Matching with Complementary Contracts*

Marzena Rostek[†] and Nathan Yoder[‡]

September 19, 2018

Abstract

In this paper, we provide existence results for matching environments with complementarities, such as markets for patent licenses, differentiated products, or multi-sided platforms. Our results apply to both nontransferable and transferable utility settings, and allow for multilateral agreements and those with externalities. Additionally, we give comparative statics regarding the way primitive characteristics are combined to form the set of available contracts. These show the impact of various contract design decisions, such as the application of antitrust law to disallow patent cross-licenses, on stable outcomes.

Keywords: Complementarities, Matching with Contracts, Stability, Contract Design

* The authors are grateful to numerous colleagues for their helpful comments. This material is based upon work supported by the National Science Foundation under Grant No. SES-1357758. Parts of this paper contain material from an earlier (2017) draft of Rostek and Yoder (2018).

† University of Wisconsin-Madison, Department of Economics; E-mail: `mrostek@ssc.wisc.edu`.

‡ University of Georgia, Terry College of Business, Department of Economics; E-mail: `nathan.yoder@uga.edu`.

1 Introduction

In many settings of economic interest, agents negotiate indivisible agreements with one another. When these agreements are substitutable, the matching literature gives us ample tools to determine which sets of them are *stable*, or robust to renegotiation (e.g., Gale and Shapley (1962), Kelso and Crawford (1982), or Hatfield and Milgrom (2005)).

However, many of these environments are characterized by complementarities. For example, agents typically derive more value from participating in a multi-sided platform (e.g., Rochet and Tirole (2006), Weyl (2010)) when a larger set of other agents also participate. If participation in one platform lowers the cost of participation in others — for instance, if a post on Twitter can easily be reposted to Facebook — this observation extends to competing platforms. Complementarities are also a defining feature of the market for patent licenses. If a firm needs to secure licenses from multiple rightsholders in order to sell a given product, those licenses are perfect complements (e.g., Shapiro (2000)). Similarly, licenses will be complementary if they lower a firm's marginal cost of production: Acquiring one will cause a firm to produce more units of output, each of which will become cheaper to produce after acquiring subsequent licenses. Finally, differentiated product markets also frequently feature complementarities — see, e.g., Nalebuff (1999).

Although these applications are important, accommodating complementary agreements in matching models has proven challenging.[1] Positive existence results have been given when there is a continuum of agents (e.g., Azevedo and Hatfield (2015), Che et al. (2017), Kojima et al. (2013), Jagadeesan (2017), Scotchmer and Shannon (2015)[2]) or outcomes (Hatfield and Kominers (2015)). Pycia (2012) shows that stable outcomes exist when preferences are *pairwise aligned*, a condition that allows for complementarities but demands strong agreement between agents. For example, in a two-sided matching market, workers must strictly prefer working with exactly those other workers that their employer prefers hiring alongside them. Existence results have also been given when preferences over agreements satisfy substitutability after the application of a suitable change of basis — most notably, under the *gross substitutes and complements* condition of Sun and Yang (2006).

To our knowledge, the literature has yet to provide existence results which accommodate more general forms of complementarities. In our view, the biggest reason for this is its implicit assumption that the most basic matching environments — the ones where existence results should hold, if they are to hold anywhere — are the classical one-to-one and many-to-one models. These settings rule out some complementarities directly: agents on at least one side

[1] In particular, these complementarities are different from those discussed in the assortative matching literature following Becker (1973). Assortative matching models consider complementarities *between types*, whereas these settings feature complementarities *between agreements*, which do not arise in a one-to-one setting.

[2] Scotchmer and Shannon (2015) additionally consider settings with incomplete information.

of the market are limited to a single agreement. If working for Hospital A rules out working for Hospital B, the two jobs cannot be complementary in any way.

Moreover, by requiring substitutability on one side of the market, the many-to-one structure also contributes to nonexistence when agents on the *other* side of the market view agreements as complementary. To understand why, consider typical maximal domain results in the literature (e.g., those of Hatfield and Kojima (2008) or Hatfield and Kominers (2016)). These are often interpreted as implying that stable outcomes do not generally exist when agreements are not substitutable. However, a closer examination of the quantifiers used in these results reveals that they do not rule out stability in the presence of complementarities. Each is of the form "if any agent has preferences outside of class $\mathcal{C}$, there exists a profile of preferences in $\mathcal{C}^{N-1}$ for the other agents such that no stable outcome exists." But such statements do not imply that existence is nongeneric when agents' preferences lie in a class $\mathcal{D}$ which does not contain $\mathcal{C}$. This is precisely the case when $\mathcal{D}$ is the class of preferences with complementarities and $\mathcal{C}$ is the class of preferences for which agreements are substitutes. Thus, the message of these converses is more nuanced than it might appear: if stable outcomes are to generically exist in the presence of arbitrary complementarities, subsitutability between agreements must be limited.

This is precisely what we do in this paper. Instead of weakening the substitutes condition, we abandon it entirely.[3] In its place, we assume complementarity between agreements — which, following Hatfield and Milgrom (2005), we refer to as *contracts*. Our results are readily applicable to two-sided many-to-many matching settings, which do not *per se* rule out such complementarities.[4] However, our framework is more general: we do not assume any two-sided market structure. In fact, we allow contracts to be multilateral and even to have externalities that affect the choices of other agents.[5]

[3]In particular, we do not consider a setting isomorphic to an exchange economy with subsitutable goods (e.g., Ostrovsky (2008), Hatfield et al. (2013), Hatfield et al. (2018), or Fleiner et al. (2018)).

[4]As noted earlier, many-to-one settings *do* rule out complementarities, since they restrict agents on one side to a single contract. In general, complementarity is incompatible with limitations on the number of contracts that agents may sign: once they reach the limit, adding contracts to their choice set is either irrelevant or causes them to reject contracts they previously accepted. However, complementarity characterizes many settings without such restrictions, such as the markets for patent licenses, differentiated products, and multi-sided platforms that we discuss throughout the paper.

[5]Relative to most of the matching literature, allowing for multilateral contracts is an innovation in its own right: Multilateral contracts have so far been investigated by Hatfield and Kominers (2015), who consider divisible agreements, and Teytelboym (2012), who utilizes Pycia's (2012) pairwise alignment condition in the contracts context. (We consider multilateral contracts in the substitutes context in Rostek and Yoder (2018).) On the other hand, several papers consider externalities in two-sided matching environments; the most general of these is Pycia and Yenmez (2017). In contrast to our setting with complementarities, externalities, like multilateral contracts, are much more challenging to accommodate in their two-sided setting with substitutes. Our results in Rostek and Yoder (2018) suggest that this is because stability in these settings relies on each side of the market having well-behaved aggregate demand, but externalities or multilateral contracts can cause the weak axiom to fail under aggregation.

Our main result, Theorem 1, shows that a unique stable outcome exists when contracts are complementary. Moreover, it characterizes that stable outcome as the largest fixed point of a monotone operator representing the market's aggregate demand for contracts. This operator has not previously appeared in the literature; in particular, it is *not* a Gale-Shapley operator. We show that this fixed point is precisely the outcome which clears the market for contracts: there is no excess supply and no excess demand.

In practice, the set of available contracts may be affected by decisions of a designer. For instance, regulators may attempt to block the formation of patent pools or cross-licencing agreements; the introduction of differentiated products; or the integration of complementary platforms (e.g., Facebook and Instagram) under antitrust law. These potential changes each combine, or *bundle*, contracts into a single agreement. In Section 5, we examine their effects on the stable outcome.

The bundling operation we consider is related to the expressiveness ordering introduced by Hatfield and Kominers (2016) in many-to-many settings with substitutes. However, it is distinct: We consider the effect of bundling contracts together and replacing them with the resulting agreement, while they analyze the impact of making new bundles available while leaving existing contracts intact. These correspond to separate interventions by a market designer. For instance, our comparative static informs us about the effect of requiring patents to be licensed individually *instead* of cross-licensed; theirs, on the other hand, concerns the requirement that patents be licensed individually *in addition* to being available for cross-licensing.

Proposition 1 shows that when a set of contracts is *not* part of the stable outcome, changes in the way that set is bundled always make agents weakly better off and never cause existing contracts to be cancelled. Thus, a designer can experiment with different regulatory regimes without risking welfare, so long as existing agreements are exempt. Proposition 2, on the other hand, considers the bundling and unbundling of contracts when they *are* part of a stable outcome. Neither intervention will add new agreements to the existing outcome; moreover, bundling existing agreements together can only destabilize an outcome if they involve different sets of agents and have negative externalities.

In Section 6, we consider settings where agents have transferable utility, and the nonpecuniary aspects of an agreement (for which we use the term *primitive contracts* (Hatfield and Kominers (2016))) can be combined with any set of transfers to create a contract. Here, we first show that the classical welfare theorems hold in the presence of complementarities between primitive contracts (Theorem 2) by using a discrete convex duality result from Fujishige (1984):[6] Every competitive equilibrium is efficient and every efficient outcome can be com-

[6]In matching or indivisible goods markets, the classical welfare theorems do not hold without restrictions on preferences such as the gross substitutes property or (as we show in Theorem 2) the gross complements

bined with a suitable price vector to form a competitive equilibrium. Theorem 3 then links competitive equilibria with stability in our setting with complementarities, externalities, and multilateral contracts. In the absence of negative externalities, the largest efficient outcome is stable when paired with transfers corresponding to competitive equilibrium prices. This result contrasts with our nontransferable utility result, Theorem 1, which places no requirements on the direction of externalities. The representation of stable outcomes as solutions to a supermodular social planner's problem allows for a comparative static: Proposition 3 shows that stronger complementarities increase the size of the stable outcome. That is, when the presence of one primitive contract increases agents' marginal utility from the others by a greater amount, more of those primitive contracts will be included in the stable outcome.

Finally, we consider an alternative solution concept from the many-to-many matching literature, *setwise stability*, (Echenique and Oviedo (2006), Klaus and Walzl (2009)) in the context of complementarities. The chief difference between this concept and stability is that the former's robustness criterion is based on *common preference* among renegotiating agents, whereas the latter's is based on *common choice*. This means that, in general, setwise stability is neither more nor less demanding than stability.[7]

As we show, setwise stability is closely linked to the *coalition-proof correlated equilibrium* (Milgrom and Roberts (1996), Moreno and Wooders (1996)) stability concept for normal form games. Specifically, Proposition 4 shows that for any matching market, we can define a game whose coalition-proof correlated equilibria correspond to setwise stable outcomes. Further, when contracts are complementary and have positive externalities, the reverse is true: each setwise stable outcome corresponds to a coalition-proof correlated equilibrium of our game. To our knowledge, this connection is new to the literature, even in the context of substitutes.

While this result is interesting on its own, we leverage it to make use of Milgrom and Roberts' (1996) result that coalition-proof equilibria exist in normal-form games with strategic complementarities and monotone externalities. We show that the game from Proposition 4 satisfies these conditions when contracts in the matching market it is based on are complementary and have positive externalities. This yields an existence result (Theorem 4) for setwise stable outcomes, and shows that they are payoff-equivalent to the stable outcomes found in Theorem 1. However, it demands more restrictive assumptions than Theorem 1; namely, externalities that are weakly positive. Intuitively, setwise stability requires not only that individuals have no incentive to remove existing contracts, but that *coalitions* have no incentive to do so either.

The paper is organized as follows. Section 2 describes the environment. Section 3 de-

property. See, e.g., Gul and Stacchetti (1999) (in goods markets) or (Hatfield et al., 2013) (in bilateral matching markets).

[7]In Example 2, we show that the two concepts may yield disjoint collections of stable outcomes.

scribes the solution concept. Section 4 provides our existence and characterization results for environments with nontransferable utility; Section 5 considers the effects of bundling in those environments. Section 6 gives our results for environments with transferable utility. Finally, Section 7 considers setwise stability. Appendix A endogenizes the objects bundled together in Section 5. Results whose proofs are omitted from the main text are proven in Appendix B.

2 Setting

There is a finite set N of agents and a finite set X of contracts they can sign with one another. Each contract $x \in X$ requires the agreement of a set of agents $N(x) \subseteq N$ to enter into force. For sets of contracts $X' \subseteq X$, we write $N(X') \equiv \bigcup_{x \in X'} N(x)$. For each agent $i \in N$, denote the set of contracts requiring i's agreement as $X_i \equiv \{x | i \in N(x)\}$. In keeping with the literature, we say that X_i is the set of contracts that *name* i. Similarly, let $X_J \equiv \bigcup_{i \in J} X_i$, let $X_{-i} \equiv X \setminus X_i$, and for sets of contracts $X' \subseteq X$, write $X'_i \equiv X' \cap X_i$ and $X'_{-i} \equiv X' \cap X_{-i}$. We assume that each contract names at least one agent. When a contract names only one agent, it represents that agent taking an action. Each agent i has preferences over sets of implemented contracts, or *outcomes*, which are represented by a utility function $u_i : 2^X \to \mathbb{R}$.[8]

Define each agent i's *choice correspondence* $C_i : 2^{X_i} \times 2^{X_{-i}} \rightrightarrows 2^{X_i}$ as follows: $C_i(X'|Y) \equiv \arg\max_S \{u_i(S \cup Y) \text{ s.t. } S \subseteq X'\}$. $C_i(X'|Y)$ gives the sets of contracts that i might choose to sign from the set of available contracts X' when she expects the other agents to sign Y.

For our purposes, the crucial property of these choice correspondences is complementarity. We say that *contracts are complements* if for all $Y \subseteq Z \subseteq X$ and $i \in N$, $Y^* \in C_i(Y_i|Y_{-i})$ and $Z^* \in C_i(Z_i|Z_{-i})$ imply $Y^* \cup Z^* \in C_i(Z_i|Z_{-i})$. In words, complements means that an agent never rejects a previously chosen contract when new contracts become available to her or other agents sign new contracts. When C_i is single-valued, complements means that it is monotone (in the usual set order, $\subseteq$) in both the set of contracts available for i to sign and the set of contracts she expects other agents to sign. In general, it is slightly weaker than monotonicity (in the strong set order, $\sqsubseteq$) of C_i in both arguments. Hence, it is implied by quasisupermodularity of the utility function u_i (Topkis (1998)).

2.1 Notes

Most papers in the matching literature assume that the market has a certain structure (e.g., a two-sided market, an acyclic network, etc.) and do not have externalities. The majority also assume that contracts are bilateral, ruling out multilateral agreements and preventing unilateral contracts (i.e., actions) from existing alongside them. We want to emphasize that

[8]Throughout, we use 2^Y to denote the power set of a set Y.

these assumptions are not present in the description of our setting. As our results show, they are not necessary to ensure the existence of stable outcomes in settings with complementarities.

3 Solution Concept

We follow the bulk of the matching literature in adopting *stability* as our primary solution concept. Below, we extend it to accommodate externalities while maintaining its core features. Namely, outcomes are stable if they are robust to unilateral deletion of contracts or multilateral addition of contracts.

Definition (Stability). A set of contracts $Y \subseteq X$ is *stable* if it is

- *Individually rational*: $Y_i \in C_i(Y_i|Y_{-i})$ for all $i \in N$.
- *Unblocked*: There does not exist $X'' \subseteq (X \setminus Y)$ such that for all $i \in N(X'')$,

$$X''_i \subseteq Z \text{ for some } Z \in C_i((X'' \cup Y)_i|(X'' \cup Y)_{-i}).$$

As is well understood, stability allows agents in a blocking coalition to disagree about the existing contracts they will keep intact. However, when contracts are complements, the only relevant blocks are those in which this disagreement is absent: If Y is individually rational, and $X''_i \subseteq Z$ for some $Z \in C_i((X'' \cup Y)_i|(X'' \cup Y)_{-i})$, then $Z \cup Y_i = X''_i \cup Y_i \in C_i((X'' \cup Y)_i|(X'' \cup Y)_{-i})$. Hence, a stronger requirement that agents in a blocking coalition have consistent beliefs about others' deletion of contracts would be redundant.

This observation is related to another that allows us to give a simple characterization of stability in terms of the fixed points of a novel operator. When contracts are complements, there is a largest element of $C_i((X'' \cup Y)_i|(X'' \cup Y)_{-i})$ — which is the *only* set that we need to check to see if it contains X''_i. Likewise, there is a largest element of $C_i(Y_i|Y_{-i})$ — which is the only set that we need to check to see if it equals Y_i. Thus, complementarities allow all relevant information about an agent's choices from a set X' to be encoded in a single *acceptance set* $A_i(X')$, rather than a collection of sets $C_i(X'_i|X'_{-i})$.

More formally, define agent i's *acceptance function* $A_i : 2^X \to 2^{X_i}$ by $A_i(X') \equiv \bigcup C_i(X'_i|X'_{-i})$. For any X', agent i's acceptance set $A_i(X')$ gives the set of contracts in X' which she is willing and able to sign, given the existence of the contracts in X' which do not name her. Lemma 1 formalizes the logic of the previous paragraph.

Lemma 1 (Complementarities ⇒ Acceptance Set is Chosen). *If contracts are complements, then $A_i(X') \in C_i(X'_i|X'_{-i})$ for all $X' \subseteq X$, $i \in N$.*

Proof. Choose $Y = Z = X'$ in the definition of complements. For any $Y^*, Z^* \in C_i(X'_i|X'_{-i})$, $Y^* \cup Z^* \in C_i(X'_i|X'_{-i})$. By induction, $A_i(X') \equiv \bigcup C_i(X'_i|X'_{-i}) \in C_i(X'_i|X'_{-i})$. $\square$

These individual acceptance functions generate an *aggregate* acceptance function $A : 2^X \to 2^X$ according to $A(X') \equiv \bigcap_{i\in N}(A_i(X') \cup X'_{-i})$. The aggregate acceptance set $A(X')$ gives the contracts $x \in X'$ that each agent $i \in N(x)$ is willing to sign given the existence of the contracts in X'_{-i}. It can thus be interpreted as the aggregate demand for the entire market when faced with the contracts X'. Note that in contrast to Hatfield and Milgrom (2005), "aggregate demand" refers to aggregation over agents rather than the cardinality of the chosen set of an individual agent.

Lemma 2 characterizes stability in terms of A.

Lemma 2 (Stability as a Fixed Point). *If contracts are complements, then $Y \subseteq X$ is stable if and only if* (1) $A(Y) = Y$ *and* (2) $A(Y') \neq Y'$ *for all* $Y' \supset Y$.

Proof. ($\Rightarrow$) Suppose Y is stable. Then for all i, $Y_i \in C_i(Y_i|Y_{-i}) \Rightarrow Y_i = A_i(Y) \Rightarrow Y = A_i(Y) \cup Y_{-i}$. Thus, $Y = A(Y)$. For (2), suppose $A(Y') = Y'$ for some $Y' \supset Y$. Then $Y'_i = A_i(Y')\forall i \in N$. By Lemma 1, $Y'_i \in C_i(Y'_i|Y'_{-i})$ for all $i \in N$. Then $X'' = Y' \setminus Y$ blocks Y, a contradiction.

($\Leftarrow$) Suppose (1) and (2) are satisfied. Since $Y = A(Y)$, $Y_i = A_i(Y)\forall i \in N$. By Lemma 1, $Y_i \in C_i(Y_i|Y_{-i})$ for all $i \in N$ and Y is individually rational.

Suppose that X'' blocks Y, i.e., for all $i \in N(X'')$, $X''_i \subseteq Z$ for some $Z \in C_i((X'' \cup Y)_i|(X'' \cup Y)_{-i})$. Since $X''_i = \emptyset$ for all $i \notin N(X'')$, we can trivially extend the quantifier: for all $i \in N$, $X''_i \subseteq Z$ for some $Z \in C_i((X''\cup Y)_i|(X''\cup Y)_{-i})$. Fix $i \in N$. By complements, since $Y_i \in C_i(Y_i|Y_{-i})$, we have $Y_i \cup Z \in C_i((X'' \cup Y)_i|(X'' \cup Y)_{-i})$. We have $Y_i \cup X''_i \subseteq Y_i \cup Z \subseteq (Y \cup X'')_i = Y_i \cup X''_i \Rightarrow Y_i \cup Z = Y_i \cup X''_i$. Then $Y_i \cup X''_i = A_i(X'' \cup Y) \Rightarrow Y \cup X'' = A_i(X'' \cup Y) \cup (Y \cup X'')_{-i}$. Since this holds for each $i \in N$, we have $Y \cup X'' = A(Y \cup X'')$, contradicting (2). So Y is unblocked. $\square$

The acceptance conditions (1) and (2) have two interpretations, one of which is economic and the other mathematical. Economically, (1) says that *there is no excess supply at Y*: all contracts available at Y are accepted by the agents they name. (2) says that *there is no excess demand at Y*: the agents would not accept a larger set of contracts were it available.[9] We can therefore think of stable outcomes as outcomes which clear the market for contracts.[10]

Mathematically, (1) says that Y *is a fixed point of A* and (2) says that *there is no fixed point of A larger than Y*. This is the backbone of our main result, Theorem 1.

[9]Note that unlike in goods markets or with transferable utility (Section 6), *no excess demand* and *no excess supply* are not equivalent here.

[10]While Lemma 2 requires complementarities, a similar market-clearing characterization of stability holds more generally; see Rostek and Yoder (2018).

4 Existence and Characterization

In this section, we show that stable outcomes exist when contracts are complements. The first step in doing so is showing that the aggregate acceptance function is monotone.

Lemma 3 (Complementarity and Monotonicity). *If contracts are complements, then A is monotone.*

Proof. Let $Y \subseteq Z \subseteq X$. From Lemma 1, for all $i \in N$, $A_i(Y) \in C_i(Y_i|Y_{-i})$ and $A_i(Z) \in C_i(Z_i|Z_{-i})$. By the definition of complements, $A_i(Y) \cup A_i(Z) \in C_i(Z_i|Z_{-i})$. Then $A_i(Y) \cup A_i(Z) \subseteq A_i(Z) \Leftrightarrow A_i(Y) \subseteq A_i(Z) \Rightarrow A_i(Y) \cup Y_{-i} \subseteq A_i(Z) \cup Z_{-i}$. Then $A(Y) \subseteq A(Z)$, as desired. □

Lemma 2 tells us that a set is stable if and only if it is a fixed point of A and there are no larger fixed points of A. Lemma 3, along with Tarski's fixed point theorem, tells us that A has a largest fixed point. This yields our main existence and uniqueness result, Theorem 1.

Theorem 1 (Stability with Complementarities). *If contracts are complements, then A has a largest fixed point X^* on 2^X, which is the unique stable outcome.*

Proof. By Lemma 3, A is monotone. Then by Tarski's fixed point theorem, its set of fixed points is a complete lattice, and so has a largest element X^*. The result follows from Lemma 2. □

When contracts are substitutes, it is well-known that stable outcomes are precisely those which can result from a two-sided deferred acceptance algorithm (Gale and Shapley (1962); Hatfield and Milgrom (2005); Hatfield and Kominers (2012)). In each stage of this algorithm, agents on one side of these contracts (e.g., colleges, hospitals, or sellers) may make new offers to agents on the other side (e.g., applicants, doctors, or buyers). From among their new offers and the offers they already hold, the recipients of these offers choose a set to reject and a set to hold until the next stage.

Theorem 1 shows that when contracts are complements, we can continue to think of stable outcomes as the results of a deferred acceptance algorithm, this time with only one side. The agents start each stage of this algorithm with a set of available contracts. In the first stage, this is the set of all contracts X. Each agent may then reject a set of available contracts that name them. When making this choice, agents take as given the existence of the available contracts that do not name them. All contracts which have not yet been rejected continue to be available in the next stage.

Both this result and its counterpart in the transferable utility section (Theorem 2) may seem similar to those showing the core is nonempty when the coalitional value function is supermodular (e.g., Sherstyuk (1999)). However, they differ in important ways. First, stability considers different deviations than the core: the core considers only deviations in which the deviating coalition stops interacting with the rest of the agents. In contrast, stability allows coalitions to maintain existing relationships when they deviate. Additionally, these nonempty core results consider complementary *agents*, whereas we consider complementary *contracts*. Finally, unlike with the core, the unique stable outcome need not be Pareto efficient. The next example illustrates this:

Example 1 (Stable Outcomes Need Not Be Pareto Efficient). Suppose there are three contracts $X = \{a, b, c\}$ which each name both of two agents $N = \{1, 2\}$ with utility functions which satisfy the following inequalities:

$$u_1(\{a, c\}) > u_1(\{a, b, c\}) > u_1(c) > u_1(a) > u_1(\{b, c\}) > u_1(\{a, b\}) > u_1(\emptyset) > u_1(b);$$
$$u_2(\{a, b\}) > u_2(\{a, b, c\}) > u_2(b) > u_2(a) > u_2(\{b, c\}) > u_2(\{a, c\}) > u_2(\emptyset) > u_2(c).$$

These utility functions satisfy quasisupermodularity, so Theorem 1 applies. The aggregate acceptance function is given by

$$A(\{a, b, c\}) = A(\{a, b\}) = A(\{a, c\}) = A(a) = a;$$
$$A(\{b, c\}) = A(b) = A(c) = A(\emptyset) = \emptyset.$$

From Theorem 1, the unique stable outcome is a, even though $\{a, b, c\}$ is Pareto efficient. Even though the agents have a profitable deviation to $\{a, b, c\}$, it does not block a: agent 1 cannot commit to signing b, whereas agent 2 cannot commit to signing c.

5 Bundling and Contract Design

Binding agreements, like the contracts in our model, are useful because they allow agents to take a certain action (e.g., granting the right to use intellectual property) contingent on other agents taking certain other actions (e.g., transferring currency or intellectual property of their own). In other words, they allow agents' individual property rights over their own actions to be combined into a joint property right over the contract's formation. If we take these actions as primitive, we can think of them as generating the set of contracts available to the agents through this process of combination, or *bundling*. Similarly, taking bilateral contracts as primitive, one can generate a set of multilateral agreements.

Importantly, there are many ways in which this bundling can take place. Patent licenses can be combined into pools or cross-licenses; platform participation can be bundled by making various platforms interoperable; and multiple attributes can be bundled into a single differentiated product. The stable outcomes which these give rise to may include very different sets of actions or other primitive objects. Clearly, if this bundling is affected by the decisions of a designer — for instance, if a regulator can block the formation of patent pools under antitrust law — that designer can exert substantial control over outcomes.

This section considers the impact of these choices on the stable outcome when contracts are complementary. In particular, we consider the impact of bundling contracts together (e.g., combining several bilateral agreements into a multilateral one), and of unbundling them into more primitive objects (e.g., the reverse).

5.1 Contracts as Bundles

Until now, we have supposed that the set of contracts X was the primitive object over which agents have preferences. In this section, we depart from this assumption. Instead, we assume that the contracts in X (e.g., patent cross-licenses) are formed by bundling elements of some set B of "primitive contracts" (e.g., patent licenses) in the sense of Hatfield and Kominers (2016).[11] That is, there exists a map $\beta : X \to 2^B$ such that for all $X' \subseteq X$,

$$u_i(X') = u_i\left(\bigcup_{x \in X'} \beta(x)\right) \text{ for all } i \in N;$$
$$N(X') = \bigcup_{x \in X'} N(\beta(x)).$$

We are primarily concerned with how stable outcomes change when the available set of contracts exhibits a higher or lower level of bundling of the primitive contracts. We call these levels of bundling *layers*. Formally, $Y \subseteq X$ is a layer of X if the collection $\{\beta(y)\}_{y \in Y}$ is a partition of B. These layers are subsets of X which can be formed by combining the primitive contracts of B in such a way that no primitive contract is used more than once and no primitive contract is left unused. For instance, if the primitive contracts are patent licenses, a layer might consist of several cross-licenses that together contain all available licenses, without duplication. To facilitate our comparison of stable outcomes across layers, we say that X' *is stable in* Y if X' is stable when the set of contracts available is restricted to Y.

It is easy to order the layers of X by how bundled they are, simply by using the refinement ordering on the partitions $\{\beta(y)\}_{y \in Y}$. That is, we say a layer Y is *more bundled than* a layer Y', or $Y \trianglerighteq Y'$, if for each $y' \in Y'$ there exists $y \in Y$ such that $\beta(y') \subseteq \beta(y)$. Intuitively, a more bundled layer can be created by bundling together elements of a less bundled one. More concretely, consider a layer in our patent license example consisting of many cross-licenses.

[11]In Appendix A, we take a more general approach: We start from the set of all contracts X and show when and how one can derive a *basis* whose elements function like the primitive contracts used here.

This layer can be converted into the more bundled layer in which all licenses are combined into a single agreement by simply bundling all of its cross-licenses together.

$\underline{\blacktriangleright}$ is a partial order on the collection of layers, since it inherits transitivity and antisymmetry from the refinement order. Moreover, higher layers inherit complementarities from lower ones, and from the set of primitive contracts itself:

Lemma 4 (Bundling Preserves Quasisupermodularity). *Suppose that $Y' \underline{\blacktriangleright} Y$. If agent i's utility function u_i is quasisupermodular on Y, it is quasisupermodular on Y'.*

Thus, assuming that utility functions are quasisupermodular on B is enough to ensure that contracts are complements on every layer of X. We can combine this result with Theorem 1 to generate an existence result on the layers of X:

Corollary 1 (Stability on Layers of X). *If each agent's utility function u_i is quasisupermodular on the set B of primitive contracts, then for any layer Y of X, the unique stable outcome in Y is given by the largest fixed point Y^* of A on 2^Y.*

By allowing us to compare stable outcomes on different layers of X, this result lets us analyze the way that bundling affects stability in the next section.

5.2 Bundling and Stability

In this section, we give two comparative statics describing the different effects of bundling contracts together inside and outside of a stable outcome. First, when we either bundle *or* unbundle contracts outside of Y^*, the resulting stable outcome contains all of the contracts in Y^* and is either payoff equivalent to it or a Pareto improvement upon it:

Proposition 1 (Effects of Bundling Contracts Not Signed in the Stable Outcome). *Suppose each agent's utility function u_i is quasisupermodular on the set B of primitive contracts. Let Y and Y' be two layers of X with $Y^* \subseteq Y'$. Then*

i. $Y^* \subseteq Y'^*$.

ii. $u_i(Y'^*) \geq u_i(Y^*)$ *for all i.*

Proof. (i) Since $Y^* \subseteq Y'$, Y^* is a fixed point of A on $2^{Y'}$. Since Y'^* is the largest fixed point of A on $2^{Y'}$, $Y^* \subseteq Y'^*$. (ii) If $u_i(Y'^*) < u_i(Y^*)$ for some i, then $A(Y'^*) \neq Y'^*$, a contradiction. □

While the proof of Proposition 1 is simple, the result is powerful: when primitive contracts are complements and have positive externalities, bundling or unbundling contracts which are *not* signed as part of a stable outcome never makes things worse for any agent and never destroys any contracts. Hence, changing the level of bundling at which agreements are negotiated can overcome obstacles to their implementation, and is weakly Pareto-improving. When agents are unable to form a new multilateral agreement, allowing its negotiation as a set of bilateral agreements instead may cause some of its benefits to be realized. Similarly, if bilateral negotiation fails, organizations which facilitate multilateral negotiation may be helpful.

Second, when we bundle contracts inside a stable outcome together, the stable outcome may include fewer primitive contracts, but will never involve more. Further, the new outcome will be equivalent to the old one so long as *primitive contracts have positive externalities*; i.e., if each u_i is nondecreasing in B_{-i}. (More formally, for each $i \in N$ and $X', X'' \subseteq B$, $u_i(X') \leq u_i(X' \cup X''_{-i})$.) The two outcomes will also be equivalent if we only bundle together contracts which involve the same agents; i.e., if the only change we make is requiring the *joint settlement* of issues involving the same group of agents. More formally, we say that a layer Y' is a joint settlement of a layer $Y \trianglelefteq Y'$ if for any $y' \in Y'$ and $y \in Y$ with $\beta(y) \subset \beta(y')$, $N(y') = N(y)$.

Finally, when we unbundle contracts inside a stable outcome, the result may include fewer primitive contracts, but never more.

Proposition 2 (Effects of Bundling/Unbundling Contracts Signed in the Stable Outcome). *Suppose each agent's utility function u_i is quasisupermodular on the set B of primitive contracts. Let Y, Y' be layers of X with $Y' \trianglerighteq Y$.*

i. If $Y \setminus Y^* \subseteq Y'$, then $Y'^* \subseteq Y' \setminus (Y \setminus Y^*)$.

ii. If $Y \setminus Y^* \subseteq Y'$ and primitive contracts have positive externalities, then $Y'^* = Y' \setminus (Y \setminus Y^*)$ and $u_i(Y^*) = u_i(Y'^*)$ for all $i \in N$.

iii. If $Y \setminus Y^* \subseteq Y'$ and Y' is a joint settlement of Y, then $Y'^* = Y' \setminus (Y \setminus Y^*)$ and $u_i(Y^*) = u_i(Y'^*)$ for all $i \in N$.

iv. If $Y' \setminus Y'^* \subseteq Y$, then $Y^* \subseteq Y \setminus (Y' \setminus Y'^*)$.

Proposition 2 tells us that when the fundamental contract attributes are complementary, bundling contracts between the same agents for joint settlement can never destabilize an outcome. If an agent has no incentive to veto a set of contracts individually, he has no incentive to veto them once they are bundled. Instead, bundling a stable set of agreements together can only make the result unstable if the bundled contracts involve different agents. Moreover, it can only do so in the presence of negative externalities, by allowing an agent veto power over a bundle containing a primitive contract that harms him.

Additionally, Proposition 2 (iv) implies that bundling an unstable set of contracts together can only make the set stable if its instability is due to the presence of an *individual* deviation (to veto one or more contracts). (Corollary 1 tells us that if $Y^* \subset Y \setminus (Y' \setminus Y'^*)$, then $Y \setminus (Y' \setminus Y'^*)$ is not a fixed point of A on 2^Y, since Y^* is the largest such fixed point. Thus, $A(Y \setminus (Y' \setminus Y'^*)) \subset Y \setminus (Y' \setminus Y'^*)$, implying that some agent has an individual deviation from $Y \setminus (Y' \setminus Y'^*)$.) This is perhaps the most important implication of Proposition 2 for the purpose of understanding bundling's role in the determination of stable outcomes. It is intuitive that bundling outcomes together can make them stable by eliminating opportunities to selectively veto contracts. What is more subtle is Proposition 2 (iv)'s conclusion that eliminating veto opportunities is the *only* way it can do so.

6 Stability in Transferable Utility Environments

In practice, agents can often make payments as part of the agreements they make with one another. In this section, we show that when the objects of negotiation are complementary and do not have negative externalities, the structure of the set of stable outcomes in these settings is similar to that in the nontransferable utility setting. In particular, there is a unique set of primitive contracts common to all stable outcomes. This is in spite of the fact that the addition of transfers introduces substitutability between contracts: two agreements which differ only in the transfers they compel are, quite naturally, perfect substitutes.

Moreover, the additional structure of the transferable utility setting allows for new insights. We are able to link the set of stable outcomes in settings with complementarities to the set of competitive equilibria, which we show is nonempty.[12] We also provide comparative statics results that show how outcomes change when complementarity between contracts increases.

6.1 Contracts with Transfers

In the previous section, we considered a set of primitive contracts B and asked what would happen if they were bundled with each other. Here, we ask what would happen if they could be bundled with *monetary transfers*.[13]

Formally, in a transferable utility environment, each element of X can be written (x, t^x), where $x \in B$ and $t^x \in \mathbb{R}^N$ is a vector of monetary transfers from each agent which sum to

[12]As is well known, the welfare theorems do not hold in matching or indivisible goods markets without restrictions on preferences. (See footnote 6.)

[13]In particular, we allow primitive contracts to be bundled with transfers to or from agents they do not name. This allows for Coasean bargaining between agents that hold a property right over the enactment of a primitive contract and agents that are affected by its externalities.

zero (i.e., $\sum_{i\in N} t_i^x = 0$).[14] Further, X contains *all* such elements:

$$X = \left\{ (x, t^x) | x \in B, \sum_{i\in N} t_i^x = 0 \right\};$$

i.e., any feasible set of transfers can be combined with any $x \in B$ to create a contract. Denote the primitive contracts associated with each $Z \subseteq X$ by $\beta(Z) \equiv \{x | (x, t^x) \in Z \text{ for some } t^x\}$.

In order to become active, a contract (x, t^x) requires not only the agreement of $N(x)$ (the group of agents needed to put x into action) but of everyone making a transfer as part of it. That is, $N((x, t^x)) = N(x) \cup \{i | t_i^x \neq 0\}$.

Agents can only sign one contract associated with each primitive contract x. This allows us to write an outcome as $(Y, t) \in 2^B \times \mathbb{R}^{N\times B}$, for arbitrary values of $t^{B\setminus Y}$, the subvector of t corresponding to $B \setminus Y$. It will occasionally be convenient to abuse notation and represent a set of primitive contracts $Y \subseteq B$ as an indicator vector $Y \in \{0,1\}^B$, so that we can write the transfers from i attached to Y as $t_i \cdot Y$. Agents have quasilinear preferences; their utility from an outcome (Y, t) is given by $u_i(Y) - t_i \cdot Y$.

In the transferable utility setting, stability is closely linked with the classical concept of *competitive equilibrium.* Making this leap requires that agents can never be harmed by contracts that do not require their agreement. As in the previous section, we say that *primitive contracts have positive externalities* if each u_i is nondecreasing in B_{-i}; that is, for each $i \in N$ and $Y, Z \subseteq B$, $u_i(Y) \leq u_i(Y \cup Z_{-i})$.

To define competitive equilibrium, we must introduce some terminology from demand theory. Given a vector $p_i \in \mathbb{R}^B$ of *prices* for primitive contracts, define agent i's *demand correspondence* $D_i(p_i) \equiv \arg\max_{S\subseteq B} u_i(S) - p_i \cdot S$. Given demand correspondences, competitive equilibrium is characterized by the absence of excess demand and excess supply:

Definition (Competitive Equilibrium). We say that the outcome (X', p) is a *competitive equilibrium* if it is a point at which the market for primitive contracts clears: $X' \in D_i(p_i)$ for each $i \in N$ and $\sum_{i\in N} p_i = 0$.

Our notion of competitive equilibrium allows all agents to participate in the market for each primitive contract, even those whom that primitive contract does not name. In other words, for each $x \in X$, we posit a market-clearing process in which agents $i \notin N(x)$ can subsidize x's enactment (the case if $p_i^x > 0$) or its veto (if $p_i^x < 0$, in which case $-p_i^x$ represents the opportunity cost to $N(x)$, and the savings to i, of the contract's enactment). Without externalities, they will be unwilling to do so, and so our definition coincides with those in Hatfield and Kominers (2015) and (in settings with bilateral agreements) Hatfield et al. (2013); more generally, it coincides with the older notion of Lindahl equilibrium.

[14] If we have $x \in B \subset X$, we can write $(x, \mathbf{0})$.

The key property of the demand correspondence for our purposes is that of *gross complementarity*: the addition of a primitive contract is more attractive when the price of other primitive contracts is lower. In particular, we say that D_i satisfies the *gross complements* condition if for any price vectors $p_i \geq q_i$ and $X' \in D_i(p_i)$, $X'' \in D_i(q_i)$, $X' \cap X'' \in D_i(p_i)$ and $X' \cup X'' \in D_i(q_i)$. As is well known, the gross complements condition on demand is equivalent to supermodularity of the utility function (Topkis (1998)).

6.2 Stability, Competitive Equilibrium, and Efficiency

We characterize the set of stable outcomes by relating stability, competitive equilibrium, and efficiency. This analysis highlights the role of positive externalities for stability with transferable utility. We begin by proving the classical welfare theorems in our environment. That is, we show that every efficient set of primitive contracts is part of a market-clearing outcome, and every market-clearing outcome is efficient.

Theorem 2 (Efficiency ⇔ Competitive Equilibrium). *If each agent's utility function u_i is a gross complements valuation, then there exists a nonempty collection P^* of price vectors $p = \{p_i\}$ such that the outcome (X', p) clears the market if and only if X' is efficient and $p \in P^*$.*

Theorem 2 states that if primitive contracts are complements, then a collection of price vectors exists which clear the market when combined with any efficient set of primitive contracts. This result relies on a Fenchel-type min-max duality theorem for supermodular functions on a lattice given by Fujishige (1984). We extend this result to show that a social planner's primal and dual problems are equivalent in our setting, so long as agents have gross complements valuations.

Though matching and indivisible goods markets are related, this result is not a special case of those of Baldwin and Klemperer (2016), who show that competitive equilibria exist in markets for complementary indivisible goods: matching settings like those we consider need not be transformable into *any* market for goods, complementary or otherwise.[15] Consequently, readers may be surprised that competitive equilibria exist when primitive contracts are gross complements, especially given the transferable utility matching literature's reliance on the gross substitutes condition and its variants. We conjecture that this has been missed by the literature because of its general reliance on the existence arguments of Kelso and Crawford

[15] The matching settings we consider are not transformable into a market for goods when contracts are multilateral or have externalities. Even when contracts are bilateral (and thus can each be represented as the trade of an indivisible good), such a transformation would not preserve complementarity, i.e., goods need not be complementary in the transformed market just because contracts are in the matching market.

(1982) rather than discrete convex duality results like those we use.[16] As mentioned in the introduction, many-to-one matching models such as theirs naturally do not consider settings where all agents have gross complements valuations, which are ruled out by the assumption that some agents will never accept more than one contract.

Next, we show that stable outcomes are precisely those formed by combining the largest efficient set of primitive contracts with a competitive equilibrium price vector satisfying a "no wasteful payments" property.[17] To understand why this property is necessary, observe that a stable outcome (Y, t) can never contain a contract (x, t^x) which pays agents $i \notin N(x)$. If it did, it would be blocked by another contract $(x, \hat{t}^x)$, where $\hat{t}^x$ redistributes those payments among agents in $N(x)$ and those with $t_i^x > 0$. This excludes some competitive equilibrium price vectors, but as we show, not all of them — so long as primitive contracts have positive externalities.

Definition (No Wasteful Payments (NWP)). (X', p) satisfies the *no wasteful payments* property (NWP) if for each $x \in X$, $p_i^x \geq 0$ for all $i \notin N(x)$.

Lemma 5 (Some Competitive Equilibrium Prices Satisfy NWP). *Suppose that primitive contracts have positive externalities. If the set of competitive equilibrium price vectors P^* is nonempty, then so is $NWP(P^*)$, the set of competitive equilibrium price vectors which satisfy NWP.*

To see why positive externalities are necessary here, consider the case where they are absent, and a primitive contract has a *negative* externality. Any competitive equilibrium which includes that contract will necessarily involve a price vector that does not satisfy NWP: otherwise, the agents affected by the externality would not demand the contract in question. Thus, if a primitive contract with a negative externality is part of any efficient outcome, Theorem 2 implies that $NWP(P^*)$ is empty.

Lemma 5 allows us to connect competitive equilibrium to stability in Theorem 3, below.

Theorem 3 (Stability with Transferable Utility). *Suppose that primitive contracts have positive externalities and each agent's demand satisfies the gross complements condition.*

[16] An exception is recent work by Candogan et al. (2016), which uses discrete duality results for $M^\natural$-concave functions to show that stable outcomes exist in the context of substitutable contracts. Our results on complementarities, in contrast, rely on duality results for supermodular functions.

[17] This contrasts with Hatfield et al. (2013), in which the authors show that all competitive equilibria are stable. This is for two reasons. First, NWP may eliminate competitive equilibria from consideration in our setting, but not in theirs, because ours allows for externalities, whereas theirs does not. The second is due to a minor difference in our definitions of stability: Hatfield et al. (2013) require that for X'' to block Y, each contract in X'' must be part of *every* set that agents in $N(X'')$ might choose from $X'' \cup Y$; i.e., $X'' \subseteq Z \forall Z \in C_i((X'' \cup Y)_i | (X'' \cup Y)_{-i}) \forall i \in N(X'')$. In contrast, our definition requires only that each contract in X'' is part of *some* set that agents in $N(X'')$ might choose from $X'' \cup Y$. This allows for blocks involving contracts which some agents are indifferent about. Consequently, only the largest set of primitive contracts that arises in competitive equilibrium can be part of a stable outcome.

There is a largest efficient set of primitive contracts B^*, and the set of stable outcomes is given by $\{(B^*, p) | p \in NWP(P^*)\}$.

With positive externalities and gross complements, Theorem 3 tells us that the willingness of agents to make coordinated deviations can be captured by their optimization against a vector of latent prices for the enactment of primitive contracts. In other words, just as Lemma 2 showed in the NTU context, stability corresponds to *no excess demand and no excess supply* — this time in the classical demand-theoretic sense. For deviations which only involve the elimination of contracts, this is intuitive in the presence of positive externalities: agents demand a primitive contract at the current transfer level if and only if they don't want to veto it. Consequently, an outcome is individually rational if and only if there is no excess supply. For deviations involving signing new contracts, it is more subtle. Theorem 3 shows that the *absence* of a vector of transfers which induces a set of agents to make *some* deviation is implied by the *presence* of a vector of transfers for which no one wants to make *any* deviations, i.e., for which there is no excess demand.

Theorem 3's requirement of positive externalities is notably absent in our existence result for nontransferable utility, Theorem 1. This difference arises from the fact that Theorem 3's arguments rely on competitive equilibria, whereas Theorem 1's do not. As noted above, competitive equilibria can only correspond to stable outcomes if they do not involve wasteful payments — a condition that cannot be satisfied if the equilibrium contains a contract with a negative externality.

6.3 Complementarity Comparative Statics

In this section, we ask how stable outcomes change when preferences change. When agents have gross complements valuations and primitive contracts are complementary, we show that as primitive contracts become more complementary, stable outcomes include more of them. More precisely, the largest efficient set of primitive contracts B^* increases.

Proposition 3 (Transferable Utility Comparative Statics). *Let each agent i's utility function be given by $u_i(\theta_i, S)$ for θ_i in some lattice Θ_i. If for all i, $u_i(\theta_i, S)$ is a gross complements valuation for each $\theta_i \in \Theta_i$ and has increasing differences in (S, θ_i), then B^* is nondecreasing in $\{\theta_i\}_{i \in N}$.*

Proof. If each u_i has increasing differences in (S, θ_i), then $\sum_{i \in N} u_i(S, \theta_i)$ has increasing differences in $(S, \{\theta_i\}_{i \in N})$. The rest follows from the fact that the maximizers of a parameterized family of supermodular functions with increasing differences in the parameter and the choice variable are increasing in the parameter; see, e.g., Topkis (1998) Theorem 2.8.1. □

The reason we use increasing differences here instead of the single crossing property is that the former aggregates well whereas the latter may not: see Quah and Strulovici (2012).

Proposition 3 produces a complementary insight to the comparative statics result of Pycia and Yenmez (2017). In a two-sided setting with substitutable preferences, they find that market conditions faced by one side of the market improve as their propensity to substitute increases. In our setting with complementarities, however, market conditions faced by *each agent* improve — in the sense that others are willing to accept lower transfers to enact a primitive contract — as the complementarity between those contracts increases. Hence, as Proposition 3 shows, the agents will agree to a larger set of primitive contracts.

7 Setwise Stability

While stability remains the most widely used solution concept in the matching literature, authors such as Echenique and Oviedo (2006) and Klaus and Walzl (2009) explore several others which diverge from stability outside of the one-to-one and many-to-one settings. Of these, the concept of *setwise stability* is perhaps the most widely considered. We extend this concept to our environment in Section 7.1. Then, in Section 7.2, we explain the link between setwise stability in matching markets and coalition-proofness in a normal form game that we introduce. This gives us an existence result for setwise stable outcomes in environments with complementarities — Theorem 4 — which also shows that in such settings, stable and setwise stable outcomes coincide (up to payoff equivalence).

7.1 Setwise Stability: Concept and Discussion

Definition (Setwise Stability). An outcome $Y \subseteq X$ is *setwise stable* if there exists no $X'' \subseteq X$ and $J \subseteq N$ such that

1. X'' is a feasible deviation for J: $X'' \setminus X_J = Y \setminus X_J$ and $N(X'' \setminus Y) \subseteq J$,
2. X'' is a profitable deviation for J: $u_i(X'') \geq u_i(Y)$ for all $i \in J$, with at least one inequality strict, and
3. Deviating to X'' is individually rational for J: $X''_i \in C_i(X''_i \cup (Y_i \setminus Y_{J\setminus i}) \cup (X_i \setminus X_{N\setminus i}) | X''_{-i})$ for all $i \in J$.

In words, an outcome is setwise stable if there is no coalition J whose members want to recontract and can convince each other to do so; i.e., no coalition has a profitable self-enforcing deviation. In their definitions, Echenique and Oviedo (2006) and Klaus and Walzl (2009) require only that the deviation be to an outcome which is individually rational among the agents of J.[18] With externalities, our criterion for a deviation's individual rationality must be more demanding, since it may not be rational for agents to delete contracts they sign with only agents in $N \setminus J$. Without externalities, this requirement does not change the set of setwise stable outcomes, since keeping such a contract as part of a deviation does not affect the other agents' decisions to go along with it.

The presence of contracts which name only one agent (i.e., actions) also requires that we move beyond the requirement that X'' be individually rational among J, since agents in J must also not have an incentive to unilaterally add these contracts. Once again, without externalities, this does not affect the setwise stability of an outcome: in this case, i's decisions about contracts in $X_i \setminus X_{N\setminus i}$ are immaterial to other agents. While we consider such contracts in the above definition, they complicate the analysis of this section. Therefore, for the remainder of this section, we shall follow the bulk of the matching literature in assuming they are absent. That is, we assume that $N(x)$ is never a singleton, and so $X_i \setminus X_{N\setminus i} = \emptyset$ for each i.

Like stability, setwise stability requires robustness to blocks formed by any set of agents. Unlike stability, it requires that a block be profitable and individually rational, rather than chosen when offered along with the existing contracts. This can matter for several reasons.

First, setwise stability requires that the agents involved in a block agree on which, if any, contracts they are deleting, whereas stability does not; Example 2 illustrates this.

Example 2 (Not All Setwise Stable Outcomes Are Stable). Suppose that there are three contracts $\{a, b, c\}$ which each name both of two agents $\{1, 2\}$ with utility functions which satisfy the following inequalities:

$$u_1(\{a,c\}) > u_1(\{a,b\}) > u_1(\{a,b,c\}) > u_1(a) > u_1(c) > u_1(b) > u_1(\{b,c\}) > u_1(\emptyset);$$
$$u_2(\{b,c\}) > u_2(\{a,b\}) > u_2(\{a,b,c\}) > u_2(b) > u_2(c) > u_2(a) > u_2(\{a,c\}) > u_2(\emptyset).$$

Clearly, contracts are not complements here. (In fact, they are substitutes.)

It is easy to see that $\{a, b\}$ is the unique setwise stable outcome. However, since c is one of the contracts chosen by both agents when faced with the choice set $\{a, b, c\}$, $\{a, b\}$ is not stable. Instead, c is the unique stable outcome. As this example shows, relaxing the requirement that each pair of agents signs only one contract creates the potential for disagreement about deletion of contracts. This causes the equivalence between setwise stability and stability to break down, even in many-to-one matching.

This potential for disagreement is absent with complementarities: as noted in Section 3, if

[18] Earlier versions of setwise stability (Roth (1985), Sotomayor (1999)) do not require deviations to be individually rational.

Y is individually rational, then each agent continues to choose Y_i when additional contracts are available. However, it may be present when some or all contracts are instead substitutable.

Second, setwise stability only requires that a block is payoff-improving for each agent, whereas stability requires that a block is part of *each agent's favorite* set of contracts when offered alongside the existing set. In general, this can make stable outcomes fail to be setwise stable. With complementarities, Theorem 4 shows that it does not.

On the other hand, this difference can also make setwise stable outcomes fail to be stable. When negative externalities are present, stability requires robustness to blocks that need not improve the payoffs of every agent in the blocking coalition. When an agent chooses contracts as part of a block, she takes the contracts enacted by the other agents in the blocking coalition as given. Thus, the block may harm her on net without causing her to deviate from it.

Finally, unlike stability, setwise stability allows for deviations to smaller sets of contracts which are conducted by more than one agent. When externalities are positive, this difference is irrelevant: only those named by a contract have any reason to eliminate it. When they are negative, however, it is not. A group of agents might choose to eliminate a set of contracts which have small benefits to agents they name but large negative externalities on the others in the group, and whose meager benefits to the former vanish when the other contracts in the set are withdrawn.

Which deviations are more plausible — and thus which stability concept is more appropriate — depends entirely on the setting. In many cases, it is inherently less plausible for a group of agents to band together to abolish existing contracts than for them to do the same to create new ones. Likewise, one agent may be able to get others to go along with a deviation that makes them worse off due to the negative externalities of his contracts. These factors would tend to favor stability, though others might make setwise stability a wiser choice.

7.2 Setwise Stability and Coalition-Proof Equilibria

Setwise stability bears more than a passing resemblance to coalition-proof correlated equilibrium (Milgrom and Roberts (1996), Moreno and Wooders (1996)). Each concept considers the robustness of an outcome to profitable, self-enforcing deviations. Might there be a link? In short, yes — though it is more subtle than one might think. We exploit it in order to provide an existence result for setwise stability with complementary contracts (Theorem 4) by way of the result of Milgrom and Roberts (1996) for games with strategic complementarities.

To formalize this link, we first need to define a game based on our matching market. In a matching setting, agents' decisions consist of whether or not to sign the contracts that name them. Intuitively, then, their actions a_i in the game should be sets of contracts which they are willing to sign. The natural formulation of their payoffs, then, is

$$\hat{v}_i(a) = u_i\left(X(a)\right), \text{ where } X(a) \equiv \bigcap_{j\in N}(a_j \cup X_{-j}).$$

That is, each agent's payoffs $\hat{v}_i$ are those from the set of contracts which are agreed to by every agent they name.

However, this particular formulation is not adequate for us to import the strategic complementarities result of Milgrom and Roberts (1996) to the matching setting. First, the resulting game does not necessarily have strategic complementarities: if

$$\hat{v}_i(a_i', a_{-i}) \geq \hat{v}_i(a_i, a_{-i}) \tag{1}$$

for $a_i' \supset a_i$, we cannot invoke quasisupermodularity of u_i to show that

$$\hat{v}_i(a_i', a_{-i}') = u_i\left((a_i' \cup X_{-i}) \cap X(X_i, a_{-i}')\right) \geq u_i\left((a_i \cup X_{-i}) \cap X(X_i, a_{-i}')\right) = \hat{v}_i(a_i, a_{-i}')$$

for all a_{-i}' such that $a_j' \supseteq a_j$ for all $j \neq i$. This is because some contracts which are in $X(X_i, a_{-i}') \setminus X(X_i, a_{-i})$ may be in a_i' but not a_i. Hence, adding $X(X_i, a_{-i}') \setminus X(X_i, a_{-i})$ to the arguments of the utility functions on either side of (1) — as we would do if we were increasing a_{-i} to a_{-i}' — need not increase those arguments by the same set of contracts.

Second, their result requires payoffs to be monotone in other agents' strategies, which is not the case here, even with positive externalities: If an agent offers to sign a contract she does not want, her payoffs decrease when other agents take her up on that offer. Instead, we must apply their result to a game where agents are penalized for making an unreciprocated offer. As it turns out, adding this penalty solves our first problem as well.

Define this game, G, as follows. Each player's set of actions is given by 2^{X_i}. Payoffs are given by

$$v_i(a) = u_i\left(X(a)\right) - \omega\left|a_i \setminus X(X_i, a_{-i})\right| \text{ for large } \omega.$$

As Lemma 6 shows, G has strategic complementarities whenever u_i are quasisupermodular, and v_i is increasing in a_{-i} whenever contracts have positive externalities (i.e., whenever u_i is nondecreasing in X_{-i}).

Lemma 6 (Complementarities and Externalities in G). *If u_i are quasisupermodular and nondecreasing in X_{-i}, then G has strategic complementarities and v_i are nondecreasing in a_{-i}.*

In the parlance of Milgrom and Roberts (1996), setwise stable outcomes correspond to pure strategy coalition-proof correlated equilibria of G with communication structure $\Sigma^* =$

$\{(J,i)|J \subseteq N, i \in J\}$: both require the outcome to be robust to coalitional deviations which are *feasible*, *profitable*, and *individually rational*. Thus, any block X'' corresponds to a payoff-improving deviation a' which is self-enforcing in Σ^* according to the rule $a'_i = X''_i$. Likewise, any *pure strategy* payoff-improving deviation a' that is self-enforcing in Σ^* corresponds to a block X'' according to the rule $X'' = X(a')$. When u_i are quasisupermodular, are nondecreasing in X_{-i}, and each contract names at least two agents, considering correlated strategy deviations is unnecessary, and so the two stability notions are equivalent. We codify this observation in Proposition 4.

Proposition 4 (Setwise Stability and CPCE). *Y is setwise stable if the pure strategy profile $\{Y_i\}_{i\in N}$ is a coalition-proof correlated equilibrium for (G, Σ^*). The converse holds if the u_i are quasisupermodular, are nondecreasing in X_{-i}, and $N(x)$ is never a singleton.*

In fact, connections between G and the matching market run deeper than this. For any pure strategy profile a_{-i} of agent i's opponents, agent i's best responses are given by

$$\begin{aligned} B_i(a_{-i}) &= \arg \max_{a_i \subseteq X(X_i, a_{-i})_i} u_i(a_i \cup X(X_i, a_{-i})_{-i}), \\ &= C_i(X(X_i, a_{-i})_i | X(X_i, a_{-i})_{-i}). \end{aligned}$$

As a consequence, the pure strategy Nash equilibria of G correspond to the individually rational outcomes of the matching market. Further, when contracts are complements, its best responses are monotone. Thus, because of the way we structure G, applying the deferred acceptance algorithm we suggest following Theorem 1 will also yield the largest action profile in G that survives iterated removal of dominated strategies: Each round of the algorithm deletes exactly those pure strategies in G that are dominated by smaller pure strategies, given the actions that remain.

Under the additional assumption of positive externalities, Lemma 6 shows that G's structure also gives it strategic complementarities and payoff functions which are nondecreasing in opponents' strategies. Thus, we can call on Milgrom and Roberts (1996) Theorem 2 to show that the set of contracts X^* at which our deferred acceptance algorithm terminates gives the unique (up to payoff equivalence) coalition proof correlated equilibrium of G. Consequently, Proposition 4 shows that the logic of Theorem 1 carries over to the alternative solution concept of setwise stability.

Theorem 4 (Setwise Stability with Complementarities and Positive Externalities). *Suppose that agents' utility functions u_i are quasisupermodular and nondecreasing in X_{-i}, and $N(x)$ is never a singleton. X^* is setwise stable, and all setwise stable outcomes are payoff equivalent to X^*.*

More direct intuition also exists for Theorem 4's equivalence between stability and setwise stability. Because of complementarities, if agents in a coalition have an incentive to create new contracts as part of a setwise block, they have an incentive to do so while keeping the old ones. This means that each would choose all contracts from a set combining the new and old ones; i.e., there is *excess demand* in the sense of Lemma 2. Moreover, due to contracts' positive externalities, if agents have an incentive to delete contracts as part of a coalition, they have an incentive to do so unilaterally; i.e., there is *excess supply*. Thus, any setwise block is also a block in the sense of stability. For the converse, note once again that with complementarities, if a set of contracts is blocked, then it is blocked by a larger set of contracts. Since contracts have positive externalities, this implies that each agent in the blocking coalition prefers that larger set to each of its subsets — and hence it is a setwise block as well.

Note that in contrast to Theorem 1, Theorem 4 requires positive externalities. Though our transferable utility existence theorem (Theorem 3) makes the same requirement, it is necessary for a different reason. As the intuition above shows, its function here is not to establish a link with competitive equilibrium, but rather to ensure that agents *choose* a set of contracts over its subsets if and only if they *prefer* it to those subsets.

8 Conclusion

This paper introduces a framework for analyzing settings where agents form complementary agreements with one another. We provide existence results which are constructive in that they show how to find stable outcomes. Further, we offer comparative statics on the effects of bundling contracts.

This opens new possibilities for future research. Though the techniques we use to prove our existence results in the nontransferable and transferable utility settings (a fixed point theorem and convex duality, respectively) are quite different, the structure of the set of stable outcomes in these settings is similar. This suggests there may be some common mathematical structure underlying both results.

Additionally, there has been recent interest in the structural estimation of matching games. (For a survey, see Chiappori and Salanié (2016).) The results from this paper suggest there might be new possibilities for the use of matching models in applied work on environments characterized by complementarities.

References

Azevedo, E. M. and J. W. Hatfield (2015): "Existence of Equilibrium in Large Matching Markets With Complementarities," Working paper.

Baldwin, E. and P. Klemperer (2016): “Understanding Preferences:’Demand Types’, and The Existence of Equilibrium with Indivisibilities,” Working paper.

Becker, G. S. (1973): “A Theory of Marriage: Part I,” *Journal of Political Economy*, 81, 813–846.

Candogan, O., M. Epitropou, and R. V. Vohra (2016): “Competitive Equilibrium and Trading Networks: A Network Flow Approach,” in *Proceedings of the 2016 ACM Conference on Economics and Computation*, New York, NY, USA: ACM, EC ’16, 701–702.

Che, Y.-K., J. Kim, and F. Kojima (2017): “Stable Matching in Large Economies,” Working paper.

Chiappori, P.-A. and B. Salanié (2016): “The Econometrics of Matching Models,” *Journal of Economic Literature*, 54, 832–61.

Echenique, F. and J. Oviedo (2006): “A Theory of Stability in Many-To-Many Matching Markets,” *Theoretical Economics*, 1, 233–273.

Fleiner, T., R. Jagadeesan, Z. Jankó, and A. Teytelboym (2018): “Trading Networks with Frictions,” Working paper.

Fujishige, S. (1984): “Theory of Submodular Programs: A Fenchel-Type Min-Max Theorem and Subgradients of Submodular Functions,” *Mathematical Programming*, 29, 142–155.

Gale, D. and L. S. Shapley (1962): “College Admissions and the Stability of Marriage,” *The American Mathematical Monthly*, 69, 9–15.

Gul, F. and E. Stacchetti (1999): “Walrasian Equilibrium with Gross Substitutes,” *Journal of Economic Theory*, 87, 95–124.

Hatfield, J. W. and F. Kojima (2008): “Matching with Contracts: Comment,” *The American Economic Review*, 98, 1189–1194.

Hatfield, J. W. and S. D. Kominers (2012): “Matching in Networks with Bilateral Contracts,” *American Economic Journal: Microeconomics*, 4, 176–208.

——— (2015): “Multilateral Matching,” *Journal of Economic Theory*, 156, 175–206.

——— (2016): “Contract Design and Stability in Many-to-Many Matching,” *Games and Economic Behavior*.

Hatfield, J. W., S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2013): "Stability and Competitive Equilibrium in Trading Networks," *Journal of Political Economy*, 121, 966–1005.

——— (2018): "Full Substitutability," Working paper.

Hatfield, J. W. and P. R. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 913–935.

Jagadeesan, R. (2017): "Complementary Inputs and the Existence of Stable Outcomes in Large Trading Networks," Working paper.

Kelso, A. S. and V. P. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica: Journal of the Econometric Society*, 1483–1504.

Klaus, B. and M. Walzl (2009): "Stable many-to-many matchings with contracts," *Journal of Mathematical Economics*, 45, 422–434.

Kojima, F., P. A. Pathak, and A. E. Roth (2013): "Matching with Couples: Stability and Incentives in Large Markets," *The Quarterly Journal of Economics*, 128, 1585–1632.

Milgrom, P. and J. Roberts (1996): "Coalition-Proofness and Correlation with Arbitrary Communication Possibilities," *Games and Economic Behavior*, 17, 113–128.

Milgrom, P. and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica: Journal of the Econometric Society*, 157–180.

Moreno, D. and J. Wooders (1996): "Coalition-proof equilibrium," *Games and Economic Behavior*, 17, 80–112.

Ostrovsky, M. (2008): "Stability in Supply Chain Networks," *American Economic Review*, 98, 897–923.

Pycia, M. (2012): "Stability and Preference Alignment in Matching and Coalition Formation," *Econometrica*, 80, 323–362.

Pycia, M. and M. B. Yenmez (2017): "Matching With Externalities," Working paper.

Quah, J. K.-H. and B. Strulovici (2012): "Aggregating the Single Crossing Property," *Econometrica*, 80, 2333–2348.

Rochet, J.-C. and J. Tirole (2006): "Two-Sided Markets: A Progress Report," *The RAND Journal of Economics*, 37, 645–667.

Rostek, M. and N. Yoder (2018): "Matching with Multilateral Contracts," In preparation.

Roth, A. E. (1985): "The College Admissions Problem Is Not Equivalent To The Marriage Problem," *Journal of Economic Theory*, 36, 277–288.

Scotchmer, S. and C. Shannon (2015): "Verifiability and Group Formation in Markets," Working paper.

Shapiro, C. (2000): "Navigating the Patent Thicket: Cross Licenses, Patent Pools, and Standard Setting," *Innovation Policy and the Economy*, 1, 119–150.

Sherstyuk, K. (1999): "Multisided Matching Games With Complementarities," *International Journal of Game Theory*, 28, 489–509.

Sotomayor, M. (1999): "Three Remarks on the Many-to-Many Stable Matching Problem," *Mathematical Social Sciences*, 38, 55–70.

Sun, N. and Z. Yang (2006): "Equilibria and Indivisibilities: Gross Substitutes and Complements," *Econometrica*, 74, 1385–1402.

Teytelboym, A. (2012): "Strong Stability in Contractual Networks and Matching Markets," in *MATCH-UP 2012: the Second International Workshop on Matching Under Preferences*, 111.

Topkis, D. M. (1998): *Supermodularity and Complementarity*, Princeton University Press.

Weyl, E. G. (2010): "A Price Theory of Multi-Sided Platforms," *American Economic Review*, 100, 1642–72.

A Endogenizing Primitive Contracts

In section 5, we discuss the ways that bundling contracts together can change (or not change) stable outcomes. But instead of starting from primitive contracts and building the set of contracts, we can also start with the set of all contracts, X. Suppose X contains some contracts which are formed by bundling together others. What is the relevant notion of equivalence between a contract and the set of contracts bundled together to form it? Can we identify a minimal set of contracts, or *basis*, which generates the others through bundling? How can we determine whether one set of contracts is more or less bundled than another? And what properties of the basis are important for all of these objects to be well-behaved? These are the questions we answer in this appendix.

In order for two sets of contracts to be equivalent to each other, they must not only yield the same payoffs when standing alone, but function identically when added to other contracts as well. Define the contract equivalence relation $\triangleq$ on 2^X as follows:

Definition (Equivalent Sets of Contracts). We say that X' is *equivalent* to X'', or $X' \triangleq X''$, whenever

$$N(X') = N(X'') \text{ and } u_i(X' \cup Y) = u_i(X'' \cup Y) \ \forall i \in N, Y \subseteq X.$$

Sets of contracts are equivalent if (1) the same set of agents have a joint property right over their enactment and (2) each agent is indifferent between them, not only alone but also when added to other sets of contracts. Trivially, the contract equivalence relation $\triangleq$ is transitive. Less trivially, it is preserved under union:

Lemma 7 (Equivalence Preserved Under Unions). *Suppose that $X' \triangleq X''$.*

i. $X' \cup X''' \triangleq X'' \cup X'''$ for all $X''' \subseteq X$.

ii. If $Y' \triangleq Y''$, then $X' \cup Y' \triangleq X'' \cup Y''$.

Proof. For any $Y' \subseteq X$, choose $Y = Y' \cup X'''$. Then by definition, $u_i(X' \cup X''' \cup Y') = u_i(X'' \cup X''' \cup Y') \ \forall i \in N$. And $N(X' \cup X''') = N(X') \cup N(X''') = N(X'') \cup N(X''') = N(X'' \cup X''')$. (i) follows. Then since $X' \triangleq X''$, $X' \cup Y' \triangleq X'' \cup Y'$; and since $Y' \triangleq Y''$, $X'' \cup Y' \triangleq X'' \cup Y''$. (ii) follows. □

Sometimes, two equivalent sets of contracts $X' \triangleq B$ can be broken down into equivalences between each contract $x \in X'$ and subsets of B. For instance, suppose $B = \{a, b, c\}$ and $X' = \{d, e\}$ with $\{a, b\} \triangleq d$ and $\{b, c\} \triangleq e$. In this case, we say that B *generates* X', and that each contract in X' is a *bundle* of contracts from B.[19]

Definition (Generators, Bundles, Basis, Uncovered). We say that the set of contracts B *generates* X' if there exists a map $\beta : X' \to 2^B$ such that $\beta(x) \triangleq x$ for each $x \in X'$, and call each $x \in X'$ a *bundle* of $\beta(x)$. B is a *basis* for X' if it generates X and there exists no $B' \subset B$ which generates X'. A basis B is *uncovered* if there exists no $B' \subset B$ such that $B' \triangleq B$.

We call the map β a *representation operator* for X' in B. It extends naturally to a map from the power set $2^{X'}$ to 2^B: Let $\beta(X'') = \bigcup_{x \in X''} \beta(x) \triangleq X''$ by Lemma 7. We call $\beta(X'')$ a *representation* of X'' in B.

When B generates X', every element in X' can be created by combining elements of B into a single contract, or *bundle*. For instance, the set of patent licenses generates the set of cross-licenses. The reader will note that the set of primitive contracts in Section 5 is characterized

[19] Here, d is a bundle of $\{a, b\}$ and e is a bundle of $\{b, c\}$.

by its generation of X. When B is a basis for X', it is a minimal set that has this property. Since every set trivially generates itself, it follows that every set has a basis.

Finally, when a basis is uncovered, it contains no contracts that become redundant when combined with the others. This is critically important for basis representations to be unique and for bases to be isomorphic (Proposition 5).

When one set of contracts X'' is made up of bundles of *disjoint* subsets of another set X', and these subsets cover all of X', we say that X'' is more bundled than X', just as we considered more and less bundled layers in Section 5. This time, however, we define the bundling order $\underline{\blacktriangleright}$ without reference to a set of primitive contracts or a specific representation operator.

Definition (Bundling Order). We say that X'' is *more bundled than* X', or $X'' \underline{\blacktriangleright} X'$, if X' generates X'' and the representations $\{\beta(x)\}_{x \in X''}$ form a partition of X'. If $X'' \underline{\blacktriangleright} X'$ and $X' \underline{\blacktriangleright} X''$ we write $X'' \overset{\blacktriangle}{=} X'$.

X'' is more bundled than X' if X'' can be formed by combining contracts in X' in such a way that no contract is used more than once and no contract is left unused. $\underline{\blacktriangleright}$ orders sets of contracts within the same payoff equivalence class in a transitive way:

Lemma 8 (Transitivity of the Bundling Order). *$\underline{\blacktriangleright}$ is a preorder.*

Proof. Reflexivity is trivial — let β be the identity.

For transitivity, if $X''' \underline{\blacktriangleright} X'' \underline{\blacktriangleright} X'$ and β'' and β' are the maps associated with $X''' \underline{\blacktriangleright} X''$ and $X'' \underline{\blacktriangleright} X'$, choose $\beta(x) = \bigcup_{y \in \beta''(x)} \beta'(y)$. $\beta(x) \triangleq x$ follows from Lemma 7. □

Note that $\underline{\blacktriangleright}$ isn't a partial order on 2^X — antisymmetry fails, for instance, between two equivalent contracts x and x'.

The next proposition shows that the structure of the set of contracts is well-behaved whenever its basis is uncovered. That is, basis representations are unique, and the basis is unique up to isomorphism. As a consequence, the collection of layers of X is invariant to the choice of basis.

Proposition 5 (Basis Invariance). *Let B be an uncovered basis for X.*

i. (Unique Basis Representation) Each $X' \subseteq X$ has a unique representation $\beta(X')$ in B.

ii. (Bases are Isomorphic) If B' is a basis for X, then $B \overset{\blacktriangle}{=} B'$.

Proof. (i) Suppose X' has two distinct basis representations $\alpha(X'), \beta(X')$. Then $\alpha(X') \triangleq X' \triangleq \beta(X')$ and so $(B \setminus \alpha(X')) \cup \beta(X') \triangleq (B \setminus \beta(X')) \cup \alpha(X') \triangleq B$. Since $\alpha(X'), \beta(X')$ are distinct, at least one of the first two sets must be a strict subset of B, a contradiction. That $\bigcup_{x \in X'} \beta(x) \triangleq X'$ follows immediately from Lemma 7.

(ii) Let β and β' be the representation operators for X in B and B' respectively. For each $x \in B$, $\beta(\beta'(x)) \triangleq x$. If $\beta(\beta'(x)) \neq x$, then $(B \setminus \beta(\beta'(x))) \cup x \triangleq B$, a contradiction. Hence, $\beta(\beta'(x)) = x$. Then$\{\beta'(x)\}_{x \in B}$ is a partition of B' and $B \underline{\blacktriangleright} B'$; likewise $B' \underline{\blacktriangleright} B$. □

B Omitted Proofs

Proof of Lemma 4 (Bundling Preserves Quasisupermodularity) Let X', X'' be two subsets of Y' with $u_i(X') - u_i(X' \cap X'') \geq 0$. Then

$$u_i\left(\bigcup_{x \in X'} \beta(x)\right) - u_i\left(\bigcup_{x \in X' \cap X''} \beta(x)\right) \geq 0,$$

$$u_i\left(\bigcup_{x \in X'} \beta(x)\right) - u_i\left(\left(\bigcup_{x \in X'} \beta(x)\right) \cap \left(\bigcup_{x \in X''} \beta(x)\right)\right) \geq 0, \quad (\{\beta(x)\}_{x \in Y'} \text{ is a partition})$$

$$u_i\left(\left(\bigcup_{x \in X'} \beta(x)\right) \cup \left(\bigcup_{x \in X''} \beta(x)\right)\right) - u_i\left(\bigcup_{x \in X''} \beta(x)\right) \geq 0, \quad (\text{quasisupermodularity on } Y)$$

$$u_i\left(\beta(X' \cup X'')\right) - u_i\left(\beta(X'')\right) \geq 0,$$

$$u_i(X' \cup X'') - u_i(X'') \geq 0,$$

and u_i is quasisupermodular on Y'. □

Proof of Proposition 2 (Effects of Bundling/Unbundling Contracts Signed in the Stable Outcome) We need the following lemma:

Lemma 9. *Let Y and Y' be layers of X. For any $Z \subseteq Y$ and $Z' \subseteq Y'$, $\bigcup_{z \in Z} \beta(z) = \bigcup_{z' \in Z'} \beta(z')$ implies $\bigcup_{y \in Y \setminus Z} \beta(y) = \bigcup_{y' \in Y' \setminus Z'} \beta(y')$.*

Proof. Follows directly from the fact that $\{\beta(y)\}_{y \in Y}$ and $\{\beta(y')\}_{y' \in Y'}$ are both partitions. □

i. Let $\hat{Y}' = Y' \setminus (Y \setminus Y^*)$.

   We need to show that there is no fixed point of A on $2^{Y'}$ larger than $\hat{Y}'$. Suppose that for some $X'' \subseteq Y \setminus Y^*$, $\hat{Y}' \cup X''$ is a fixed point of A. By Lemmas 4 and 1, $X''_i \cup \hat{Y}'_i \in C_i(X''_i \cup \hat{Y}'_i | X''_{-i} \cup \hat{Y}'_{-i})$ for each i. Hence for each i,

$$u_i\left(\bigcup_{x \in \hat{Y}' \cup X''} \beta(x)\right) \geq u_i\left(\bigcup_{x \in Z'' \cup X''_{-i} \cup \hat{Y}'} \beta(x)\right) \text{ for all } Z'' \subseteq X''_i. \qquad (2)$$

Since $A(Y^*) = Y^*$, for each i,

$$u_i\left(\bigcup_{x\in Y^*}\beta(x)\right) \geq u_i\left(\bigcup_{x\in Z\cup Y^*_{-i}}\beta(x)\right) \text{ for all } Z \subseteq Y^*_i.$$

By quasisupermodularity, we have

$$u_i\left(\bigcup_{x\in Y^*\cup X''_{-i}\cup Z''}\beta(x)\right) \geq u_i\left(\bigcup_{x\in Z\cup Y^*_{-i}\cup X''_{-i}\cup Z''}\beta(x)\right) \text{ for all } Z \subseteq Y^*_i, Z'' \subseteq X''_i.$$

Using Lemma 9 to replace $\hat{Y}'$ with Y^* in (2), we have

$$u_i\left(\bigcup_{x\in Y^*\cup X''}\beta(x)\right) \geq u_i\left(\bigcup_{x\in Z\cup Y^*_{-i}\cup X''_{-i}\cup Z''}\beta(x)\right) \text{ for all } Z \subseteq Y^*_i, Z'' \subseteq X''_i.$$

Thus $(Y^*\cup X'')_i = A_i(Y^*\cup X'')$ for each i and $Y^*\cup X''$ is a fixed point of A, a contradiction.

ii. Let $\hat{Y}' = Y' \setminus (Y \setminus Y^*)$. From $Y^*_i = A_i(Y^*)$ and positive externalities we have

$$u_i\left(\bigcup_{x\in Y^*}\beta(x)\right) \geq u_i\left(\bigcup_{x\in Z}\beta(x)\right) \text{ for all } Z \subseteq Y^*.$$

By Lemma 9 and the fact that $\{\beta(y)\}_{y\in Y}$ is a finer partition than $\{\beta(y')\}_{y'\in Y'}$,

$$u_i\left(\bigcup_{x\in \hat{Y}'}\beta(x)\right) \geq u_i\left(\bigcup_{x\in Z}\beta(x)\right) \text{ for all } Z \subseteq \hat{Y}'.$$

It follows that $\hat{Y}'_i = A_i(\hat{Y}')$ for each i and thus $\hat{Y}'$ is a fixed point of A, so $Y'^* \supseteq \hat{Y}'$. The proposition follows from part (i).

iii. Let $\hat{Y}' = Y' \setminus (Y \setminus Y^*)$. By Lemma 9, $\bigcup_{y'\in\hat{Y}}\beta(y') = \bigcup_{y\in Y^*}\beta(y)$. Then since Y' is a joint settlement of Y, for all $y \in Y^*$, there exists $y' \in \hat{Y}$ with $\beta(y) \subseteq \beta(y')$ and $N(y') = N(y)$. Then $\bigcup_{y'\in\hat{Y}_{-i}}\beta(y') \supseteq \bigcup_{y\in Y^*_{-i}}\beta(y)$ and $\bigcup_{y'\in\hat{Y}_i}\beta(y') \supseteq \bigcup_{y\in Y^*_i}\beta(y)$; it follows that both inclusions are equalities.

From $Y^*_i = A_i(Y^*)$ we have

$$u_i\left(\bigcup_{x\in Y^*}\beta(x)\right) \geq u_i\left(\bigcup_{x\in Z\cup Y^*_{-i}}\beta(x)\right) = u_i\left(\bigcup_{x\in Z}\beta(x)\cup\bigcup_{x\in Y^*_{-i}}\beta(x)\right) \text{ for all } Z \subseteq Y^*_i.$$

For any $Z' \subseteq \hat{Y}_i$, choose $Z = \{y \in Y | \beta(y) \subseteq \beta(y') \text{ for some } y' \in Z'\} \subseteq Y_i^*$. Since $\{\beta(y)\}_{y \in Y}$ is a finer partition of B than $\{\beta(y')\}_{y' \in Y'}$, we have $\bigcup_{y' \in Z'} \beta(y') = \bigcup_{y \in Z} \beta(y)$. Then we have

$$u_i\left(\bigcup_{x \in Y^*} \beta(x)\right) \geq u_i\left(\bigcup_{x \in Z' \cup \hat{Y}_{-i}} \beta(x)\right) = u_i\left(\bigcup_{x \in Z'} \beta(x) \cup \bigcup_{x \in \hat{Y}_{-i}} \beta(x)\right) \text{ for all } Z' \subseteq \hat{Y}_i.$$

Thus $\hat{Y}_i = A_i(\hat{Y})$ for each i. The proposition follows.

iv. The proof proceeds similarly to (i). Let $\hat{Y} = Y \setminus (Y' \setminus Y'^*)$. We need to show that there is no fixed point of A on 2^Y larger than $\hat{Y}$. Suppose that for some $X'' \subseteq Y' \setminus Y'^*$, $\hat{Y} \cup X''$ is a fixed point of A. By Lemmas 4 and 1, $X_i'' \cup \hat{Y}_i \in C_i(X_i'' \cup \hat{Y}_i | X_{-i}'' \cup \hat{Y}_{-i})$ for each i. Hence for each i,

$$u_i\left(\bigcup_{x \in \hat{Y} \cup X''} \beta(x)\right) \geq u_i\left(\bigcup_{x \in Z'' \cup X_{-i}'' \cup \hat{Y}} \beta(x)\right) \text{ for all } Z'' \subseteq X_i''. \tag{3}$$

Since $A(Y'^*) = Y'^*$, for each i,

$$u_i\left(\bigcup_{x \in Y'^*} \beta(x)\right) \geq u_i\left(\bigcup_{x \in Z \cup Y_{-i}'^*} \beta(x)\right) \text{ for all } Z \subseteq Y_i'^*.$$

By quasisupermodularity, we have

$$u_i\left(\bigcup_{x \in Y'^* \cup X_{-i}'' \cup Z''} \beta(x)\right) \geq u_i\left(\bigcup_{x \in Z \cup Y_{-i}'^* \cup X_{-i}'' \cup Z''} \beta(x)\right) \text{ for all } Z \subseteq Y_i'^*, Z'' \subseteq X_i''.$$

Using Lemma 9 to replace $\hat{Y}$ with Y'^* in (3), we have

$$u_i\left(\bigcup_{x \in Y'^* \cup X''} \beta(x)\right) \geq u_i\left(\bigcup_{x \in Z \cup Y_{-i}'^* \cup X_{-i}'' \cup Z''} \beta(x)\right) \text{ for all } Z \subseteq Y_i'^*, Z'' \subseteq X_i''.$$

Hence, $(Y^{*\prime} \cup X'')_i = A_i(Y^{*\prime} \cup X'')$ for each i and thus $Y^{*\prime} \cup X''$ is a fixed point of A, a contradiction. □

Proof of Theorem 2 (Efficiency ⇔ Competitive Equilibrium) The strategy is as follows: First, show that supermodularity gives us a Fenchel-type duality theorem. Then, use the theorem to show that the social planner's primal and dual problems are equivalent. Finally, we show that for a solution X^* to the primal problem and a solution p^* to the dual problem, (X^*, p^*) clears the market.

Define the convex and concave conjugate functions of $f : 2^X \to \mathbb{R}$ as follows:

$$f^\circ(p) = \min_{S \subseteq X} \{p \cdot S - f(S)\},$$
$$f^\bullet(p) = \max_{S \subseteq X} \{p \cdot S - f(S)\}.$$

Recall that $\pi_i(p) \equiv \max_{S \subseteq X} u_i(S) - p \cdot S$ and note that $(-u_i)^\bullet(-p) = -u_i^\circ(p) = \pi_i(p)$.

Lemma 10 (Fujishige (1984) Theorem 3.3). *For a supermodular function $g : 2^X \to \mathbb{R}$ and a submodular function $f : 2^X \to \mathbb{R}$,*

$$\min_{S \subseteq X} \{f(S) - g(S)\} = \max_{p \in \mathbb{R}^X} \{g^\circ(p) - f^\bullet(p)\}.$$

Corollary 2. *For two supermodular functions $f, g : 2^X \to \mathbb{R}$,*

$$\max_{S \subseteq X} \{f(S) + g(S)\} = \min_{p \in \mathbb{R}^X} \{-g^\circ(p) - f^\circ(-p)\}.$$

Now that we have shown Fenchel duality for both classes of utility functions, we proceed to convert them into Lagrange duality results.

Noting that the class of supermodular functions is closed under affine transformations and replacing $g(S)$ with $g(S) - q \cdot S$ for some $q \in \mathbb{R}^X$ in the statement of Corollary 2 yields the following more general version:

Corollary 3. *For two supermodular functions $f, g : 2^X \to \mathbb{R}$,*

$$-(f(S) + g(S))^\circ(q) = \min_{p \in \mathbb{R}^X} \{-g^\circ(p + q) - f^\circ(-p)\}.$$

Picking f and g as the sums $\sum_{i \in N_1} u_i$, $\sum_{i \in N_2} u_i$ for some partition N_1, N_2 of N in the statements of Corollary 2 and inductively applying Corollary 3 along with the fact that sums of supermodular functions are supermodular yields

Lemma 11. *If each u_i is a gross complements valuation, then*

$$\max_{S \subseteq X} \left\{ \sum_{i \in N} u_i(S) \right\} = \min_{\{p_i\}_{i \in N} \in \mathbb{R}^{X \times N}} \left\{ \sum_{i \in N} \pi_i(p_i) \ \ s.t. \ \sum_{i \in N} p_i = 0 \right\}.$$

In keeping with the literature, we call the left-hand side of this equation the social planner's *primal* problem (the set of solutions to which is precisely the collection of efficient sets of primitive contracts) and the right-hand side the social planner's *dual* problem.

From Lemma 11, we can complete the proof of Theorem 2. Let X^* be a solution to the social planner's primal problem and p^* a solution to the dual problem. By definition, for each i we have $u_i(X^*) - p_i^* \cdot X^* \leq \pi_i(p_i^*)$. Since the p_i^* sum to zero, summing over the i yields $\sum_{i \in N} u_i(X^*) \leq \sum_{i \in N} \pi_i(p_i^*)$. We know from Lemma 11 that this holds with equality, which can only be true if $u_i(X^*) - p_i^* \cdot X^* = \pi_i(p_i^*)$ for each i and hence $X^* \in D_i(p_i^*)$ for each i.

Conversely, suppose that (X', p') clears the market for some $X' \notin \arg\max_{S \subseteq X} \left\{\sum_{i \in N} u_i(S)\right\}$. Then we have $\sum_{i \in N} p_i' = 0$ as well as $\pi_i(p_i') = u_i(X') - p_i' \cdot X'$ for each i, and hence

$$\sum_{i \in N} \pi_i(p_i') = \sum_{i \in N} u_i(X') < \max_{S \subseteq X} \left\{ \sum_{i \in N} u_i(S) \right\} = \min_{\{p_i\}_{i \in N} \in \mathbb{R}^{X \times N}} \left\{ \sum_{i \in N} \pi_i(p_i) \text{ s.t. } \sum_{i \in N} p_i = 0 \right\},$$

a contradiction. □

Proof of Lemma 5 (Some Competitive Equilibria Satisfy NWP) In order to prove Lemma 5, we introduce an operator $\rho : \mathbb{R}^{B \times N} \to \mathbb{R}^{B \times N}$:

$$\rho[p]_i^x \equiv \begin{cases} \max\{0, p_i^x\}, & i \notin N(x); \\ p_i^x + \frac{1}{|N(x)|} \sum_{j \notin N(x)} \min\{0, p_j^x\}, & i \in N(x). \end{cases}$$

Observe that $\sum_{i \in N} \rho[p]_i = 0$, and that p satisfies NWP if and only if $p = \rho[p]$. We use the following lemma:

Lemma 12 (ρ Preserves Competitive Equilibrium). *Suppose that primitive contracts have positive externalities. If the outcome (X', p) is a competitive equilibrium, then so is the outcome $(X', \rho[p])$.*

Proof. For each $Y \subseteq B$, let $\rho^Y[p] \equiv p^{B \setminus Y} \oplus \rho[p]^Y$. The proof proceeds by induction on $Y \subseteq B$. By positive externalities, since (X', p) is a competitive equilibrium, we must have $p^x = \rho[p]^x$ for each $x \notin X'$. Otherwise, we would have $u_i(x \cup X') - p_i \cdot (x \cup X') > u_i(X') - p_i \cdot (X')$ for some $i \notin N(x)$, a contradiction. Thus, $X' \in D_i(\rho^{B \setminus X'}[p]_i)$ for all i. For each $Y \supseteq B \setminus X'$ and each $y \in X' \setminus Y$, if $X' \in D_i(\rho^Y[p]_i)$ then $X' \in D_i(\rho^{Y \cup y}[p]_i)$:

- If $i \notin N(y)$, then either $\rho^{Y \cup y}[p]_i = \rho^Y[p]_i$ and $X' \in D_i(\rho^{Y \cup y}[p]_i)$; or $\rho^{Y \cup y}[p]_i^y = 0$ and for all $X'' \ni y$ we have

$$\begin{aligned} u_i(X') - X' \cdot \rho^{Y \cup y}[p]_i &= u_i(X') - X' \cdot \rho^Y[p]_i + \rho^Y[p]_i^y, \\ &\geq u_i(X'') - X'' \cdot \rho^Y[p]_i + \rho^Y[p]_i^y, \\ &= u_i(X'') - X'' \cdot \rho^{Y \cup y}[p]_i, \\ &\geq u_i(X'' \setminus y) - (X'' \setminus y) \cdot \rho^{Y \cup y}[p]_i, \end{aligned}$$

by positive externalities, and so from the last two lines, $X' \in D_i(\rho^{Y \cup y}[p]_i)$.

- If $i \in N(y)$ then $\rho^{Y \cup y}[p]_i \leq \rho^Y[p]_i$, and so for all X'',

$$\begin{aligned} u_i(X') - X' \cdot \rho^{Y \cup y}[p]_i &= u_i(X') - X' \cdot \rho^Y[p]_i - (\rho^{Y \cup y}[p]_i^y - \rho^Y[p]_i^y), \\ &\geq u_i(X'') - X'' \cdot \rho^Y[p]_i - (\rho^{Y \cup y}[p]_i^y - \rho^Y[p]_i^y), \\ &\geq u_i(X'') - X'' \cdot \rho^{Y \cup y}[p]_i, \end{aligned}$$

  and so $X' \in D_i(\rho^{Y \cup y}[p]_i)$.

Thus, $X' \in D_i(\rho^B[p]_i) = D_i(\rho[p]_i)$ for all i, and $D_i(\rho^B[p]_i)$ is a competitive equilibrium. □

Thus, $NWP(P^*) = \{p \in P^* | p = \rho[p]\}$ is nonempty so long as $P^* \neq \emptyset$. □

Proof of Theorem 3 (Stability with Transferable Utility) First, since $\sum_{i \in N} u_i$ is supermodular, its set of maximizers on 2^B is a complete lattice, and so has a largest element B^*.

To prove the rest of our characterization, we define a price vector for agent i that is equivalent to a given choice set: Define $\tau_i : 2^X \to \mathbb{R}^B$ by the rule

$$\tau_i(Z) \equiv \left(\bigoplus_{x \in \beta(Z)} \min\{t_i^x | (x, t^x) \in Z\} \right) \oplus \omega \mathbf{1}^{B \setminus Z} \text{ for large } \omega.$$

Lemma 13. *Suppose that agent i's demand satisfies the gross complements property and that primitive contracts have positive externalities. Then $D_i(\tau_i(Z)) = \{\beta(Z' \cup Z_{-i}) | Z' \in C_i(Z_i | Z_{-i})\}$.*

Proof. Clearly, $(x, t^x) \notin Z'$ for any $Z' \in C_i(Z_i | Z_{-i})$ if $t_i^x > \hat{t}_i^x$ for $(x, \hat{t}^x) \in Z_i$, or if $t_i^x > 0$ and $x \in \beta(Z_{-i})$ (and thus there exists $(x, \hat{t}^x) \in Z$ with $\hat{t}_i^x = 0$). It follows that

$$\begin{aligned} \{\beta(Z' \cup Z_{-i}) | Z' \in C_i(Z_i | Z_{-i})\} &= \{\beta\left(Z' \cup Z_{-i}\right) | Z' \in C_i((\beta(Z_i), \tau_i(Z)) | (\beta(Z_{-i}), \tau_i(Z)))\}, \\ &= \left\{ \beta(Z_{-i}) \cup S' | S' \in \arg\max_{S \subseteq \beta(Z_i)} u_i(S \cup \beta(Z_{-i})) - S \cdot \tau_i(Z) \right\}, \\ &= \arg\max_{S \subseteq \beta(Z)} u_i(S) - S \cdot \tau_i(Z), \text{ (by postive externalities)} \\ &= \arg\max_{S \subseteq B} u_i(S) - S \cdot \tau_i(Z), \text{ (for large enough } \omega) \\ &= D_i(\tau_i(Z)). \end{aligned}$$

□

This also yields a useful corollary concerning monotonicity:

Lemma 14. *Suppose that agent i's demand satisfies the gross complements property and that primitive contracts have positive externalities. Then* $\{\beta(Z' \cup Z_{-i}) | Z' \in C_i(Z_i|Z_{-i})\}$ *is monotone (in the strong set order) in* Z.

Proof. Observe that $\tau_i(Z)$ is weakly decreasing in Z. The statement follows from Lemma 13 and the gross complements property. □

Now, we show that for $p \in NWP(P^*)$, (B^*, p) is stable.

First, it is individually rational: By Theorem 2, (B^*, p) is a competitive equilibrium, and so $B^* \in D_i(p_i)$ for each i. Since $\tau_i((B^*, p))$ differs only from p_i on $R^{B \setminus B^*}$, where it is higher, we must have $B^* \in D_i(\tau_i((B^*, p)))$ as well. Then by Lemma 13,

$$B^* \in \{\beta(Z' \cup (B^*, p)_{-i}) | Z' \in C_i((B^*, p)_i | (B^*, p)_{-i})\},$$

and we must have $(B^*, p)_i \in C_i((B^*, p)_i | (B^*, p)_{-i})$.

Second, it is unblocked by Z with $\beta(Z) \cap B^* \neq \emptyset$: Suppose $(x, t^x) \in Z$ and $x \in B^*$. For agents $i \in N((x, t^x)) \cap N((x, p^x))$ to choose (x, t^x) when (x, p^x) is available, it must be that $p_i^x \geq t_i^x$. Likewise, for agents $i \in N((x, t^x)) \setminus N((x, p^x))$ to choose (x, t^x) when (x, p^x) is chosen by the other agents, it must be that $t_i^x \leq 0 = p_i^x$. Thus $p_i^x \geq t_i^x$ for all $i \in N((x, t^x))$. Summing across these inequalities yields $\sum_{i \in N((x,t^x))} p_i^x \geq 0$, which holds strictly if $p_i^x \neq t_i^x$ for any $i \in N((x, t^x))$. Since p satisfies NWP, we must have $p_j^x \geq 0$ for all $j \notin N(x) \subseteq N((x, t^x))$; since transfers must sum to zero, this implies $0 \geq \sum_{i \in N((x,t^x))} p_i^x \geq 0 \Rightarrow \sum_{i \in N((x,t^x))} p_i^x = 0$. Hence, $p_i^x = t_i^x$ for all $i \in N((x, t^x))$ and $p_i^x = 0$ for all $i \notin N((x, t^x))$. Then $p^x = t^x$ and $(x, t^x) \in (B^*, p)$ — a contradiction since (x, t^x) is part of the blocking set Z.

Finally, it is unblocked by Z with $\beta(Z) \cap B^* = \emptyset$: Note that Z cannot contain distinct contracts (x, t^x) and $(x, \hat{t}^x)$ associated with the same primitive contract x, since each agent in $N(x)$ must choose between them, and can only choose one. So we can write $Z = (\beta(Z), t)$ for some t. Obviously, if Z blocks (B^*, p), for each $i \in N(Z)$ we must have $\beta(Z_i) \subset Y$ for some $Y \in \{\beta(Z' \cup (B^*, p)_{-i}) | Z' \in C_i((B^*, p)_i \cup Z_i | (B^*, p)_{-i} \cup Z_{-i})\}$. Since (B^*, p) is individually rational, by Lemma 14, we have

$$\begin{aligned} \beta(Z) \cup B^* &\in \{\beta\left(Z' \cup (B^*, p)_{-i} \cup Z_{-i}\right) | Z' \in C_i((B^*, p)_i \cup Z_i | (B^*, p)_{-i} \cup Z_{-i})\}, \\ &\in D_i(\tau_i((B^*, p) \cup Z)). \text{ (by Lemma 13)} \end{aligned}$$

Thus,

$$\begin{aligned} u_i(\beta(Z) \cup B^*) - B^* \cdot p_i - \beta(Z) \cdot t_i &\geq u_i(B^*) - B^* \cdot p_i & (\forall i \in N(Z)), \\ \Rightarrow \sum_{i \in N(Z)} u_i(\beta(Z) \cup B^*) &\geq \sum_{i \in N(Z)} u_i(B^*). & \left(\text{since} \sum_{i \in N(Z)} t_i^{\beta(Z)} = \mathbf{0}\right) \end{aligned}$$

By positive externalities,

$$
\begin{aligned}
u_i(\beta(Z) \cup B^*) &\geq u_i(B^*), && (\forall i \notin N(\beta(Z)) \subseteq N(Z)) \\
\Rightarrow \sum_{i \in N} u_i(\beta(Z) \cup B^*) &\geq \sum_{i \in N} u_i(B^*),
\end{aligned}
$$

a contradiction, since B^* is the largest efficient set of primitive contracts.

Now we show that if (Y, t) is stable, $Y = B^*$ and $p^Y = t^Y$ for some $p \in NWP(P^*)$.[20]

Suppose $B^* \not\subseteq Y$ and choose $p \in NWP(P^*)$. Since (Y, t) is stable, it is individually rational: for all $i \in N$,

$$
u_i(Y) - t_i \cdot Y \geq u_i(S \cup \beta((Y,t)_{-i})) - t_i \cdot (S \cup \beta((Y,t)_{-i})) \qquad (\forall S \subseteq \beta((Y,t)_i)).
$$

Since the gross complements property implies that u_i is supermodular,

$$
u_i(Y \cup S) - t_i \cdot Y \geq u_i(S \cup \beta((Y,t)_{-i})) - t_i \cdot (S \cap Y \cup \beta((Y,t)_{-i})) \quad (\forall S \subseteq \beta((Y,t)_i) \cup (B^* \setminus Y)). \tag{4}
$$

By Theorem 2, (B^*, p) is a competitive equilibrium: For all $i \in N$,

$$
u_i(B^*) - p_i \cdot B^* \geq u_i(S') - p_i \cdot S' \qquad (\forall S \subseteq B).
$$

For $S \subseteq \beta((Y,t)_i) \cup (B^* \setminus Y)$, choose $S' = B^* \cap (Y \cup S)$: For all $i \in N$,

$$
\begin{aligned}
u_i(B^*) - p_i \cdot B^* &\geq u_i(B^* \cap (Y \cup S)) - p_i \cdot B^* \cap (Y \cup S), && (\forall S \subseteq \beta((Y,t)_i) \cup (B^* \setminus Y)) \\
\Leftrightarrow u_i(B^*) - p_i \cdot (B^* \setminus Y) &\geq u_i(B^* \cap (Y \cup S)) - p_i \cdot (B^* \cap S). && (\forall S \subseteq \beta((Y,t)_i) \cup (B^* \setminus Y))
\end{aligned}
$$

By supermodularity, since $(S \cap B^*) \cup Y = S \cup Y$,

$$
\begin{aligned}
u_i(B^* \cup Y) - p_i \cdot (B^* \setminus Y) &\geq u_i(S \cup Y) - p_i \cdot (B^* \cap S), && (\forall S \subseteq \beta((Y,t)_i) \cup (B^* \setminus Y)) \\
\Leftrightarrow u_i(B^* \cup Y) - p_i \cdot (B^* \setminus Y) - t_i \cdot Y &\geq u_i(S \cup Y) - t_i \cdot Y - p_i \cdot (B^* \cap S). && (\forall S \subseteq \beta((Y,t)_i) \cup (B^* \setminus Y))
\end{aligned}
$$

Applying (4), for all $i \in N$ and all $S \subseteq \beta((Y,t)_i) \cup (B^* \setminus Y)$,

$$
u_i(B^* \cup Y) - p_i \cdot (B^* \setminus Y) - t_i \cdot Y \geq u_i(S \cup \beta((Y,t)_{-i})) - t_i \cdot (S \cap Y \cup \beta((Y,t)_{-i})) - p_i \cdot (B^* \cap S).
$$

It follows that $(B^* \setminus Y, p)$ blocks (Y, t), a contradiction.

[20] Recall that this is all that is necessary for (Y, t) and (Y, p) to represent the same outcome, since $t^{B \setminus Y}$ is arbitrary.

Now suppose $B^* \subseteq Y$. From individual rationality of (Y, t), for all $i \in N$, we have

$$u_i(Y) - t_i \cdot Y \geq u_i(S \cup \beta((Y,t)_{-i})) - t_i \cdot (S \cup \beta((Y,t)_{-i})). \qquad (\forall S \subseteq \beta((Y,t)_i))$$

From positive externalities, and since $t_i^x = 0$ for $x \in \beta((Y,t)_{-i})$,

$$\begin{aligned} u_i(S \cup \beta((Y,t)_{-i})) - t_i \cdot (S \cup \beta((Y,t)_{-i})) &\geq u_i(S) - t_i \cdot S, && (\forall i \in N)(\forall S \subseteq Y) \\ \Rightarrow u_i(Y) - t_i \cdot Y &\geq u_i(S) - t_i \cdot S, && (\forall i \in N)(\forall S \subseteq Y) \quad (5) \\ \Rightarrow \sum_{i \in N} u_i(Y) &\geq \sum_{i \in N} u_i(S). && (\forall S \subseteq Y) \end{aligned}$$

Since B^* is the largest efficient set of primitive contracts, we must have $B^* = Y$.

Now choose $p \in NWP(P^*)$. $t^{B^*} \oplus p^{B \setminus B^*}$ satisfies NWP: if for some $x \in B^*$, $t_i^x < 0$ for some $i \notin N(x)$, then clearly $(x, \rho[t]^x)$ blocks (B^*, t), a contradiction.

$t^{B^*} \oplus p^{B \setminus B^*} \in P^*$: From (5),

$$u_i(B^*) - t_i \cdot B^* \geq u_i(S \cap B^*) - t_i \cdot (S \cap B^*). \qquad (\forall i \in N)(\forall S \subseteq B)$$

By supermodularity,

$$u_i(S \cup B^*) - t_i \cdot B^* \geq u_i(S) - t_i \cdot (S \cap B^*). \qquad (\forall i \in N)(\forall S \subseteq B) \qquad (6)$$

Since (B^*, p) is a competitive equilibrium,

$$\begin{aligned} & u_i(B^*) - p_i \cdot B^* \geq u_i(S \cup B^*) - p_i \cdot (S \cup B^*). && (\forall i \in N)(\forall S \subseteq B) \\ \Leftrightarrow\ & u_i(B^*) - t_i \cdot B^* \geq u_i(S \cup B^*) - t_i \cdot B^* - p_i \cdot (S \setminus B^*). && (\forall i \in N)(\forall S \subseteq B) \end{aligned}$$

Combining this with (6),

$$\begin{aligned} & u_i(B^*) - t_i \cdot B^* \geq u_i(S) - t_i \cdot (S \cap B^*) - p_i \cdot (S \setminus B^*), && (\forall i \in N)(\forall S \subseteq B) \\ & \Leftrightarrow B^* \in D_i(t_i^{B^*} \oplus p_i^{B \setminus B^*}), && (\forall i \in N) \\ & \Leftrightarrow t^{B^*} \oplus p^{B \setminus B^*} \in P^*, \end{aligned}$$

as desired. □

Proof of Lemma 6 (Complementarities and Externalities in G) v_i are nondecreasing in a_{-i}: For all i, all $j \neq i$, and all $a_j \subseteq a'_j \subseteq X_j$, we have $v_i((a'_j \cap X_{-i}) \cup a_j, a_{-j}) \geq v_i(a)$

because u_i is nondecreasing in X_{-i}. Further, we have

$$v_i((a'_j \cap X_i) \cup (a'_j \cap X_{-i}) \cup a_j, a_{-j}) - v_i((a'_j \cap X_{-i}) \cup a_j, a_{-j}) \geq 0,$$

since if adding contracts in $(a'_j \cap X_i)$ changes i's payoffs, it must reduce $|a_i \setminus X(X_i, a_{-i})| = \left|a_i \setminus \left(\bigcap_{j \neq i}(a_j \cup X_{-j})\right)\right|$; the resulting decrease in the penalty term is greater than any change in the u_i term.

v_i are quasisupermodular in a_i for each a_{-i}: Follows directly from quasisupermodularity of u_i.

v_i have single crossing in a_i, a_{-i}: Suppose that $v_i(a'_i, a_{-i}) \geq v_i(a_i, a_{-i})$ for $a'_i \supset a_i$. Since ω is assumed very large, this is only possible if $a'_i \setminus X(X_i, a_{-i}) = a_i \setminus X(X_i, a_{-i})$. So we must have

$$u_i\left((a'_i \cup X_{-i}) \cap X(X_i, a_{-i})\right) \geq u_i\left((a_i \cup X_{-i}) \cap X(X_i, a_{-i})\right).$$

Since u_i is quasisupermodular and $a'_i \setminus X(X_i, a_{-i}) = a_i \setminus X(X_i, a_{-i})$, this implies

$$u_i\left((a'_i \cup X_{-i}) \cap X(X_i, a'_{-i})\right) \geq u_i\left((a_i \cup X_{-i}) \cap X(X_i, a'_{-i})\right),$$

and hence, $v_i(a'_i, a'_{-i}) \geq v_i(a_i, a_{-i})$ for all a'_{-i} such that $a'_j \supseteq a_j$ for all $j \neq i$. □

Proof of Proposition 4 (Setwise Stability and CPCE) (CPCE⇒ SW) Suppose that Y has a setwise block X'' by coalition J. Since X'' is a feasible deviation for J in the matching market, it follows that $\{\{X''_i\}_{i \in J}, \{Y_i\}_{i \notin J}\}$ is a feasible deviation for J from $\{Y_i\}_{i \in N}$ in G. Since X'' is a profitable deviation for J in the matching market, $\{X''_i\}_{i \in N}$ is a profitable deviation for J in G. Each agent in J receives the same payoffs from $\{X''_i\}_{i \in N}$ and $\{\{X''_i\}_{i \in J}, \{Y_i\}_{i \notin J}\}$, since $X'' \setminus X_J = Y \setminus X_J$ and therefore $\bigcap_{i \in N}(X''_i \cup X_{-i}) \supseteq \bigcap_{i \in J}(X''_i \cup X_{-i}) \cap \bigcap_{i \notin J}(Y_i \cup X_{-i})$, with equality since $N(X'' \setminus Y) \subseteq J$. Thus, $\{\{X''_i\}_{i \in J}, \{Y_i\}_{i \notin J}\}$ is a profitable deviation for J from $\{Y_i\}_{i \in N}$ in G. Also, since deviating to X'' is individually rational for J in the matching market, for any $i \in J$ we have $u_i(X'') \geq u_i(a_i \cup X''_{-i})$ for all a_i such that

$$a_i \subseteq X''_i \cup (Y_i \setminus Y_{J \setminus i}) \cup (X_i \setminus X_{N \setminus i}) = \left(\bigcap_{j \in J \setminus i}(X''_j \cup X_{-j}) \cap \bigcap_{j \notin J}(Y_j \cup X_{-j})\right)_i.$$

Hence, for all $a_i \subseteq X_i$,

$$u_i\left(\bigcap_{j \in J}(X''_j \cup X_{-j}) \cap \bigcap_{j \notin J}(Y_j \cup X_{-j})\right) \geq u_i\left((a_i \cup X_{-i}) \cap \bigcap_{j \in J \setminus i}(X''_j \cup X_{-j}) \cap \bigcap_{j \notin J}(Y_j \cup X_{-j})\right).$$

Thus, there is no feasible, profitable deviation from $\{\{X''_i\}_{i \in J}, \{Y_i\}_{i \notin J}\}$ for any $i \in J$ in G. It follows that $\{Y_i\}_{i\in N}$ is not a CPCE for (G, Σ^*).

(SW$\Rightarrow$ CPCE) Suppose that the u_i are quasisupermodular, and that ν is a feasible, profitable, self-enforcing deviation for J from $\{Y_i\}_{i\in N}$ in G. Then there is a pure strategy a^* that is also a feasible, profitable, self-enforcing deviation for J from $\{Y_i\}_{i\in N}$ in G.

Let $a' = \left\{\bigcup_{a_i \in \operatorname{supp}\nu_i} a_i\right\}_{i\in N}$, and for $c = \{c_i\}_{i\in J}$, let

$$a'(c) = \left\{\left\{c_i \cup \bigcup_{a_i\in\operatorname{supp}\nu_i} a_i\right\}_{i\in J}, \left\{\bigcup_{a_i\in\operatorname{supp}\nu_i} a_i\right\}_{i\notin J}\right\}.$$

Now construct a^* as follows. Let $c^0 = \{\emptyset\}_{i\in J}$. Since v_i are quasisupermodular in a_i, $\arg\max_{Z_i\subset(Y_i\setminus Y_J)} v_i(Z_i \cup a'_i, a'_{-i}(c))$ is a lattice; let $h_i(c)$ be its largest element. Since v_i has the single crossing property, $h_i(c)$ is nondecreasing in c. For each $n \in N$, let $c_i^n = h_i(c^{n-1})$. c^n is a nondecreasing sequence: trivially, $c_i^0 \subseteq c_i^1$ for each i, and $c_i^{n-1} \subseteq c_i^n$ for each $i \implies h_i(c^{n-1}) \subseteq h_i(c^n)$ for each $i \implies c_i^n = h_i(c^{n-1}) \subseteq h_i(c^n) = c_i^{n+1}$ for each i. Then since $(Y_i \setminus Y_J)$ is finite for each i, $c_n = c_{n+1}$ for some n; then choose $a^* = a'(c^n)$. By construction, $v_i(a^*) \geq v_i(a'_i \cup \hat{a}_i, a^*_{-i})$ for all $\hat{a}_i \in a'_i \cup (Y_i \setminus Y_J)$.

First, deviating to a^* is feasible for J: For $i \notin J$, $\nu_i = \mathbf{1}\{a_i = Y_i\} = a_i^*$.

Next, deviating from $\{Y_i\}_{i\in N}$ to a^* is self-reinforcing for J: Since ν is self-enforcing for J, for each $i \in J$ and each $a_i \in \operatorname{supp}\nu_i$,

$$\sum_{a_{-i}} \nu(a_{-i}|a_i) v_i(a_i, a_{-i}) \geq \sum_{a_{-i}} \nu(a_{-i}|a_i) v_i(a''_i, a_{-i}) \forall a''_i \subseteq X_i.$$

Then for every $a''_i \subseteq a_i$, for some $a_{-i} \in \operatorname{supp}\nu_{-i}$, we must have $v_i(a_i, a_{-i}) \geq v_i(a''_i, a_{-i})$. Since v_i has the single crossing property, this implies $v_i(a_i, a^*_{-i}) \geq v_i(a''_i, a^*_{-i})$.

Label the elements of $\operatorname{supp}\nu_i$ $a_i^1, a_i^2, \ldots, a_i^K$, and let $b_i^k = \bigcup_{\ell=1}^k a_i^\ell$. Then for any $\hat{a}_i \subseteq a_i^* \cup (Y_i \setminus Y_J)$, we have $v_i(a_i^k, a'_{-i}) - v_i((a_i^k \cap (\hat{a}_i \cup b_i^{k-1}), a^*_{-i}) \geq 0$ for all k.

Using quasisupermodularity to increase each strategy by $(b_i^{k-1} \cup \hat{a}_i) \setminus a_i^k$ yields

$$\begin{aligned}
v_i\left((a_i^k \cup b_i^{k-1} \cup \hat{a}_i), a^*_{-i}\right) - v_i\left((b_i^{k-1} \cup \hat{a}_i), a^*_{-i}\right) &\geq 0, && (\forall \hat{a}_i \subseteq a'_i \cup (Y_i \setminus Y_J)) \\
\Rightarrow \sum_{k=1}^K v_i\left((b_i^k \cup \hat{a}_i), a^*_{-i}\right) - v_i\left((b_i^k \cup \hat{a}_i), a^*_{-i}\right) &\geq 0, && (\forall \hat{a}_i \subseteq a'_i \cup (Y_i \setminus Y_J)) \\
\Leftrightarrow v_i\left(a'_i \cup \hat{a}_i, a^*_{-i}\right) - v_i\left(\hat{a}_i, a^*_{-i}\right) &\geq 0. && (\forall \hat{a}_i \subseteq a'_i \cup (Y_i \setminus Y_J)) \\
\Rightarrow v_i(a^*) - v_i\left(\hat{a}_i, a^*_{-i}\right) &\geq 0. && (\forall \hat{a}_i \subseteq a'_i \cup (Y_i \setminus Y_J)) \qquad (7)
\end{aligned}$$

Because ν is self-enforcing for J, for each $i \in J$ and each $a_i \in \operatorname{supp}\nu_i$, we must have $a_i \subseteq X(X_i, a^*_{-i})$; otherwise $X(X_i, a^*_{-i})_i \cap a_i$ would be a profitable deviation for i after an instruction to play a_i in ν, since it would produce the same set of contracts but avoid the certain penalty associated with $a_i \setminus X(X_i, a^*_{-i})$. Then for each $i \in J$, $a'_i \subseteq X(X_i, a^*_{-i})$.

Now note that for all $i \in J$, $a'_i \cup (Y_i \setminus Y_{J\setminus i}) = X(X_i, a^*_{-i})_i$: Suppose not, and $a'_i \cup (Y_i \setminus Y_{J\setminus i}) \subset X(X_i, a^*_{-i})_i$. Then for some $x \in X(X_i, a^*_{-i})_i$, $x \notin a'_i$ and $x \notin (Y_i \setminus Y_{J\setminus i})$. $N(x)$ is not a singleton, so $\exists j \neq i$, $j \in N(x)$; since $x \notin (Y_i \setminus Y_{J\setminus i}) = X(X_i, a'_{-i})_i \setminus Y_{J\setminus i}$, it follows that $j \in J$. Since $x \in X(X_i, a^*_{-i})$, we must have $x \in a^*_j$; then since $i \in N(x)$, we must have $x \in a'_j$. But $x \notin a'_i$, $x \notin (Y_i \setminus Y_{J\setminus i})$, and $x \in X_i \Rightarrow x \notin X(X_j, a^*_{-j})$, a contradiction since $a'_j \subseteq X(X_j, a^*_{-j})$. Thus, because of the penalty ω, $v_i(a') > v_i(\hat{a}_i, a'_{-i})$ for all $\hat{a}_i \not\subseteq a'_i \cup (Y_i \setminus Y_{J\setminus i})$. It follows from (7) that a^* is self-reinforcing.

In addition, a^* is a profitable deviation for J: From (7), we have

$$
\begin{aligned}
v_i(a^*) &\geq \sum_{a_{-i}} \nu(a_{-i}|a_i) v_i(a_i, a^*_{-i}), \\
&\geq \sum_{a_{-i}} \nu(a_{-i}|a_i) v_i(a_i, a_{-i}), \\
&\geq v(Y_i, \{Y_j\}_{j\neq i}),
\end{aligned}
$$

where the second line follows from v_i nondecreasing in a_{-i}, and the last line follows from ν being a profitable deviation.

Now to show that $X(a^*)$ is a setwise block of Y by coalition J: Since $a^*_i = Y_i$ for $i \notin J$, it follows that $N(X(a^*) \setminus Y) \subseteq J$, and furthermore that $X(a^*) \setminus X_J = Y \setminus X_J$. Thus $X(a^*)$ is a feasible deviation for J. That $X(a^*)$ is profitable follows from profitability of a^*: no penalty is incurred at a^* since $a^*_i \subseteq X(X_i, a^*_{-i})_i \cup (Y_i \setminus Y_{J\setminus i})$, so $u_i(X(a^*)) = v_i(a^*)$, and likewise $v_i(Y_i, \{Y_j\}_{j\neq i}) = u_i(Y)$. Finally, since a^* is self-reinforcing, $u_i(X(a^*)) \geq u_i\left(X(a_i, a^*_{-i})\right)$ for all $a_i \subseteq X_i$; equivalently, $u_i(X(a^*)) \geq u_i\left(a_i \cup X(a^*)_{-i}\right)$ for all $a_i \subseteq X(X_i, a^*_{-i})_i = a^*_i \cup (Y_i \setminus Y_{J\setminus i}) = X(a^*)_i \cup (Y_i \setminus Y_{J\setminus i})$. Thus, $X(a^*)_i = a^*_i \in C_i(X(a')_i \cup (Y_i \setminus Y_{J\setminus i}) | X(a^*)_{-i})$, and deviating to $X(a^*)$ is individually rational for J. $\square$

Proof of Theorem 4 (Setwise Stability with Complementarities and Positive Externalities)

For each $a_i \subseteq X_i$, let $G(a)$ be the game G with each agent i's strategies restricted to 2^{a_i}. When $a_i = Y_i$ for some $Y \subseteq X$ for each i, we write $G(a) = G(Y)$.

For each i, any $a_i \not\subseteq A_i(Y)$ is strictly dominated in $G(Y)$ by $A_i(Y) \cap a_i$: Suppose not and there is some a_{-i} such that $v_i(a) \geq v_i(A_i(Y) \cap a_i, a_{-i})$. Since $|a_i \setminus X(Y_i, a_{-i})| \geq |A_i(Y) \cap a_i \setminus X(Y_i, a_{-i})|$, we must have $u_i(X(a_i, a_{-i})) \geq u_i(X(A_i(Y) \cap a_i, a_{-i}))$. In addition, we must have $a_i \setminus A_i(Y) \subseteq X(Y_i, a_{-i})$ or else $|a_i \setminus X(Y_i, a_{-i})| > |A_i(Y) \cap a_i \setminus X(Y_i, a_{-i})|$ (which is not possible if $v_i(a) \geq v_i(A_i(Y) \cap a_i, a_{-i})$).

Then quasisupermodularity implies

$$u_i(X(a_i, a_{-i}) \cup (A_i(Y) \cap a_i)) \geq u_i(A_i(Y) \cap a_i), \quad \text{(adding } (A_i(Y) \cap a_i) \setminus X(Y_i, a_{-i}) \text{ to both arguments)}$$
$$u_i(X(a_i, a_{-i}) \cup A_i(Y)) \geq u_i(A_i(Y)), \qquad \text{(adding } A_i(Y) \setminus a_i \text{ to both arguments)}$$

a contradiction.

For each i, any $a_i \nsubseteq A(Y)$ is strictly dominated in $G(\{A_i(Y)\}_{i\in N})$ by $a_i \cap A(Y)$: For all $a_{-i} \in \{A_j(Y)\}_{j\neq i}$, $X((a_i \cap A(Y)), a_{-i}) = X(a)$. So $u_i(X(a_i, a_{-i})) = u_i(X(a_i \cap A(Y), a_{-i}))$. But $a_i \setminus X(a) \supset (a_i \cap A(Y)) \setminus X(a) = (a_i \cap A(Y)) \setminus X((a_i \cap A(Y)), a_{-i})$, so $|a_i \setminus X(a)| > |(a_i \cap A(Y)) \setminus X((a_i \cap A(Y)), a_{-i})|$. Thus $v_i(a) < v_i(a_i \cap A(Y), a_{-i})$.

It follows that any $a_i \nsubseteq A(Y)$ does not survive iterated strict dominance in $G(Y)$. Now let

$$Y_0 = X, \qquad Y^{n+1} = A(Y^n). \qquad (\forall n \in \mathbb{Z}_+)$$

It follows by induction that for all n, any $a_i \nsubseteq Y^n$ does not survive iterated strict dominance in G. Now since the sequence $\{Y^n\}_{n=0}^{\infty}$ is monotone and 2^X is finite, it must converge for finite n^*. Then Y^{n^*} is a fixed point of A, and $Y^{n^*} \subseteq X^*$. Since A is monotone, so is A^{n^*}, so $X^* = A^{n^*}(X^*) \subseteq A^{n^*}(X) = Y^{n^*}$. Thus $Y^{n^*} = X^*$. It follows that for each i, any $a_i \nsubseteq X_i^*$ does not survive iterated strict dominance in G.

Since X^* is individually rational, $X_i^* \in C_i(X_i^*|X_{-i}^*) = B_i(\{X_j^*\}_{j\neq i})$, so $\{X_i^*\}_{i\in N}$ is a Nash equilibrium of G. Thus, for each i, X_i^* survives iterated strict dominance in G. Thus, $\{X_i^*\}_{i\in N}$ is the largest element of the serially undominated set of G. Then by Lemma 6 and Milgrom and Shannon (1994) Theorem 2, it is the unique CPCE of G up to payoff equivalence. The proposition follows from Proposition 4. □