Decomposing long bond returns: A decentralized modeling approach*

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Abstract

We develop a new, decentralized theory that determines the fair value of the yield to maturity on a bond or bond portfolio based purely on the near-term dynamics of its own yield, without the need to make assumptions on the instantaneous interest rate dynamics, nor the need to know whether and how the yield dynamics will change in the future. The new theory decomposes the yield into three components: near-term expectation, risk premium, and convexity effects. We propose to estimate the convexity effect with its recent time series and determine the expectation from either statistical models or economists forecasts, leaving the remaining component of the yield as a risk premium estimate. Empirical analysis on US and UK swap rates shows that this risk premium component can predict future bond excess returns. We also propose to perform comparative analysis of the yield curve via common factor structure assumptions on the rate of change across the yield curve. The extracted rate of change factor strongly predicts future changes in the swap curve slope.

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1. Introduction

The literature on interest rate modeling is vast, with different approaches targeting different challenges. The traditional literature on expectation hypothesis focuses on predicting changes in short-term interest rates with the slope of the interest rate term structure. The shape of the yield curve shape contains information on not only the expectation but also risk premium and convexity; nevertheless, the expectation information dominates the short end of the yield curve (Longstaff (2000)) and can be used to predict future short rate movements. During the past decade, the literature on no-arbitrage dynamic term structure models (DTSMs) has experienced tremendous growth in terms of both theoretical characterization[1] and empirical analysis.[2] By specifying the instantaneous interest rate dynamics and applying the principle of no dynamic arbitrage, these models generate fair values on the whole yield curve and are thus capable of pricing bonds of all maturities within one centralized view of the short rate dynamics. The centralization has played important roles in practical applications, such as generating interpolated valuation in between observed maturities and identifying relative valuation opportunities based on the deviations between market observation and model valuation (Bali, Heidari, and Wu (2009)). By contrast, to price interest rate options, the literature (e.g., Heath, Jarrow, and Morton (1992)) often takes the observed yield curve as given and focuses on modeling the interest rate volatility. This approach highlights the contribution of interest rate volatility to the option valuation while proposing to delta hedge the yield curve exposure.

All these existing frameworks, however, have limited capabilities in explaining the short-term return behavior of long-dated bonds. The literature on expectation hypothesis uses long rates to predict short rate changes, not the other way around. Modeling long rates with DTSMs also stretches the modeler's imagination on how short rate should move in the far distant future. The starting point for this literature is often some mean-reverting dynamics assumption on the short rate. Yet, mean reversion calibrated to the

[1] See Duffie and Kan (1996), Duffie, Pan, and Singleton (2000), and Duffie, Filipović, and Schachermayer (2003) for a progressively more general characterization of the affine class of models, Leippold and Wu (2002) for a characterization of the quadratic class, and Cheng and Scaillet (2007) for an integration of the two classes.

[2] Prominent examples include Dai and Singleton (2000), Dai and Singleton (2002), Duffee (2002), Leippold and Wu (2003), Aït-Sahalia and Kimmel (2010), Adrian, Crump, and Moench (2013), Hamilton and Wu (2012), Collin-Dufresne, Goldstein, and Jones (2008), Joslin, Singleton, and Zhu (2011).

short end of the yield curve often implies much smaller movements for long rates than actually observed from data (Giglio and Kelly (2017)). Long rates are neither easily predictable, nor converging to a constant any time soon. They tend to move randomly and with substantial volatility, a behavior that is difficult to be reconciled within existing DTSMs. Furthermore, the centralized approach of DTSMs can experience stability issues in practical implementation. The centralization dictates that adding or removing one security from the estimation or an accidental data error on one security can alter the fair valuations on all other securities, making the approach vulnerable to error contagion.

It is important to realize that investors can choose to hold a very long-term bond for a very short period of time. In this case, investors are mainly concerned with the short-term movement of the long-term yield rather than the long-term movement of the short rate. Indeed, even long-term investors must be concerned with the short-term fluctuations for risk management purposes, such as value at risk calculations.

In this paper, we propose a new, decentralized theory that provides pricing insights for a particular bond or bond portfolio of interest based on the short-term behavior of the yield on that particular bond or bond portfolio. The new theory compliments and contrast with the centralized approach as it determines the fair value of the yield to maturity on a bond (portfolio) based purely on its own near-term dynamics, without the need to make assumptions on how its dynamics will change in the future, or how the instantaneous interest rate or any other bond yields behave.

The new theory starts by performing a short-term profit and loss (P&L) attribution to a bond investment through its yield representation. Taking expectation on the P&L attribution under the risk-neutral measure and setting the expected instantaneous return to the instantaneous interest rate by no-arbitrage leads to a simple pricing equation on the bond yield. The pricing equation decomposes the fair valuation of the bond yield into three components: near-term expectation, risk premium, and convexity effects. The expectation component is determined by the current forecast on the yield's rate of change, with no reference to where the forecast comes from and how the forecast varies in the future. One can there generate the forecast either via statistical models or by directly borrowing from economists forecasts, without worrying about how their forecasts are formulated. The convexity effect is determined by the current volatility estimate on the yield changes, again with no reference to its future variation. Accordingly, a simple yield volatility estimator can be used to determine the convexity effect, leaving the remaining component of the yield as a risk premium estimate.

Since the yield on each bond or portfolio can be analyzed on its own, there is no error contagion effect from one security to another. Since the theory only relies on the yield's near-term dynamics, one does not need to make assumptions on how the yield dynamics will change in the future. In particular, when pricing a 60-year bond, the focus is to generate the best conditional mean and variance forecasts on the movements of its yield, rather than making 60-year projections on an unobserved instantaneous interest rate. The decomposition can be performed, locally and with equal ease, on the yield of a zero-coupon bond, a coupon bond, or a bond portfolio. While the standard centralized approach is better suited to perform relative valuation across bonds of different maturities, our new decentralized theory can be used to analyze each bond on its own by linking its pricing at a point in time directly to its own conditional risk estimates at that time.

When an investor desires to analyze and compare a selected basket of bonds, the new theory can be used for the comparative analysis by directly comparing the risk behaviors of the underlying yields. In particular, one can impose common factor structures on the yield changes and generate comparative pricing implications based on the common risk structure. Empirical results from the literature on risk factor analysis of bond yields can thus be readily incorporated into this new pricing framework, making them useful not only for risk analysis, but also for fair pricing of the bonds under consideration.

As an application, we consider the pricing of long-dated bonds with the assumption of no directional prediction on its underlying yield. It is extremely difficult to predict the directional movement of long-term yields. In this application, we take this hard-to-predict feature as our starting point, and infer the risk premium component in the long bond yield while controlling for the convexity component based on historical variance estimators on the yield changes. We perform empirical analysis on US and UK long-term swap rates and treat them as the coupon rates for par bonds. The analysis shows that the risk premium extracted from each long-term bond can be used for out-of-sample prediction of the future excess returns on the bond.

We also explore the new theory's application to comparative yield curve analysis via common factor structure assumptions on the near-term dynamics. In particular, we propose to estimate the convexity effects based on its recent time series behavior, and assume a common factor structure on the yield's rate of change and market price of risk. We propose an estimation framework that extracts the common factors from the observed yield curve and estimated convexity effects. Estimating the common factor structure on the US and UK swap curve shows that the extracted common factor on the expected rate of yield change shows strong predictive power on future changes in the yield curve.

The remainder of this paper is organized as follows. Section 2 develops our new pricing theory, and contrasts it with the classic DTSM. Section 3 describe the data used for the empirical analysis and its general behaviors. Section 4 discusses the results. Section 5 provides concluding remarks and directions for future research.

2. A decentralized theory of bond yields

We consider an infinite-horizon continuous-time economy. Uncertainty is represented by a filtered probability space $\{\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\}$, where $\mathbb{P}$ is the physical measure. We assume that the usual conditions of right continuity and completeness with respect to the null sets of $\mathbb{P}$ are satisfied. We further assume the existence of a money market account (MMA) associated with an instantaneous interest rate $r_t \geq 0$. Since the value of the MMA is always strictly positive, this MMA is a numeraire. The assumption of no dynamic arbitrage implies the existence of an equivalent martingale measure $\mathbb{Q}$ associated to this MMA numeraire.

Let B_t denote the time-t value of a riskfree bond (or bond portfolio) that pays a stream of N fixed cash flows $\{C_j\}_{j=1}^N$ at times $\{t+\tau_j\} \geq t$ for $j = 1, 2, \cdots, N$, with τ_j denoting the time to maturity of the jth cash flow. Traditional dynamic term structure models start by modeling the dynamics of the instantaneous interest rate r_t and value the bond via the following expectation operations,

$$
\begin{aligned}
B_t &= \sum_{j=1}^{N} C_j \mathbb{E}_t^{\mathbb{P}} \left[M_{t,t+\tau_j} \right] && (1)\\
&= \sum_{j=1}^{N} C_j \mathbb{E}_t^{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) e^{-\int_t^{t+\tau_j} r_u du} \right] && (2)\\
&= \sum_{j=1}^{N} C_j \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^{T} r_u du} \right]. && (3)
\end{aligned}
$$

where $\mathbb{E}_t^{\mathbb{P}}[\cdot]$ and $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ denote the expectation operator conditional on time-t filtration $\mathcal{F}_t$ under the physical measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$, respectively, $M_{t,T}$ denotes the pricing kernel linking value at time t to value at time T, and $\frac{d\mathbb{Q}}{d\mathbb{P}}$ defines the measure change from $\mathbb{P}$ to $\mathbb{Q}$. The measure change represents the martingale component of the pricing kernel that defines the pricing of various risks. The three equations (1)-(3) represent the bond valuation with different starting points. Through the expectation operation, bonds with cash flows at all times are linked together through the centralized modeling of the pricing kernel or the instantaneous interest rate.

Given the price of a bond B_t, its yield to maturity y_t is defined via the following equality,

$$
B_t \equiv \sum_{j=1}^{N} \exp\left(-y_t \tau_j\right) C_j. \tag{4}
$$

The yield to maturity can be regarded as the continuous compounding internal rate of return for holding the bond to expiration.

2.1. Centralized yield decomposition under the classic setting

Before we introduce our new pricing framework, we decompose the yield of a zero-coupon bond under the classic setting as a reference point. Let $B_t(T)$ denotes the time-t price of a zero-coupon bond that pays \$1 at expury $T \geq t$. With a single cash flow, its yield to maturity $y_t(T)$ can be explicitly solved from the bond price as,

$$y_t(T) \equiv -\frac{\ln B_t(T)}{T-t}. \tag{5}$$

Substituting the bond pricing formula (3) into the yield equation in (5) reveals the link between the T-maturity yield observed at time $t \in [0,T]$ and future short rates r_u realized at times $u \in [t,T]$:

$$y_t(T) \equiv -\frac{1}{T-t} \ln \mathbb{E}_t^{\mathbb{Q}} e^{-\int_t^T r_u du}, \qquad t \in [0,T]. \tag{6}$$

Furthermore, by adding and subtracting the same term twice, we can decompose the zero-coupon bond yield into three distinct terms,

$$y_t(T) = \mathbb{E}_t^{\mathbb{P}} \frac{\int_t^T r_u du}{T-t} + \mathbb{E}_t^{\mathbb{P}} \left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}} - 1 \right) \frac{\int_t^T r_u du}{T-t} \right] - \frac{1}{T-t} \left[\ln \mathbb{E}_t^{\mathbb{Q}} e^{-\int_t^T (r_u - E_t^{\mathbb{Q}} r_u) du} \right]. \tag{7}$$

The first term in the decomposition (7) represents the expectation of the average short rate $\frac{\int_t^T r_u du}{T-t}$ over the life of the bond between now t and the expiry T.

The second term is the risk premium as captured by the covariance under $\mathbb{P}$ of this average short rate with the random variable, $\frac{d\mathbb{Q}}{d\mathbb{P}} - 1$, which has zero mean under $\mathbb{P}$. If interest rates are stochastic and if bond returns are thought to have a positive risk premium, this covariance will also be positive.

The third term represents the convexity effect. As the term $C \equiv \frac{1}{T-t} \left[\ln \mathbb{E}_t^{\mathbb{Q}} e^{-\int_t^T (r_u - E_t^{\mathbb{Q}} r_u) du} \right]$ is non-negative, the convexity effect C can only lower the yield. One can interpret C as a *non-standard* deviation under $\mathbb{Q}$ of the zero mean random variable $-\int_t^T (r_u - E_t^{\mathbb{Q}} r_u) du$. When compared to the standard deviation, the non-standard deviation replaces the quadratic function with an exponential. When the average future short rate $\frac{1}{T-t} \int_t^T r_u du$ is normally distributed under $\mathbb{Q}$ with variance V, the convexity term is equal to $C = \frac{1}{2} V(T-t)$, proportional to the variance of the average future short rate. When the distribution is non-normal, the non-standard distribution also captures contributions from higher-order moments.

The relative importance of the three terms in the yield decomposition varies across maturities. As the maturity date T approaches the current time, t, the last two terms vanish, so that the current yield-to-maturity approaches the short rate:

$$\lim_{T \downarrow t} y_t(T) = r_t, \qquad t \geq 0. \tag{8}$$

As we increase the time to maturity $\tau = T - t$, the second and the third terms both start affecting the yield, but at different speeds. In the very special example where the instantaneous interest rate follows a random walk under $\mathbb{P}$,

$$dr_t = \sigma dW_t \tag{9}$$

and the market price of the Brownian risk is a negative constant $\gamma < 0$, the risk premium component increases linearly with maturity as $-\frac{1}{2}\gamma\sigma\tau$, while the convexity effect increases quadratically with maturity $\frac{1}{6}\sigma^2\tau^2$. Thus, as maturity increases, the convexity term will ultimately dominate and drive the yield to negative territory. Researchers often impose mean reversion in the short rate dynamics, which allows both the risk premium and the convexity terms to asymptote to finite constants.

The three-term decomposition of a yield is generic. What's particular about the classic decomposition in (7) is that the three components are linked to the expectation, risk, and pricing of one single variable, the instantaneous interest rate, and they are all tied to the expectation over the life span of the bond. Thus, to decompose the yield under the classic setting on a long-dated bond, one would need to make projection far into the future about the risk and pricing of the instantaneous interest rate. In practice, an investor can invest in very long-dated bonds for a very short period of time. In this case, the investor worries more about the near-term value fluctuation of the bond than any long-run projections. Even for investors with a long investment horizon, managing the daily P&L fluctuation of their investment is still vitally important. Given these practical considerations, our new pricing framework does not rely on long-run projections of a centralized variable (e.g., the instantaneous interest rate), but builds on a decentralized, short-run P&L attribution of the bond investment.

2.2. Decentralized short-run P&L attribution of bond investments

To perform short-run P&L analysis on a bond investment, we focus on the bond value change over the next instant. To decentralize the P&L attribution, we examine how the bond value varies with its own yield to maturity. First, we follow industry practice by characterizing the risk of the bond by its duration and convexity, which capture the first- and second-order interest rate sensitivity of the bond value. While there are many variations in the definition, we take the following particular definition that measures the sensitivity of the bond price against its own yield to maturity,

$$\tau \equiv -\frac{\partial B_t}{B_t \partial y_t} = \sum_{j=1}^{N} w_j \tau_j, \tag{10}$$

$$\tau^2 \equiv \frac{\partial^2 B_t}{B_t \partial y_t^2} = \sum_{j=1}^{N} w_j \tau_j^2, \tag{11}$$

where the weights w_j are given by

$$w_j = \frac{\exp\left(-y_t \tau_j\right) C_j}{\sum_{i=1}^{N} \exp\left(-y_t \tau_i\right) C_i}. \tag{12}$$

According to this definition, the duration (τ) and the convexity (τ^2) are simply the value-weighted average maturity and maturity squared of the cash flows from the bond. The weight on each cash flow is based on its value as a fraction of the total bond worth. For a zero-coupon bond, its duration is simply its time to maturity, and its convexity is the maturity squared.

The industry quotes the yield to maturity of a bond instead of its price for stability and comparability across different bonds and over different time periods. The duration and convexity measures capture how much the bond value varies when the yield varies. While both the yield to maturity in (4) and the duration/convexity risk measures in (10) and (11) can be compared cross-sectionally across different bonds, they are decentralized measures whose calculation depends only on the particular bond itself.

There are also other duration/convexity measures that are calculated by shocking the yield curve in a particular way and thus lose the decentralized feature. Our particular choice of the duration and convexity definition not only has simple forms and interpretations, but is also local to the bond itself, without the need of stripping a yield curve.

With the yield to maturity definition and the decentralized risk exposure measures, we can attribute the short-term investment P&L of a bond with respect to the movement of its own yield to maturity via a Taylor expansion,

$$dB_t = \frac{\partial B_t}{\partial t}dt + \frac{\partial B_t}{\partial y}dy + \frac{1}{2}\frac{\partial^2 B_t}{\partial y^2}(dy)^2 + o(dt), \tag{13}$$

where $o(dt)$ denotes higher-order terms of dt when yield moves diffusively. When the yield can jump, the jump induces more significant higher-order terms,

$$JC = \int \left(B(y_t e^x) - B(y_t)\right)\nu(x,t)dxdt$$

where $\nu(x,t)$ counts the jump of size x in the logarithm of the yield at time t. We henceforth assume that the next move for the yield to maturity of the bond is continuous. Accordingly, we can attribute the bond investment P&L solely to time decay and first and second-order effects from the yield to maturity movement. Since the P&L attribution focuses on the bond value change over the next instant, the continuity assumption is only for the next instant. The results hold even if the yield can jump at any other times.

Compared to classic centralized approach of bond pricing, equation (13) focuses on the short-term variation of the bond value regardless of the bond maturity. Furthermore, the P&L attribution is decidedly local and is based on the variation of its own yield. Dividing both sides of equation (13) by $B_t dt$, and pluging in the definition of yield in (4), the definition of duration in (10), and the definition of convexity in (11), we have an attribution of the annualized investment return as,

$$\frac{dB_t}{B_t dt} = y_t - \tau\frac{dy}{dt} + \frac{1}{2}\tau^2\frac{(dy)^2}{dt}. \tag{14}$$

The first term in (14) denotes the carry — If the bond yield does not change, the instantaneous bond return is simply the yield to maturity. The second term highlights the directional impact of the yield change on the bond return. The negative bond-yield relation dictates that the bond return declines when yield goes up. The sensitivity is measured by the bond's duration τ. The third term captures the convexity of the bond-yield relation. Larger yield moves of either direction increases the bond return due to the convex bond-yield relation. The magnitude of this exposure is captured by the bond's convexity measure τ^2.

Taking expectation on (14) under the statistical measure $\mathbb{P}$, we can attribute the expected bond investment return to three sources,

$$\mathbb{E}_t^{\mathbb{P}}\left[\frac{dB_t}{B_t dt}\right] = y_t - \mu_{t,y}\tau + \frac{1}{2}\sigma_{t,y}^2\tau^2, \tag{15}$$

where $\mu_{t,y} = \mathbb{E}_t^{\mathbb{P}}[dy_t/dt]$ denotes the time-t expected rate of change on the yield, and $\sigma_{t,y}^2 = \mathbb{E}_t\left[(dy)^2/dt\right]$ denotes the time-t conditional variance rate of the yield. Equation (15) decomposes the expected bond return into three sources. The first term captures the expected return from carry. Bonds with a higher yield generates a higher return on average due to carry. Second, due to the negative bond-yield relation, expected yield increase reduces the expected bond return. Third, due to the convexity of the bond-yield relation, higher volatility on the yield movements leads to higher expected bond return. This convexity effect is analogous to the effect of volatility on the value of an option. Due to the convex relation between the option value and the underlying security price, a higher volatility increases the option value just as a higher yield volatility increases the expected bond return. A duration neutral bond portfolio that is long convexity is analogous to a delta-neutral long options position.

The decomposition highlights the key risk and return sources of bond investments. If an investor has no view on the direction of the yield movement, the investor can form duration-neutral bond portfolios with bonds of nearby maturities. Assuming that yields at nearby maturities strongly co-move, the duration-neutral portfolio will have minimal exposure to common directional movements of the bond yields. Then, the long-short positioning of the two bonds will be driven by the difference between the carry and convexity benefits of the two bonds.

To illustrate the contributions from the different components, let us imagine a situation where zero-coupon bond yields at long maturities (e.g., 10, 15, 30) are flat and move in parallel by substantial amount (i.e., σ^2 is large).[3] In this case, we can form a self-financing and riskless portfolio (under our parallel movement assumption) that makes money from convexity. First, since the yield is the same, a dollar-neutral portfolio would be self-financing. Second, since the yields move in parallel, a duration-neutral portfolio will have no directional exposure. Then, if such a portfolio can be formed with positive convexity, one would expect to market positive money in the future. For example, if we are long \$300 10-year zero-coupon bond, long \$100 30-year zero-coupon bond, and short \$400 15-year zero-coupon bond, the butterfly portfolio will cost zero dollar (dollar-neutral), has zero duration, and contain a positive convexity of $\tau_f^2 = 75$. By cancelling out carry and duration while retaining positive convexity, the instantaneous P&L on the fly is positive and proportional to the squared of the yield change,

$$dFly_t = \frac{1}{2}\tau_f^2 (dy)^2 \geq 0. \tag{16}$$

Therefore, observing a flat and parallel moving yield curve presents an arbitrage opportunity.

2.3. Decentralized no-arbitrage pricing and yield decomposition

The P&L attribution analysis highlights the local risk sources and return opportunities for the bond investment over the next instant. To generate pricing implications, we take expectation under the risk-neutral measure $\mathbb{Q}$ on the attribution in (14),

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{dB_t}{B_t dt}\right] = y_t - \mu_{t,y}^{\mathbb{Q}}\tau + \frac{1}{2}\sigma_{t,y}^2\tau^2, \tag{17}$$

[3] One can perform similar analysis on coupon bonds with specific duration and convexity estimates. Using zero-coupon bonds make the duration and convexity numbers explicit.

where $\mu_{t,y}^{\mathbb{Q}} = \mathbb{E}_t^{\mathbb{Q}}[dy_t/dt]$ denotes the time-t expected rate of change on the yield under the risk-neutral measure. Given the diffusive assumption over the next instant with the instantaneous volatility $\sigma_{t,y}$, if we use λ_t to denote the market pricing of the bond Brownian risk (i.e., the negative of the yield Brownian risk), we can link the expected rate of yield change under the two measures by

$$\mu_{t,y}^{\mathbb{Q}} = \mu_{t,y} + \lambda_t \sigma_{t,y}. \tag{18}$$

The market price of bond risk is positive $\lambda_t > 0$ if bond returns are thought to contain a positive risk premium.

No dynamic arbitrage dictates that the risk-neutral expected instantaneous rate of return on any investment is equal to the instantaneous interest rate r_t. Applying this no-dynamic-arbitrage condition to the risk-neutral expectation in (17) leads to a simple pricing relation for the bond yield spread over the instantaneous interest rate,

$$y_t - r_t = \mu_{t,y}\tau + \lambda_t \sigma_{t,y}\tau - \frac{1}{2}\sigma_{t,y}^2\tau^2. \tag{19}$$

The fair value of the yield spread $(y_t - r_t)$ on the bond investment is determined by its expected rate of change foreacst $(\mu_{t,y})$, risk premium $(\lambda_t\sigma_{t,y})$, and its volatility forecast $(\sigma_{t,y})$.

Theorem 1 *If the yield of a bond is moving continuously over the next instant, no dynamic arbitrage dictates that the fair spread of this yield over the instantaneous interest rate is linked to its expected rate of change* $(\mu_{t,y})$*, its risk premium* $(\lambda_t\sigma_{t,y})$*, and its variance rate* $(\sigma_{t,y}^2)$ *through the bond's duration* τ *and convexity* τ^2 *by*

$$y_t - r_t = \mu_{t,y}\tau + \lambda_t \sigma_{ty}\tau - \frac{1}{2}\sigma_{t,y}^2\tau^2. \tag{20}$$

Compared to classic centralized bond pricing, the new pricing relation in (20) is highly decentralized. The fair valuation of the bond investment in (20) only depends on the behavior of its own yield to maturity, with no direct dependence on the short rate dynamics or the dynamics of any other yields. In fact, the pricing does not even rely on the full dynamics of its own yield, but only depend on the conditional expectation estimators of its rate of change, its volatility, and its market pricing. All three estimates can change over time, so can the dynamics of the yield, but none of these changes enter the pricing of the current yield spread. Thus, in the end, the pricing relation does not depend on any particular dynamics assumptions, but only rely on three conditional forecasts. One can bring in the forecasts from any outside sources and directly examine the pricing implication under our new theory. These forecasts can come from any model assumptions or algorithms, allowing maximum flexibility and cross-field collaboration.

The pricing relation in (20) also provides a decentralized version of the yield decomposition. Similar to the centralized yield decomposition, equation (20) also decomposes the yield into expectation, risk premium, and convexity. The difference is that the expectation, risk premium, and convexity in (20) are all measured on the yield of this particular bond. Furthermore, they reflect the expected behavior of the yield over the next instant, rather than the behavior of the short rate over the whole life span of the bond.

To distinguish our new pricing framework from the classic pricing framework of DTSMs, we henceforth label our new pricing theory as **Dynamic Duration Convexity Model** (DDCM).

2.4. Contrast with DTSMs

Our theory links the time-t value of the yield spread of a particular bond to the current forecasts of its rate of change, risk premium, and variance rate, with no reference to the exact dynamics of the instantaneous interest rate or any other interest rates or even the future dynamics changes of this particular yield, thus making the analysis completely decentralized to the present and on the particular bond of interest. The decentralized feature also dictates that the no-arbitrage relations in (20) only guarantee dynamic no-arbitrage between the particular bond in consideration and the money market account given the assumptions on the bond yield's rate of change, risk premium, and volatility levels, but with no direct implications on cross-sectional relations across yields on different bonds.

By contrast, DTSMs derive the fair value for the whole yield curve based on the full dynamic specification of a centralized instantaneous interest rate, and thus guarantees dynamic consistency across the whole yield curve. Therefore, while our theory allows us to compare the level of a yield to its own near-term behavior forecasts, without referencing to other parts of the yield curve, DTSMs are built to analyze the yield curve shape and to make cross-sectional comparisons of yields across different maturities. The two theories are complimentary in focusing on different aspects of dynamic no-arbitrage.

To compare yields on different bonds under our new framework, we must first compare our forecasts on their near-term behaviors, i.e., their rate of change, their volatility, and how market prices the risk on each bond. This effort amounts to centralize our decentralized model. In practice, investors can be interested in performing comparative analysis on a selected number of bond yields without making inference on the whole yield curve. In this case, we can make common factor assumption on the selected set of bond yields and derive relative valuations among them, without defining the whole yield curve.

Traditional dynamic term structure models derive implications on the term structure based on risk-neutral dynamics assumptions on the instantaneous interest rate. By deriving everything from one dynamics, it guarantees cross-sectional consistency among the yields across different maturities, and thus provides a framework for cross-sectional comparison and relative value. In particular, when one performs statistical arbitrage trading based on the relative valuation, i.e., the deviation between market quotes and the fair value, the assumed dynamics do not play any direct role in prediction, but play important roles in forming the hedge to neutralize the assumed factors.

By contrast, since our approach focus on the relation between the value of one particular yield on a bond and its current rate of change and volatility forecasts. The forecasts play a direct role in our assessment of the yield's fair value. Furthermore, by focusing on the short-term return of a bond instead of the long-term projection of a short rate, our new framework strives to derive a pricing relation from standard risk-return analysis.

2.4.1. Decentralizing DTSM to DDCM

Since DTSMs are derived based on dynamic no arbitrage of all yields relative to one centralized instantaneous interest rate process, the level of each derived yield is naturally consistent with the derived dynamics of this yield. Thus, the level of the derived yield and the levels of the derived rate of change and volatility of the yield must satisfy our DDCM pricing relation.

As an example, consider the general diffusion-affine dynamic term structure model of Duffie and Kan (1996), who assume that the instantaneous interest rate is an affine function of K factors with affine continuous dynamics under the risk-neutral measure:

$$r_t = a_r + b_r^{\top} X_t \tag{21}$$

$$dX_t = \kappa(\theta - X_t)\,dt + \sqrt{\Sigma(X_t)}dZ_t, \tag{22}$$

with $\Sigma(X_t)_{ii} = \alpha_i + \beta_i^{\top} X_t$ and $\Sigma_{ij} = 0$ for $i \neq j$. Under these dynamics assumptions, the yields on all zero-coupon bonds are affine in the K factors,

$$y_t(T) = \frac{a(\tau)}{\tau} + \left[\frac{b(\tau)}{\tau}\right]^{\top} X_t, \tag{23}$$

for all $\tau = T - t > 0$, where the coefficients $(a(\tau), b(\tau))$ are solutions to the following set of ordinary differential equations,

$$a'(\tau) = a_r + b(\tau)^{\top}\kappa\theta - \frac{1}{2}\sum_i b(\tau)_i^2\,\alpha_i, \tag{24}$$

$$b'(\tau) = b_r - \kappa^{\top} b(\tau) - \frac{1}{2}\sum_i b(\tau)_i^2\,\beta_i, \tag{25}$$

starting at $a(0) = 0$ and $b(0) = 0$.

Equation (23) centralizes the yields of all maturities by linking them as affine functions of a common set of factors X_t. To decentralize the affine model, we take one particular zero-coupon bond with an expiry T as an example and derive the risk-neutral dynamics of the yield on this zero-coupon bond. Applying Ito's lemma to (23) and (22), we have the risk-neutral rate of change and variance of the yield $y_t(T)$ as

$$\begin{aligned}
\mu_t^{\mathbb{Q}} &= -\left[\frac{a'(\tau)}{\tau} - \frac{a(\tau)}{\tau^2}\right] - \left[\frac{b'(\tau)}{\tau} - \frac{b(\tau)}{\tau^2}\right]^\top X_t + \left[\frac{b(\tau)}{\tau}\right]^\top \kappa(\theta - X_t), \\
\sigma_t^2 &= b(\tau)^\top \Sigma(X) b(\tau) \frac{1}{\tau^2},
\end{aligned}$$

with which we can write the risk-neutral dynamics of the yield as

$$dy_t(T) = \mu_t^{\mathbb{Q}} dt + \sigma_t dW_t,$$

where dW_t denotes the change of a newly constructed Brownian motion that is linked to the factor Brownian shocks by

$$dW_t = \frac{b(\tau)^\top \sqrt{\Sigma(X_t)}}{\sigma_t} dZ_t.$$

Multiplying the rate of change by τ and the variance by τ^2, and collecting terms, we have

$$\begin{aligned}
\mu_t^{\mathbb{Q}} \tau &= -a'(\tau) + \frac{a(\tau)}{\tau} + b(\tau)^\top \kappa\theta - \left[b'(\tau) - \frac{b(\tau)}{\tau} + b(\tau)\kappa\right]^\top X_t, \\
\sigma_t^2 \tau^2 &= b(\tau)^\top \Sigma(X) b(\tau) = \sum_i b(\tau)_i^2 \alpha_i + \sum_i b(\tau)_i^2 \beta_i^\top X_t.
\end{aligned}$$

Applying the DDCM relation in (20) and the instantaneous interest rate function in (21),

$$\begin{aligned}
y_t(T) &= r_t + \mu_t^{\mathbb{Q}} \tau - \frac{1}{2} \sigma_t^2 \tau^2 \\
&= a_r + b_r^\top X_t - a'(\tau) + \frac{a(\tau)}{\tau} + b(\tau)^\top \kappa\theta - \left[b'(\tau) - \frac{b(\tau)}{\tau} + b(\tau)\kappa\right]^\top X_t \\
&\quad - \frac{1}{2} \sum_i b(\tau)_i^2 \alpha_i - \frac{1}{2} \sum_i b(\tau)_i^2 \beta_i^\top X_t,
\end{aligned}$$

which is affine in X_t. Applying (23) and collecting terms, we have

$$
\begin{aligned}
\frac{a(\tau)}{\tau} &= a_r - a'(\tau) + \frac{a(\tau)}{\tau} + b(\tau)^\top \kappa\theta - \frac{1}{2}\sum_i b(\tau)_i^2 \alpha_i, \\
\frac{b(\tau)}{\tau} &= b_r - \left[b'(\tau) - \frac{b(\tau)}{\tau} + b(\tau)\kappa \right] - \frac{1}{2}\sum_i b(\tau)_i^2 \beta_i.
\end{aligned}
$$

Rearrage, our DDCM leads to the same ordinary differential equation as the affine model in (24)-(25),

$$
\begin{aligned}
a'(\tau) &= a_r + b(\tau)^\top \kappa\theta - \frac{1}{2}\sum_i b(\tau)_i^2 \alpha_i, \\
b'(\tau) &= b_r - b(\tau)\kappa - \frac{1}{2}\sum_i b(\tau)_i^2 \beta_i.
\end{aligned}
$$

The DDCM pricing relation starts with $\mu_t^{\mathbb{Q}}$ and σ_t^2 and derive their linkage to the yield level via dynamic no-arbitrage arguments, without referring to other parts of the curve. The standard DTSM starts with the instantaneous interest rate dynamics and derives the whole yield curve as well as their dynamics via no arbitrage arguments. The derived yield level and the derived yield dynamics naturally satisfy the no-arbitrage relation that we derive. By deriving everything from a centralized instantaneous interest rate dynamics, DTSM allows one to compare the cross-sectional behavior of whole yield curve. By focusing on the rate of change and volatility of a particular yield, the DDMC relation allows one to link the level of one particular yield to its own near-term dynamics.

2.4.2. Centralizing DDCM to DTSM

The DDCM approach derives the no-arbitrage relation between the level of one particular yield and its near-term dynamics. By imposing a functional linkage on the near-term dynamics across the whole yield curve, we can centralize the DDCM relation to arrive something close to a dynamic term structure model. This centralization process, however, is not always easy because it is not straightforward to simultaneously assume the near-term dynamics of all yields without introducing arbitrages among them. In what follows, we show one particularly simple example that allows us to do the centralization without introducing cross-sectional arbitrage.

Assume that the continuously compounding yield curve goes up and down in parallel under the physical measure $\mathbb{P}$. If we use $y_t(\tau)$ to denote the yield at a fixed time to maturity τ, we can write its dynamics as,

$$dy_t(\tau) = \sigma dW_t^{\mathbb{P}}, \tag{26}$$

for all $\tau \geq 0$, where we assume zero drift and the same volatility σ for all maturities τ.

Further assume that the time-t market price of the Brownian risk $(-dW_t)$ on the bond is λ. We can derive the yield dynamics under risk-neutral measure $\mathbb{Q}$ as,

$$dy_t(\tau) = \lambda\sigma dt + \sigma dW_t. \tag{27}$$

Now consider a zero-coupon bond with fixed expiry T. The parallel shifting yield curve implies that the risk-neutral dynamics for the yield of this zero-coupon bond $y_t(T)$ can be written as

$$dy_t(T) \quad = \quad dy_t(\tau) - y_t'(\tau)\,dt = \left[\lambda\sigma - y_t'(\tau)\right] dt + \sigma dW_t, \tag{28}$$

where the $y'(\tau)$ term accounts for the sliding of the yield for this bond along the yield curve.

Starting with the drift and diffusion in (28), we can apply our DDCM pricing relation in (20), and represent the yield with fixed time to maturity as,

$$y_t(\tau) = r_t + \left(\lambda\sigma - y_t'(\tau)\right)\tau - \frac{1}{2}\sigma^2\tau^2, \tag{29}$$

for all τ.

Define $z_t(\tau) \equiv y_t(\tau)\tau = -\ln B_t(T)$, equation (29) implies

$$z_t'(\tau) = r_t + \lambda\sigma\tau - \frac{1}{2}\sigma^2\tau^2, \tag{30}$$

for all τ. Thus, under this particular parallel shift assumption and taking logs on the zero-coupon bond price curve, the short rate is simply the negative of the slope of this log price curve at zero maturity, the risk premium is the curvature of this log price at zero maturity, and the yield variance is just the third derivative of the log price curve.

We can solve for the whole yield curve by integrating equation (30) over maturity,

$$y_t(\tau) \quad = \quad \frac{z(\tau)}{\tau} = \frac{1}{\tau}\int_0^\tau \left(r_t - \lambda\sigma u - \frac{1}{2}\sigma_t^2 u^2\right) du = r_t - \frac{1}{2}\lambda\sigma\tau - \frac{1}{6}\sigma^2\tau^2, \tag{31}$$

for all τ. By assuming parallel shift on the yield curve and by specifying the full dynamics of all yields, equation (31) centralizes the DDCM pricing relation to arrive at a term structure model.

If we instead only assume the near-term dynamics by allowing σ_t and λ_t to vary over time with unknown dynamics, the local differential equation in (29) remains valid from our DDCM, but we can no longer perform the integration in (31) without knowing the full path of σ_t and λ_t from t to T. To derive the full term structure model necessitates the specification of the full dynamics.

Merton (1973) considers in a footnote a similar model with $dr = \sigma dW_t^{\mathbb{P}}$ and arrives at a similar term structure that excludes arbitrage. This particular example not only guarantees no arbitrage between the particular zero-coupon bond and the instantaneous rate, but also guarantees that bonds across all finite maturities do not allow arbitrage. To verify, we can start with $dr = \lambda\sigma dt + \sigma dW_t$. Then the relation in (31) between $y_t(\tau)$ and r_t suggest that $dy_t(\tau) = dr_t = \lambda\sigma dt + \sigma dW_t$, just as we have assumed to begin with.

3. Data and summary statistics

We perform empirical analysis using US and UK swap rates. The financing leg of the swap contracts for both currencies are the 6-month LIBOR rate. We can treat the swap rates as the coupon of a par bond. We obtain the LIBOR and swap rate data from Bloomberg, daily from January 3, 1995 to December 29, 2017, spanning 5,790 business days. The swap maturities include 2, 3, 4, 5, 7, 10, 20, 30, 40, and 50 years. Swap rates at short maturities are available over the whole sample period. Longer-term swap rates start at a later date. For the US, 40- and 50-year swap rates become available starting November 12, 2004. For the UK, 20- and 30-year swap rates become available starting January 19, 1999; 40- and 50-year swap rates become available starting August 8, 2003. When computing summary statistics and estimating models, we sample the data weekly every Wednesday from January 3rd, 1996 to December 27, 2017, for 1148 weeks. The weekly sampling is to avoid week-day effects, and we leave the first-year of the sample for constructing historical variance estimators.

Table 1 reports the summary statistics of the swap rate series (in percentage points), including the sample average ("Mean"), standard deviation ("Stdev"), skewness ("Skew"), excess kurtosis ("Kurt"), and weekly autocorrelation ("Auto"). Panel A reports the statistics on the swap rate levels. The mean swap rate term structure is initially upward sloping for both economies. The mean estimates become lower at very long maturities partly because the samples start at later dates for these maturities. The standard deviation of the swap rate series show a declining term structure, suggesting that longer rates vary within a narrower range. For both economies, the skewness estimates are small and the excess kurtosis estimates are negative. The weekly autocorrelation estimates for all swap series are very close to one (0.994 to 0.999), suggesting that the swap series are highly persistent, if stationary at all.

[Table 1 about here.]

Panel B of Table 1 reports the summary statistics on the weekly changes of the swap rates. The mean and standard deviation estimates of the weekly changes are annualized. The annualized sample averages of the weekly changes are negative for all series, suggesting that interest rates have been showing a declining trend over the past two decades for both economies.

The annualized standard deviation estimates of the weekly swap rate changes represent the unconditional volatility estimators for the interest rate changes. The estimates show remarkable stability across maturities. For the US swap rates, the volatility estimates range from 0.82 at two-year maturity to 0.95 at seven-year maturity. The volatility estimates for swap rates from 20 to 50 years are virtually the same around 0.86 of a percentage point. For the UK, the volatility estimates range from 0.60 at 50-year maturity to 0.78 at four-year maturity, showing again a very flat volatility term structure. The skewness of the weekly changes remain small, but the excess kurtosis estimates of weekly changes are all positive. The autocorrelation estimates on the weekly changes are all close to zero.

Figure 1 plots the time series of the swap rates at three selected maturities: two-years (solid lines), 10 years (dashed lines), and 30 years (dash-dotted lines), for the US dollar in Panel A and the UK pound in Panel B. In line with the negative mean swap rate change estimates in Panel B of Table 1, the swap rates show a distinct downward trend for both economies during the 20-plus year sample period, reflecting the downward trend in inflation during this sample period. We overlay the swap rate time series with the recession band for each economy. During our sample period, the US economy has experienced two recessions, a minor one in 2001 and the more severe one often dubbed as the great recession in 2008-2009. The UK economy did not experience a recession in 2001, but shared the great recession in 2008-2009. Usually at the start of a recession, the central bank starts to cut the short-term interest rate in an effort to stimulate the economy. Such actions are reflected in the sharp short rate drops during the shaded recession period. The US economy recovered quickly after the 2001 minor recession. The short-term swap rate also went up quickly after the initial drop. After the great recession, however, the short-term rate stayed low for a much longer period for both economies.

[Fig. 1 about here.]

Based on the daily changes of the swap rates series, we construct a simple volatility rate estimator on each series with a one-year rolling window. Figure 2 plots the time series of the rolling volatility estimates, overlayed with the recession bands. For each swap rate series, the rolling volatility estimates vary strongly over time, reaching its peak during the 2009 financial crises but having been calming down since then. Across maturities, the volatility estimates show both upward and downward sloping term structure patterns. The term structure tends to be downward sloping when the volatility level is high, and upward sloping during more quiet periods. The volatility estimates tend to be high during transition periods in a business cycle when the central bank is actively cutting or raising rates. During these periods with active central bank activities, the volatility estimates for short-term rates tend to be higher than the estimates for long-term rates, leading to downward sloping volatility term structure.

[Fig. 2 about here.]

On the other hand, during periods with few central bank actions on the short rate, such as during the past eight years since 2010 as the short rate was trapped at virtually zero, the volatility term structure becomes distinctively upward sloping. The upward sloping volatility term structure is interesting and can prove challenging for classic models that assume mean-reverting dynamics on the instantaneous interest rate. Our new pricing theory allows us to directly take the volatility estimators as inputs without making explicit assumptions on the short rate dynamics. Furthermore, the persistently high volatility for daily changes of very long dated swap rates suggest that the convexity effects can be large on long rates.

4. Applications

We explore practical applications of our new pricing theory from several angles. First, we propose to predict future bond excess returns while assuming no predictability on long-dated floating interest rate series. Second, we perform comparative analysis of the yield curve via common factor structure assumptions on the yield's rate of change.

4.1. Predicting long bond returns with no rate predictability

It is extremely difficult to predict long-term interest rate movements. Historically, the literature has strived to predict future short-rate changes based on the slope of the year curve, but that literature does not provide much help in predicting long-term rates. As an application of our new pricing theory, in this section, we take no-predictability on long-term rate as a starting point, and infer bond risk premium from the observed interest rate level and interest rate volatility estimates. We examine the predictability of the extracted bond risk premium on future bond excess returns.

We start by assuming that the constant-maturity floating yield $y_t(\tau)$ at some long fixed time to maturity τ moves diffusively like a random walk over the next instant,

$$dy_t(\tau) = \sigma_t(\tau) dW_t^{\mathbb{P}}, \tag{32}$$

where $\sigma_t(\tau)$ denotes the time-t conditional forecast of the volatility rate of this yield. The conditional volatility can vary over time with unspecified dynamics. It is also possible that the volatility forecasts differ for yields of different maturities. The key assumption underlying (32) is the absence of predictability on the long-term floating yield as the expected rate of change is assumed to be zero.

If we denote the market price of the risk for the corresponding bond ($-dW_t$) as λ_t,[4] we can derive the risk-neutral dynamics for the yield as

$$dy_t(\tau) = \lambda_t \sigma_t(\tau) dt + \sigma_t(\tau) dW_t. \tag{33}$$

The risk-neutral drift for the fixed-expiry yield $y_t(T)$ is further adjusted by the local shape of the yield curve

[4] Given our notation, the market price of the interest rate risk would be $-\lambda_t$. Throughout this paper, even if we start with the yield dynamics, we deliberately model the market price of the Brownian risk on the bond, which is simply the negative of the Brownian risk on the yield. This way, the market price is more in line (in sign) with the bond excess return that our analysis focuses on.

as it slides along the yield curve,

$$\mu_{t,y}^{\mathbb{Q}} = \lambda_t \sigma_t(\tau) - y_t'(\tau). \tag{34}$$

Plugging the no-prediction dynamics into the DDCM pricing relation in (20), we have

$$y_t = r_t + \lambda_t \sigma_t(\tau)\tau - y_t'(\tau)\tau - \frac{1}{2}\sigma_t(\tau)^2\tau^2. \tag{35}$$

Rearrange, we have,

$$y_t + y'(\tau)\tau = r_t + \lambda_t \sigma_t(\tau)\tau - \frac{1}{2}\sigma_t(\tau)^2\tau^2. \tag{36}$$

For zero-coupon bonds, $y_t + y'(\tau)\tau = \frac{\partial(y_t\tau)}{\partial\tau} = f_t(\tau)$ is the instantaneous forward rate. For coupon bonds, we can directly estimate the yield curve slope against the bond duration based on observed yields at nearby maturities.

It is convenient to define the *volatility-weighted duration and convexity* as

$$D_t(\tau) = \sigma_t(\tau)\tau, \quad C_t(\tau) = \frac{1}{2}\sigma_t(\tau)^2\tau^2. \tag{37}$$

Equation (36) shows that in the absence of rate prediction, positive market price of bond risk drives the yield curve up with increasing duration whereas convexity drives the curve down. Based on observed yield curve time series, we can construct volatility estimators for yield changes across different maturities to generate a volatility term structure curve $\sigma_t(\tau)$ at each date. We can also use the observed yield curve to infer the interest rate level and slope at the corresponding maturity. Combining these observations and estimates with the financing cost r_t, we can infer the common market price of bond risk.

Proposition 1 *Assuming that the long-term constant-maturity yields are not predictable as in (32), we can estimate the market price of the bond risk based on the observed yield curve shape and the volatility estimator of the yield changes,*

$$\lambda_t = \frac{y_t + y'(\tau)\tau - r_t + C_t}{D_t}, \tag{38}$$

where D_t and C_t as defined in (37) are the volatility-weighted duration and convexity estimators, respectively.

We assume that long-dated US and UK swap rates with maturities 10 years and longer are not predictable. We apply the proposition and estimate the market price of bond risk from these long-dated rates. In the estimation, we take the financing rate (6-month LIBOR) as the short rate r_t. We construct volatility estimators on daily changes of each swap rate series with a one-year rolling window as shown in the data section. We treat the swap rates as par bond yields and estimate the duration and convexity of these par bonds, and we estimate the swap rate curve slope against the duration at each maturity using a local linear regression.

Without rate prediction, Equation (38) sets the bond risk premium to the forward yield spread adjusted for the convexity contribution (C_t). Figure 3 plots the time series of the estimated convexity contribution at selected maturities. The convexity contribution is negligible at short maturities, but become significant at long maturities. In the US, the convexity contribution estimates are over 100 basis points for 30- and 50-year swap rates during the volatile period of 2009. In the UK, the swap rate volatility estimates are lower. The convexity contribution for 30- and 50-years swaps varies between 20 and 40 basis points.

[Fig. 3 about here.]

Figure 4 plots the time series of the extracted market price of bond risk (λ_t) from each swap series from 10 to 50 years. The market price of bond risk extracted from different swap rate series are similar in magnitude and move closely together. Over the common sample, the cross-correlation estimates among the different λ_t series average 99.44% for the US swap rates and 99.48% for the UK swap rates. Similar to findings in Cochrane and Piazzesi (2005), our evidence supports a one-factor structure for the bond risk premium.

[Fig. 4 about here.]

On average, the market price of bond risk estimates are positive for both economies, supporting the hypothesis of positive bond risk premium. Nevertheless, the estimates vary strongly over time. In the US, the market price of risk estimates become close to zero right before the start of the two recessions in 2000 and 2007, respectively, but the estimates become the most positive at the end of each recession. In the UK, the market price of bond risk went into negative territory in 1998 and again between 2007 and 2008, but otherwise show positive co-movements with the US economy.

To examine whether the ex-ante risk premium estimate ($\lambda_t \sigma_t \tau$) on each long-term swap rate series predicts the future ex post excess return on the corresponding par bond, we compute six-month and one-year ahead excess return on the par bond and measure the forecasting correlation between the ex ante risk premium and ex post excess returns on each par bond. Table 2 reports the forecasting correlation estimates. The ex ante risk premium estimates show strongly positive predictive power on the ex post bond return. At six-month horizon, the forecasting correlation estimates for the US range from 0.23 to 0.31. The estimates for the UK range from 0.17 to 0.21. At one-year horizon, the forecasting correlation estimates become higher and more uniform, ranging from 0.30 to 0.37 for the US and 0.31 to 0.35 for the UK.

[Table 2 about here.]

Paradoxically, the assumption of no prediction on long-dated swap rates leads to significant prediction on bond excess returns. The forecasting correlation estimates are quite high, particularly when recognizing that the forecasting correlation estimates are fully out of sample as the risk premium estimates at each time t depends only on information up to time t. The risk premium estimates at time t are constructed based on the observed yield curve at that time and the yield curve's volatility over the past year, with no further calibration or forecasting regression involved.

4.2. Comparative yield analysis via common factor structures

The new theory is built to analyze the relation between the yield of a particular bond and its own near-term dynamics. To perform comparative yield analysis across different maturities, we can assume common factor structures in their dynamics and derive the implications of the common factor dynamics on the yield curve structure. In what follows, we consider a particular simple common factor structure for the swap rate curve dynamics and explore its implications.

4.2.1. Common factor structures along the term structure

We make the following commonality assumptions on the expected rate of change, market price, and volatility of the swap rates across different maturities:

Assumption 1 (Expected rate of change term structure). *The expected rate of change on the constant-maturity yields vary across duration via an exponential form,*

$$\mu_t(\tau) = e^{-\kappa_t \tau}(\mu_{t,0} - \mu_{t,\infty}) + \mu_{t,\infty}. \tag{39}$$

Under the assumption, the expected rate of change converges to μ_t^0 as duration approaches zero and converges to μ_t^∞ as the duration approaches infinity. Furthermore, we maintain the no-rate predictability assumption in the long-run limit,

Assumption 2 (No predictability on long rates). *The expected rate of change approaches zero as the duration approaches infinity:*

$$\mu_{t,\infty} = 0. \tag{40}$$

Based on the empirical evidence from the previous section and the literature findings, we maintain a one-factor structure on the risk premium by assuming identical market price of bond risk across maturities:

Assumption 3 (Market price of risk). *The market price of bond risk is identical across maturities,*

$$\lambda_t(\tau) = \lambda_t. \tag{41}$$

Finally, we use daily changes on the yields over the past year to generate a historical volatility estimator $V_t(\tau)$ for $\sigma_t(\tau)$:

Assumption 4 (Volatility). *The volatility rates of yield changes are equal to their corresponding historical estimators $V_t(\tau)$.*

With assumptions (1) to (4), we can write the term structure of the yield curve as

$$o_t(\tau) = y_t(\tau) + y'(\tau)\tau - r_t + \frac{1}{2}V_t^2(\tau)\tau^2 = e^{-\kappa_t\tau}\mu_{t,0}\tau + \lambda_t V_t(\tau)\tau + e_t, \tag{42}$$

where we move observable quantities to the left hand side of the equation and leave the parametric component of the yield curve to the right hand side. The yield level $y_t(\tau)$ and the financing rate r_t are directly observable. We estimate the yield curve slope $y_t'(\tau)$ using a local linear regression, and we construct the volatility estimator $V_t(\tau)$ using one-year of historical daily yield changes. We label the yield spread adjusted by the curve slope and convexity as $o_t(\tau)$ and treat it as an observable quantity, potentially with noise e_t.

The right hand side of equation (42) includes the parametric specification on the expected rate of change and the market pricing of bond risk. At each date t, the specification governs the yield curve term structure via three variables $(\mu_{t,0}, \kappa_t, \lambda_t)$. Intuitively, a positive market price of bond risk λ_t contributes to a positive slope to the term structure. The expected rate of change on the short end $\mu_{t,0}$ further adjusts the slope through the expectation difference across maturities, and the speed of decay κ_t controls the curvature of the slope and the speed by which the expectation contribution declines as maturity increases.

In principle, one can have non-zero expectations for long-term rates as well, in which case we can include a nonzero $\mu_{t,\infty}$ in the specification and rewrite the measurement equation as

$$o_t(\tau) = e^{-\kappa_t \tau}(\mu_{t,0} - \mu_{t,\infty})\tau + \mu_{t,\infty}\tau + \lambda_t V_t(\tau)\tau + e_t. \tag{43}$$

Nevertheless, without other sources of information, equation (43) makes it clear that it is difficult to separately identify $\mu_{t,\infty}$ and λ_t from the yield curve alone. By setting $\mu_{t,\infty} = 0$, we use λ_t to capture both the risk premium and potentially a common component of the yield change expectation while using $\mu_{t,0}$ to capture the yield's rate of change difference across maturities. Separate identification of $\mu_{t,\infty}$ is possible by incorporating other sources of information such as economist forecasts on future yield curve levels.

4.2.2. Identifying the common factors from the observed yield curve

With the common factor structure assumptions, the yield curve at any given date t is governed by three variables $(\mu_{t,0}, \kappa_t, \lambda_t)$. One particular feature of the new theory is that the yield curve at time t depends on the levels of these variables at time t, but does not depend on the particular dynamics specification for these variables. Therefore, the emphasis of the empirical analysis involves the extraction of the state variables from the yield curve, without knowing the state dynamics. Based on this unique feature, we cast the model into a state-space form, where we treat the variates as the hidden states and treat the observed yield curve as measurements with errors.

We define the state vector as X_t,

$$X_t \equiv [\mu_{t,0}, \ln(\kappa_t), \lambda_t]^\top, \tag{44}$$

where the logarithm transform on κ_t guarantees that it stays strictly positive and the state vector can take values on the whole real line. Since how the state variables vary over time does not affect the pricing, we can specify the state propagation equation without worrying about their pricing implications. For the state extraction, we make the particularly simple assumption of a random walk dynamics,

$$X_t = X_{t-1} + \sqrt{\Sigma_x}\varepsilon_t. \tag{45}$$

where the standardized error vector ε_t is assumed to be normally distributed with zero mean and unit variance. We further assume that the covariance matrix is a diagonal matrix with distinct diagonal values so that the states can have different degrees of variation but the movements are independent of each other.

We define the measurement equations on the observed yield spread,

$$o_t = h(X_t, \tau) + e_t, \tag{46}$$

where $o_t \in \mathbb{R}^{10}$ denotes the ten yield spread series and $h(X_t)$ denotes the value of the yield spread as a function of the states X_t and yield maturity τ, as defined by the specification in equation (42), and e_t denotes the measurement error on the observed yield. We assume that the pricing errors are iid normally distributed with error variance σ_y^2.

When the state-space model is Gaussian linear, the Kalman (1960) filter provides efficient forecasts and updates on the mean and covariance of the state and observations. Our state-propagation equations are constructed to be Gaussian and linear, but the measurement functions $h(X_t)$ are not linear in the state vector. We use the unscented Kalman filter (Wan and van der Merwe (2001)) to handle the nonlinearity.

The setup introduces four auxiliary parameters that define the covariance matrices of the state propagation errors and the measurement errors. The relative magnitude of the state propagation error variance versus the measurement error variance controls the speed with which we update the states based on new observations. Intuitively, if the states vary a lot (large Σ_x) and the observations are accurate (small σ_y^2), one would want to update the states faster to better match the new observations. If on the other hand the states vary slowly over time and the observations are very noisy, one would want to update the states more slowly to smooth out the noise in the observation. We choose these auxiliary parameters, and accordingly the optimal state updating speed, by minimizing the sum of squared forecasting errors in a quasi maximum likelihood setting.

In pricing the swap rates across different maturities, we assume some common structures on their rate of change and market price of bond risk. The assumed common structures lead to common pricing on the swap rate curve; nevertheless, it is important to understand that the pricing is consistent with the assumed dynamics, but there is no direct guarantee that the assumed common factor structure is fully consistent by itself. Hence, to achieve internal consistency across the whole term structure, the traditional approach of DTSM remains the most convenient approach. By contrast, our theory focuses on the link between the near-term dynamics and the pricing of a bond, without saying much about long-run variation. As a result of this particular feature, under our common factor structure, the swap curve is priced by a set of state variables, but the specification has no fixed model parameters. The absence of fixed model parameters greatly simplifies model estimation and removes potential consistency issues encountered in model re-calibration: A model with re-calibrated model parameters represents essentially a different model and thus generates different pricing and hedging implications from previous calibrations. Such consistency issues do not show up in our model as the pricing relation contains no fixed parameters. In our state-space approach to extract the state variables, we introduce four auxiliary parameters to define the state-propagation error variance and the measurement error variance. These variance estimates control the updating speed of the states based on new observations, and we use maximum likelihood estimation to determine the magnitudes of these parameters and accordingly the optimal updating speed. Altering the state-propagation equation specification and/or the variance estimates does not induce consistency issues for the pricing relation, but can nevertheless change the state updates and thus change our valuation.

4.2.3. Pricing performance of the common factor structures

Table 3 reports the summary statistics of the model's pricing errors, defined as the basis point difference between the observed swap rates and the model-generated values. The statistics include the sample average of the pricing error ("Mean"), root mean squared pricing error ("RMSE"), weekly autocorrelation ("Auto"), and the model's explained variation ("EV") on each series, defined as one minus the variance ratio of the model's pricing error to the original volatility series. For both economies, the mean pricing errors are positive at intermediate maturities from 10 to 20 years, but negative at both short and very long maturities, suggesting that the assumed factor structure is not flexible enough to fully capture the curvature of the swap rate curve.

[Table 3 about here.]

The specification assumes a particularly simple common factor structure, with the market price of risk component generating a common slope and an exponentially decaying rate of change accommodating the slope difference at different maturities. This simple structure has difficulties capturing the strong curvature of the swap curve during an extended period of our sample when the short rate hits the lower bound. Figure 5 plots the time series of the pricing error on the 15-year swap rate (solid line), which show the most positive mean pricing error, and contrasts the pricing error behavior with that of the 6-month LIBOR rate (dashed line). For both economies, the pricing errors on the 15-year swap rate become the most positive when the LIBOR rate hit the lower bound between 2010 and 2015. When the short rate hits the lower bound, the shadow rate can be very negative[5] and the observed term structure for the expected rate of change is more S-shaped than exponential.

[Fig. 5 about here.]

Table 3 shows that the root mean squared pricing errors average around 10 basis points. The explained variation is somewhat lower at short maturities, but over 99% for maturities at five years and longer. The autocorrelation estimates for the pricing errors are much smaller than that for the original swap series reported in Table 1, suggesting that the factor structure is reasonably successful in separating the systematic common

[5]Black (1995) proposes the idea of shadow rate and treats observed nominal interest rate as options on the shadow rate. Krippner(2012, 2013), among others, proposes tractable implementations of shadow rate models. Christensen and Rudebusch (2015) estimate a three-factor shadow rate on Japanese yields.

movements on the yield curve from temporal dislocations at particular maturities. The highest autocorrelation estimates come from 15- and 50-year maturities, coinciding with the highest root mean squared pricing error.

4.2.4. Predicting changes in yield curve slope with the extracted rate of change

Figure 6 plots the time series of the common market price of bond risk λ. The behavior is very similar to those extracted from long-term swap rates in Figure 4 based on no-rate-predictability assumption.

[Fig. 6 about here.]

With the common factor structure, not only can we extract the common market price of risk, but we can also separate out an expectation component from the swap rate curve. Figure 7 plots the time series of the extracted state variable μ_0, which measures the relative expected rate of change at short maturities versus long maturities. The expected annualized rate of change varies within a band of $(-1.84\%, 1.82\%)$ for the US and within a slightly narrower range $(-1.72\%, 1.23\%)$ for the UK. The time series show both large variations following the business cycle and shorter-term more temporal variations. For example, for the US, the expected rate of change switched signs several times between 1996 to 2006.

[Fig. 7 about here.]

Figure 8 plots the time series of the reciprocal of the decay estimates, $1/\kappa$. κ measures the speed at which the expected rate of change exponentially declines with maturity. The reciprocal of κ provides an intuitive time measure. For the US, the decay is below five years before the financial crisis. After the financial crisis, as the short rate is hitting the lower bound, the decay becomes much slower to about ten years. For the UK, the decay is within ten years except the spike in late 2010. Thus overall, the contribution of the expectation component is limited to the first segment of the swap curve below ten years. After that, one can largely ignore the contribution of the short-rate expectation and focus on the risk premium and convexity contribution, as we have done in the previous section.

[Fig. 8 about here.]

For identification reason, we set the expected rate of change for long-term rates to zero in the limit (μ_∞). As such, μ_0 captures more of the relative component of the expected rate of change between short- and long-maturity swap rates, and thus the swap curve slope changes. To examine whether the extracted rate of change is informative about future swap curve slope changes, Table 4 reports the forecasting correlation estimates between the expected rate of change estimates (μ_0) and changes in the swap curve slope over different horizons (h, in weeks), from one month ($h = 4$ weeks), to one quarter ($h = 13$ weeks), to half a year ($h = 26$) and one year ($h = 52$ weeks). We measure the swap curve slope using the swap rate difference between two year and other maturities from three to ten years. The forecasting correlation estimates are all strongly positive. For the US, the forecasting correlation on changes of the 2-3swap slope is 27% at one-month horizon, 48% at quarterly horizon, 59% at half-year horizon, and 61% at one-year horizon. The forecasting correlation estimates are similar on other slope measures. For the UK, the forecasting correlation estimates are equally high, from 22% at monthly horizon to 64% at annual horizon on the 2-3 swap slope. The high correlation estimates suggest that our simple common factor structure allows us to separate out an expectation component that is highly predictive of future swap curve slope changes.

5. Concluding remarks

In this paper, we propose a new modeling framework that is particularly suited for analyzing returns on a bond or bond portfolio. The framework does not try to model the full dynamics of an instantaneous short rate, but focus squarely on the behavior of the bond yield in question. It does not even ask for the full dynamics specification of this bond yield, but only needs estimates of its current expectation, risk premium, and volatility. It can readily accommodate results from other models and algorithms.

The model framework decomposes each yield into three components: expectation, risk premium, and volatility. One can estimate the volatility from historical time series, or infer it from the curvature of the yield curve, or interest rate options. We show that we can predict bond excess returns on long-term bonds, without running predictive regressions, even by assuming no prediction on interest rates. We also show how to perform comparative analysis on the yield curve via common factor structures on the near-term dynamics. For future research, separating risk premium from expectation via common factor structure assumptions and new information sources can be a very challenging, but very fruitful endeavor.

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Table 1
Summary statistics of swap rates.
Entries report the summary statistics of the US and UK swap rates. Data are weekly from January 3rd, 2996 to December 27th, 2017, for 1148 weeks. US 40- and 50-year swap rates start at a later date with 685 weekly observations each. Longer-maturity UK swap rates also start at later dates, with 1081, 1074, 989, 956, 751 weekly observations at 15-, 20-, 30-, 40-, and 50-year maturity, respectively. The statistics include the sample average ("Mean"), the standard deviation ("Stdev"), the skewness ("Skew"), the excess kurtosis ("Kurt"), and weekly autocorrelation ("Auto"). Panel A reports the statistics on the interest rate levels and Panel B reports the statistics on weekly differences in the swap rates, with the mean and standard deviation of the weekly changes being annualized.

	US					UK				
Maturity	Mean	Stdev	Skew	Kurt	Auto	Mean	Stdev	Skew	Kurt	Auto
Panel A. Statistics on swap rate levels										
2	3.10	2.21	0.31	-1.41	0.999	3.79	2.37	-0.08	-1.56	0.999
3	3.35	2.13	0.24	-1.38	0.999	3.93	2.30	-0.12	-1.49	0.998
4	3.57	2.04	0.19	-1.35	0.998	4.04	2.22	-0.13	-1.41	0.998
5	3.76	1.97	0.15	-1.32	0.998	4.13	2.15	-0.14	-1.32	0.998
7	4.05	1.85	0.09	-1.27	0.998	4.26	2.03	-0.13	-1.12	0.997
10	4.34	1.75	0.03	-1.22	0.997	4.40	1.90	-0.07	-0.82	0.997
15	4.61	1.67	-0.05	-1.18	0.997	4.28	1.57	-0.33	-0.75	0.995
20	4.72	1.64	-0.09	-1.18	0.997	4.25	1.46	-0.39	-0.60	0.995
30	4.78	1.61	-0.11	-1.17	0.997	3.99	1.28	-0.64	-0.55	0.996
40	3.76	1.14	0.18	-1.29	0.994	3.86	1.23	-0.69	-0.40	0.995
50	3.74	1.15	0.19	-1.29	0.994	3.46	1.12	-0.77	-0.56	0.995
Panel B. Statistics on swap rate weekly changes										
2	-0.15	0.82	0.20	3.62	0.013	-0.26	0.74	0.09	3.66	0.005
3	-0.15	0.89	0.25	2.51	-0.003	-0.27	0.77	0.33	3.04	-0.028
4	-0.15	0.92	0.26	1.90	-0.012	-0.28	0.78	0.32	2.28	-0.047
5	-0.15	0.94	0.23	1.75	-0.020	-0.28	0.77	0.22	1.80	-0.049
7	-0.16	0.95	0.22	1.74	-0.029	-0.29	0.75	0.19	1.52	-0.069
10	-0.16	0.94	0.19	1.99	-0.035	-0.30	0.73	0.14	1.35	-0.074
15	-0.17	0.91	0.22	1.83	-0.052	-0.32	0.70	0.10	1.45	-0.094
20	-0.17	0.88	0.17	1.60	-0.052	-0.29	0.67	0.07	1.75	-0.102
30	-0.17	0.86	0.17	1.82	-0.035	-0.20	0.64	-0.15	2.49	-0.105
40	-0.21	0.87	0.09	2.71	-0.048	-0.22	0.64	-0.17	2.78	-0.100
50	-0.21	0.88	0.04	2.78	-0.053	-0.24	0.60	-0.41	3.61	-0.077

Table 2
Forecasting future bond excess returns under no-predictability assumption on long-term swap rates.
Entries report the forecasting correlation between the bond risk premium extracted from each swap rate and the future excess return of the corresponding par bond over the next six months (panel A) and one year (panel B). Each column denotes one swap series.

Maturity	10	15	20	30	40	50
Panel A. Bond return horizon: 6-month						
US	0.31	0.28	0.26	0.23	0.28	0.26
UK	0.20	0.19	0.17	0.21	0.19	0.19
Panel B. Bond return horizon: One year						
US	0.37	0.36	0.33	0.30	0.35	0.33
UK	0.35	0.34	0.31	0.33	0.31	0.33

Table 3

Pricing performance of the common factor structure

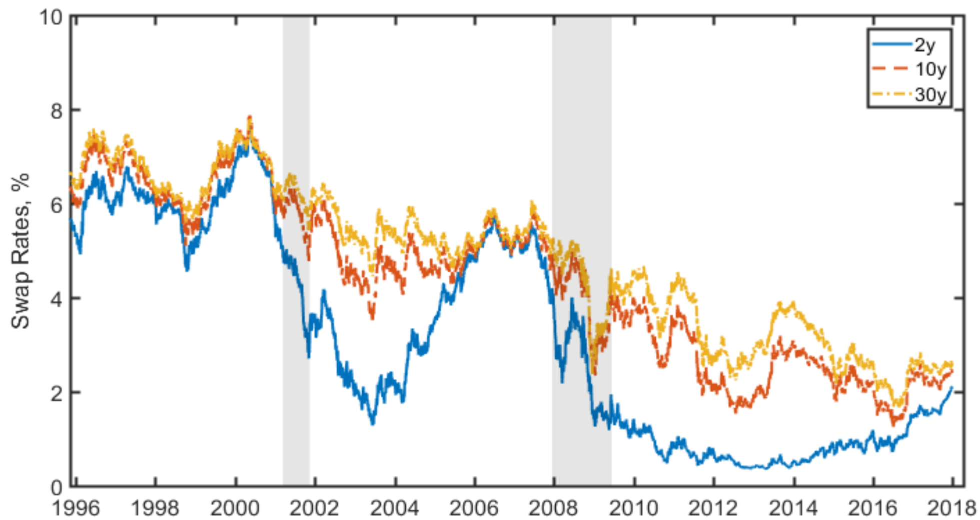
Entries report the summary statistics of the pricing errors of the common factor structure on the US and UK swap rates, including the sample average ("Mean") in basis points, root mean squared pricing error ("RMSE") in basis points, the weekly autocorrelation ("Auto"), and the model's explained variation (EV), defined as one minus the variance ratio of the pricing error to the original data series, and

	US				UK			
Maturity	Mean	RMSE	Auto	EV	Mean	RMSE	Auto	EV
2	-1.79	9.02	0.848	0.966	-0.55	13.58	0.923	0.914
3	-2.76	10.12	0.896	0.981	-2.26	9.19	0.870	0.983
4	-5.16	10.60	0.893	0.989	-3.95	7.98	0.847	0.993
5	-9.42	12.66	0.884	0.994	-5.38	10.56	0.932	0.992
7	-3.46	8.04	0.884	0.997	-0.73	8.19	0.941	0.996
10	3.43	10.29	0.949	0.996	5.44	9.93	0.949	0.997
15	12.17	16.34	0.972	0.996	10.95	15.07	0.964	0.996
20	8.03	11.59	0.954	0.998	8.27	10.91	0.927	0.998
30	5.14	7.96	0.878	0.999	0.99	6.51	0.897	0.999
40	-4.62	7.64	0.910	0.999	-6.16	10.57	0.934	0.998
50	-17.36	21.89	0.973	0.995	-12.43	17.55	0.962	0.996

Table 4
Predicting changes in yield curve slope with the extracted rate of change.
Entries report the forecasting correlation between the extracted rate of change μ_0 and future changes in the swap curve slope over different horizons h, measured in number of weeks. The swap curve slopes are measured as swap rate difference between the two-year and other maturities from three to ten years.

	Swap-curve slope				
Horizon, h	2-3	2-4	2-5	2-7	2-10
Panel A. US					
4	0.27	0.27	0.26	0.25	0.24
13	0.48	0.45	0.44	0.41	0.39
26	0.59	0.56	0.54	0.51	0.48
52	0.61	0.58	0.56	0.53	0.50
Panel B. UK					
4	0.22	0.19	0.16	0.12	0.09
13	0.40	0.35	0.29	0.22	0.15
26	0.59	0.52	0.45	0.35	0.26
52	0.64	0.60	0.55	0.48	0.39

Panel A. US



Panel B. UK

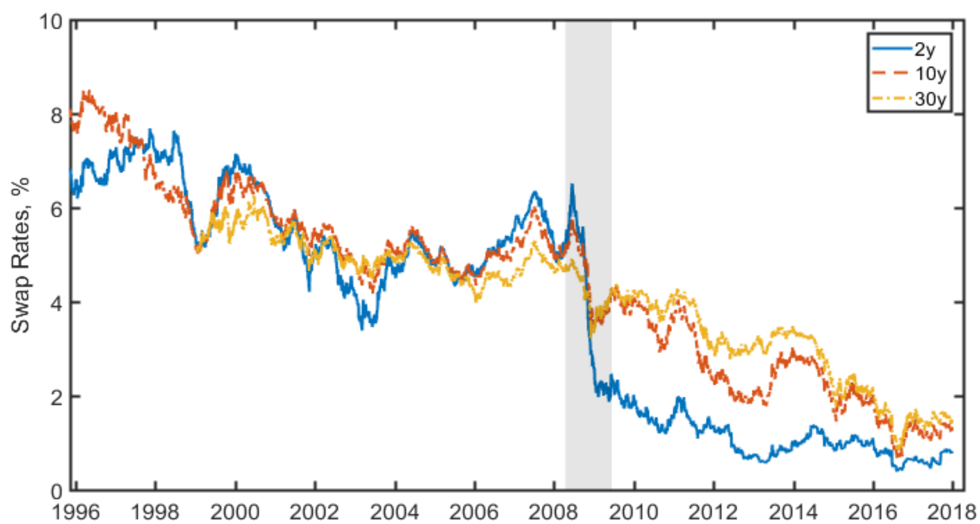
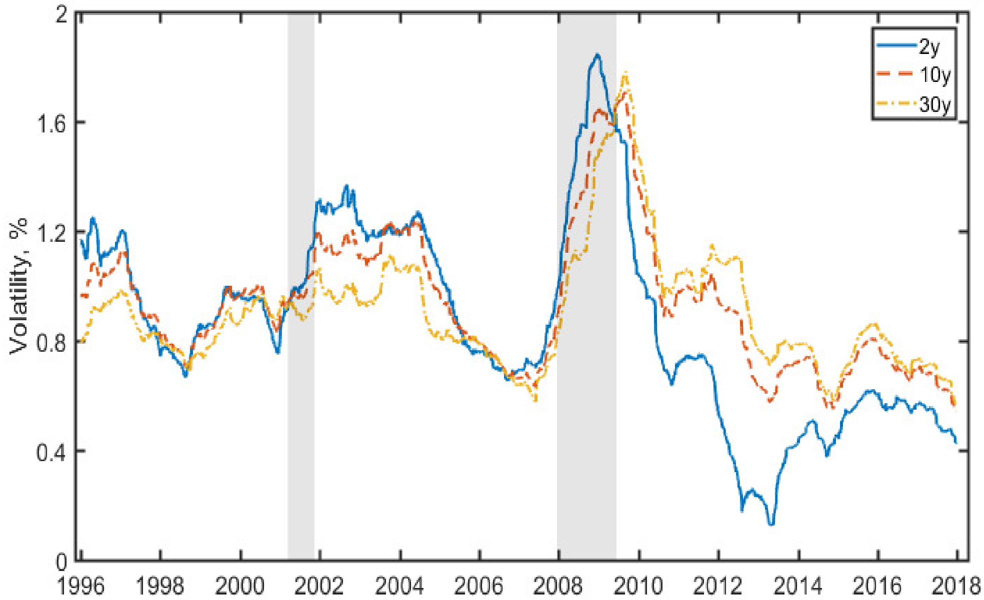


Fig. 1. The time-series variation of US and UK swap rates. Each panel plots the time series of swap rates at three selected maturities: 2-year (solid line), 10-year (dashed line), and 30-year (dash-dotted line), overlayed with the recession band of the corresponding economy. Panel A represents the US swap rates and Panel B the UK swap rates.

Panel A. US



Panel B. UK

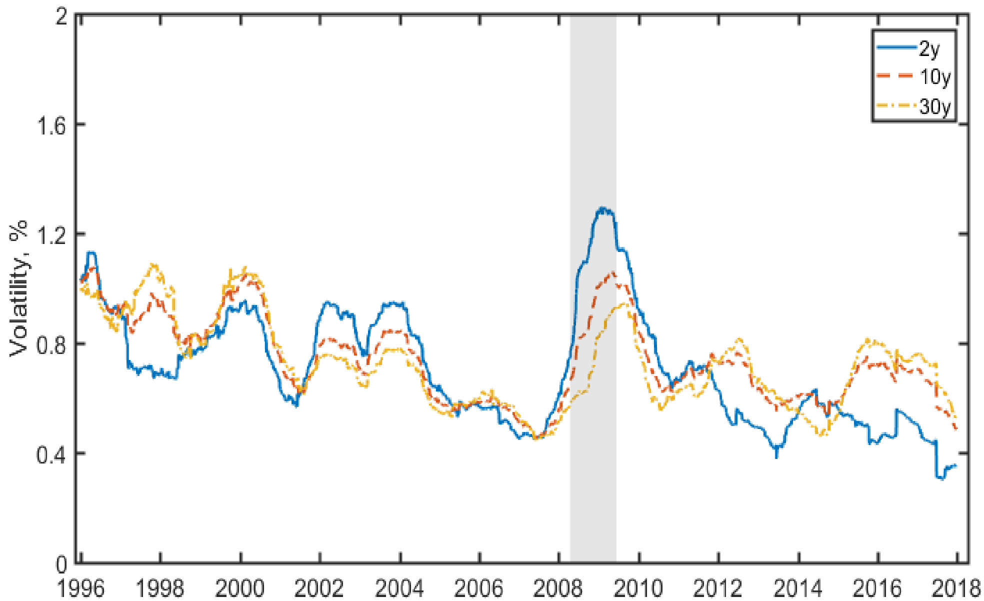
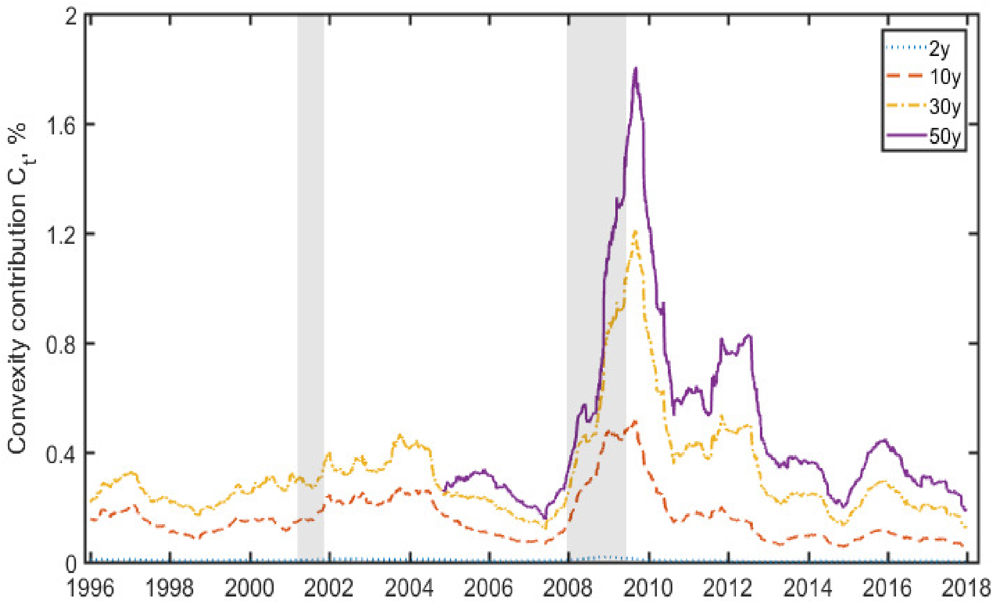


Fig. 2. Volatility estimators on US and UK swap rates. Each panel plots the time series of the one-year rolling volatility estimators on the daily changes of the swap rate series at selected maturities: 2-year (solid line), 10-year (dashed line), and 30-year (dash-dotted line), overlayed with the recession band of the corresponding economy. Panel A represent the US swap rates and Panel B the UK swap rates.

Panel A. US



Panel B. UK

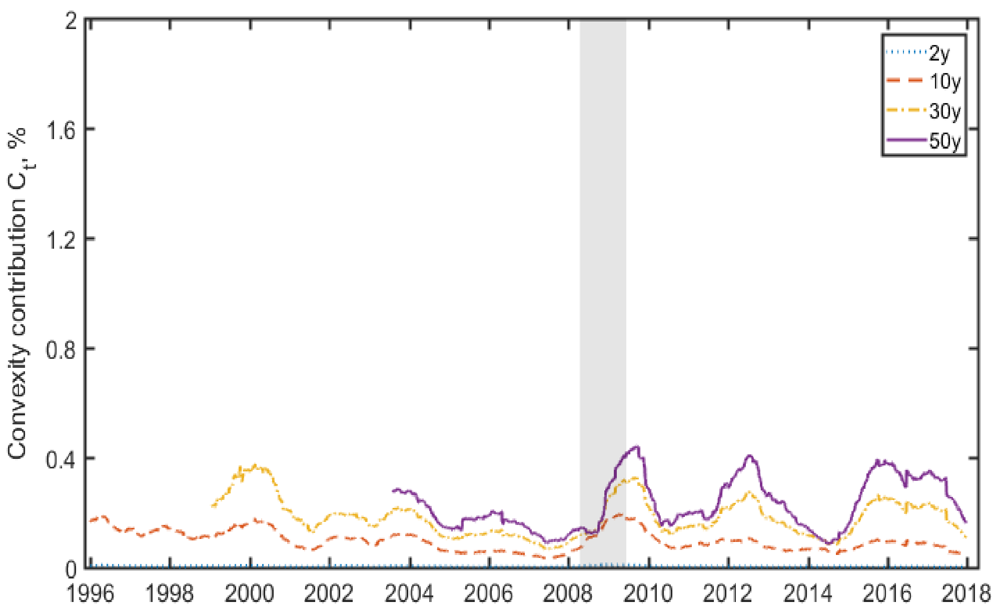
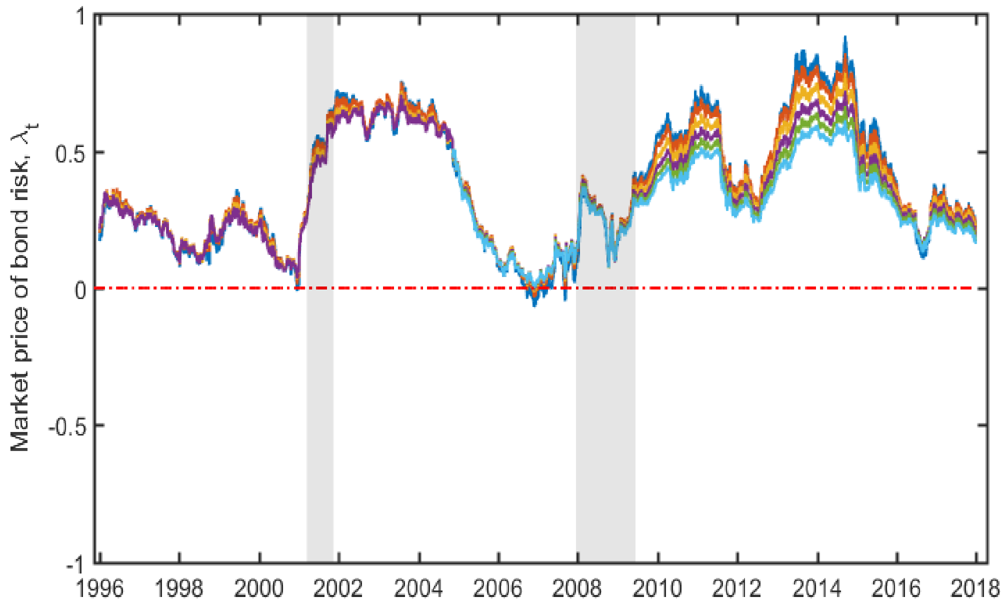


Fig. 3. Convexity contribution at selected maturities. Lines denote the time series of the estimated convexity contribution to swap rates at selected maturities: 2-year (dotted line), 10-year (dashed line), 30-year (dash-dotted line), and 50-year (solid line), overlayed with the recession band of the corresponding economy, US in panel A and UK in panel B.

Panel A. US



Panel B. UK

Fig. 4. Market price of bond risk. Lines denote the time series of the market price of bond risk extracted from swap rates with maturities 10 years and longer, overlayed with the recession bands. Panel A is for the US and Panel B is for the UK.

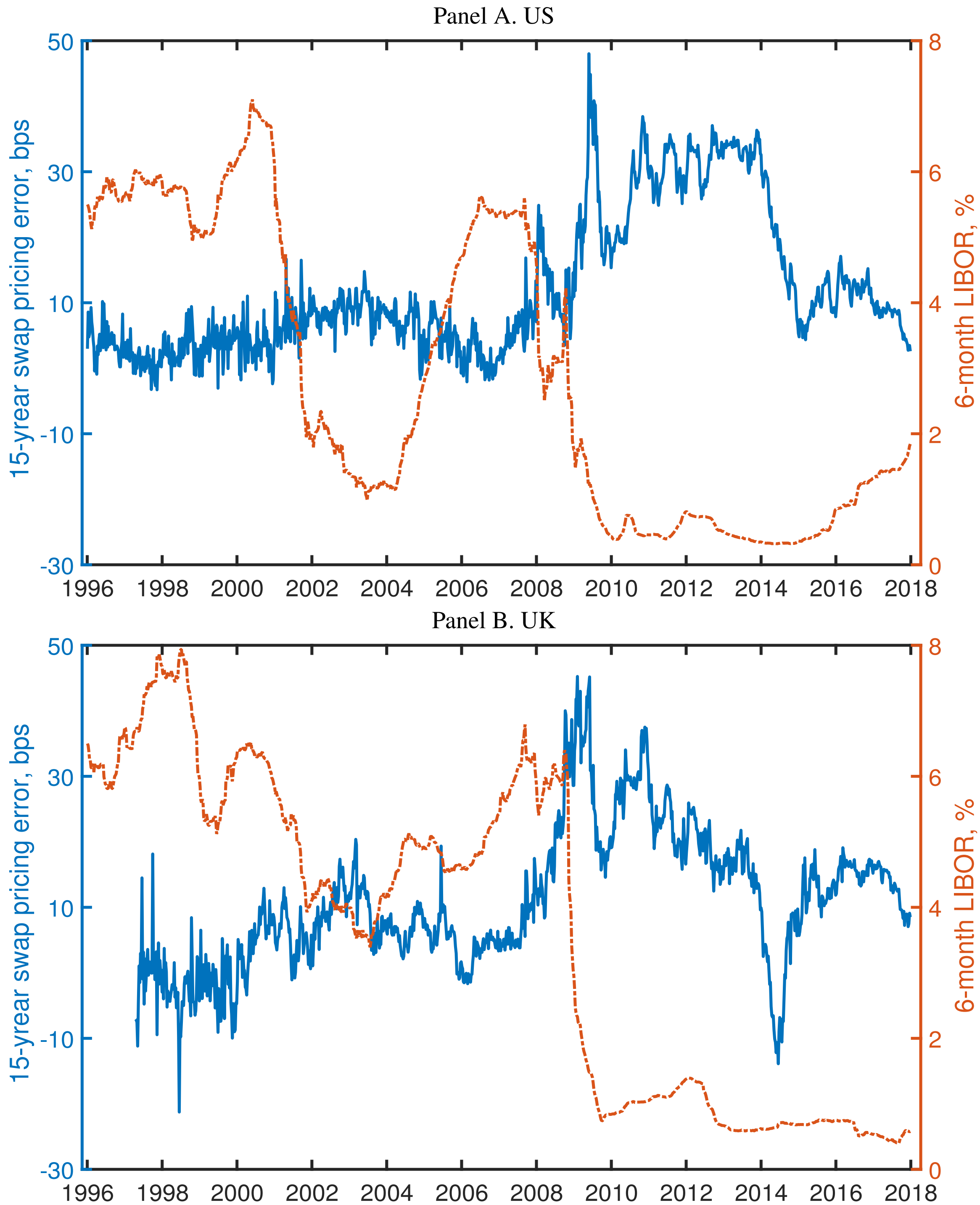
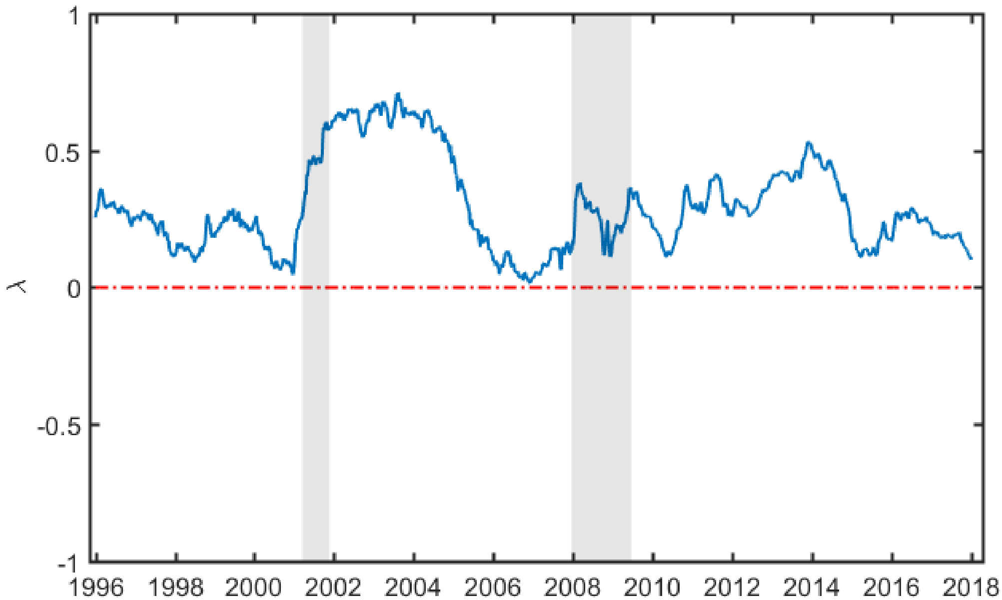


Fig. 5. Positive pricing errors on the 15-year swap rate when short rate hits the lower bound. Solid lines denote the pricing errors on the 15-year swap rate with scales on the left hand size. Dashed lines denote the time series of the 6-month LIBOR rate, with scales on the right hand size.Panel A is for the US and Panel B is for the UK.

Panel A. US



Panel B. UK

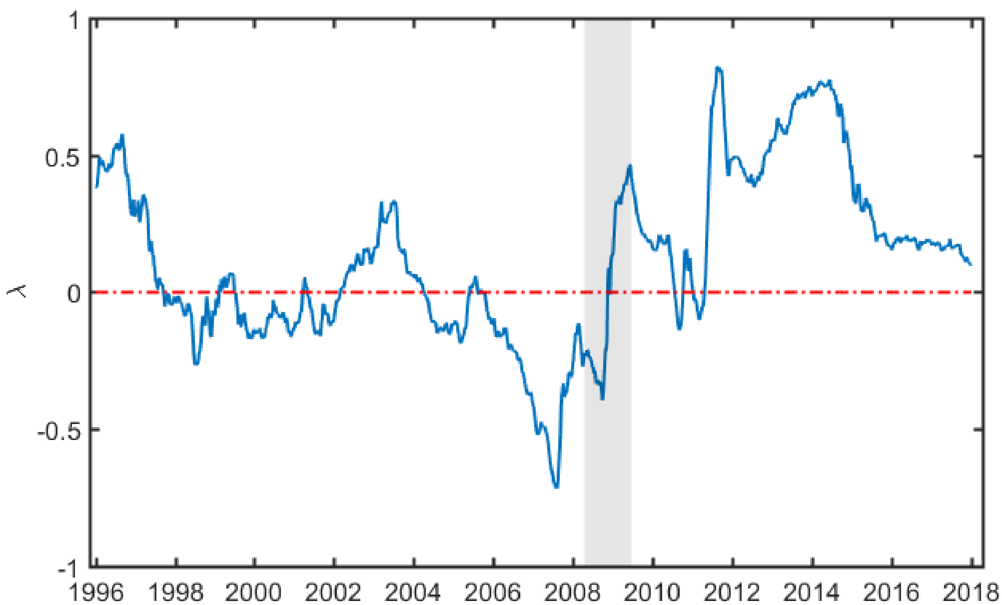


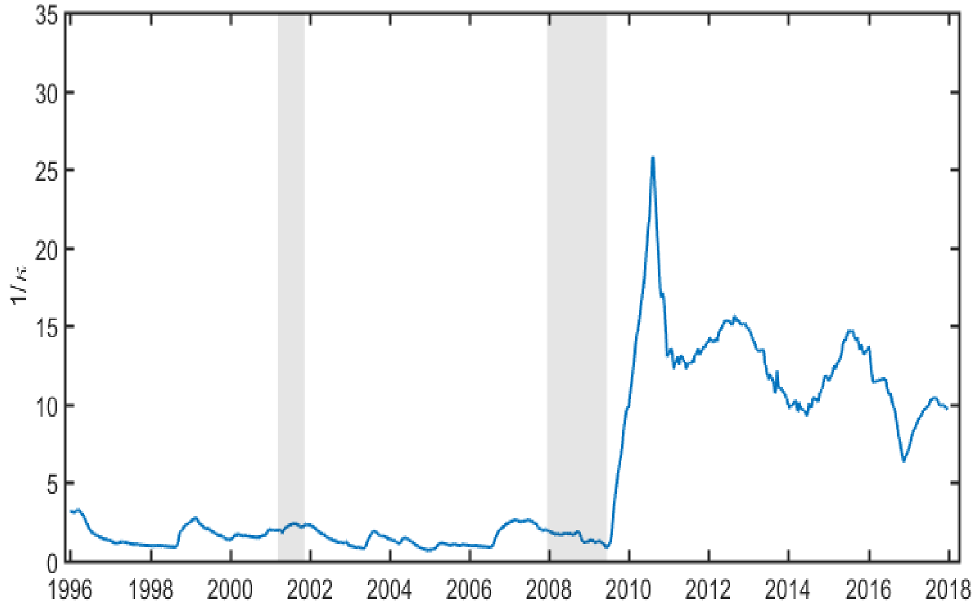
Fig. 6. The market price of bond risk λ

Panel A. US

Panel B. UK

Fig. 7. The expected rate of change μ_0.

Panel A. US



Panel B. UK

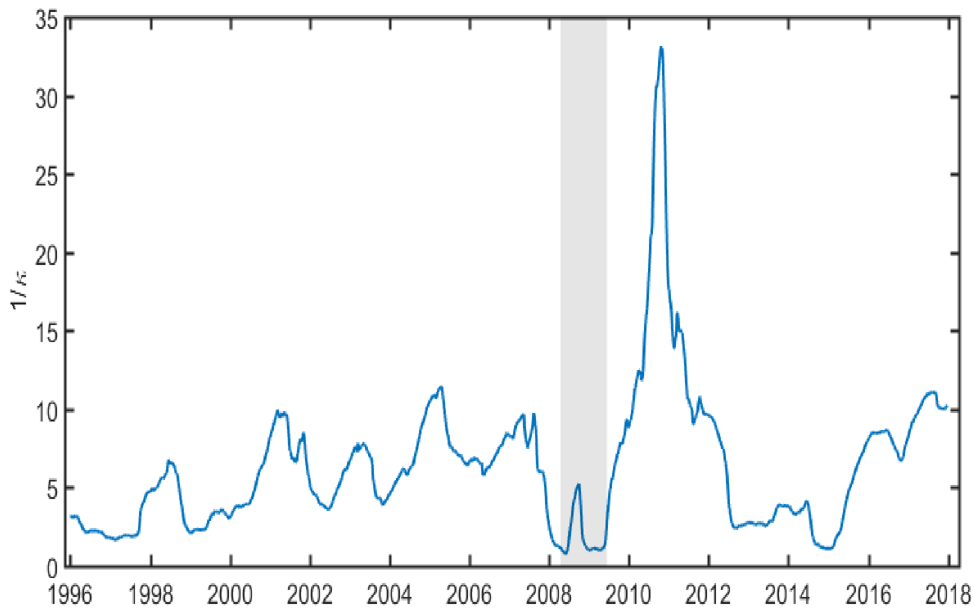


Fig. 8. The reciprocal of the decay speed, $1/\kappa$.