modeled economy.

Compared to when the ISTC shock hits the economy as shown in Figure 3, however, agents update their beliefs about ISTC more quickly under the noise shock. On impact, the difference between $q_{t|t}$ and q_t is 53% of its standard deviation, dropping to 1% by the fourth quarter, whereas this discrepancy is still 23% at the fourth quarter following the ISTC shock. That is why investment is more persistent in response to the ISTC shock than the noise shock. The misperception of TFP, however, lasts longer than that of ISTC. The initial difference between $z_{t|t}$ and z_t is 14% and only declines to 12% by the $4th$ quarter. This also contributes to the persistence of macroeconomic variables in response to the noise shock, despite the fact that agents learn quickly about ISTC.

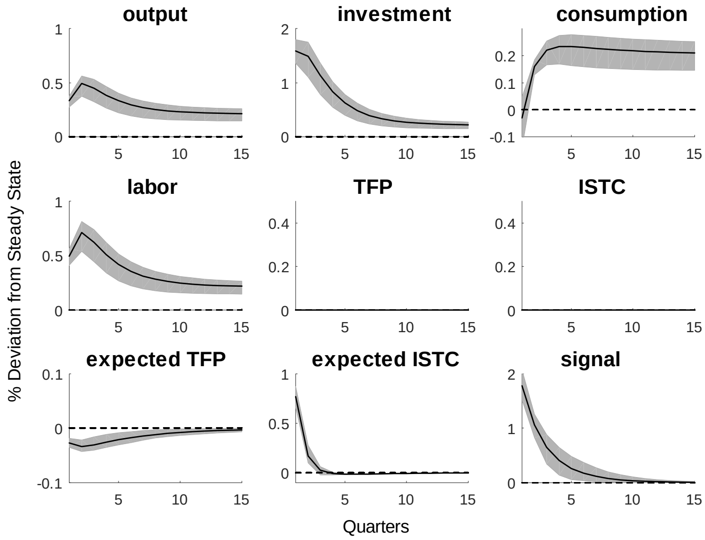

Figure 4 – Effect of a noise shock

The solid line is the mean impulse response in the benchmark model with uncertain technology; the shaded area is the 10 percent and 90 percent posterior intervals; he dotted line is the zero line.

5.6 Biased beliefs and fluctuations

In the previous simulation we compute impulse response functions under the assumption that agents have no expectation errors at the moment of shock impact – Assumption 2. However, in our analytical study, we show that any deviation of the expectation from the actual value of the uncertain economic fundamentals could persistently affect economic dynamics, even after the discrepancy converges to zero. Thus, we also study the impact of shocks when beliefs are "biased" at the moment of impact, i.e. when the expected value of q is not equal to its actual value.

We illustrate the impact of biased beliefs by assuming the economy starts with an expectation of ISTC that differs from the actual value. Initially, rather than at 0, $q_{t|t}$ is assumed to be at a value equivalent to one standard deviation of the ISTC shock, i.e., $q_{0|0} = 1.83$, whereas the initial expectations of TFP and efficient capital are still accurate. At the same time, the economy experiences the same one standard deviation ISTC shock as in Figure 3. Under this assumption, we calculate the dynamics of macroeconomic variables in response to the ISTC shock, and compare it to the situation when the initial expectation about ISTC is accurate.

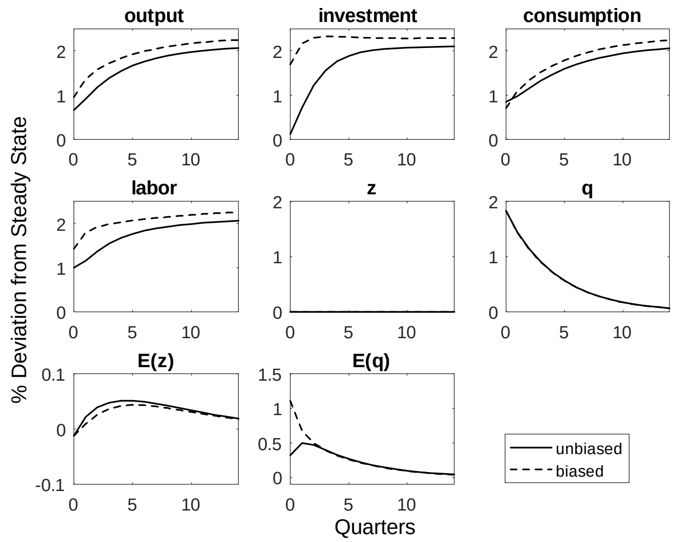

Figure 5 – Impact of positively biased beliefs

In Figure 5, when the initial belief about ISTC exceeds the actual value, the dynamics of beliefs back to the steady state and the impact on beliefs of the shock become convolved. Expected ISTC is significantly larger when the initial beliefs are positively biased, than when the initial beliefs are accurate, until the fourth quarter. Expected TFP, however, is not as big, due to the fact that higher beliefs about ISTC slightly decrease beliefs about TFP in future periods. Overall, the positive change in ISTC dominates the negative change in TFP. As a result, the overall effect is that output, investment, and consumption are all higher than in Figure 3. In addition, during the initial periods when the positive bias has the largest effects on expected TFP and ISTC relative to the "unbiased" benchmark, the difference in investment and output is also the largest. The difference reduces gradually as the impact of initial bias diminishes after the fourth quarter.

What if the initial belief of ISTC is *smaller* than its actual value at the time there is a positive shock to ISTC? Figure 6 shows this scenario. During the initial periods, expected ISTC is much smaller compared to that in Figure 3, where the simulated economy also experiences an ISTC shock but the initial belief about ISTC is correct. Not surprisingly, we get the opposite results compared to the positive bias scenario in Figure 5: expected TFP is bigger due to lower beliefs about ISTC. As ISTC dominates agents' choices, output, investment, and consumption all fall short of their counterparts under accurate initial beliefs.

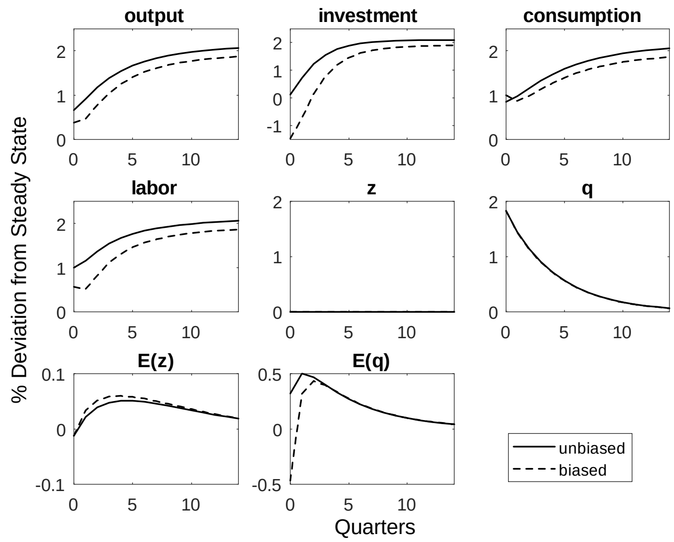

Figure 6 – Impact of negatively biased beliefs

5.7 The "co-movement" curse

Having studied the role of various shocks on the model economy, we now use our model to address a key challenge that faces general equilibrium business cycle models: the "co-movement curse." Barro and King (1984) argue that models with a real business cycle core have difficulty generating positive co-movement between consumption and investment, when shocks other than TFP shocks, such as news shocks, uncertainty shocks, or financial shocks, are the main driving forces of fluctuations. This problem is particularly stark for models with investment-specific shocks, as these shocks are important demand-side shocks, as stated in Justiniano et al. (2010, 2011), which change the incentive to invest instead of consuming without affecting the current capacity of the economy, inducing a strong negative correlation between consumption and investment: see Campbell (1998) for an early example. The literature tends to address the co-movement issue by adding more features to standard models, such as roundabout production and stochastic trends in neutral and investment technology as in Ascari et al. (2016), countercyclical markups and sticky prices as in Basu and Bundick (2017), a two-sector sticky-price model with durable goods consumption as in Chen and Liao (2018), among others. In contrast, our model implies that, even without additional features, uncertainty about the nature of technological progress in an otherwise standard model (i.e. without additional frictions or changes to the model structure) can aid the resolution of the curse.

Notice first of all that in Figures 2, 3 and 4, the response of the model economy to *any* type of shock is to increase (or decrease) both consumption and investment together. As illustrated in Figure 5, the investment reaction of agents to an investment-specific shock is more muted than for a TFP shock because agents are unsure about the nature and value of the shock – yet even then consumption and investment both rise together. In addition, besides its positive impact on investment, the ISTC shock also leads to a persistently rising and higher consumption, as agents, after observing the signals, not only expect ISTC to rise, but also expect a gradual increase in neutral productivity, as implied by Proposition 3. Quantitatively, given the estimated parameter values, if we assume the ISTC shock were the only shock affecting the economy, the model implied conditional correlation between investment and consumption following the ISTC shock would be 0.88. When taking the dynamics generated by all the stochastic shocks (TFP, ISTC, and noise shocks) into consideration, our model with uncertainty still generates an unconditional correlation at 0.62. This is quite close to the correlation between (cyclical) consumption and investment observed in our data (1968Q3 to 2015Q3), which is 0.74, and a significant improvement compared to the model without uncertainty, which displays a correlation between consumption and investment of 0.26. Note that 0.26 is obtained using model NU, where we simulate the standard model without uncertainty using the parameter values estimated when we allow the parameters in the standard model to freely move, i.e., we conduct a Bayesian estimation for the standard model with the same shock processes and observables as when we estimate the model with uncertainty. If we remove the uncertainty from the benchmark model and simulate it using the benchmark parameter values instead of re-estimating them, the correlation is -0.48.

These results show that our model is successful at reproducing the positive correlation between consumption and investment. In this way, our model contributes to the RBC literature by providing an alternative but simple solution to the Barro-King puzzle: the introduction of uncertainty about the nature of technologies alone is enough to generate co-movement between investment and consumption comparable to that observed in the US economy, without resort to new elements that are not already a core part of the shocks and frictions in the business cycle literature.

6 Concluding remarks

In this paper, we first show micro-level evidence that uncertainty is indeed greater in high-ISTC industries. We then propose a general dynamic stochastic model with uncertain productivity values and Bayesian learning to study the effects of technology shocks and a noise shock on macroeconomic fluctuations in an environment with uncertainty about current fundamentals. We show that TFP shocks generate more volatile responses under uncertainty than certainty, whereas ISTC shocks generate smaller but more persistent responses. In addition, the introduction of uncertainty about technologies yields a co-movement of consumption and investment in response to ISTC shocks, which cannot be obtained from the standard model with no uncertainty. Finally, a non-persistent noise shock regarding the investment specific technology changes agents' behavior persistently, even though the underlying fundamentals governing the economy remain unchanged. This is because the noise shock not only affects beliefs about investment specific technology, but also about neutral productivity.

At a broad level, the paper delivers an important generalization result: the response to ISTC shocks, which the literature has increasingly emphasized as sources of variation in aggregate time series, and to *noise* regarding ISTC, are both persistent. Thus, uncertainty about fundamentals introduces increased persistence in macroeconomic time series. This persistence is something that a policy authority can do little to ameliorate unless they have access to information that is not available to private agents. Ma (2015) studies monetary policy in an environment with uncertainty such as ours, but with additional features of preferences and technology that are typically incorporated into models for policy analysis. In addition, an extension of the model might be suitable for addressing is the fact that GDP *itself* may be observed with error, which could introduce even more persistence. Collard et al (2009) study an environment with error, but ISTC is not a feature of that model and, as suggested by the current paper, the implications when ISTC is observed with error are likely to be significantly different. In addition, it could be that the extent of noise in the signalling process is time-varying, along the lines of Bloom (2009), which would introduce possibly interesting interactions between volatility and learning dynamics. Finally, we do not explore how uncertainty about fundamentals might affect dynamics when agents' information sets might be different. These extensions are left for future work.

Third, the methodology we propose to solve the benchmark model with uncertain technologies assumes that the variance-covariances of the unobserved fundamentals are constant. This approach can be applied to the situation where the variance of beliefs about the unobserved technologies is time varying. It is interesting to explore the implications for business cycles under this scenario, which could be either due to economic agents still learning about the signal-to-noise ratio that has not yet settled down to the steady state level, or to the presence of stochastic shocks to the variance of noise that lead to time-varying higher moments of beliefs.

Last but not least, the finding that "fake news" (noise in signals) can have a persistent impact on beliefs and thus on economic outcomes – even when the fake news shocks themselves are not persistent – could have implications for understanding other environments where informational rigidities or inefficiencies are important. This may apply not just in macroeconomics but perhaps in industrial organization – where learning is widely believed to be important for productivity dynamics, entry and exit – or in political economy environments with imperfect information.

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8 Appendix

8.1 Appendix 1: Measuring Uncertainty by ISTC

We hypothesize that, If uncertainty is indeed linked to the rate of ISTC, uncertainty measured in sectors where ISTC is more rapid, i.e. which use capital goods that experience particularly rapid ISTC – should bear more importance in accounting for economic fluctuations than uncertainty measured in other sectors of the economy. Specifically, if we measure uncertainty using forecast errors as before, forecast errors in high-ISTC industries should have macroeconomic impact, whereas forecast errors in low-ISTC industries are due to idiosyncratic or other factors that are not as critical for macroeconomic aggregates.

To obtain sector-specific measures of uncertainty, we draw on recent work by Ma and Samaniego (2018). The premise behind the Ma and Samaniego (2018) strategy for measuring uncertainty is that changes in the uncertainty, and thus the unpredictability of the economic environment – at the aggregate and at the industry level – will be reflected in the analyst forecasts being less accurate than usual on average. Ma and Samaniego (2018) thus develop a measure of aggregate uncertainty based on the median forecast errors of the swarm of analysts making predictions every day about financial data for different firms[21]. Since their uncertainty measure is constructed using firm level data, and since the industry codes of most of these firms are available in the CRSP database, one can measure uncertainty about a particular industry or sector by simply focusing on forecasts made about the relevant subset of firms. We will use this strategy to measure uncertainty in industries where the rate of ISTC is high, and uncertainty in industries where the rate of ISTC is not high.

Here we focus on US firms. This yields almost 2 million forecasts issued by roughly 1300 different brokers. For each firm at each date we compute the average forecast error.[22] Then we take the median forecast error across firms within each month, starting in September 1981.[23] Thus it is the median forecast error by firm-day pair. We calculate the absolute values of these median forecast errors and deflate them by the monthly CPI in order to ensure our measures are reported in real terms[24].

We then obtain industry codes from CRSP.[25] Thus, we can define the set of firms of interest I as being either high-ISTC firms or low-ISTC firms. We define high-ISTC firms using the data in Cummins and

[21] They also draw I/B/E/S forecasts of the earnings-per-share ratios (EPS) of individual companies, and measure sector uncertainty using the absolute value of the median EPS forecast error for firms in that sector within each month.

[22] 86 percent of them are single forecasts about a firm on a given day. The rest have 2 forecasters making forecasts about a firm on the same day, except for 0.29 percent of the sample which has $3-5$ forecasts.

[23] This is the first month after which continuous series may be computed. The date is based on the month and year of the variable *anndats*.

[24] Notice that the uncertainty is defined based on the *absolute value* of the median forecast error. This way uncertainty is measured as lack of correctness – regardless of the direction. Not doing so would lead to a measure of relative optimism or pessimism compared to the realization, not uncertainty.

[25] CRSP reports the NAICS and SIC codes of these firms. We use SIC codes because NAICS codes did not exist early in our sample.

Violante (2002). They partition the private economy into 62 non-overlapping industries, and construct ISTC rates for each industry by weighing measures of ISTC for each type of capital using the composition of investment in each industry from the US capital flow tables. We measure high-ISTC industries as those above the median.[26] The remainder are low-ISTC industries.

Ma and Samaniego (2018) find that uncertainty in the financial sector is more important for aggregates than uncertainty outside the financial sector. Again, the financial sector is among the high-ISTC industries so that, in order to ensure that our results are not simply detecting the impact of financial uncertainty, we remove all firms that are in the financial sector before computing our uncertainty measures.

Figure $A1$ displays the HP filtered uncertainty series for high- and low-ISTC industries. Both series are somewhat similar in that certain key events are visible, including the recession of the early 1980s, the popping of the tech bubble in 2001 and the financial crisis of 2008. The contemporaneous correlation is 0.20, significant at the 5 percent level. At the same time, it is also notable that the two series are by no means identical, so the two may have different time series properties. First, there are several events identified as uncertainty shocks by the low-ISTC measure to which the high-ISTC measure does not respond. Second, intriguingly, when we examine the cross-correlogram of the two uncertainty series, we find that the highest correlation is 0.3430 when high-ISTC uncertainty is lagged by 3 months. This suggests that high-ISTC uncertainty lags low-ISTC uncertainty, consistent with the notion that uncertainty shocks originate in the high-ISTC sectors, or that high-ISTC industries experience uncertainty earlier.

[26] These industries are air transportation; pipeline transportation; motion picture and sound recording; broadcasting and telecommunications; information and data processing services; credit intermediation and related activities; securities, commodity contracts, and investments; insurance carriers and related activities; funds, trusts, and other financial vehicles; rental and leasing services and lessors of intangible assets; computer systems design and related services; miscellaneous professional, scientific, and technical services; administrative and support services; ambulatory health care services; and hospitals.

Table 4: Correlations between different measures of Uncertainty

Correlations between the high- and low-ISTC uncertainty measures based on I/B/E/S/ forecasts on the one hand, and uncertainty measures drawn from the literature on the other. See text for definitions. Two and three asterisks represent statistical significance at the five and one percent levels respectively. Sources: I/B/E/S database, authors' calculations, Bloom (2009), Bachmann et al (2013) and Jurado et al (2015).

Uncertainty measure	High ISTC	Low ISTC
$Dispersion_t$	.118**	.146
$Unpredict_t$	.319***	.095
$Google_t$	.047	.128
$Stock_t$	-.075	-.020

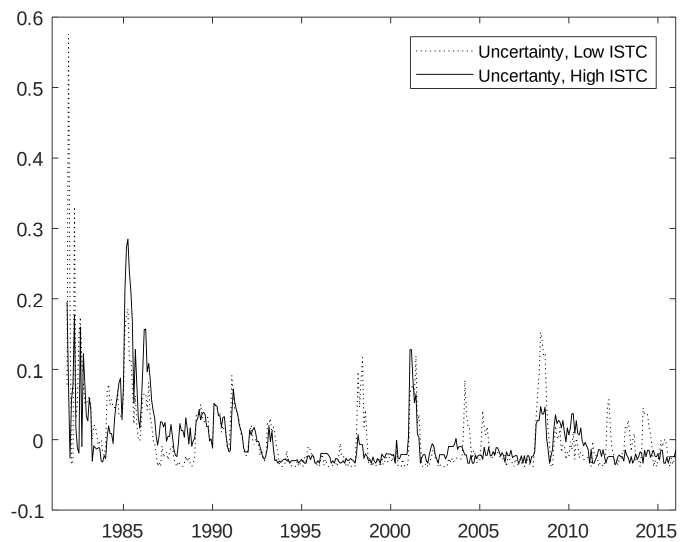

Figure $A1$ – Monthly uncertainty measured separately for high- and low- ISTC industries.

In addition, we find that the correlation between the high- and low-ISTC uncertainty measures and the other aggregate uncertainty measures mentioned earlier are informative. The high-ISTC uncertainty is statistically significantly related to some of the aggregate measures. In contrast, the low-ISTC measure is not related to these other uncertainty measures.

We then use standard, recursively identified VAR model to investigate the dynamic responses of key macro variables to innovations in our uncertainty measures. For brevity in discussing the results, we will often refer to these innovations to uncertainty as *uncertainty shocks*. As is the case in all VAR analyses, the impulse responses and variance decompositions depend on the identification scheme, which is based on the ordering of the variables in our exercise.

Existing empirical research on uncertainty has often found important dynamic relationships between real activity and various uncertainty proxies. In particular, these proxies are often countercyclical and VAR estimates suggest that they have a significant impact on output and employment in the months after an innovation in these measures. A key finding is that a rise in some proxies of aggregate uncertainty depresses real activity in the short run, consistent with the predictions of some theoretical models where uncertainty is a driving force of macroeconomic fluctuations[27]. Instead, we wish to see whether the high- and low-ISTC uncertainty measures behave differently in the sense that one has a more significant aggregate effect than the other. For this purpose, we choose a specification similar to that studied in Bloom (2009), as to which variables to include in the VAR and how to order the variables. Following Bloom (2009), we use 12 lags of monthly data of the log S&P 500 index, federal funds rate, log wages, log CPI, log hours worked in the manufacturing sector, log employment for the manufacturing sector, and log industrial production. The macroeconomic dynamics of these variables have been extensively studied in the literature. All variables are included in levels. The variables are ordered as follows:

$$\begin{bmatrix} \log(\text{S\&P 500 Index}) \\ uncertainty_1 \\ uncertainty_2 \\ \text{federal funds rate} \\ \log(\text{wages}) \\ \log(\text{CPI}) \\ \log(\text{hours}) \\ \log(\text{employment}) \\ \log(\text{industrial production}) \end{bmatrix}$$

The difference with a standard VAR in the uncertainty literature is the inclusion of two uncertainty measures in the VAR in place of one. The measure of uncertainty will be computed using either high-ISTC firms or low-ISTC firms. The reason we include both of them is that we wish to see the role of each uncertainty measure in macroeconomic dynamics conditional on the other. Thus, we perform our estimation twice: once with high-ISTC ordered first, and once with low-ISTC ordered first.

Figure $A2$ displays results when we put high-ISTC uncertainty before low-ISTC uncertainty. An increase in both uncertainty measures reduces the stock index, but the magnitude of the high-ISTC uncertainty impact is larger and considerably more persistent. The same is true of industrial production: the impact of low-ISTC uncertainty is smaller and wears off after about 15 months, whereas the impact of

[27]See Ma and Samaniego (2018) for a survey. In particular, they find that that an aggregate uncertainty measure that uses all firms (instead of splitting the sample as we do) behaves in this way.

high-ISTC uncertainty lasts much longer and has a much larger peak magnitude.

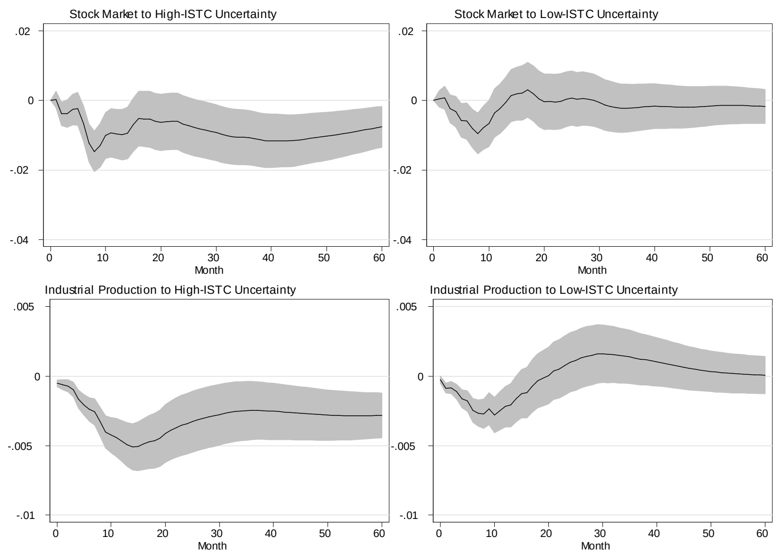

Figure $A2$ – Impulse response of stock market and industrial production from estimation of VAR with high-ISTC uncertainty and low-ISTC uncertainty. The VAR includes both forms of uncertainty, with high-ISTC uncertainty ordered before low-ISTC uncertainty.

Figure $A3$ repeats the estimation, but ordering low-ISTC uncertainty before high-ISTC uncertainty. Results are broadly similar. Now low-ISTC uncertainty has a larger short term impact on the stock market, but it wears off rapidly. In contrast, the high-ISTC uncertainty impact takes longer to develop and is considerably more persistent. Both types of uncertainty measures have similar peak impact on industrial production, but again the impact of low-ISTC uncertainty wears off after about 18 months, whereas the impact of high-ISTC uncertainty lasts much longer. We conclude that uncertainty measured in high-ISTC industries is more economically significant.

To summarize, we find evidence that ISTC is the source or a conduit of uncertainty. This is interesting because it suggests that uncertainty may have a technological origin – not with productivity, as hypothesized in Bloom et al (2012), but with ISTC. In what follows we develop a model where this uncertainty is due to the presence of imperfectly observed ISTC and neutral technological shocks.

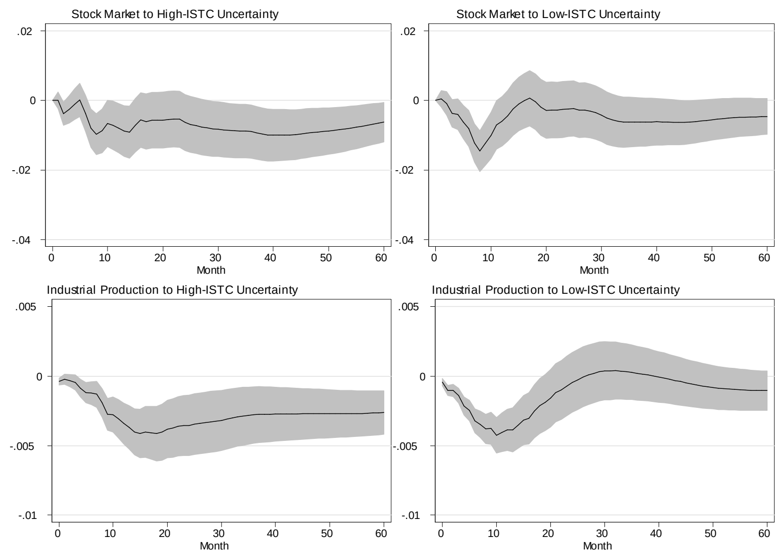

Figure $A3$ – Impulse response of stock market and industrial production from estimation of VAR with high-ISTC uncertainty and low-ISTC uncertainty. The VAR includes both forms of uncertainty, with low-ISTC uncertainty ordered before high-ISTC uncertainty.

8.2 Appendix 2: Calculation of the update of beliefs

For the ISTC and TFP, given their posterior beliefs, and their known true processes of evolution that follow AR(1) processes:

$$z_{t+1} = \psi z_t + \varepsilon_{t+1}$$

$$q_{t+1} = \rho q_t + w_{t+1}$$

We can derive the prior belief of TFP for period $t+1$, h^Z_{t+1}, and the prior belief of ISTC for period $t+1, h^Q_{t+1}$by conjugate distribution calculation as:

$$z_{t+1}\tilde{}N(\psi z_{t|t}, \psi^2\sigma_z^2 + \sigma_\varepsilon^2)$$

$$q_{t+1}\tilde{}N(\rho q_{t|t}, \rho^2\sigma_q^2 + \sigma_w^2)$$

For the capital, given the posterior belief, and the known true process of evolution:

$$k_{t+1}e^{g_k} = e^{q_t}\left[1 - S\left(e^g \frac{i_t}{i_{t-1}}\right)\right] i_t + (1-\delta)k_t$$

We can calculate the cumulative density function for a random variable $K_{t+1} = \frac{e^{Q_t}\left[1-S\left(e^g \frac{i_t}{i_{t-1}}\right)\right]i_t+(1-\delta)K_t}{e^{g_k}} \leq k'$as

$$\begin{aligned}
F^K_{t+1}(k'|m^K_t, m^q_t) &= Prob\left(\frac{e^{Q_t}\left[1-S\left(e^g \frac{i_t}{i_{t-1}}\right)\right]i_t+(1-\delta)K_t}{e^{g_k}} \leq k'\right) \\
&= \int\int_{\frac{e^q\left[1-S\left(e^g \frac{i_t}{i_{t-1}}\right)\right]i_t+(1-\delta)k}{e^{g_k}} \leq k'} m^K_t m^q_t dkdq \\
&\quad \text{where } k \text{ is the realized value of } K_t, q \text{ is the realized value of } Q_t \\
&= \int_{q=0}^{\infty}\int_{k=0}^{\frac{e^{g_k}k'-e^q\left[1-S\left(e^g \frac{i_t}{i_{t-1}}\right)\right]i_t}{(1-\delta)}} m^K_t m^Q_t dkdq \\
&= \int_{k=0}^{\infty}\int_{q=0}^{\log\left(\frac{e^{g_k}k'-k(1-\delta)}{\left[1-S\left(e^g \frac{i_t}{i_{t-1}}\right)\right]i_t}\right)} m^K_t m^Q_t dkdq
\end{aligned}$$

and therefore the prior belief of capital for period $t+1$, h^K_{t+1}, the probability density function is

$$\begin{aligned}
h^K_{t+1}(k') &= \int_{k=0}^{\infty}\frac{e^{g_k}}{e^{g_k}k'-k(1-\delta)}m^K_t m^Q_t\left(\log\left(\frac{e^{g_k}k'-k(1-\delta)}{\left[1-S\left(e^g \frac{i_t}{i_{t-1}}\right)\right]i_t}\right)\right)dk \\
&= \frac{e^{g_k}}{(1-\delta)}\int_{q=0}^{\infty}m^K_t\left(\frac{e^{g_k}k'-\left[1-S\left(e^g \frac{i_t}{i_{t-1}}\right)\right]i_t e^q}{(1-\delta)}\right)m^Q_t dq
\end{aligned}$$

Meantime, we can also derive an expectation for the next period signals (the likelihood of the signals conditional on prior distributions of unobserved variables, $h(y_{t+1}, \phi_{t+1}|z, q, k)$). That is, for next period output, the cumulative density function is

$$
\begin{aligned}
F_{t+1}^{Y}\left(y'|h_{t+1}^{Z},h_{t+1}^{K},n'\right) &= Prob\left(e^{Z_{t+1}}K_{t+1}^{\alpha}n_{t+1}^{1-\alpha}\leq y'\right)\\
&= \int\int_{e^{z'}k'^{\alpha}n'^{1-\alpha}\leq y'} h_{t+1}^{K}h_{t+1}^{Z}dk'dz'\\
&= \int_{z'=0}^{\infty}\int_{k'=0}^{\left(\frac{y'}{e^{z'}n'^{1-\alpha}}\right)^{1/\alpha}} h_{t+1}^{K}h_{t+1}^{Z}dk'dz'\\
&= \int_{k'=0}^{\infty}\int_{z'=0}^{\log\left(\frac{y'}{k'^{\alpha}n'^{1-\alpha}}\right)} h_{t+1}^{K}h_{t+1}^{Z}dk'dz'
\end{aligned}
$$

and therefore the prior belief of output for period $t+1$, h_{t+1}^{Y}, the probability density function is

$$
\begin{aligned}
h_{t+1}^{Y}(y') &= \frac{1}{\alpha}y'^{\frac{1}{\alpha}-1}n'^{\frac{\alpha-1}{\alpha}}\int_{z'=0}^{\infty}\left(\frac{1}{e^{z'}}\right)^{1/\alpha}h_{t+1}^{K}\left(\left[\frac{y'}{e^{z'}n'^{1-\alpha}}\right]^{1/\alpha}\right)h_{t+1}^{Z}dz'\\
&= \int_{k'=0}^{\infty}\frac{k'^{\alpha}n'^{1-\alpha}}{y'}h_{t+1}^{K}h_{t+1}^{Z}\left(\log\left(\frac{y'}{k'^{\alpha}n'^{1-\alpha}}\right)\right)dk'
\end{aligned}
$$

8.3 Appendix 3: Calculation of evolution of linearized beliefs

The calculation can be done through the Kalman filter as follows.

Measurement update: Since $\tilde{y}_t = z_t + \alpha\tilde{k}_t + (1-\alpha)\tilde{n}_t$, $\tilde{\phi}_t = \pi\tilde{\phi}_{t-1} + (1-\pi)q_t + v_t$, the conditional vector

$$
\begin{pmatrix} z_t \\ q_t \\ \tilde{k}_t \\ \tilde{y}_t-(1-\alpha)\tilde{n}_t \\ \tilde{\phi}_t-\pi\tilde{\phi}_{t-1} \end{pmatrix} | Y_{t-1},\Phi_{t-1},\tilde{n}_t
$$

is Gaussian, with mean and variance:

$$\left[\begin{pmatrix} z_{t|t-1} \\ q_{t|t-1} \\ \tilde{k}_{t|t-1} \end{pmatrix}, \begin{pmatrix} z_{t|t-1}+\alpha\tilde{k}_{t|t-1} \\ q_{t|t-1} \end{pmatrix}\right],$$

$$\left[\begin{array}{cc} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{pmatrix}, & \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-\pi \\ \alpha & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1-\pi & 0 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{pmatrix}, & \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1-\pi & 0 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-\pi \\ \alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v \end{pmatrix} \end{array}\right]$$

To compute $\begin{pmatrix} z_t \\ q_t \\ \tilde{k}_t \end{pmatrix} |Y_t, \Phi_t, \tilde{n}_t = \begin{pmatrix} z_t \\ q_t \\ \tilde{k}_t \end{pmatrix} |y_t, Y_{t-1}, \phi_t, \Phi_{t-1}, \tilde{n}_t$, which are the posterior beliefs on z_t, q_t, and $\tilde{k}_t$, we apply the formula for conditional expectation of Gaussian random variables, with everything preconditioned on Y_t and Φ_t. It follows that $\begin{pmatrix} z_t \\ q_t \\ \tilde{k}_t \end{pmatrix} |Y_t, \Phi_t, \tilde{n}_t$ is Gaussian, with mean

$$
\begin{aligned}
\begin{pmatrix} z_{t|t} \\ q_{t|t} \\ \tilde{k}_{t|t} \end{pmatrix} &= E\left(\begin{bmatrix} z_t \\ q_t \\ \tilde{k}_t \end{bmatrix} |Y_t, \Phi_t, \tilde{n}_t \right) \\
&= \begin{bmatrix} z_{t|t-1} \\ q_{t|t-1} \\ \tilde{k}_{t|t-1} \end{bmatrix} + \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-\pi \\ \alpha & 0 \end{bmatrix} \\
&\quad \left[\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1-\pi & 0 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-\pi \\ \alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v \end{pmatrix} \right]^{-1} \\
&\quad \begin{bmatrix} \tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha \tilde{k}_{t|t-1} \\ \tilde{\phi}_t - q_{t|t-1} - \pi \tilde{\phi}_t \end{bmatrix} \\
&= \begin{bmatrix} z_{t|t-1} \\ q_{t|t-1} \\ \tilde{k}_{t|t-1} \end{bmatrix} + P_t \begin{bmatrix} \tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha \tilde{k}_{t|t-1} \\ \tilde{\phi}_t - q_{t|t-1} - \pi \tilde{\phi}_{t-1} \end{bmatrix}
\end{aligned}
$$

and covariance

$$
\begin{aligned}
\begin{pmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{pmatrix} &= cov\left(\begin{bmatrix} z_t \\ q_t \\ \tilde{k}_t \end{bmatrix} |Y_t, \Phi_t, \tilde{n}_t \right) \\
&= \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{bmatrix} - \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-\pi \\ \alpha & 0 \end{bmatrix} \\
&\quad \left[\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1-\pi & 0 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-\pi \\ \alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v \end{pmatrix} \right]^{-1} \\
&\quad \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1-\pi & 0 \end{bmatrix} \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{bmatrix}
\end{aligned}
$$

where

$$P_t = \begin{bmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1-\pi \\ \alpha & 0 \end{bmatrix} \times$$

$$\left[\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1-\pi & 0 \end{pmatrix} \begin{pmatrix} \Sigma_t^Z & \Sigma_t^{Z,Q} & \Sigma_t^{\tilde{K},Z} \\ \Sigma_t^{Z,Q} & \Sigma_t^Q & \Sigma_t^{\tilde{K},Q} \\ \Sigma_t^{\tilde{K},Z} & \Sigma_t^{\tilde{K},Q} & \Sigma_t^{\tilde{K}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-\pi \\ \alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_v \end{pmatrix} \right]^{-1}$$

P_t converges very quickly to a stationary matrix P. Therefore we will use the converged values (steady state) of P for the calculation and simulation.

Time update: Recall that $z_{t+1} = \psi z_t + \varepsilon_{t+1}$. $q_{t+1} = \rho q_t + w_{t+1}, \tilde{k}_{t+1} = \frac{(1-\delta)}{e^{g_k}} \tilde{k}_t + \frac{\bar{\imath}}{\bar{k} e^{g_k}} \tilde{\imath}_t + \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}} q_t$. Furthermore, z_{t+1} and ε_{t+1} are independent, and q_{t+1} and w_{t+1} are independent given Y_t, Φ_t. Therefore, posterior beliefs of the means are

$$\begin{bmatrix} z_{t+1|t} \\ q_{t+1|t} \\ \tilde{k}_{t+1|t} \end{bmatrix} = E\left(\begin{bmatrix} z_{t+1} \\ \tilde{k}_{t+1} \\ q_{t+1} \end{bmatrix} |Y_t, \Phi_t, \tilde{n}_t \right) = \begin{bmatrix} \psi & 0 & 0 \\ 0 & \rho & 0 \\ 0 & \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}} & \frac{(1-\delta)}{e^{g_k}} \end{bmatrix} \begin{bmatrix} z_{t|t} \\ q_{t|t} \\ \tilde{k}_{t|t} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{\bar{\imath}}{\bar{k}e^{g_k}} \end{bmatrix} \tilde{\imath}_t$$

posterior beliefs of the covariances are

$$\begin{bmatrix} \Sigma_{t+1}^Z & \Sigma_{t+1}^{Z,Q} & \Sigma_{t+1}^{\tilde{K},Z} \\ \Sigma_{t+1}^{Z,Q} & \Sigma_{t+1}^Q & \Sigma_{t+1}^{\tilde{K},Q} \\ \Sigma_{t+1}^{\tilde{K},Z} & \Sigma_{t+1}^{\tilde{K},Q} & \Sigma_{t+1}^{\tilde{K}} \end{bmatrix} = cov\left(\begin{bmatrix} z_{t+1|t} \\ q_{t+1|t} \\ \tilde{k}_{t+1|t} \end{bmatrix} |Y_t, \Phi_t, \tilde{n}_t \right)$$

$$= \begin{bmatrix} \psi & 0 & 0 \\ 0 & \rho & 0 \\ 0 & \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}} & \frac{(1-\delta)}{e^{g_k}} \end{bmatrix} \begin{bmatrix} \sigma_t^Z & \sigma_t^{Q,Z} & \sigma_t^{\tilde{K},Z} \\ \sigma_t^{Q,Z} & \sigma_t^Q & \sigma_t^{\tilde{K},Q} \\ \sigma_{t,}^{\tilde{K},Z} & \sigma_t^{\tilde{K},Q} & \sigma_t^{\tilde{K}} \end{bmatrix} \begin{bmatrix} \psi & 0 & 0 \\ 0 & \rho & \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}} \\ 0 & 0 & \frac{(1-\delta)}{e^{g_k}} \end{bmatrix} + \begin{bmatrix} \sigma_\varepsilon & 0 & 0 \\ 0 & \sigma_w & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The conditional covariance matrices do not depend on the measurement $\tilde{y}_t$ and $\tilde{\phi}_t$. They can therefore be computed in advance, given the noise variances and model parameters.

We can simplify this process, by plugging in the posterior beliefs to the next period prior beliefs, and we get

$$\begin{aligned} z_{t+1|t} &= \psi z_{t|t} \\ &= \psi[z_{t|t-1} + P_{1,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{1,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})] \end{aligned}$$

$$\begin{aligned} q_{t+1|t} &= \rho q_{t|t} \\ &= \rho[q_{t|t-1} + P_{2,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{2,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})] \end{aligned}$$

$$\begin{aligned} \tilde{k}_{t+1|t} &= \frac{(1-\delta)}{e^{g_k}}\tilde{k}_{t|t} + \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}q_{t|t} + \frac{\bar{\imath}}{\bar{k}e^{g_k}}\tilde{\imath}_t \\ &= \frac{(1-\delta)}{e^{g_k}}[\tilde{k}_{t|t-1} + P_{3,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{3,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})] \\ &\quad + \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}[q_{t|t-1} + P_{2,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{2,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})] \\ &\quad + \frac{\bar{\imath}}{\bar{k}e^{g_k}}\tilde{\imath}_t \end{aligned}$$

8.4 Appendix 4:

We have the posterior beliefs as:

$$z_{t|t} = z_{t|t-1} + P_{1,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{1,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})$$

$$q_{t|t} = q_{t|t-1} + P_{2,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{2,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})$$

$$\tilde{k}_{t|t} = \tilde{k}_{t|t-1} + P_{3,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{3,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})$$

The difference between the actual value z_t and the posterior belief $z_{t|t}$ is then given by:

$$
\begin{aligned}
& z_{t|t} - z_t \\
= \;& -P_{1,2}\pi\tilde{\phi}_{t-1} + z_{t|t-1} + P_{1,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{1,2}(\tilde{\phi}_t - q_{t|t-1}) - z_t \\
= \;& -P_{1,2}\pi\tilde{\phi}_{t-1} + z_{t|t-1} - z_t + P_{1,1}[z_t + \alpha\tilde{k}_t + (1-\alpha)\tilde{n}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}] \\
& ... + P_{1,2}(q_t + v_t - q_{t|t-1}) \\
= \;& -P_{1,2}\pi\tilde{\phi}_{t-1} + z_{t|t-1} - z_t + P_{1,1}(z_t - z_{t|t-1}) + P_{1,1}(\alpha\tilde{k}_t - \alpha\tilde{k}_{t|t-1}) + P_{1,2}(q_t - q_{t|t-1} + v_t) \\
= \;& -P_{1,2}\pi\tilde{\phi}_{t-1} + (1-P_{1,1})\{[E_{t-1}(\psi z_{t-1} + \varepsilon_t)] - (\psi z_{t-1} + \varepsilon_t)\} + P_{1,2}\{\rho q_{t-1} + \epsilon_t \\
& ... - [E_{t-1}(\rho q_{t-1} + \epsilon_t)] + v_t\} + \alpha[\frac{(1-\delta)}{e^{g_k}}\tilde{k}_{t-1} + \frac{\bar{\imath}}{\bar{k}e^{g_k}}\tilde{\imath}_{t-1} + \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}q_{t-1} - \frac{(1-\delta)}{e^{g_k}}\tilde{k}_{t-1|t-1} \\
& ... - \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}q_{t-1|t-1} - \frac{\bar{\imath}}{\bar{k}e^{g_k}}\tilde{\imath}_t] \\
= \;& -P_{1,2}\pi\tilde{\phi}_{t-1} + (1-P_{1,1})[\psi(z_{t-1|t-1} - z_{t-1}) - \varepsilon_t] + P_{1,2}[\rho(q_{t-1} - q_{t-1|t-1}) + \epsilon_t + v_t] + \\
& ... + P_{1,1}\alpha[\frac{(1-\delta)}{e^{g_k}}(\tilde{k}_{t-1} - \tilde{k}_{t-1|t-1}) + \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}(q_{t-1} - q_{t-1|t-1})] \\
= \;& -P_{1,2}\pi\tilde{\phi}_{t-1} + (1-P_{1,1})\psi(z_{t-1|t-1} - z_{t-1}) + (P_{1,2}\rho + P_{1,1}\alpha\frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}})(q_{t-1} - q_{t-1|t-1}) \\
& ... + P_{1,1}\alpha\frac{(1-\delta)}{e^{g_k}}(\tilde{k}_{t-1} - \tilde{k}_{t-1|t-1}) - (1-P_{1,1})\varepsilon_t + P_{1,2}(\epsilon_t + v_t)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
z_{t|t} - z_t \;=\; & -P_{1,2}\pi\tilde{\phi}_{t-1} + (1-P_{1,1})\psi(z_{t-1|t-1} - z_{t-1}) - (P_{1,2}\rho + P_{1,1}\alpha\frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}})(q_{t-1} - q_{t-1|t-1}) \\
& ... - P_{1,1}\alpha\frac{(1-\delta)}{e^{g_k}}(\tilde{k}_{t-1} - \tilde{k}_{t-1|t-1}) - (1-P_{1,1})\varepsilon_t + P_{1,2}(\epsilon_t + v_t)
\end{aligned}
$$

Similarly, the difference between the actual value q_t and the posterior belief $q_{t|t}$ is

$$
\begin{aligned}
& q_{t|t} - q_t \\
= \quad & -P_{2,2}\pi\tilde{\phi}_{t-1} + q_{t|t-1} + P_{2,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - q_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{2,2}(\tilde{\phi}_t - q_{t|t-1}) - q_t \\
= \quad & -P_{2,2}\pi\tilde{\phi}_{t-1} + q_{t|t-1} - q_t + P_{2,1}[z_t + \alpha\tilde{k}_t + (1-\alpha)\tilde{n}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}] + P_{2,2}(q_t + v_t - q_{t|t-1}) \\
= \quad & -P_{2,2}\pi\tilde{\phi}_{t-1} + q_{t|t-1} - q_t + P_{2,2}(q_t - q_{t|t-1}) + P_{2,1}(\alpha\tilde{k}_t - \alpha\tilde{k}_{t|t-1}) + P_{2,1}(z_t - z_{t|t-1}) + P_{2,2}v_t \\
= \quad & -P_{2,2}\pi\tilde{\phi}_{t-1} + (1-P_{2,2})(\rho q_{t-1|t-1} - \rho q_{t-1} - \epsilon_t) + P_{2,1}\alpha[\frac{(1-\delta)}{e^{g_k}}(\tilde{k}_{t-1} - \tilde{k}_{t-1|t-1}) + \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}(q_{t-1} - q_{t-1|t-1} \\
& ... + P_{2,1}(\psi z_{t-1} + \varepsilon_t - \psi z_{t-1|t-1}) + P_{2,2}v_t \\
= \quad & -P_{2,2}\pi\tilde{\phi}_{t-1} + [(1-P_{2,2})\rho - P_{2,1}\alpha\frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}](q_{t-1|t-1} - q_{t-1}) + P_{2,1}\psi(z_{t-1} - z_{t-1|t-1}) \\
& ... + P_{2,1}\alpha\frac{(1-\delta)}{e^{g_k}}(\tilde{k}_{t-1} - \tilde{k}_{t-1|t-1}) + P_{2,1}\varepsilon_t - (1-P_{2,2})\epsilon_t + P_{2,2}v_t
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
q_{t|t} - q_t \quad = \quad & -P_{2,2}\pi\tilde{\phi}_{t-1} + [(1-P_{2,2})\rho - P_{2,1}\alpha\frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}](q_{t-1|t-1} - q_{t-1}) - P_{2,1}\psi(z_{t-1|t-1} - z_{t-1}) \\
& -P_{2,1}\alpha\frac{(1-\delta)}{e^{g_k}}(\tilde{k}_{t-1|t-1} - \tilde{k}_{t-1}) + P_{2,1}\varepsilon_t - (1-P_{2,2})\epsilon_t + P_{2,2}v_t
\end{aligned}
$$

Similarly, the difference between the actual value $\tilde{k}_t$ and the posterior belief $\tilde{k}_{t|t}$ is

$$
\begin{aligned}
& \tilde{k}_{t|t} - \tilde{k}_t \\
= \quad & -P_{3,2}\pi\tilde{\phi}_{t-1} + \tilde{k}_{t|t-1} - \tilde{k}_t + P_{3,1}(\tilde{y}_t - (1-\alpha)\tilde{n}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{3,2}(\tilde{\phi}_t - q_{t|t-1}) - \tilde{k}_t \\
= \quad & -P_{3,2}\pi\tilde{\phi}_{t-1} + \tilde{k}_{t|t-1} - \tilde{k}_t + P_{3,1}\alpha(\tilde{k}_t - \tilde{k}_{t|t-1}) + P_{3,1}(z_t - z_{t|t-1}) + P_{3,2}(q_t - q_{t|t-1} + v_t) \\
= \quad & -P_{3,2}\pi\tilde{\phi}_{t-1} + (1-P_{3,1}\alpha)(\tilde{k}_{t|t-1} - \tilde{k}_t) + P_{3,1}(z_t - z_{t|t-1}) + P_{3,2}(q_t - q_{t|t-1} + v_t) \\
= \quad & -P_{3,2}\pi\tilde{\phi}_{t-1} + (1-P_{3,1}\alpha)[\frac{(1-\delta)}{e^{g_k}}\tilde{k}_{t-1|t-1} + \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}q_{t-1|t-1} + \frac{\bar{\imath}}{\bar{k}e^{g_k}}\tilde{\imath}_{t-1} - \frac{(1-\delta)}{e^{g_k}}\tilde{k}_{t-1} - \frac{\bar{\imath}}{\bar{k}e^{g_k}}\tilde{\imath}_{t-1} \\
& ... - \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}q_{t-1}] + P_{3,1}\psi(z_{t-1} - z_{t-1|t-1}) + P_{3,2}\rho(q_{t-1} - q_{t-1|t-1}) + P_{3,1}\varepsilon_t + P_{3,2}(\epsilon_t - v_t) \\
= \quad & -P_{3,2}\pi\tilde{\phi}_{t-1} + (1-P_{3,1}\alpha)\frac{(1-\delta)}{e^{g_k}}(\tilde{k}_{t-1|t-1} - \tilde{k}_{t-1}) + P_{3,1}\psi(z_{t-1} - z_{t-1|t-1}) \\
& ... + [(1-P_{3,1}\alpha)\frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}} - P_{3,2}\rho](q_{t-1|t-1} - q_{t-1}) + P_{3,1}\varepsilon_t + P_{3,2}(\epsilon_t - v_t)
\end{aligned}
$$

Therefore, we have

$$\tilde{k}_{t|t} - \tilde{k}_t = (1 - P_{3,1}\alpha)\frac{(1-\delta)}{e^{g_k}}(\tilde{k}_{t-1|t-1} - \tilde{k}_{t-1}) - P_{3,1}\psi(z_{t-1|t-1} - z_{t-1})$$
$$... + [(1 - P_{3,1}\alpha)\frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}} - P_{3,2}\rho](q_{t-1|t-1} - q_{t-1}) + P_{3,1}\varepsilon_t + P_{3,2}(\epsilon_t - v_t)$$

8.5 Appendix 5: Linearized equations characterizing the economy

Since the evolution of beliefs are in log-linearized form, it will be convenient if the FOCs and budget constraints are also in log-linearized form. Thus we transform the equations before doing the estimation and simulation into the following linearized system of equations, after assuming the utility function follows $U = \log c_t - An_t$:

Evolution of beliefs:

$z_{t+1|t} = \psi[z_{t|t-1} + P_{1,1}(z_t + \alpha\tilde{k}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{1,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})]$

$q_{t+1|t} = \rho[q_{t|t-1} + P_{2,1}(z_t + \alpha\tilde{k}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{2,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})]$

$\tilde{k}_{t+1|t} = \frac{(1-\delta)}{e^{g_k}}[\tilde{k}_{t|t-1} + P_{3,1}(z_t + \alpha\tilde{k}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{3,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})] +$
$\quad + \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}[q_{t|t-1} + P_{2,1}(z_t + \alpha\tilde{k}_t - z_{t|t-1} - \alpha\tilde{k}_{t|t-1}) + P_{2,2}(\tilde{\phi}_t - \pi\tilde{\phi}_{t-1} - q_{t|t-1})] + \frac{\bar{\imath}}{\bar{k}e^{g_k}}\tilde{\imath}_t$

Resources:

$\tilde{y}_t = \frac{\bar{c}}{\bar{y}}\tilde{c}_t + \frac{\bar{\imath}}{\bar{y}}\tilde{\imath}_t$

$\tilde{y}_{t|t-1} = \frac{\bar{c}}{\bar{y}}\tilde{c}_{t|t-1} + \frac{\bar{\imath}}{\bar{y}}\tilde{\imath}_{t|t-1}$

Household optimization:

$0 = \tilde{c}_t - E_t\tilde{c}_{t+1} + \frac{\beta}{e^{g_k}}(\bar{r}E_t\tilde{r}_{t+1} - (1-\delta)q_{t+1|t}) + q_{t|t}$

$0 = \tilde{c}_{t|t-1|} + \tilde{w}_{t|t-1}$

$\tilde{\imath}_t = \frac{1}{1+\beta}\tilde{\imath}_{t-1} + (1 - \frac{1}{1+\beta})E_t\tilde{\imath}_{t+1} + \frac{1}{(1+\beta)e^{2g}\varphi}\tilde{p}_t + q_{t|t}$

$\tilde{p}_t = \frac{\beta}{1-\delta}E_t\tilde{p}_{t+1} + (1 - \frac{\beta}{1-\delta})E_t\tilde{r}_{t+1} - r_t$

Production optimization:

$\tilde{y}_t = z_t + \alpha\tilde{k}_t + (1-\alpha)\tilde{n}_t$

$\tilde{y}_{t|t-1} = z_{t|t-1} + \alpha\tilde{k}_{t|t-1} + (1-\alpha)\tilde{n}_t$

$\tilde{r}_{t|t-1} = (1 - \beta(1-\delta))(z_{t|t-1} + (\alpha - 1)\tilde{k}_{t|t-1} + (1-\alpha)\tilde{n}_t)$

$\tilde{w}_{t|t-1} = z_{t|t-1} + \alpha\tilde{k}_{t|t-1} - \alpha\tilde{n}_t$

$\tilde{r}_t = (1 - \beta(1-\delta))(z_t + (\alpha - 1)\tilde{k}_t + (1-\alpha)\tilde{n}_t)$

$\tilde{w}_t = z_t + \alpha\tilde{k}_t - \alpha\tilde{n}_t$

Capital evolution

$\tilde{k}_t = \frac{(1-\delta)}{e^{g_k}}\tilde{k}_{t-1} + \frac{\bar{\imath}}{\bar{k}e^{g_k}}\tilde{\imath}_{t-1} + \frac{\bar{\imath}(1+\beta)e^{2g}\varphi}{\bar{k}e^{g_k}}q_{t-1}$

Shock processes:

$\tilde{\phi}_t = \pi\tilde{\phi}_{t-1} + (1-\pi)q_t + v_t$

$$
z_t = \psi z_{t-1} + \varepsilon_t
$$
$$
q_t = \rho q_{t-1} + w_t
$$