Optimal Sequential Decision with Limited Attention

Yeon-Koo Che and Konrad Mierendorff

KAEA Microeconomics

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Introduction

- We revisit Wald's (1947) and Arrow/Blackwell/Girshick's (1949) sequential decision problem: *DM decides sequentially on information acquisition before making a decision.*
- Classical feature: Information incurs delay and/or costs. Question: *How long* should you acquire information?
- New feature: Different types of information are received, and the DM allocates limited attention on them for processing. Question: *What kind* of information should you acquire?
- Applications:
  - Investment Decision
  - Recruiting
  - **Deliberation of a jury**
  - Prosecutorial investigation (in an inquisitorial system)
  - **Selection of news media**
  - Deliberation/research strategy: "Prove" or "disprove"?

Model

Baseline Model

- Two States: $\omega \in \{A, B\}$
- One DM — Two actions: a, b
- Payoffs conditional on state and action:

State:	A	B
a	u_a^A *	u_a^B
b	u_b^A	u_b^B *

  - Assume $u_a^A \geq u_b^A$, $u_b^B \geq u_a^B$.
- Prior probability of state A: $p_0 \in (0, 1)$.
- At each point in time, the DM can take a **final irreversible action** (a or b), or **acquire information**.
  - Continuous time $t \geq 0$: **flow cost** $c \geq 0$, and/or **discount rate** $r \geq 0$. (At least one $\neq 0$.)

Model

Information Acquisition

- At each t: DM has one unit of "Attention" to divide between
  - If DM seeks A-evidence
    - **discovers the state** at the Poisson rate of $\lambda > 0$ **in state** A**,**
    - receives **no signal in state** B**.**
  - If DM seeks B-evidence
    - **discovers the state** at the Poisson rate of $\lambda > 0$ **in state** B**,**
    - receives **no signal in state** A**.**
- Attention Choice: When choosing $(\alpha, \beta = 1 - \alpha)$, the DM
  - learns $\omega = A$ at rate $\alpha\lambda$ in $\omega = A \ \Rightarrow p = 1$
  - learns $\omega = B$ at rate $\beta\lambda$ in $\omega = B \ \Rightarrow p = 0$
- No signal — Bayesian updating:

$$\dot{p}_t = -\lambda(\alpha - \beta)p(1 - p).$$

Generalization:

Non-Conclusive Signals

- "Correct Signal" has arrival rate $\overline{\lambda}$
- "Noise" has arrival rate $\underline{\lambda} < \overline{\lambda}$
- Results generalize if the noise is not too high.

Two Learning Strategies:

- Confirmatory strategy: Details
  - Try to confirm what is likely
  - Choose $\alpha = 1$ for a high p and $\alpha = 0$ for a low p.
  - Use until absorbing belief p^* reached, then stationary strategy

$$\Big|\underbrace{\longrightarrow\longrightarrow\longrightarrow\longrightarrow}_{\alpha=0}\underbrace{p^*}_{\alpha=\frac{1}{2}}\underbrace{\longleftarrow\longleftarrow\longleftarrow\longleftarrow}_{\alpha=1}\Big|$$

- Contradictory strategy: Details
  - Seek evidence for the unlikely.
  - Choose $\alpha = 0$ for a high p and $\alpha = 1$ for a low p.
  - Use until sufficiently certain so that immediate action optimal.

$$\Big|\underbrace{\text{— — —}}_{b}\underline{p}^*\underbrace{\longleftarrow\longleftarrow\longleftarrow}_{\alpha=1}\check{p}\underbrace{\longrightarrow\longrightarrow\longrightarrow}_{\alpha=0}\bar{p}^*\underbrace{\text{— — —}}_{a}\Big|$$

- Optimal Policy:
  - combines these strategies optimally for different beliefs.

Structure of Value Function and Optimal Policy

Theorem

Fix r, λ, u_x^ω. There exist $0 \leq \underline{c} \leq \overline{c}$ such that

(a) *No information acquisition: $V(p) = U(p), \forall p$ if $c \geq \overline{c}$.*

(b) *Only "contradictory evidence" if $\underline{c} \leq c < \overline{c}$.*

$$\Big|\underbrace{\text{———————}}_{b}\ \underline{p}^*\ \overbrace{\underbrace{\longleftarrow\longleftarrow\longleftarrow\longleftarrow}_{\alpha=1}\ \check{p}\ \underbrace{\longrightarrow\longrightarrow\longrightarrow\longrightarrow}_{\alpha=0}}^{\textit{contradictory}}\ \bar{p}^*\ \underbrace{\text{———————}}_{a}\Big|$$

(c) *"Contradictory" and "Confirmatory" evidence if $c < \underline{c}$.*

$$\Big|\underbrace{\text{————}}_{b}\ \underline{p}^*\ \overbrace{\underbrace{\longleftarrow\longleftarrow}_{\alpha=1}}^{\textit{contradictory}}\ \underline{p}\ \overbrace{\underbrace{\longrightarrow\longrightarrow}_{\alpha=0}\ p^*\ \underbrace{\longleftarrow\longleftarrow}_{\alpha=1}}^{\textit{confirmatory}}\ \bar{p}\ \overbrace{\underbrace{\longrightarrow\longrightarrow}_{\alpha=0}}^{\textit{contradictory}}\ \bar{p}^*\ \underbrace{\text{————}}_{a}\Big|$$

(N.B.: All p-cutoffs are distinct.)

Proof

Intuition

- Trade-off between Confirmatory and Contradictory Strategy:
  - Confirmatory is effective in full learning, but may take a long time.
  - Contradictory is effective in ruling out unlikely and reaching a fast decision.
    - When close to $\underline{p}^*$ or $\overline{p}^*$, contradictory more effective.
    - When far away from $\underline{p}^*$ or $\overline{p}^*$, confirmatory more effective.
- "Skepticism fosters deliberation."

Application 1: Grand Jury vs Trial Jury

Assumptions

- Juror is deciding either to indict ("grand jury") or convict ("trial jury") a suspect; collective decision ignored.
- States: guilty A and innocent B
- **Actions:** indict/convict (a) or acquit (b)

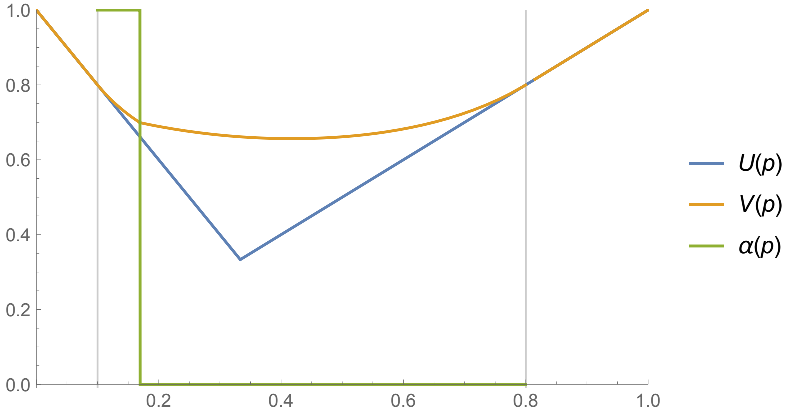
State:	Guilty A	Innocent B)
a: (indict/convict)	1	u_a^B
b: (acquit)	u_b^A	1

- Two payoff structures
  - Grand jury faces a higher cost of "not indicting a guilty": $u_b^A \ll u_a^B < 1$.
  - Trial jury faces a high cost of "convicting an innocent": $u_a^B \ll u_b^A < 1$.

Detailed Comparative Statics

Application 1: Grand Jury

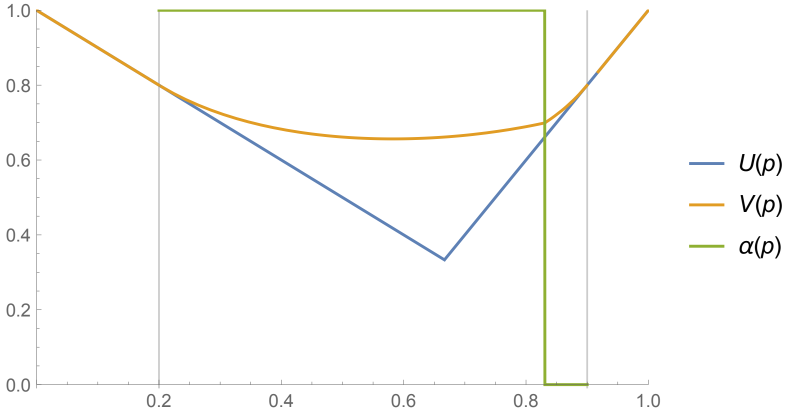
$$(\lambda = 1, r = 0,\ c = 0.2,\ u_a^A = u_b^B = 1,\ u_b^A = -1,\ u_a^B = 0)$$



$$|\underbrace{\text{—}}_{b}\ \underline{p}^*\ \underbrace{\longleftarrow}_{\alpha=1}\ \check{p}\ \underbrace{\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow\longrightarrow}_{\alpha=0}\ \bar{p}^*\ \underbrace{\text{— — —}}_{a}|$$

Application 1: Trial Jury

$$(\lambda = 1, r = 0,\ c = 0.2,\ u_a^A = u_b^B = 1,\ u_a^B = -1,\ u_b^A = 0)$$



$$\Big|\underbrace{— — —}_{b}\ \underline{p}^*\ \underbrace{\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow\longleftarrow}_{\alpha=1}\ \check{p}\ \underbrace{\longrightarrow}_{\alpha=0}\ \bar{p}^*\ \underbrace{— —}_{a}\Big|$$

Application 2: Choice of News Media

- A citizen decides between *a* and *b*—two candidates (e.g., Trump vs Hillary) or two policies (e.g., "Brexit" vs "Stay")

Candidates and Payoffs

- Candidate *a*: Right-wing
  - In state *A* (e.g. "immigration is harmful"), *a* is better.
- Candidate *b*: Left-wing
  - In state *B* (e.g., "immigration is beneficial"), *b* is better.

News Media

- Interpret α as a bias of a news medium.
- There are continuum of (exogenous) news media indexed by $\alpha \in [0, 1]$.
- $\alpha =$ fraction of left-leaning journalists hired by the medium,

Bias of News Media

- Now interpret "non-arrival of evidence" as a news report by a medium involving particular bias.
- $\alpha = 0$: Right-wing medium (e.g., Fox) that hires right-leaning journalists who
  - report in favor of B *only* in state B only if backed up by facts. report in favor of A *always* in state A but also in B.
- $\alpha = 1$: Left-wing medium (e.g., MSNBC) that hires only left-leaning journalists

(a) right-leaning journalists (b) left-leaning journalists

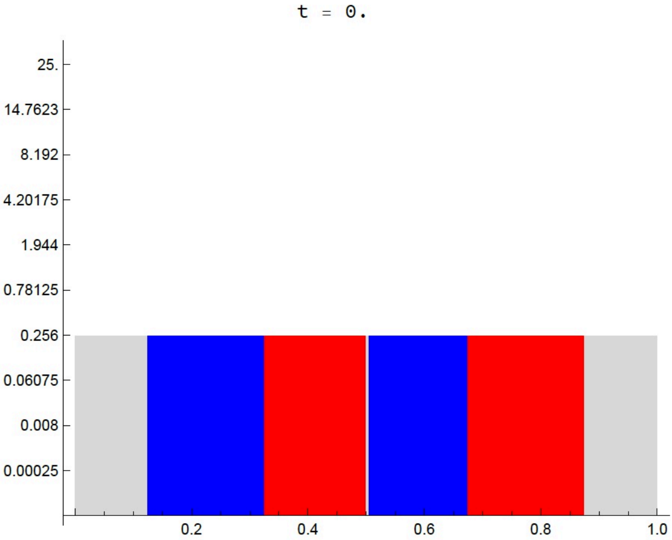
Strategy $\alpha \in (0, 1)$ "corresponds to" (subscribing to) a medium hiring fraction α of left-leaning jounrnalists.

Implications: Static

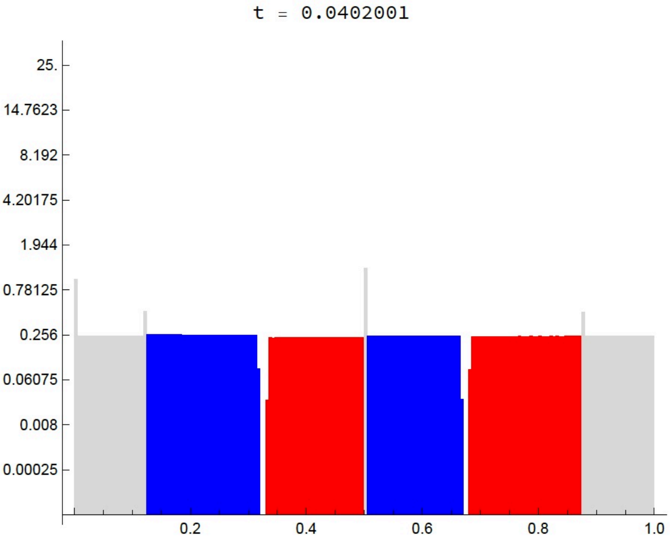
$$\Big|\underset{b}{\text{———}}\ \underline{p}^*\ \overbrace{\underset{\alpha=1}{\longleftarrow\longleftarrow}}^{\text{left}}\ \underline{p}\ \overbrace{\underset{\alpha=0}{\longrightarrow\longrightarrow}}^{\text{right}}\ \overbrace{\ p^*\ }^{\text{unbiased}}\ \overbrace{\underset{\alpha=1}{\longleftarrow\longleftarrow}}^{\text{left}}\ \bar{p}\ \overbrace{\underset{\alpha=0}{\longrightarrow\longrightarrow}}^{\text{right}}\ \bar{p}^*\underset{a}{\text{———}}\Big|$$

- Citizens with **extreme prior beliefs** choose "own-biased" medium
- Citizens with **moderate prior beliefs** choose "opposite-biased" medium
- Citizens with **middle belief** p^* choose "unbiased" medium $\alpha = 1/2$.

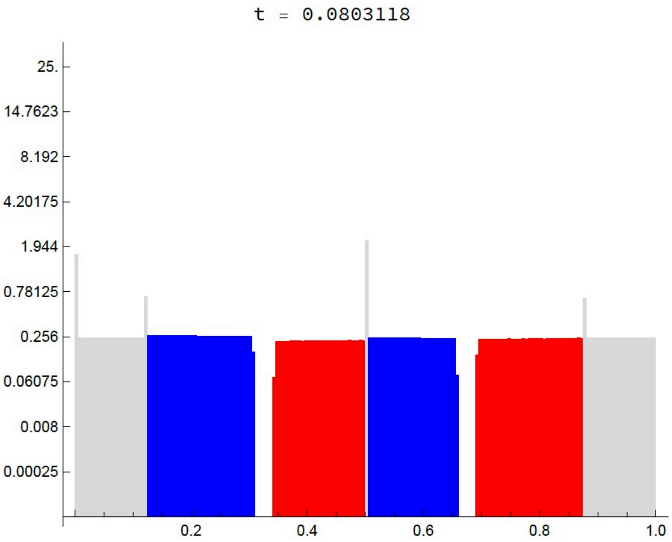
Dynamic Evolution of Beliefs: $\omega = B$ and uniform beliefs initially



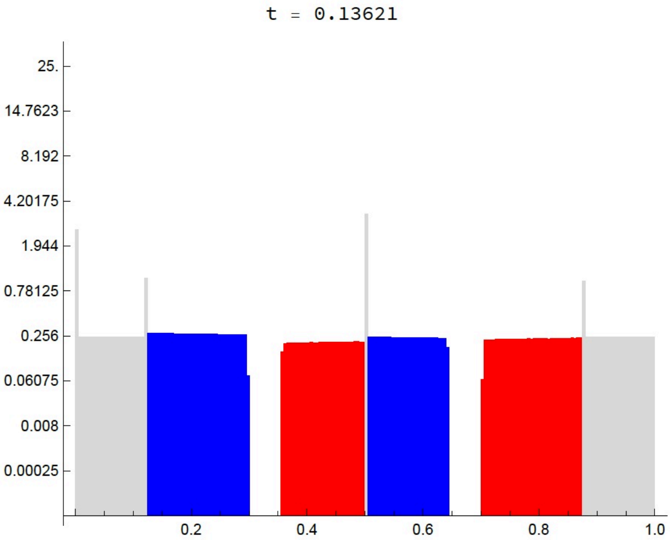
Dynamic Evolution of Beliefs: $\omega = B$



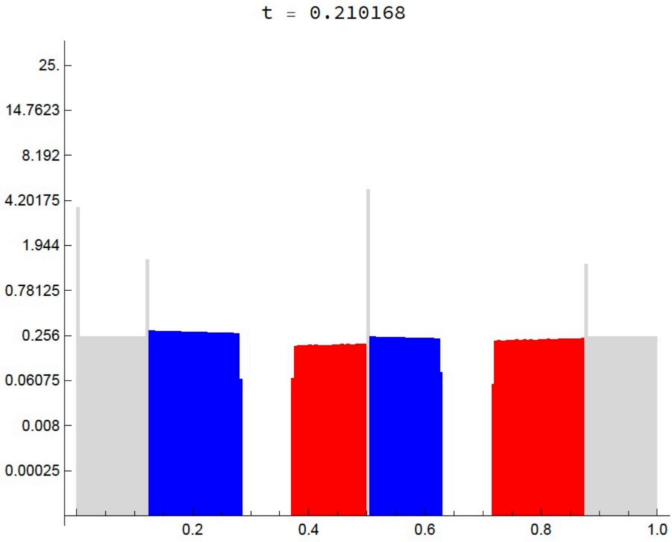
Dynamic Evoluation of Beliefs: $\omega = B$



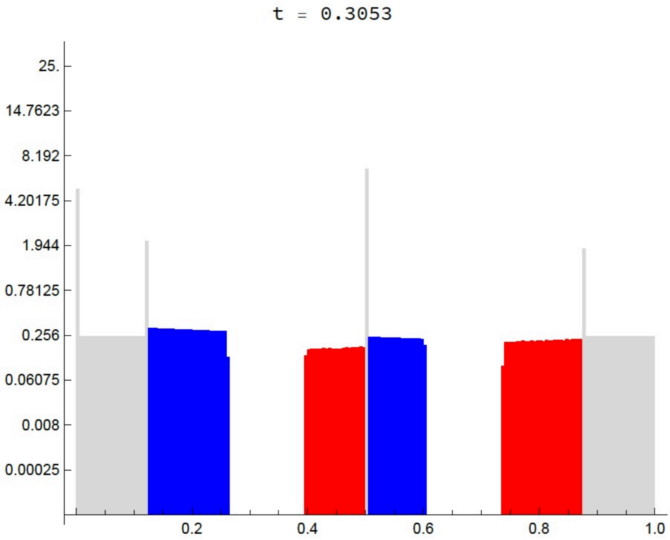
Dynamic Evolution of Beliefs: $\omega = B$



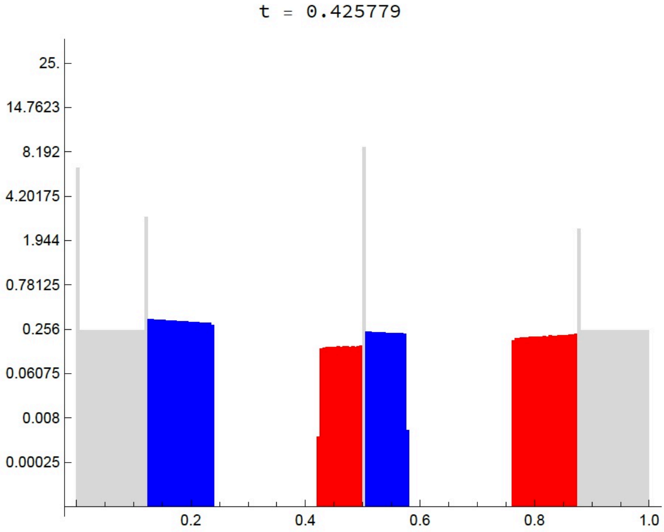
Dynamic Evolution of Beliefs: $\omega = B$



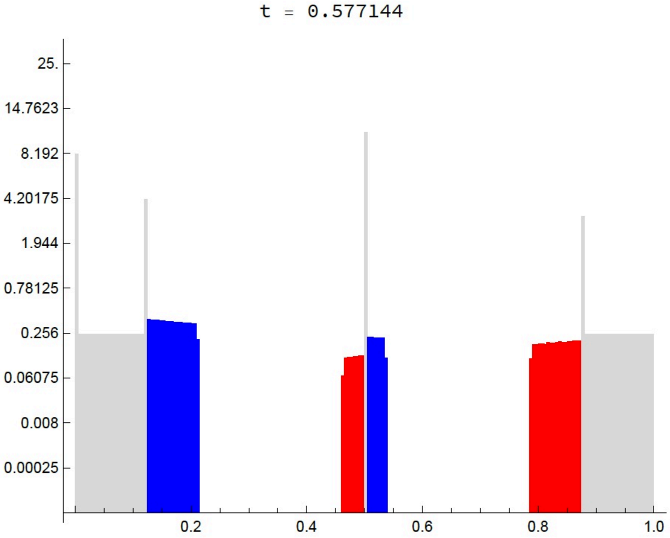
Dynamic Evoluation of Beliefs: $\omega = B$



Dynamic Evolution of Beliefs: $\omega = B$



Dynamic Evoluation of Beliefs: $\omega = B$



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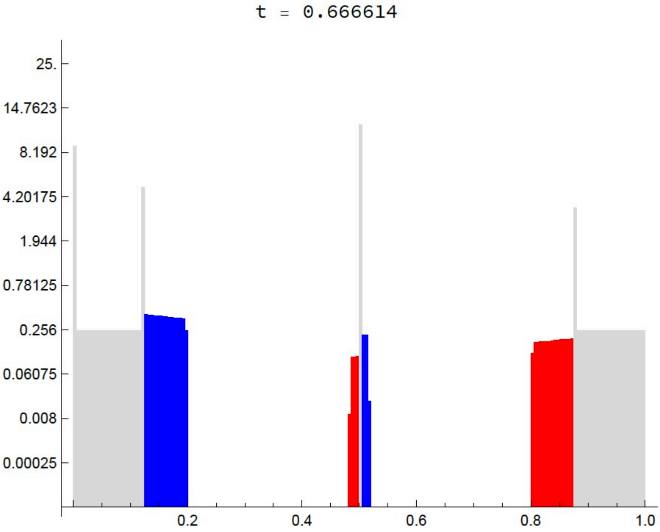
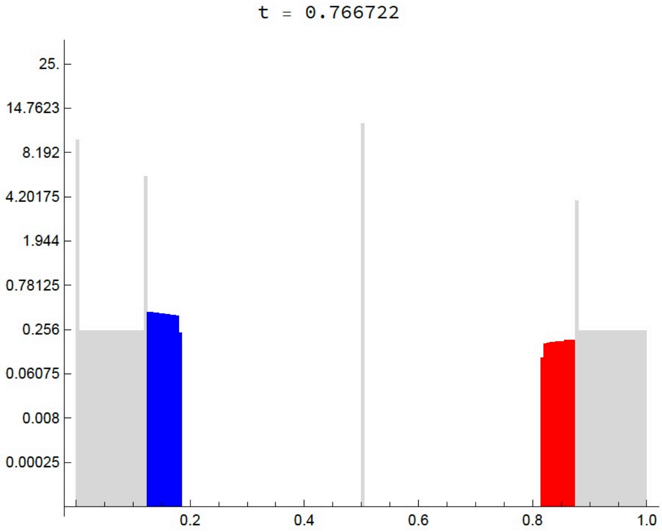


Dynamic Evolution of Beliefs: $\omega = B$



Implications: Dynamic

$$\Big| \underset{b}{\overline{\qquad\quad}} \ \underline{p}^* \ \overbrace{\underset{\alpha=1}{\longleftarrow\longleftarrow}}^{\text{left}} \ \underline{p} \ \overbrace{\underset{\alpha=0}{\longrightarrow\longrightarrow}}^{\text{right}} \ \overbrace{\ p^*\ }^{\text{unbiased}} \ \overbrace{\underset{\alpha=1}{\longleftarrow\longleftarrow}}^{\text{left}} \ \bar{p} \ \overbrace{\underset{\alpha=0}{\longrightarrow\longrightarrow}}^{\text{right}} \ \bar{p}^* \underset{a}{\overline{\qquad\quad}} \Big|$$

Over time,

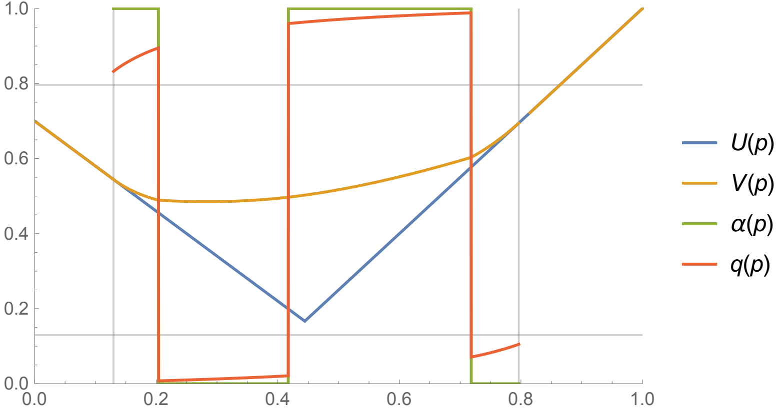
- Citizens with **extreme prior beliefs** become more polarized: "Echo-chamber" effect.
- Citizens with **moderate prior beliefs** become more undecided. "Anti Echo-chamber" effect.

Generalization: Non-conclusive signals

- DM can divide attention between seeking
  - A-**evidence** which arrives
    - at rate $\overline{\lambda}$ in state A
    - at rate $\underline{\lambda} \in (0, \overline{\lambda})$ in state B.
  - B-**evidence** which arrives
    - at rate $\overline{\lambda}$ in state B
    - at rate $\underline{\lambda} \in (0, \overline{\lambda})$ even in state A.
- Results generalize, modulo single experimentation property (SEP)—*i.e.,any successful experimentation is immediately followed by an action*—, which requires the "noise" $\underline{\lambda}$ to be sufficiently low.
- Without SEP, difficult to characterize... we have some examples.

Example: SEP holds

$$(\overline{\lambda} = 1,\ \underline{\lambda} = 0.03,\ r = \frac{3}{10},\ u_a^A = 1,\ u_b^B = .7,\ u_a^B = u_b^A = -\frac{1}{2})$$



Implications: Stochastic Choice and Response Time

Choice Rule (between subjects, comparing different priors)

- Skeptics (moderate beliefs) make more accurate decisions but at a longer delay than believers (extreme beliefs)

Response Time (within subject, fixed prior)

- Longer deliberation produces less accurate decision ("speed-accuracy complementarity")
  - consistent with cognitive pschology experiments (cf: DDM, Fudenberg et al (2016))

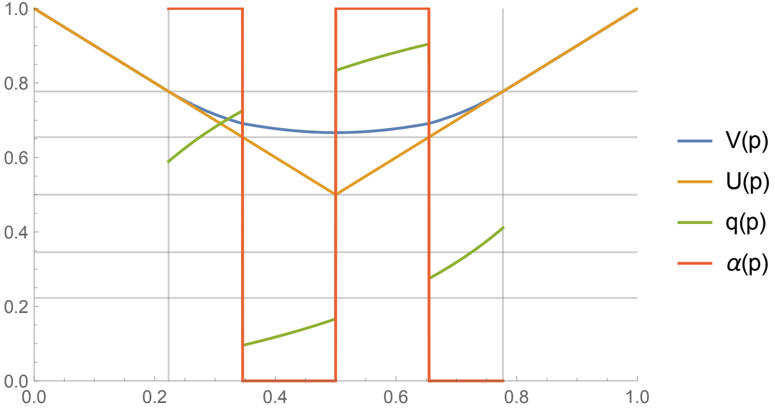
Summary

- In a class of Poisson signal environments, the optimal learning strategy combines
  - immediate action
  - contradictory learning
  - confirmatory learning
- DM with near certain belief takes immediate action.
- DM with extreme belief seeks contradictory evidence.
- DM with moderate belief may seek confirmatory evidence;
- Predictions for:
  - Jury deliberation
    (evidentiary standards, which evidence is scrutinize)
  - Choice of news media
    (preferences for bias, polarization, difference between moderates and extremists)
  - Stochastic choice function
    (delay, accuracy, speed-accuracy complementarity)

Thank you!

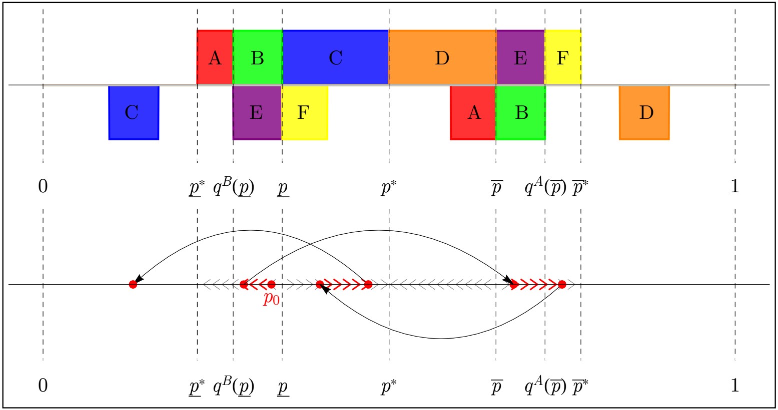
What happens if SEP fails: example

$$(\overline{\lambda} = 1,\ \underline{\lambda} = .2,\ r = 0,\ c = 0.1,\ \overline{u} = 1,\ \underline{u} = 0)$$



$$\underset{p=0}{|}\underbrace{\text{———}}_{\text{action } b}\ \underline{p}^*\ \underbrace{\longleftarrow\ q^B(\overline{p})\ \longleftarrow}_{\alpha=1}\ \underline{p}\ \underbrace{\longrightarrow\longrightarrow}_{\alpha=0}\ p^*\ \underbrace{\longleftarrow\longleftarrow}_{\alpha=1}\ \overline{p}\ \underbrace{\longrightarrow\ q^A(\underline{p})\ \longrightarrow}_{\alpha=0}\ \bar{p}^*\ \underbrace{\text{———}}_{\text{action } a}\underset{1}{|}$$

What happens if SEP fails: example



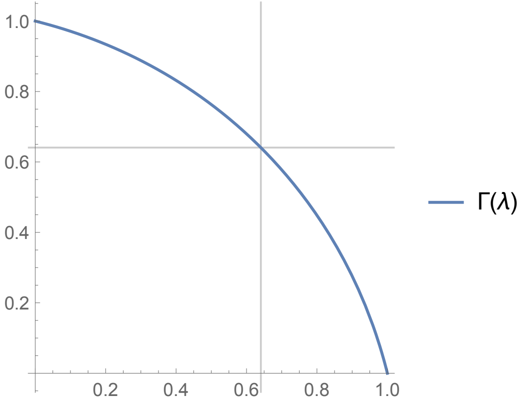
Application 2: Choice of News Media

Balanced Outlets are More Informative

- Normalize $\lambda = 1$ and index media by $\lambda^A \in [0, 1]$:
- So far: Arrival rate of articles in favour of
  - right-wing candidate: $\lambda^A = \alpha\lambda = \alpha$
  - left-wing candidate: $\lambda^B = (1-\alpha)\lambda = (1-\alpha) = 1 - \lambda^A$
  - Any (λ^A, λ^B) with $\lambda^B = 1 - \lambda^A$ was feasible
- Now: Any (λ^A, λ^B) with $\lambda^B = \Gamma\left(\lambda^A\right)$ is feasible
- Assumptions on $\Gamma(\lambda^A)$:
  - decreasing and concave,
  - symmetric $(\Gamma(\lambda^A) = 1 - \Gamma(1 - \lambda^A))$,
  - and $\Gamma(1) = 0, \Gamma(0) = 1, \Gamma(\gamma) = \gamma$, for some $\gamma > 1/2$.
- **Tradeoff between skewness and informativeness.**

Application 2: Choice of News Media

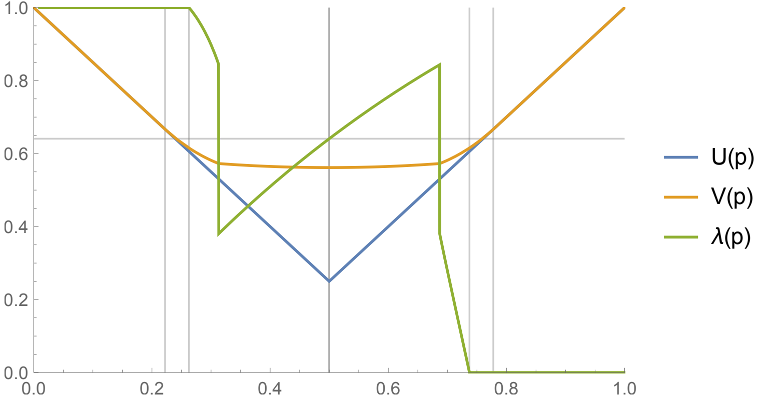
$$\rho\left(\lambda^A + \Gamma(\lambda^A)\right) + (1-\rho)\sqrt{(\lambda^A)^2 + \Gamma(\lambda^A)^2} = 1$$



(Plot Parameter $\rho = \frac{1}{4}$)

Application 2: Choice of News Media

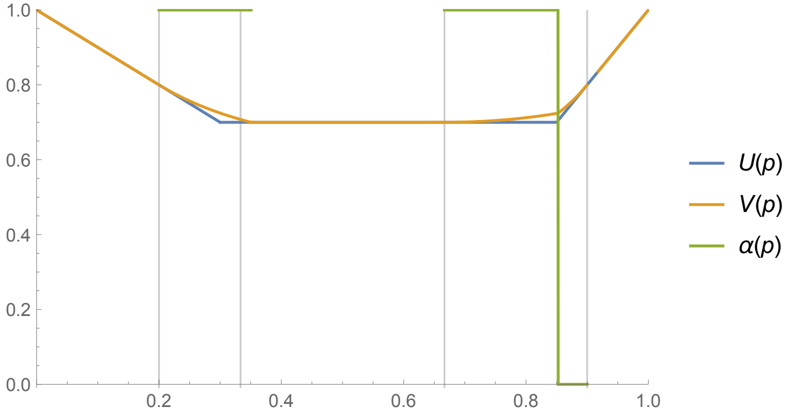
$$\left(\text{Parameters: } r = \tfrac{1}{2},\ u_a^A = u_b^B = 1,\ u_a^B = u_b^A = -\tfrac{1}{2}, \rho = \tfrac{1}{4}\right)$$

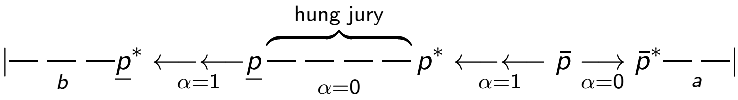


$$\Big|\!\!\underset{b}{-}\ \underline{p}^* \overbrace{\underset{\lambda^A=1}{\longleftarrow}\ \underline{p}^{**}\ \underset{\lambda^A>\gamma}{\longleftarrow}}^{\text{contradictory}} \underline{p} \overbrace{\underset{\lambda^A<\gamma,\text{rising}}{\longrightarrow\longrightarrow}\ p^*\ \underset{\lambda^A>\gamma,\text{rising}}{\longleftarrow\longleftarrow}}^{\text{confirmatory}} \bar{p} \overbrace{\underset{\lambda^A<\gamma}{\longrightarrow}\ \bar{p}^{**}\ \underset{\lambda^A=0}{\longrightarrow}}^{\text{contradictory}} \bar{p}^*\ \underset{a}{-}\!\!\Big|$$

Application 1: Effect of Hung Jury (a third action)

$(\lambda = 1, r = 0,\ c = 0.2,\ u_a^A = u_b^B = 1,\ u_a^B = -1,\ u_b^A = 0,\ u_c^A = u_c^B = 0.7\ )$





Construction: Contradictory Strategy

$$\Big| \underbrace{— — —}_{b} \underline{p}^* \underbrace{\longleftarrow \longleftarrow \longleftarrow}_{\alpha=1} \check{p} \underbrace{\longrightarrow \longrightarrow \longrightarrow}_{\alpha=0} \bar{p}^* \underbrace{— — —}_{a} \Big|$$

- $\underline{p}^*$ — Indifference between:
  - Immediate action b
  - Short period attention to A for followed by action b.
  - This yields boundary condition: $U(\underline{p}^*) = \frac{\lambda}{r+\lambda} U^*(\underline{p}^*)$.
- Obtain $\underline{V}_{ct}(p)$ on $(\underline{p}^*, 1)$ from (??) and boundary cond.
- Similar: $\overline{V}_{ct}(p)$ on $(0, \overline{p}^*)$ from (??) and boundary cond.
- Define

$$V_{ct}(p) := \begin{cases} U(p) & \text{if } p \notin [\underline{p}^*, \overline{p}^*], \\ \max\left\{\underline{V}_{ct}(p), \overline{V}_{ct}(p)\right\} & \text{otherwise.} \end{cases}$$

equals value of contradictory strategy if $\underline{V}_{ct}(p)$ and $\overline{V}_{ct}(p)$ have a unique intersection $\check{p}$.

goback

Construction: Confirmatory Strategy

$$|\underbrace{\longrightarrow\longrightarrow\longrightarrow\longrightarrow}_{\alpha=0}\underbrace{\ p^*\ }_{\alpha=\frac{1}{2}}\underbrace{\longleftarrow\longleftarrow\longleftarrow\longleftarrow}_{\alpha=1}|$$

- At p^*: use stationary strategy $\alpha = 1/2$.
- This yields a boundary condition:
  - Value at p^*: $V(p^*) = \frac{\lambda}{2r+\lambda} U^*(p^*)$
  - Tangency: $V'(p^*) = \frac{\lambda}{2r+\lambda} U^{*\prime}(p^*)$
  - yields $p^* = \frac{u_b^B}{u_a^A + u_b^B}$.
- Get $\underline{V}_{cf}(p)$ on $(0, p^*)$ from (??) and boundary condition.
- Get $\overline{V}_{cf}(p)$ on $(p^*, 1)$ from (??) and boundary condition.
- Define

$$V_{cf}(p) := \begin{cases} \underline{V}_{cf}(p) & \text{if } p \leq p^*, \\ \overline{V}_{cf}(p) & \text{if } p > p^*. \end{cases}$$

equals value of confirmatory strategy.

goback

Proofs of Theorem 1 and Proposition 1

Lemma (Lower bound)

$V_{cf}(p)$ is convex and $V_{cf}(p) \geq \overline{U}(p)$.

- Let V_0 and V_1 be solutions to (**??**) and (**??**).

Lemma (Unimprovability of Branches)

For $i = 0, 1$, if $V_i(p) \geq \overline{U}$ then $V_i(p)$ satisfies the HJB equation.

Lemma (Crossing Lemma)

If $V_0(p) = V_1(p) > \overline{U}$, then $V_1'(p) < V_0'(p)$.

◂ Theorem ◂ Proposition

Proofs of Theorem 1 and Proposition 1

- It is easy to show that $V_{ct}(\underline{p}^*) > V_{cf}(\underline{p}^*)$ and $V_{ct}(\overline{p}^*) > V_{cf}(\overline{p}^*)$.
- Proposition 1: The Crossing Lemma shows that the experimentation region must be of the form

$$\underline{p}^* \underbrace{\longleftarrow\longleftarrow\longleftarrow}_{\alpha=1} \check{p} \underbrace{\longrightarrow\longrightarrow\longrightarrow}_{\alpha=0} \bar{p}^*$$

or

$$\underline{p}^* \underbrace{\longleftarrow\longleftarrow}_{\alpha=1} \underline{p} \underbrace{\longrightarrow\longrightarrow}_{\alpha=0} p^* \underbrace{\longleftarrow\longleftarrow}_{\alpha=1} \bar{p} \underbrace{\longrightarrow\longrightarrow}_{\alpha=0} \bar{p}^*$$

- Theorem 1:
  - $V(p)$ solves HJB whenever it is differentiable.
  - Verification Theorem requires that kinks are convex.
    - $V(p) = \max\{V(p), V_{ct}(p)\}$ is a viscosity solution of the HJB equation.

Theorem Proposition

Example Rich News: Confirmatory and Contradictory

$$\Big|\underset{b}{\text{—}}\ \underline{p}^* \overbrace{\underset{\lambda^A=1}{\longleftarrow}\ \underline{p}^{**}\ \underset{\lambda^A>\gamma}{\longleftarrow}}^{\text{contradictory}} \underline{p} \overbrace{\underset{\lambda^A<\gamma,\text{rising}}{\longrightarrow\longrightarrow}\ p^*\ \underset{\lambda^A>\gamma,\text{rising}}{\longleftarrow\longleftarrow}}^{\text{confirmatory}} \bar{p} \overbrace{\underset{\lambda^A<\gamma}{\longrightarrow}\ \bar{p}^{**}\ \underset{\lambda^A=0}{\longrightarrow}}^{\text{contradictory}} \bar{p}^*\ \underset{a}{\text{—}}\Big|$$

Observations

- Direction of bias of optimal outlet as in baseline model.
- Citizens with more moderate beliefs choose more balanced and more informative outlets than citizens with extreme beliefs.
- **Proposition:** At $\underline{p}^*$, $\overline{p}^*$, purely contradictory evidence ($\lambda \in \{0, 1\}$) is optimal (even with Inada condition).

Comparison with baseline (linear) model shows:

- Most citizens will only choose balanced news outlets if they are more informative than outlets with extreme bias.