IO in I-O: Size, Industrial Organization and the Input-Output Network Make a Firm Structurally Important *

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Abstract

There is a growing literature suggesting that firm level productivity shocks can help understand macroeconomic level outcomes. However, existing models are very restrictive regarding the nature of competition within sector and its implication for the propagation of shocks across the input-output (I-O) network. This paper offers a more comprehensive understanding of how firm level shocks can shape aggregate dynamics. To this end, I build a tractable multi-sector heterogeneous firm general equilibrium model featuring oligopolistic competition and an I-O network. It is shown that a positive shock to a large firm increases both the productivity and the markup at the sector level. By reducing the sector price, the change in productivity propagates only to downstream sectors. Conversely, the change in markup, by increasing price and reducing demand for intermediate inputs, propagates both to downstream and upstream sectors. The sensitivity of aggregate output to firms' shocks is determined by the sector's (i) Herfindahl Index, which measures the competition intensity of the sector, (ii) position in the input-output network, which measures the direct and indirect importance of this sector for the household, and (iii) the profit share along the supply chain, which relates to the changes in demand to upstream sectors.

Keywords: Oligopoly, Imperfect Competition, Input-Output Network, Industrial Organization, Firm Heterogeneity, Random Growth, Granularity, Volatility, Micro-Origin of Aggregate Fluctuations, Shocks Propagation, Production Network

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1 Introduction

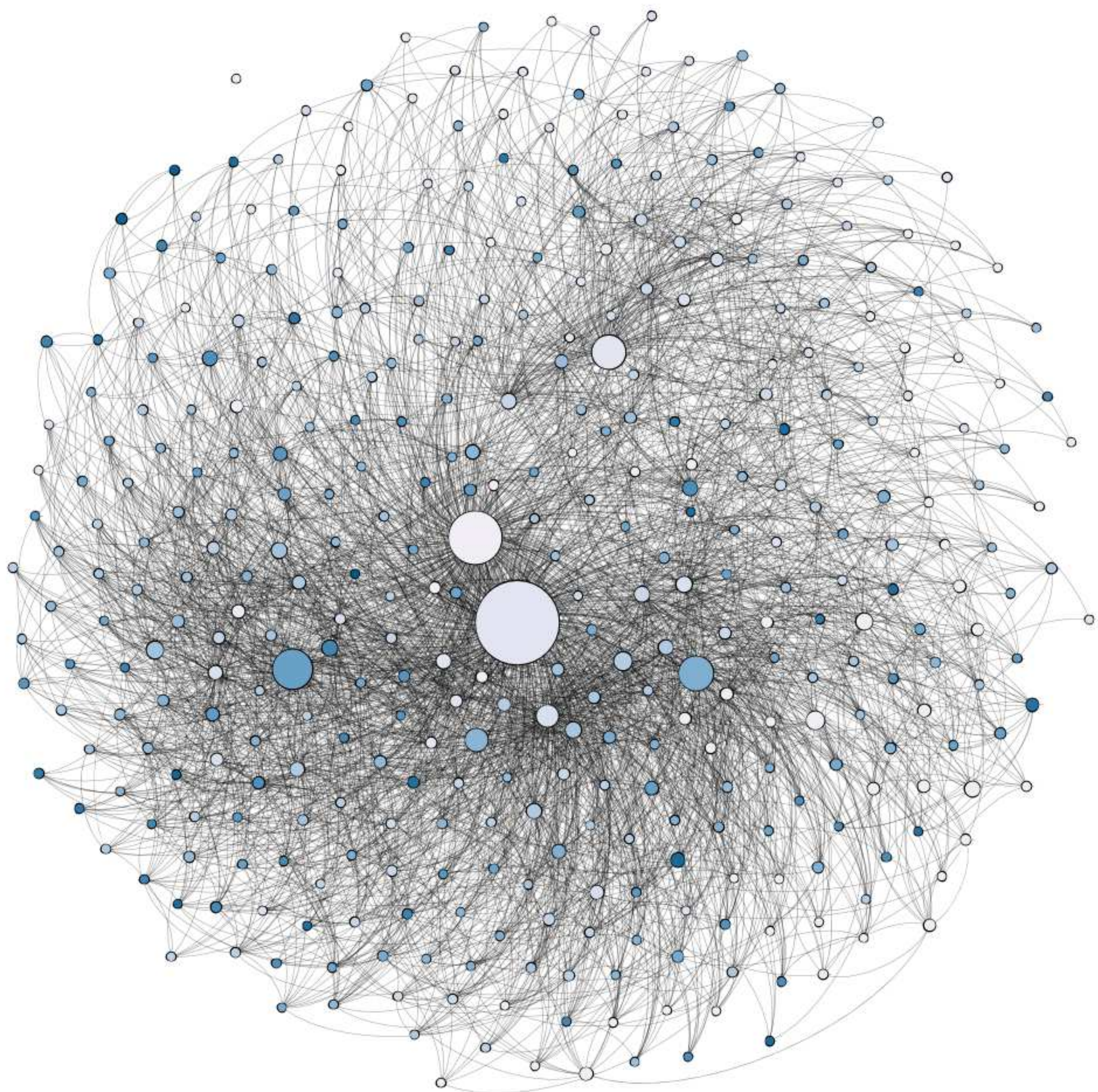
Firm-level productivity shocks can explain an important part of movement in prices and output at the sector and macroeconomic level[1]. The idea is that a handful of large firms represents a large share of a sector, and thus shocks hitting these large firms cannot be balanced out by those affecting smaller firms. However, typical models are very restrictive regarding the nature of competition within a sector: firms are large enough to have a systemic importance but these firms do not internalized it when they make their decisions. This paper explores the alternative oligopolistic market structure where firms do take into account the effect of their decisions on sector-level price and quantity in order to study the propagation of firm-level shocks to other sectors through the Input-Output (I-O) network. The properties of the propagation that arises under oligopolistic competition are shown to be dramatically different from the monopolistic case both at the sector and macroeconomic level.

Table 1 and Figure 1 motivate this paper: sectors are concentrated and linked through a "small world" I-O network. Table 1 shows summary statistics of the top four firms' share of industry revenue in 2002, 2007 and 2012 for around 970 industries. Industry revenue accounted for by the top four firms varies from almost zero to close to 100% with a median value close to 33% in 2007. The first thing to note is that large firms represent an important share of revenue of the median sector. Secondly, as concentration is a widely used measure of a sector's competition intensity, this table also suggests that different sectors have different competition levels. For the bottom 25% of these sectors the top four firms account for less than 18% of the total industry revenues, while for the top 25% of theses sectors, only four firms account for more than 50% of the total industry revenues. While confirming the "granular" nature of these sectors, this table emphasizes the heterogeneity across sectors of the intensity of competition. Besides, these sectors are not independent from each other: production in one sector relies on a complex and interlocking supply chain. Figure 1 displays the I-O network among 389 sectors for the US in 2007. This is a "small world" network: a few nodes are connected to many other nodes. In such production networks, as shown by Acemoglu et al. (2012), Carvalho (2010, 2014) and Baqaee (2016), sector-level shocks translate into aggregate volatility. In this paper, I study how firm-level shocks affect sector-level productivity and competition and how these shocks propagate in the I-O network and thus shape the aggregate dynamics.

To this end, I build a tractable multi-sector heterogeneous firm general equilibrium model featuring oligopolistic competition and an I-O network. Within each sector, a finite number of heterogeneous firms are subject to oligopolistic competition and set variable markups à la Atkeson and Burstein (2008). Up to an approximation, two sector-level sufficient statistics, the sum and Herfindahl index of the firms' productivity entirely characterize the equilibrium of this economy.

[1]An important paper in the growing literature on the micro-origin of aggregate fluctuations is the seminal work by Gabaix (2011) where he shows that when the firm-size distribution is fat-tailed, firm-level shocks do not wash out at the aggregate. Building on this seminal work, Carvalho and Grassi (2017) show that firm dynamic models contain a theory of business cycle as soon as the continuum of firms' assumption is relaxed. Acemoglu et al. (2012), Carvalho (2010, 2014) and Baqaee (2016) build on the multi-sector business cycle framework of Long and Plosser (1983) to show how shocks on sectors linked through an I-O network can translate into aggregate fluctuations. Earlier contributions include Jovanovic

Figure 1: The US Input-Output Network in 2007



NOTE: Larger nodes of the network represent sectors supplying inputs to many other sectors. A darker color represents higher top four firms' share of total revenues in 2007 (sectors without available data are left white). There are 389 sectors. Source: Bureau of Economic Analysis, detailed I-O table for 2007 and Census Bureau. The figure is drawn with the software package Gephi.

Table 1: Top 4 firms' share of total industry revenues

Year	Mean	Q1	Q2	Q3	Std
2002	35.4	17.6	31.1	51.0	23.0
2007	37.2	18	32.9	53.2	23.8
2012	35.0	16.2	31.2	50.1	22.9

NOTE: Summary statics of the distribution of top 4 firms' share of total industry revenues across 970 industries. The second column is the unweighted mean, the third column is the first quartile, the fourth column is the median, the fifth column is the third quartile and the sixth column is the standard deviation. Source: US Census Bureau, 6 digits NAICS industries, all sectors except 11 , 21, 23, 55 ,92 .

The mechanism is as follows. Firm-level shocks affect both the sector's average productivity and concentration. To see this, take a sector with a finite number of heterogeneous firms and assume that an already large firm is subject to a positive productivity shock. Following this shock, the sector's average productivity becomes larger since the productivity of one firm has increased. Since this firm was already large before the shock hit, the sector becomes even more concentrated. This firm-level shock has two opposite effects on price and output at the sector level. First, because of the increase in average productivity, the sector good is cheaper and output increases. Second, because of the increases in concentration, competition in the sector decreases: this large firm is larger and can use its size to extract even more profit. It follows that the sector price increases and output decreases. These changes in prices and output propagate to the other sectors through the I-O network. The increase in productivity, resulting in a decrease in price, reduces the marginal cost of downstream sectors. Indeed the downstream sectors use this good as an input to produce. The decrease in competition, resulting in an increase in price, propagates downstream as it increases the marginal cost of downstream sectors. But it also propagates to upstream sectors as it reduces the share of sector's income used to pay for intermediate inputs and thus the demand for upstream sectors' goods. The propagation of this shock downstream ultimately affects the price of goods purchased by the household and thus the real wage. The stronger is the effect, the more the sector's good is directly and indirectly (through other sectors) consumed by the household. The propagation of shocks upstream ultimately affects the profit rebated to the households as it reduces demand for upstream goods. The stronger is the effect, the higher is the sector's profit share along its supply chain. The above example described the effect of one shock on an already large firm but, in this paper, each firm's productivity is subject to persistent idiosyncratic shocks which make the two sufficient statistics follow $AR(1)$-type processes, as in Carvalho and Grassi (2017). Each sector's price and quantity are thus stochastic, which translate into aggregate volatility due to the "small world" nature of the I-O network.

I show that the effect of the change in productivity of a firm in a given sector on aggregate output is a function of four characteristics. First, the sector's concentration, which determines the competition intensity in that sector and thus how much shock to a firm translates into change of sector level

(1987), Durlauf (1993) and Bak et al. (1993).

markup and price. Second, the sector centrality, which measures that sector's direct and indirect importance in the household's consumption bundle. This characteristic relates to the transmission of firm-level shocks to downstream sectors. Third, the sector's profit share over its whole supply chain, which measures how much profit is captured directly and indirectly (through the I-O network) by that sector. This characteristic relates to the propagation of firm level shocks to upstream sectors. Finally, the firm size which interacts with these characteristic and determines the strength of the downstream and upstream propagation.

Furthermore, I show that a change in productivity of a firm in a given sector propagates to price of downstream sectors and to sales share of upstream sectors. Because of oligopolistic competition, a change in productivity of one firm does not pass through fully on sector-level price while changing the profit and cost share. The change in price propagates to downstream sectors but is either reduced or magnified, relative to the monopolistic competition case, depending on the identity of the firm subject to the shock. The change in profit and thus cost share propagates to upstream sectors and affects their sales as a share of output. The latter mechanism requires both oligopolistic competition and an I-O network.

Thanks to the high tractability of the model and the fact that the equilibrium is characterized by two sector-level sufficient statistics, I calibrate this economy by relying on the choice of a few deep parameters, the Census' concentration data that pin downs sectors' competition intensity and the Bureau of Economic Analysis (BEA)'s Input-Output data. For the benchmark calibration, the output volatility that comes out of simulated data is 34% of what is observed in the data. Furthermore, thanks to the high tractability of the model, I decompose the aggregate volatility in the contribution of the "downstream" and "upstream" effect. The "downstream" effect contributes to 89%, the "upstream" to 1.3% and the remaining 9.64% are due to the covariance. In a version of this model with monopolistic competition rather than oligopolistic competition all the aggregate volatility is due to the "downstream" effect.

Related Literature: This paper contributes to the literature on the micro-origin of aggregate fluctuations. This literature is based on two mains ideas: the "granular hypothesis" and the network origin. For the former, seminal work by Gabaix (2011) shows that whenever the firm-size distribution is fat-tailed, idiosyncratic shocks do not average out quickly enough and therefore translate into sizable aggregate fluctuations. Carvalho and Grassi (2017) ground the "granular hypothesis" in a well-specified firm dynamic setup. For the latter, Acemoglu et al. (2012) and Carvalho (2010) show that when the distribution of sectors' centrality in the I-O network is fat tailed then sector level perturbations also generate sizable aggregate fluctuations. Relative to these papers, I present the first framework that includes both components explicitly. The "granular hypothesis" leads to sector-level fluctuations whereas the I-O network structure translates sector-level fluctuations into aggregate fluctuations.[2] An important drawback of this literature is that firms are supposed to be large enough to influence the aggregate but also small enough not to be strategic. In Carvalho and Grassi (2017)

[2]Notable contribution in this literature include but are not limited to di Giovanni et al. (2014), Magerman et al. (2016) and Baqaee and Farhi (2017a).

framework such assumptions were made because firms interacted in a perfectly competitive labor market. Here, I present the first model of strategic pricing where aggregate fluctuations arise from purely idiosyncratic shocks. When they are taking their decisions, firms do take into account the fact that they have market power and can influence their sector's output and price.

Recently, Baqaee and Farhi (2017a) revisit the famous and influential result by Hulten (1978) which states that for efficient economies the first order impact of a productivity shock to a firm on aggregate output is equal to that firm's sales as a share of output. The framework presented here is not subject to this result as the economy is not efficient. Therefore it is closer to Basu (1995), Basu and Fernald (2002), Jones (2011, 2013), Bigio and La'O (2016) or Baqaee and Farhi (2017b) who study the introduction of distortions in multi-sector macroeconomic model with production networks. Contrary to all of these papers, here, firm level productivity shocks endogenously affect markups, the distortions in this economy.

An important and growing literature studies the transmission of shocks across sectors through the I-O network: Acemoglu et al. (2015) look at the transmission of well identified supply and demand shocks, Carvalho et al. (2016) and Boehm et al. (2016) study the firm level impact of supply chain disruptions occurring in the aftermath of the Great East Japan Earthquake in 2011, while Barrot and Sauvagnat (2016) look at the effect of natural disasters. Baqaee (2016) studies theoretically the effect of shocks on entry cost. In this paper I introduce a new propagation mechanism of firm-level shocks in the I-O network through change in sector-level competition, which act as supply shocks to downstream sectors and demand shocks to upstream sectors. Firm-level shocks propagate both downstream and upstream despite the Cobb-Douglas assumption, furthermore firm size affects the sign of the upstream propagation.

This paper also contributes to the literature on imperfect competition among heterogeneous firms. Krugman (1979), Ottaviano et al. (2002), Melitz and Ottaviano (2008), Bilbiie et al. (2012) and Zhelobodko et al. (2012) study demand-side pricing complementary whereas I look at supply-side pricing complementaries as in Atkeson and Burstein (2008) but in an I-O context. Furthermore I show, up to an approximation, that such a model is highly tractable and that firm heterogeneity can be summarized at the sector level by just two sufficient statistics.

Finally, this paper relates to a recent and growing empirical literature that documents and analyses the macroeconomic consequences of the rise in market concentration in the US. Barkai (2017), Autor et al. (2017) and Kehrig and Vincent (2017) explain the secular declines in labor share from the increase of sector level and firm level concentration, while Loecker and Eeckhout (2017) document an increase in firm level markup which they relate to a number of secular trends in the last three decades. Even if the focus of this paper is different, it contributes to this literature by providing a simple and tractable model to analyze the aggregate consequence of market concentration. For example, market concentration is shown here to be driving sector-level markup and therefore the profit and labor share.

Outline: The paper is organized as follows. In Section 2, I describe and solve the household's and firm's problem. In Section 3, I first aggregate firms' behavior at the sector level and show that firm heterogeneity can be summarized by two sufficient statistics. I then solve for the dynamics of these two statistics. In Section 4, I show that a firm's sector market structure, its role in the input-output network and the firm's size jointly determine its structural importance. In Section 5, I look at how firm-level productivity shocks propagate to other sectors through the I-O network. In Section 6, I calibrate the model and perform some quantitative exercises. Finally, Section 7 concludes.

2 Model

In this section, I describe the structure of the economy and I solve for the household and firms' problem. There are two types of agent. First, a representative household consumes and supplies labor. Second, there is a finite number of firms distributed across a finite number of sectors that are linked by a production network. In each sector, firms set their price (or quantity) strategically. Each firm is subject to independent persistent idiosyncratic shocks independent.

2.1 Household

The representative household lives a discrete and infinite number of periods. Preferences are given by $\mathbb{E}_0 \sum_{t=0}^{\infty} \rho^t u(C_t, L_t)$ where $u(C_t, L_t)$ is the instantaneous utility, ρ is the discounted rate, C_t is the composite consumption good, and, L_t is the number of hours worked at time t.

The composite consumption good C_t is a Cobb-Douglas aggregation of $N \in \mathbb{N}$ sector-level goods: $C_t = \theta \prod_{k=1}^{N} C_{k,t}^{\beta_k}$ where $C_{k,t}$ is the amount of good k consumed by the household at time t and where θ is a normalization constant.[3] The Cobb-Douglas weights, β_k, are equal to the expenditure shares of each good $\frac{P_{k,t}C_{k,t}}{P_t^C C_t}$ where $P_{k,t}$ is the price of good k and P_t^C is the aggregate price index which satisfies $P_t^C = \prod_{k=1}^{N} P_{k,t}^{\beta_k}$. Note that N is an integer number.

In a sector k, there is an integer number, N_k, of varieties index by i. These varieties are aggregated with a constant elasticity of substitution $\varepsilon_k > 1$ such that $C_{k,t} = \left(\sum_{i=0}^{N_k} C_t(k,i)^{\frac{\varepsilon_k - 1}{\varepsilon_k}}\right)^{\frac{\varepsilon_k}{\varepsilon_k - 1}}$ where $C_t(k,i)$ is the amount of sector k's variety i consumed by the household at time t. Finally, the price of good k satisfies $P_{k,t} = \left(\sum_{i=1}^{N_k} P_t(k,i)^{1-\varepsilon_k}\right)^{\frac{1}{1-\varepsilon_k}}$ where $P_t(k,i)$ is the price of variety i in sector k at time t. Each variety is produced by exactly one firm, and all the firms are owned by the representative household.

The above household preferences and the assumption that $\varepsilon_k > 1$ capture the idea that as one is dissagregating further, from sectors to firms, it is easier for the household to substitute between two dissagregating units. Furthermore, the degree of substitution between two varieties of the same good is higher that between two varieties of two different goods.

[3]The normalization constant θ makes the mathematics simpler and is equal to $\theta = \prod_{k=1}^{N} \beta_k^{-\beta_k}$.

2.2 Firms

An integer number of firms are splited in N sectors. In sector k, there are N_k firms and each variety is produced by exactly one firm. Firms are heterogeneous in their (labor-augmenting) productivity. A sector is defined as a technology and a market structure: (i) firms in the same sector have access to the same production function (ii) these firms compete with each other in a differentiated Bertrand or Cournot game. At the end of this section, I show that the implied firm dynamics in this model is consistent with recent empirical evidences.

Technology: The firm i in sector k combines labor, $L_t(k,i)$, and other sectors' goods, $x_t(k,i,l)$, to produces $y_t(k,i)$ units of its variety using the constant return to scale Cobb-Douglas technology $y_t(k,i) = \alpha_k \Big(Z_t(k,i)L_t(k,i)\Big)^{\gamma_k} \prod_{l=1}^{N} x_t(k,i,l)^{\omega_{k,l}}$ where γ_k is the labor share in the production, α_k is a normalization constant,[4] $Z_t(k,i)$ is the labor-augmenting productivity specific to the firm i in sector k, $\omega_{k,l}$ is the input share of sector l's goods needed in sector k's production. The $(N \times N)$ matrix $\Omega = \{\omega_{k,l}\}_{k,l}$ represents the input-output network.[5] Thanks to constant return to scale the kth rows of Ω sum to $1-\gamma_k$: $\sum_{l=1}^{N} \omega_{k,l} = 1-\gamma_k$. Furthermore, $x_t(k,i,l)$ is a composite of sector l's varieties such that $x_t(k,i,l) = \Big(\sum_{j=1}^{N_l} x_t(k,i,l,j)^{\frac{\varepsilon_l - 1}{\varepsilon_l}}\Big)^{\frac{\varepsilon_l}{\varepsilon_l - 1}}$ where $x_t(k,i,l,j)$ is the quantity of the variety j of sector l's good that is used for the production of variety i of sector k's good. Note that the elasticity of substitution among varieties in a sector is the same for firms and for the household.

The input-output network Ω is assumed to be fixed across time and state because it is a sector-level network. Here the input-output linkages are interpreted as technology: the input bundle needed to produce the variety of a good. At the business cycle frequency, this technology is not affected by labor-augmenting firm-level productivity shocks that are considered here. However, if one firm in a sector increases massively the price of its variety, its customers are able to substitute away from this variety thanks to the double-nested constant elasticity of substitution demand system. Therefore even if the sector-level input-output linkages are fixed, the transaction network between firms is not and varies across time and state.[6]

The productivity of firm i in sector k, $Z_t(k,i)$, is identically and independently distributed across firms but not across time. It follows a sector specific Markov chain over the discrete state space $\Phi_k = \{1, \varphi_k, \varphi_k^2, \cdots, \varphi_k^n, \cdots, \varphi_k^{M_k}\} = \{\varphi_k^n\}_{n\in\{0,1,\ldots,M_k\}}$ for $\varphi_k > 1$ which is evenly distributed in logs.[7] This Markov chain is described by the matrix of transition probabilities $\mathcal{P}^{(k)} = \{\mathcal{P}^{(k)}_{n,n'}\}_{n,n'}$ where $\mathcal{P}^{(k)}_{n,n'} = \mathbb{P}\left(Z_{t+1}(k,i) = \varphi_k^{n'} | Z_t(k,i) = \varphi_k^n\right)$ is the probability that a firm i in sector k jumps from productivity level φ_k^n to $\varphi_k^{n'}$ between time t and time $t+1$. In some cases, I assume a specific Markovian chain which is a discretization of a random growth process and is taken from Córdoba (2008). Figure 2 and

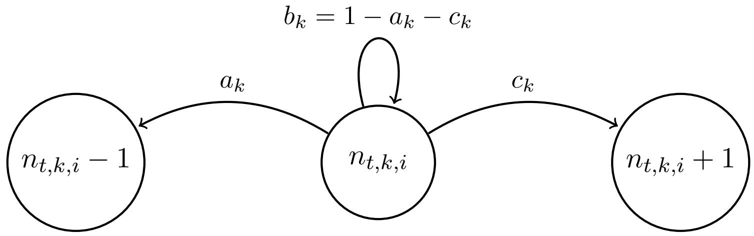
[4]The normalization constant α_k makes the mathematics simpler and is equal to $\alpha_k = \gamma_k^{-\gamma_k} \prod_{l=1}^{N} \omega_{k,l}^{-\omega_{k,l}}$.

[5]The notation $U = \{u_{k,l}\}_{k,l}$ means that U is the matrix where the element k,l is equal to $u_{k,l}$, while I denote $v = \{v_k\}_k$ the vector where the element k equal to v_k.

[6]This differs from Magerman et al. (2016) who study the micro-origin of firm-level shocks in a fixed firm-to-firm transaction network.

[7]This means that $\varphi_k^{n+1}/\varphi_k^n = \varphi_k$.

Figure 2: Productivity Process



NOTE: A representation of the transition probabilities in assumption 1 of a firm i in sector k for $M_k > n_{t,k,i} > 0$.

Assumption 1 describe its transition probabilities.

Assumption 1 ***(Random Growth)*** *For $a_k + b_k + c_k = 1$, firm level productivity in sector k follows a Markov chain over $\Phi_k = \{\varphi_k^n\}_{n \in \{0,1,\ldots,M_k\}}$ with transition probabilities such that:*

$$\mathcal{P}^{(k)} = \begin{pmatrix} a_k + b_k & c_k & 0 & \cdots & \cdots & 0 & 0 \\ a_k & b_k & c_k & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_k & b_k & c_k \\ 0 & 0 & 0 & \cdots & 0 & a_k & b_k + c_k \end{pmatrix}$$

Pricing: A sector is also defined as a market where firms are engaged in imperfect competition. Sector's goods are imperfect substitute and varieties within a sector are more substitutable: $\varepsilon_k > 1$. Each firm produce exactly one variety of its sector's good and customers cannot perfectly substitute between two varieties: $\varepsilon_k < \infty$. Following Atkeson and Burstein (2008), I assume that firms play a static game where firm i in sector k chooses its price $P_t(k,i)$ taking as given the prices chosen by other firms in the economy, the other sectors' price and quantities, the wage, and, aggregate prices and quantities. Importantly note that this firm does recognized that sector k's price and quantity is affected when it changes its price.

To understand this assumption, let us imagine that General Motors (GM) has a way to produce a car that cost $10000 less than its competitors. The above assumption implies that, when GM is taking its pricing decision, it is internalizing the impact of its decision on the quantity and price of the "Automobile Manufacturing" sector but not on the "Amusement Parks and Arcades" sector. Note that with these assumptions in place, GM is not internalizing the impact of its pricing decision on the real wage and on the prices and quantities of its upstream or downstream sectors. Relaxing these assumptions might create effects that will go beyond the scope of this paper and I leave these questions for future research.[8]

[8]Another interpretation of the assumptions made here is that firms have limited ability to compute the effect of their decision on any variable outside their sector's price and quantity. This is fundamentally different from the Atkeson and Burstein (2008) framework because, in their paper, they assume a continuum of sectors: even if firms are not atomistic within a sector, a firm's sector is atomistic with respect to the aggregate economy.

Note that here I assume a competition in price (labeled as *Bertrand*). In most of the results below, I compare the baseline case of *Bertrand* competition with (i) the case of *Cournot* competition where firms compete in quantity and (ii) with the benchmark case of *monopolistic* Dixit and Stiglitz (1977) competition. When it does not create confusions, I abstract from the time t subscript.

As a result of cost-minimization, firm i in sector k face a marginal cost $\lambda(k,i) = Z(k,i)^{-\gamma_k} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}$ where w is the wage rate in this economy. Note that due to the presence of input-output linkages, this marginal cost is a function of other sectors' prices. The sector level gross output is defined as $Y_k = \left(\sum_i^{N_k} y(k,i)^{\frac{\varepsilon_k - 1}{\varepsilon_k}}\right)^{\frac{\varepsilon_k}{\varepsilon_k - 1}}$. Proposition 1 characterizes the pricing decision of a firm i in sector k.

Proposition 1 *(**Firm's Pricing**) The firm i in sector k sets a price $P(k,i)$, a markup $\mu(k,i)$ and has a sale share $s(k,i)$ that satisfy the following system of equations:*

$$
\begin{aligned}
P(k,i) &= \mu(k,i)\lambda(k,i) \\
s(k,i) &= \frac{P(k,i)y(k,i)}{P_k Y_k} = \left(\frac{P(k,i)}{P_k}\right)^{1-\varepsilon_k} \\
\mu(k,i) &= \begin{cases} \frac{\varepsilon_k}{\varepsilon_k - 1} & \text{Under Monopolistic Competition} \\ \frac{\varepsilon_k - (\varepsilon_k - 1)s(k,i)}{\varepsilon_k - 1 - (\varepsilon_k - 1)s(k,i)} & \text{Under Bertrand Competition} \\ \frac{\varepsilon_k}{\varepsilon_k - 1 - (\varepsilon_k - 1)s(k,i)} & \text{Under Cournot Competition} \end{cases}
\end{aligned}
$$

Proof *See Atkeson and Burstein (2008).* □

The first thing to note in the above proposition is that firms charge a markup over their marginal cost. Under monopolistic Dixit and Stiglitz (1977) competition the markup is constant and equal to $\varepsilon_k/(\varepsilon_k - 1)$. Under oligopolistic competition (Bertrand or Cournot), the markup charged is increasing in the sales share of the firm: larger firms charge a higher markup.

Note that in both the Bertrand and Cournot competition case, the markup charged by a firm is converging to a constant as the size of this firm goes to zero. Indeed, for firm i in sector k, we have $\mu(k,i) \to \varepsilon_k/(\varepsilon_k - 1)$ as $s(k,i) \to 0$. As a firm gets atomistic, its markup is getting closer to the one under monopolistic competition. Because the system of equation in Proposition 1 does not admit an analytical solution, aggregating firms' behavior at the sector level in a tractable way turns out to be impossible. To circumvent this issue, Proposition 2 is approximating the sales share of a firm under oligopolistic competition by the sales share of this firm under monopolistic competition. In Section 3, this result is used to collapse firms' heterogeneity to two sector-level statistics.

Proposition 2 *(**Firm's Approximation**) The sales share of firm i in sector k under monopolistic competition is a function of its marginal cost $\lambda(k,i)$ and the sector k price index: $\widehat{s}(k,i) = \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k}$. When $\widehat{s}(k,i) \to 0$, the sales share of this firm under oligopolistic competition, $s(k,i)$, satisfies*

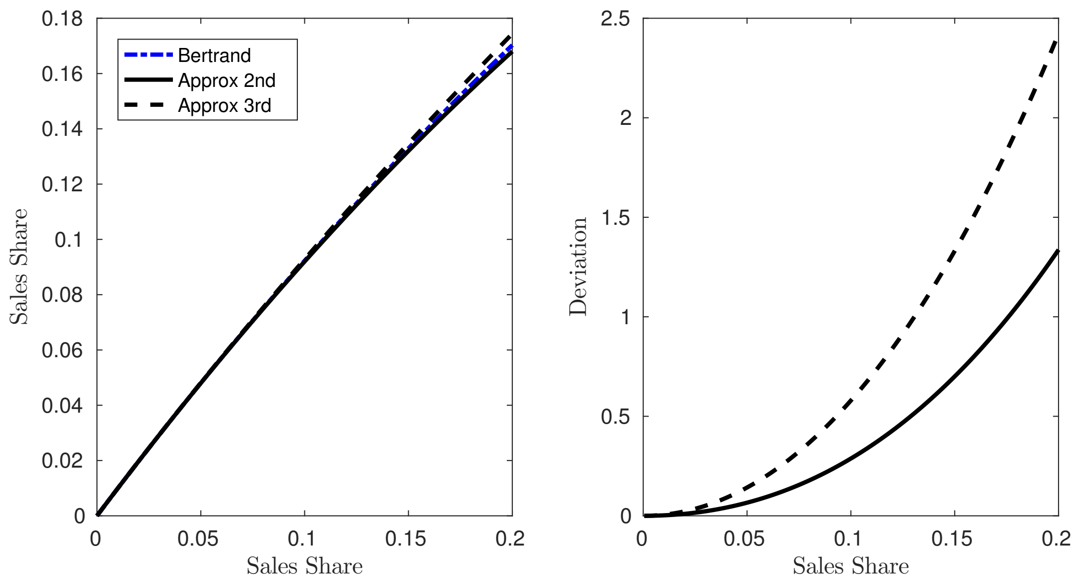
$$
s(k,i) = \begin{cases} \widehat{s}(k,i) - \left(1 - \varepsilon_k^{-1}\right)\widehat{s}(k,i)^2 + o\Big(\widehat{s}(k,i)^2\Big) & \text{Under Bertrand Competition} \\ \widehat{s}(k,i) - (\varepsilon_k - 1)\,\widehat{s}(k,i)^2 + o\Big(\widehat{s}(k,i)^2\Big) & \text{Under Cournot Competition} \end{cases}
$$

where the notation $f(x) = o\Big(g(x)\Big)$ means $f(x)/g(x) \to 0$ when $x \to 0$.

Proof *See Appendix A.1.* □

In this proposition, the sales share of firm i in sector k is approximated by the sales share under monopolistic competition $\widehat{s}(k,i)$. Since there is a one to one mapping between the marginal cost and $\widehat{s}(k,i)$ for a fixed sector price index P_k, one can think of this result as a approximation of the sales share of firms by a function of their marginal cost. Similarly, the above results holds when $\widehat{s}(k,i)$ is small or, since $\varepsilon_k > 1$, when the marginal cost $\lambda(k,i)$ is large.

Figure 3: Approximation of Firms' Sales Share



NOTE: For $\varepsilon_k = 5$. The left panel shows the Bertrand sales share using a numerical solver, the second and the third order approximation as a function of the monopolistic sales share. The right panel shows percentage deviation of both approximations with respect to the numerical solution.

The framework derived in this paper is designed to capture the aggregate effect of shocks on "large" firms while the results in Proposition 2 holds for "small" firms in their market. Therefore, it is important to know if "small" in the sense of the approximation in Proposition 2 is "small" economically. Figure 3 displays on the left panel the sales share under Bertrand competition as a function of the sales share under monopolistic competition along with the second and third order approximation. The right panel of this figure displays percentage deviations of the approximations with respect to the exact solution. For sales share up to 20%, the error made by the second order approximation is less than 1.5%: "small" for the approximation in Proposition 2 is thus not small economically. In order to aggregate the firms behavior at the sector level, in some cases, I assume this approximation to hold. [9, 10]

[9] For conciseness, the third order approximation is not reported in the formula in Proposition 2 but can be found in the proof of this proposition in Appendix A.1.

[10] A concern could be that this approximation holds well only in levels and not in term of slopes. Figure 10 in Online Appendix E shows that for sales share up to 20%, the error made on the slope is less than 5%. An other concern might be the quality of this approximation depends on the value of the elasticity across varieties in a sector ε_k. Figures 11 and 12 in Online Appendix E shows that the quality of the approximation is of the same order for different values of ε_k.

Assumption 2 ***(Approximation)*** *In Proposition 2's approximation, higher order terms are negligible*

Firm Dynamics: To conclude the description of the model, let us look at its implication in term of firm dynamics. Under Assumption 1, the productivity of a given firm satisfies Gibrat's law: the growth rate is independent of the level. Especially, the level of productivity does not affect the mean and the volatility of its growth rate. However, here Gibrat's law is violated for a firm's sales: larger is a firm, larger is its market power and less sensitive are its sales to a change in its marginal cost. Indeed, the firm's level markup adjusts and thus the pass-through of shock to price is incomplete. Proposition 3 shows this in a formal way.[11]

Proposition 3 *(**Size-Volatility**) Under Assumption 1 and 2, the (conditional) variance of the growth rate of firm i in sector k's productivity and sales share satisfies:*

$$\mathbb{V}ar_t\left[\frac{Z_{t+1}(k,i)-Z_t(k,i)}{Z_t(k,i)}\right]=\sigma_k^2 \qquad \textit{and} \qquad \mathbb{V}ar_t\left[\frac{s_{t+1}(k,i)-s_t(k,i)}{s_t(k,i)}\right]=g_k(\hat{s}_t(k,i))\sigma_k^2$$

where $\sigma_k^2 = a\left(\varphi_k^{-1}-1\right)^2 + c\left(\varphi_k-1\right)^2 - \left(a\varphi_k^{-1}+b+c\varphi_k-1\right)^2$ and $g_k : x \mapsto g_k(x)$ is a decreasing function. Furthermore, the slope of g_k is increasing in ε_k.

Proof *See Online Appendix F.1.* □

It is a well established fact that larger firms tends to be less volatile. Recently, Yeh (2017) explores empirically the possible mechanisms that could give rise to such negative relationship between size and volatility at the firm level. After ruling out diversification among establishments or products, he concludes that large firms face smaller price elasticities and therefore respond less to a given-sized productivity shock than small firms do. In the current framework, the reason behind the negative size-volatility relationship of Proposition 3 is exactly the one identified by Yeh (2017).[12]

The simple demand system and the market structure assumed here together with the random growth process for productivity implies a rich and empirically relevant firm dynamics. As it is shown in the rest of this paper, non-negligible sector-level and aggregate fluctuations arise from firm-level productivity shocks despite the fact that larger firm are less volatility.

3 Sectors Aggregation

The model derived above describes an economy where a finite number of firms, subject to productivity shocks, evolve and compete in their sector. The behavior and the dynamics of these firms shape the sector-level variables. This section characterizes the mapping between firm-level and sector-level variables. It shows that the latter are related to a few moments of the distribution of firms whose dynamics is solve for.

[11] Gibrat's law was first introduce by Gibrat (1931). See also Sutton (1997) for a review.

[12] Empirical studies that identified a negative size-volatility relationship at the firm level are Comin and Philippon (2006) and Comin and Mulani (2006) for publicly listed US firms, Fort et al. (2013) and Foster et al. (2008) for US manufacturing firms, and di Giovanni et al. (2014) using a census of French firms.

This section is organized as follow. First, I introduce two key sector-level statistics. Second, I derive the relationship between sector level markup and concentration before described the equilibrium under Assumption 2. Finally, I describe the sector dynamics under Assumptions 1 and 2.

3.1 Two Statistics

In the model described in Section 2, given the distribution of productivity $Z_t(k,i)$ at each time t, one can solve for the equilibrium allocation. The distributions of productivity in each sector are the state variables of this economy. Let me introduce two moments of these distributions that turn out to be key to describe the equilibrium allocation under Assumption 2.

For a given sector k, the first statistic is the sum of the productivity of sector k's firms raise at a power that take into account the downward slopping demand and the decreasing return in labor:

$$\overline{Z_{t,k}} = \sum_{i=1}^{N_k} Z_t(k,i)^{(\varepsilon_k - 1)\gamma_k}$$

This statistic is proportional to the unweighted average of firm-level productivity (raised at a power) in sector k and is therefore related to the first moment of the firm's productivity distribution in that sector. Note that in sector k, there is an integer number of firms N_k therefore when the productivity of one firm changes this finite sum of productivity changes. More precisely, the elasticity of $\overline{Z_{t,k}}$ with respect to the productivity of firm i in sector k is $\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)} = (\varepsilon_k - 1)\gamma_k \frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} > 0$. If there was a continuum of firms rather than an integer number then this elasticity would always be zero.

The second statistic is related to the second moment of the firms' productivity distribution in sector k. It is the sum of the square of firms' productivity shares in $\overline{Z_{t,k}}$: the Herfindahl index of productivities in sector k:

$$\Delta_{t,k} = \sum_{i=1}^{N_k} \left(\frac{Z_t(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_{t,k}}} \right)^2$$

This statistic captures the dispersion of productivity across firms in a sector. The Herfindahl index is a widely used measure of concentration. Note that this Herfindahl index is among firm level productivity and therefore not directly observable. Because of the finite number of firms in sector k, when the productivity of firm i in sector k changes then the concentration measure $\Delta_{t,k}$ changes too: $\frac{\partial \log \Delta_k}{\partial \log Z(k,i)} = \frac{2}{\Delta_k} \left(\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} - \Delta_k \right) \frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$. Note that the elasticity of $\Delta_{t,k}$ with respect to $Z(k,i)$ can be positive or negative depending on the productivity level of firm i in sector k relative to the concentration measure $\Delta_{t,k}$. This is very intuitive, because when the productivity of a "large" firm increases, i.e for $\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} > \Delta_k$, the concentration of productivity increases. Conversely, when the productivity of a "small" firm increases, i.e for $\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} < \Delta_k$, the concentration of productivity decreases.

Before describing the dynamics of these statistics under Assumption 1, I show that these two statistics are sufficient to characterize the equilibrium allocation under Assumption 2.

3.2 Sector's Allocation

In this subsection, I solve for the sector-level allocation. I start by defining the sector-level markup and productivity before characterizing the sector-level allocation under Assumption 2. Since firms' decisions are static, I abstract from the time t subscript in this section.

Markup: An important variable is the sector-level markup. This markup is defined as the sector-level price divided by the sector-level marginal cost. For a given sector k, the marginal cost is defined as $\lambda_k = \frac{dTC_k}{dY_k}$ where TC_k is the total cost in sector k: $TC_k = \sum_{i=1}^{N_k} \lambda(k,i) y(k,i)$. Note that in the context of constant return to scale the marginal cost is also equal to the average cost therefore $\lambda_k = \frac{TC_k}{Y_k} = \sum_{i=1}^{N_k} \lambda(k,i) \frac{y(k,i)}{Y_k}$. After using the fact that firm-level price is a markup over the marginal cost, it is easy to see that the sector-level markup μ_k is

$$\mu_k = \frac{P_k}{\lambda_k} = \left(\sum_{i=1}^{N_k} \mu(k,i)^{-1} s(k,i) \right)^{-1} \tag{1}$$

The sector's markup is a sales share weighted harmonic average of firm level markups. This expression is valid as long as firms charge a markup over the marginal cost. It therefore applies to any of the market structure assumed here: monopolistic, Bertrand and Cournot.
Proposition 4 shows that the sector-level markup is a function of sector-level concentration index. Especially, the directly observable Herfindahl-Hirchman-Index (HHI), the sum of the sales share squared, plays an important role.

Proposition 4 *(**Sector Level Markup**) The sector k's markup is equal to*

$$\mu_k = \begin{cases} \frac{\varepsilon_k}{\varepsilon_k - 1} & \text{Under Monopolistic competition} \\ \frac{\varepsilon_k}{\varepsilon_k - 1} \left(1 - \frac{1}{\varepsilon_k - 1} \sum_{m=2}^{\infty} \left(\frac{\varepsilon_k - 1}{\varepsilon_k} \right)^{m-1} (HK_k(m))^m \right)^{-1} & \text{Under Bertrand competition} \\ \frac{\varepsilon_k}{\varepsilon_k - 1} (1 - HHI_k)^{-1} & \text{Under Cournot competition} \end{cases}$$

where $HHI_k = \left(\sum_{i=1}^{N_k} s(k,i)^2 \right)$ is the sector k's Herfindahl-Hirchman-Index (HHI), and $HK_k(m) = \left(\sum_{i=1}^{N_k} s(k,i)^m \right)^{1/m}$ is the Hannah and Kay (1977) concentration index. NB: $HK_k(2)^2 = HHI_k$ the HHI is the square of the second Hannah and Kay (1977) concentration index.

Proof *See Appendix A.2.* □

The above proposition shows that under monopolistic competition the sector-level markup is constant and equal to the firm-level markup. This is obvious since the sector's markup is an average of firms' markups and under monopolistic competition all the firms in a given sector charge the same markup. As soon as pricing becomes strategic, under Bertrand or Cournot competition, the sales share distribution in the sector plays a crucial role. Under Cournot competition for example, the Herfindahl-Hirchman-Index entirely determines the sector's markup. The intuition is as follows, when the sector's concentration is high, i.e the Herfindahl-Hirchman-Index is high, large firms have a higher market share and thus they can use this higher market power to charge higher markups which in turn aggregate to a higher sector's markup. An important implication of Proposition 4 is that it links empirically observable variables, such as the Herfindahl-Hirchman-Index, to the sector level markup.

Using the result in the above proposition, it is easy to derive some comparative statics of the markup with respect to the Herfindahl-Hirchman-Index while keeping everything else constant.

$$\frac{\partial \mu_k}{\partial HHI_k} = \begin{cases} 0 & \text{Under Monopolistic competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k^2}\mu_k^2 > 0 & \text{Under Bertrand competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k}\mu_k^2 > 0 & \text{Under Cournot competition} \end{cases}$$

Under Bertrand and Cournot competition, a higher sector's Herfindahl-Hirchman-Index always implies a higher sector's markup. This relationship is stronger for low competitive, high markup sectors. The sensitivity of the sector's markup to the sector's Herfindahl-Hirchman Index is stronger under Cournot than under Bertrand competition. In this framework, given the demand system and the assumed market structure, sector concentration is a measure of sector competition.

Productivity: The other important variable to define is the sector-level productivity. This is defined as the sector-level labor-augmenting productivity. As it was shown above, the sector-level marginal cost is $\lambda_k = \sum_{i=1}^{N_k} \lambda(k,i)\frac{y(k,i)}{Y_k}$. After substituting for the firm-level marginal cost $\lambda(k,i) = Z(k,i)^{-\gamma_k} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}$, the sector-level marginal cost is equal to $\lambda_k = Z_k^{-\gamma_k} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}$ where

$$Z_k^{-\gamma_k} = \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} \frac{y(k,i)}{Y_k}$$

is the sector k's (labor-augmenting) productivity, an output weighted sum of firm-level productivity in sector k. It is entirely determined by the joint distribution of output and productivities across firms in a sector, while the sector-level markup was entirely determined by the distribution of sales share.

Allocation: The previous results were relating endogenous variables with each other and were not linking equilibrium allocation to the state variable in this economy. Proposition 5 first solves for the sector allocation given the sectors' markup and productivity, before explicitly describing how the two statistics $\overline{Z_k}$ and Δ_k entirely characterize the sector-level variables under Assumption 2.

Proposition 5 *(**Sector Allocation**) Sectors' prices are equal to:*

$$\{\log P_k\}_k = (I-\Omega)^{-1}\left\{\log \mu_l \left(\frac{w}{Z_l}\right)^{\gamma_l}\right\}_l \tag{2}$$

where the $(N\times N)$*-matrix* Ω *is such that* $\Omega = \{\omega_{k,l}\}_{1\leq k,l\leq N}$ *and* I *is the* $(N\times N)$*- identity matrix. Sectors' sales share are equal to:*

$$\left\{\frac{P_kY_k}{P^CC}\right\}_k' = \beta'(I-\widetilde{\Omega})^{-1} \tag{3}$$

where the $(N\times N)$*-matrix* $\widetilde{\Omega}$ *is such that* $\widetilde{\Omega} = \{\mu_k^{-1}\omega_{k,l}\}_{1\leq k,l\leq N}$ *and the* $(N\times 1)$*- vector* β *is such that* $\beta = \{\beta_k\}_k$*. Under Assumption 2, sector* k*'s markup and productivity are equal to:*

$$\mu_k = \frac{\varepsilon_k}{\varepsilon_k - f_k(\Delta_k)} \quad \textit{and} \quad Z_k = \left(\overline{Z_k}\right)^{\frac{1}{\gamma_k(\varepsilon_k-1)}}\left(f_k(\Delta_k)\right)^{\frac{-1}{\gamma_k(\varepsilon_k-1)}}\left(\frac{\varepsilon_k - f_k(\Delta_k)}{\varepsilon_k-1}\right)^{\frac{-1}{\gamma_k}}$$

where

$$f_k(x) = \begin{cases} 1 & \textit{Under Monopolistic Competition} \\ \frac{1-\sqrt{1-4\left(1-\varepsilon_k^{-1}\right)x}}{2\left(1-\varepsilon_k^{-1}\right)x} \quad \textit{for } x\in\left[0,\frac{1}{4\left(1-\varepsilon_k^{-1}\right)}\right] & \textit{Under Bertrand Competition} \\ \frac{1-\sqrt{1-4(\varepsilon_k-1)x}}{2(\varepsilon_k-1)x} \quad \textit{for } x\in\left[0,\frac{1}{4(\varepsilon_k-1)}\right] & \textit{under Cournot Competition} \end{cases}$$

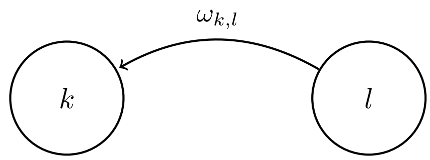
Proof *See Appendix A.3.* □

The above proposition characterizes the sectors' allocation for a given wage w. The System 2 of N equations relates the sectors' prices with sectors' productivities Z_l, sectors' markups μ_l, wage w and the input-output matrix Ω. To understand these equations, let us assume that there is no input-output linkages, i.e $\Omega = 0$ and $\gamma_k = 1$. In this case the price in sector k is just the sector's markup μ_k over the marginal cost in this sector $(w/Z_k)^{\gamma_k}$, which is standard under imperfect competition. Now let us assume that the input-output structure is the one described in Figure 4, i.e sector k is using labor and sector l' good to produce, while sector l is using only labor as input. Under imperfect competition, the price in sector k is equal to the sector's markup μ_k over the marginal cost. However, the sector k's marginal cost is $\left(\frac{w}{Z_k}\right)^{\gamma_k}(P_l)^{\omega_{k,l}}$, a combination of the marginal cost of labor and the price of the upstream sector l's good. The sector l's price is itself equal to the markup in sector l over the marginal cost of labor in sector l: $P_l = \mu_l\left(\frac{w}{Z_l}\right)$. To solve for the prices of sector k and l, one just need to solve a system of two unknowns and two equations. The System 2 is a generalization of this reasoning for any input-output network Ω.[13]

The System 3 of N equations solves for the sectors' sales share as a function of the household expenditure share β, the markups and the input-output network through $\widetilde{\Omega}$. To understand the intuition

[13] For the case of perfect competition, i.e for $\mu_k = 1$, an implication of the system of Equations 2 is that the sector's price is equal to the marginal cost of labor used directly and indirectly, through the input-output network.

Figure 4: A Simple Input-Output Structure



NOTE: In this simple input-output structure, firms in sector k are using labor and sector l's good to produce their variety, while firms in sector l are using only labor.

behind this matrix, let us assume that the input-output structure is the one described in Figure 4 and let us compute the income share that sector l captures from a dollar spend on sector k's good. Some of that dollar, a share $1-\mu_k^{-1}$, is rebated directly as profit to the household by sector k, the remaining is used to pay for inputs among which sector l's good. Therefore, sector l receives a share $\mu_k^{-1}\omega_{k,l}$ of this dollar. For any input-output network, the element (k,l) of the matrix $\widetilde{\Omega}$ is the share of income that flows directly from sector k to sector l and is equal to $\mu_k^{-1}\omega_{k,l}$. Equations 3 shows that the sales share of a sector is given by the vector $\beta'(I-\widetilde{\Omega})^{-1} = \beta' + \beta'\widetilde{\Omega} + \beta'\widetilde{\Omega}^2 + \ldots$ which captures the fact that the total sales of a sector is the sum of the direct and indirect sales to the household. A given sector receives income directly from the sales to the household. This is captured by the term β' in Equations 3. In addition, this sector's good is also sold to its downstream sectors that used it as inputs and serve the household. This first-degree indirect income is equal to the term $\beta'\widetilde{\Omega}$ in Equations 3. Furthermore, these downstream sectors' good are also used as inputs by their own downstream sectors that sell to the household. This second-degree indirect income share is equal to the term $\beta'\widetilde{\Omega}^2$ in Equations 3. Higher degree indirect income share are captured in the same way by the remaining terms. The sales share of a given sector is therefore the infinite sum of these terms which is then equal to the product of the household expenditure share β' and the Leontieff inverse of the matrix of income flow $\widetilde{\Omega}$.

While the first part of Proposition 5 (Equations 2 and 3) does not need any specific assumption, the rest of this proposition shows that under Assumption 2 the sectors' markups and productivities are entirely determined by the two statistics $\overline{Z_k}$ and Δ_k. Under this assumption, all the firm heterogeneity is summarized by these two statistics. Furthermore, under oligopolistic competition, the sector's markup is increasing in the concentration measure Δ_k. This is very intuitive. In a given sector when firms' productivities concentration is higher, the most productive firms have even more market power. It follows that the markup charged by these firms is even higher, which is reflected in a higher sector-level markup. An interesting results is that when the concentration measure Δ_k is converging to zero, i.e. when firms become homogeneous, the markup μ_k is converging to $\varepsilon_k/(\varepsilon_k-1)$ i.e. the markup under monopolistic competition. The same is true for the sector-level productivity Z_k which converges, when Δ_k goes to zero, to $\left(\overline{Z_k}\right)^{\frac{1}{\gamma_k(\varepsilon_k-1)}}$ i.e. the sector-level productivity under monopolistic competition. Therefore the concentration measure Δ_k is capturing the intensity of competition in a sector and how much this sector market structure deviates from the Dixit and Stiglitz

(1977) monopolistic competition.

Proposition 5 is important in three ways. First, this proposition solves for sector-level allocation given an equilibrium wage, nominal output, and sector-level productivities and markups. Second, it reduces firm's heterogeneity at the sector level by showing that under Assumption 2 the two statistics $\overline{Z_k}$ and Δ_k are sufficient to describe the sector level allocation. Third, it gives a natural and simple interpretation to the concentration measure of productivity Δ_k which can be think of as a measure of the competition intensity in sector k.[14]

3.3 Dynamics

In the section 3.2, the two statistics $\overline{Z_{t,k}}$ and $\Delta_{t,k}$ have been shown to entirely described the sector-level allocation under Assumption 2. In this section, I show that the dynamics of these two sufficient statistics can be summarized by a simple stochastic process under random growth at the firm-level (Assumption 1). Below, I solve for the law of motion of the firm productivity distribution before turning to the dynamics of the two statistics $\overline{Z_{t,k}}$ and $\Delta_{t,k}$.

The first step is to solve for the dynamics of the distribution of productivity in each sector, the state variables of this model. Let us define the vector $g_t^{(k)} = \{g_{t,n}^{(k)}\}_{0\leq n\leq M_k}$ where $g_{t,n}^{(k)}$ is the number of firms at productivity level φ^n at time t in sector k. The vector $g_t^{(k)}$ is thus the firm's productivity distribution at time t in sector k.[15] Recall that in sector k there is an integer number of firms N_k, following Carvalho and Grassi (2017), this assumption implies that the productivity distribution is a stochastic object. To understand the intuition behind this result, let us study a simple example.

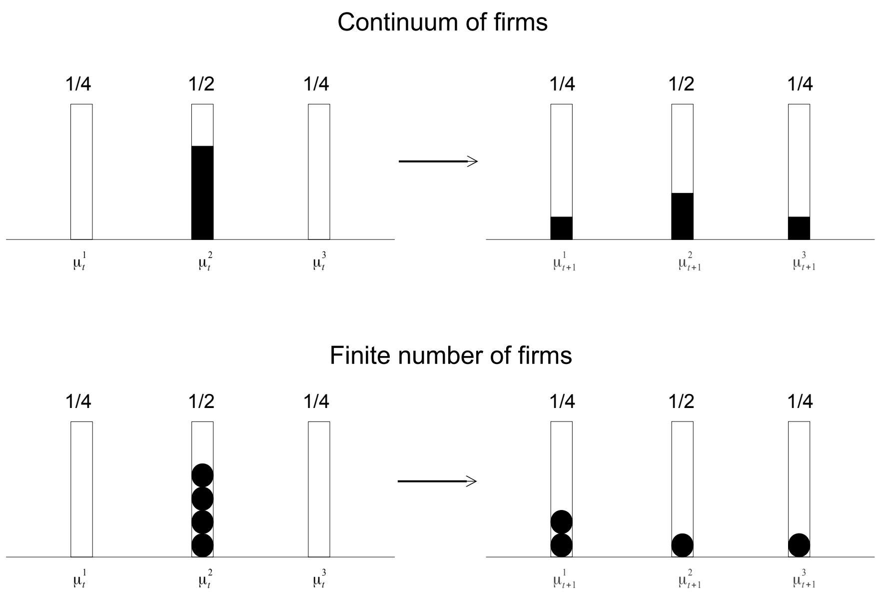
Assume there are only three levels of productivity and four firms. At time period t these firms are distributed according to the bottom-left panel of Figure 5, i.e. all four firms produce with the intermediate level of productivity. Further assume that these firms have an equal probability of $1/4$ of going up or down in the productivity ladder and that the probability of staying at the same intermediate level is $1/2$. That is, the transition probabilities are given by $(1/4, 1/2, 1/4)'$. First note that, if instead of four firms we had assumed a continuum of firms, the law of large numbers would hold such that at $t+1$ there would be exactly $1/4$ of the (mass of) firms at the highest level of productivity, $1/2$ would remain at the intermediate level and $1/4$ would transit to the lowest level of productivity (top panel of Figure 5). This is not the case here, since the number of firms is finite. For instance, a distribution of firms such as the one presented in the bottom-right panel of Figure 5 is possible with a positive probability. Of course, many other arrangements would also be possible outcomes. Thus, in this example, the number of firms in each productivity bin at $t+1$ follows a multinomial distribution with a number of trials of 4 and an event probability vector $(1/4, 1/2, 1/4)'$.

In this simple example, all firms are assumed to have the same productivity level at time t. It is easy however to extend this example to any initial arrangement of firms over productivity bins. Indeed,

[14] The measure of competition intensity Δ_k is fundamentally different from other measure of competition intensity used in macroeconomics as in Aghion et al. (2014). Indeed, Δ_k is also measuring the dispersion of firm-level productivity in a sector, a fundamental of the economy.

[15] Since firm's productivity evolves on a discrete space, the vector $g_t^{(k)}$ is the histogram of firm's productivity at time t in sector k.

Figure 5: An illustrative example of the productivity distribution dynamics



NOTE: Top panel, with a continuum of firms the transition is deterministic. Bottom Panel, with a finite number of firms the transition is stochastic.

for any initial number of firms *at a given* productivity level, the distribution of *these* firms across productivity levels next period follows a multinomial. Therefore, the *total* number of firms in each productivity level next period, is simply a sum of multinomials, i.e. the result of transitions from *all* initial productivity bins. The following proposition generalizes this example to determine the dynamics of the distribution of firms' productivity for any firm-level productivity process.

Proposition 6 ***(Sector k's Productivity Distribution Dynamics)*** *The Sector k's Productivity Distribution satisfies the following law of motion*

$$g_{t+1}^{(k)} = (\mathcal{P}^{(k)})' g_t^{(k)} + \epsilon_t^{(k)} \tag{4}$$

where $\mathcal{P}^{(k)}$ is the matrix of transition probabilities of the firm-level productivity process in sector k and where $\epsilon_t^{(k)} = \left\{\epsilon_{t,n}^{(k)}\right\}_{0\leq n\leq M_k}$ is a mean zero random vector.
Furthermore, under Assumption 1, the stationary distribution (when $\forall t, \epsilon_t^{(k)} = 0$) is Pareto and equal to $g_n^{(k)} = N_k K_k (\varphi_k^n)^{-\delta_k}$ where K_k is a normalization constant and $\delta_k = \log \frac{a_k}{c_k} / \log \varphi_k$ is the tail index.

Proof *See Online Appendix F.2.* □
The above proposition described the law of motion of a sector's productivity distribution and capture exactly the intuition of the example in Figure 5. The first term of the right hand side of the law of motion 4 is the *average* behavior of the sector's productivity distribution. If there were an infinite number of firms, this *average* behavior would be exactly the next period sector's productivity distribution. Note that this term is solely a function of the current period productivity distribution $g_t^{(k)}$

and the transition probabilities of the firm level productivity process $\mathcal{P}^{(k)}$. Because there are a finite and integer number of firms in each sector, there is a extra term, $\epsilon_t^{(k)}$. This second term is the deviation of the actual realization of $g_{t+1}^{(k)}$ and its average behavior. As described in the example above, for a given period t, this random vector is a sum of demeaned multinomial random vectors.

A direct implication of this proposition is that sectors' productivity distribution, the state variables of this framework, are stochastic vectors. It follows that every sectors' variables are themselves stochastic and fluctuate. Finally, these sectors' productivity distribution hover around their stationary value which are, under Assumption 1, Pareto distributed with a tail index determined by the probabilities a_k and c_k.[16]

Note that no aggregate or sector-level shocks are assumed, instead this sector (and aggregate) level fluctuations arise from independent firm-level shocks. The quantitative importance of such fluctuations is not formally discussed here and is addressed numerically below where the above model is calibrated to the US economy. However, the diversification among these firm-level independent shocks is weak as soon as the stationary Pareto distributions is fat-tail. As it is shown by Gabaix (2011), when it exists a small number of very productive firms, it is very unlikely that shocks to these firms cancel out and therefore they translate into quantitatively important fluctuations.[17]

After the characterization of the dynamics of the sector's productivity distribution (Proposition 6), the second step is to describe the law of motion of the two statistics $\overline{Z_{t,k}}$ and $\Delta_{t,k}$ that are sufficient under Assumption 2. Proposition 7 below shows that under random growth (Assumption 1) the law of motion of these two statistics can be described by a simple process.

Proposition 7 *(**Dynamics of $\overline{Z_{t,k}}$ and** $\Delta_{t,k}$) Under Assumption 1, the two statistics $\overline{Z_{t,k}}$ and $\Delta_{t,k}$ of the sector k's productivity distribution satisfy the following dynamics:*

$$\overline{Z_{t+1,k}} = \rho_k^{(Z)}\overline{Z_{t,k}} + o_{t,k}^{(Z)} + \sqrt{\varrho_k^{(Z)}\Delta_{t,k} + O_{t,k}^{(Z)}} \quad \overline{Z_{t,k}} \quad \varepsilon_{t+1,k}^{(Z)}$$

$$\left(\frac{\overline{Z_{t+1,k}}}{\overline{Z_{t,k}}}\right)^2 \Delta_{t+1,k} = \rho_k^{(\Delta)}\Delta_{t,k} + o_{t,k}^{(\Delta)} + \sqrt{\varrho_k^{(\Delta)}\chi_{t,k} + O_{t,k}^{(\Delta)}} \quad \Delta_{t,k} \quad \varepsilon_{t+1,k}^{(\Delta)}$$

where $\varepsilon_{t+1,k}^{(Z)}$ and $\varepsilon_{t+1,k}^{(\Delta)}$ are random variables following a $\mathcal{N}(0,1)$ with a non-zero covariance and where $\chi_{t,k}$, $o_{t,k}^{(Z)}$, $o_{t,k}^{(\Delta)}$, $O_{t,k}^{(Z)}$ and $O_{t,k}^{(\Delta)}$ are predetermined at time t, while, $\rho_k^{(Z)}$, $\rho_k^{(\Delta)}$,$\varrho_k^{(Z)}$ and $\varrho_k^{(\Delta)}$ are constant.

Proof *See Online Appendix F.3.* □

Proposition 7 is similar to Theorem 2 of Carvalho and Grassi (2017). It shows that the dynamics of the two moments of the sector k's productivity distribution are persistent. The intuition is that since the firm-level productivity is itself persistent, this persistence is aggregated at the sector level. The higher is the firm-level persistence, higher is the sector-level persistence as shown in Carvalho and

[16] The concept of a stationary distribution is the same as in Hopenhayn (1992) and Hopenhayn and Prescott (1992).

[17] In a one sector model with perfect competition and entry/exit à la Hopenhayn (1992), Carvalho and Grassi (2017) study the behavior for an increasingly large number of firms of the volatility arising from idiosyncratic independent shocks on firms.

Grassi (2017).[18]

Moreover, the (conditional) variance of the sum of the productivity of sector k's firms, $\overline{Z_{t,k}}$, is time varying and is determined by the Hearfindahl index of firms' productivity $\Delta_{t,k}$. Here as in Gabaix (2011) and Carvalho and Grassi (2017), any volatility at the sector level is due to idiosyncratic shocks at the firm level. When a sector is concentrated, shocks to firms with a large productivity do not wash out at the aggregate level. A higher concentration implies a higher importance of these firms and thus more volatility due to idiosyncratic shocks.

4 Structural Firms

In this section, I show that the structural importance of a firm is determined by the firm's size, the firm's sector industrial organization and its role in the input-output network. The structural importance of a firm is defined here as the elasticity of aggregate output with respect to the productivity of one firm in one sector.

In order to compute this elasticity, the first step is to solve for the aggregate output and the equilibrium wage. I assume that the household supplies inelastically one unit of labor and I normalized the price of the composite consumption good to one. The following proposition describes the equilibrium allocation given sector-level markups and productivities.

Proposition 8 *(**Equilibrium Allocation**) For given sector-level markups μ_k and productivities Z_k, the wage is*

$$\log w = -\beta'(I-\Omega)^{-1}\left\{\log \mu_k Z_k^{-\gamma_k}\right\}_k = -\sum_{k=1}^{N}\overline{\beta_k}\log \mu_k Z_k^{-\gamma_k} \tag{5}$$

where β is the $(N\times 1)$-vector of the household expenditure share $\{\beta_k\}_k$ and $\left\{\overline{\beta_k}\right\}_k' = \beta'(I-\Omega)^{-1}$ are the sectors' centrality. The share of aggregate profit in nominal output is

$$\frac{Pro}{P^C C} = \beta'(I-\widetilde{\Omega})^{-1}\left\{1-\mu_k^{-1}\right\}_k = \sum_{k=1}^{N}\beta_k\left(1-\widetilde{\mu_k}^{-1}\right) \tag{6}$$

where $\widetilde{\Omega} = diag\left(\left\{\mu_k^{-1}\right\}_k\right)\Omega$ with $diag\left(\{x_k\}_k\right)$ is the diagonal matrix whose non-zero elements are the x_k and where $\widetilde{\mu_k}$ is such that $\left\{1-\widetilde{\mu_k}^{-1}\right\}_k = (I-\widetilde{\Omega})^{-1}\left\{1-\mu_l^{-1}\right\}_l$. Finally, aggregate output is

$$\log Y = \log w - \log\left(1-\frac{Pro}{P^C C}\right) \tag{7}$$

Proof *See Appendix A.4.* □

The equilibrium wage of Equation 5 comes from sectors' price (Equations 2) and the normalization $P^C = 1$. Note that the log of wage can be rewritten as a weighted sum of sector level markups and

[18] For $n\in\mathbb{N}^*$, the sequences $v_{k,n} = a_k\varphi_k^{-n(\varepsilon_k-1)\gamma_k} + b_k + c_k\varphi_k^{n(\varepsilon_k-1)\gamma_k}$ and $w_{k,n} = a_k\varphi_k^{-2n(\varepsilon_k-1)\gamma_k} + b_k + c_k\varphi_k^{2n(\varepsilon_k-1)\gamma_k} - (\rho_k^{(n)})^2$ are respectively the mean and variance of the growth rate of firm i in sector k productivity measure $Z(k,i)^{n(\varepsilon_k-1)\gamma_k}$. We have that, $\rho_k^{(Z)} = v_{k,1}$, $\rho_k^{(\Delta)} = v_{k,2}$, $\varrho_k^{(Z)} = w_{k,1}$ and $\varrho_k^{(\Delta)} = w_{k,2}$.

productivities where the weights are $\{\overline{\beta_k}\}_k' = \beta'(I-\Omega)^{-1} = \beta'(I+\Omega+\Omega^2+\ldots)$ the sectors' *centrality*. The centrality measures the direct and indirect importance of a sector in the household consumption bundle. A sector's good contributes to the consumption bundle by the direct consumption of this good by the household. This is governed by the shares β. This good is also used as input by other sectors that are themselves consumed by the household. This first-degree indirect contribution to the household consumption bundle is captured by the therm $\beta'\Omega$. Furthermore, this good is also used as inputs for other goods that are themselves inputs of other goods that are consumed by the household. This second-degree indirect importance is captured by the term $\beta'\Omega^2$. Higher degree linkages are captured in the same way. The centrality is then the infinite sum of these terms which is then equal to the product of the share β and the Leontieff inverse $(I-\Omega)^{-1}$. The centralities $\overline{\beta}_k$ take into account the direct and indirect consumption of a sector's good through the input-output network.

The aggregate profit share is a function of sectoral markups and the input-output network. To understand the intuition behind Equation 6, let us compute the profit share of one dollar spend on sector k in the simple input-output network of Figure 4. The sector-level markup determines the profit share: a share $1-\mu_k^{-1}$ of this dollar is directly rebated to the household as profit. The remaining, μ_k^{-1}, is used to pay for inputs among which the sector l's good. Therefore, sector l receives $\mu_k^{-1}\omega_{kl}$ of income of the dollar spend on sector k, from which a share $1-\mu_l^{-1}$ is rebated to the household. The total profit rebated to the household of this dollar spend on sector k is then equal to $1-\mu_k^{-1}+\mu_k^{-1}\omega_{kl}\left(1-\mu_l^{-1}\right)$. Equation 6 is a generalization of this intuition to any input-output structure. The element $\mu_k^{-1}\omega_{k,l}$ of the matrix $\widetilde{\Omega}$ is the income share that goes from sector k to sector l. The Leontieff inverse of this matrix, $(I-\widetilde{\Omega})^{-1}$, gives the direct and indirect income share that goes from one sector to another while the vector $\left\{1-\mu_k{}^{-1}\right\}_k$ gives the income share of each sector that is directly rebated to the household. The aggregate profit share can also be rewritten as a weighted sum of the expenditure share β_k where the weights $1-\widetilde{\mu_k}^{-1}$ are the direct and indirect profit share of each sector k. Note also that $\widetilde{\mu_k}^{-1}$ is the direct and indirect labor share of each sector and it is such that $\{\widetilde{\mu_k}\}_k = (I-\widetilde{\Omega})^{-1}\{\gamma_k\mu_k^{-1}\}_k$.[19]

The aggregate output equation comes from the household budget constraint and the inelastic labor supply. Note that this equation will be different for different utility function. Appendix D derives the case of elastic labor supply for both separable and Greenwood–Hercowitz–Huffman (GHH) preferences. Under Assumption 2, the results in Propositions 8 and 5 describe entirely the equilibrium allocation as a function of the two sufficient statistics $\overline{Z_k}$ and Δ_k. The first part of Proposition 5 and Proposition 8 solve for the equilibrium allocation as a function of sector-level markups and productivities, while the second part of Proposition 5 gives the sectors' markup and productivity as a function of these two sufficient statistics.

Let us decompose the effect of an increase in productivity of one firm in one sector on aggregate output into the "downstream" and "upstream" part of aggregate output. The "downstream" part of aggregate output is defined as the first term of the right hand side of Equation 7: $\log Y^d = \log \frac{w}{P}L = \log w$. This is minus the (log) real labor income, since total labor and the composite good price are

[19] This can be shown using the definition of $\widetilde{\mu_k}^{-1}$ and the fact that $\widetilde{\Omega}\{1\}_k = diag(\{\mu_k^{-1}\}_k)\Omega\{1\}_k = \{\mu_k^{-1}(1-\gamma_k)\}_k$.

normalized to one. The "upstream" part is defined as $\log Y^u = -\log\left(1 - \frac{Pro}{P^C C}\right) = -\log\left(\frac{wL}{P^C C}\right)$ this is the (log of) the labor share. Therefore, this decomposition of aggregate output is just a decomposition in term of real labor income and the labor share. The terminology "downstream" comes from the fact that any change in a sector's price impacts the downstream sectors and is reflected in the wage. The "upstream" terms comes from the fact any change in markups and thus cost share impact the income share received by the upstream sectors and is ultimately reflected in the aggregate profit/labor share.[20] The elasticity of aggregate output to the productivity $Z(k,i)$ of firm i in sector k is then the sum of the effect on the "downstream" and "upstream" part of aggregate output:

$$\underbrace{\frac{\partial \log Y}{\partial \log Z(k,i)}}_{\text{change in GDP}} = \underbrace{\frac{\partial \log Y^d}{\partial \log Z(k,i)}}_{\text{change in labor income}} + \underbrace{\frac{\partial \log Y^u}{\partial \log Z(k,i)}}_{\text{-change in labor share}}$$

First, let us look at the effect of a change in productivity of one firm in one sector on the labor income. The change of the "downstream" part of aggregate output captures any change on the real wage. These changes are themselves due to changes on sectoral prices. Changes in sectoral prices propagate to downstream sector. To understand this, let us once again look at the simple input-output of Figure 4. Recall that in this simple case the sector k's price is

$$\log P_k = \log \mu_k + \log\left(\frac{w}{Z_k}\right)^{\gamma_k} + \omega_{k,l}\log P_l \quad \text{with} \quad \log P_l = \log \mu_l + \log\left(\frac{w}{Z_l}\right)$$

Following a change of the productivity of one firm in sector l, the two statistics $\overline{Z_l}$ and Δ_l are affected as described in Section 3.1. These changes affect the markup μ_l and the productivity Z_l in sector l (Proposition 5) which in turn affect the price in sector l. Any change in sector l's price impacts the marginal cost of the downstream sector k and therefore the price of sector k. Any shocks on firms in a sector propagate to downstream sectors through the price. This is ultimately affecting price of all downstream sectors and thus the real wage i.e the "downstream" part of aggregate output. The strength of this effect depends on (i) the pass-through in sector l, i.e on how much the sector l's price changes after the increase in productivity of one of its firms, and, on (ii) the input-output linkage between sector l and sector k. The market structure of the sector and the identity of the firms whose productivity increases determine the strength of the pass-through. Proposition 9 computes the elasticity of the "downstream" part of aggregate output with respect to the productivity of one firm in one sector for any input-output network.

Proposition 9 *(**Elasticity "downstream"**) Assume 2, the elasticity of the "downstream" part of aggregate output with respect to the productivity of firm i in sector k is*

$$\frac{\partial \log Y^d}{\partial \log Z(k,i)} = \frac{\overline{\beta_k}}{\varepsilon_k - 1}\left(1 + \frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}}\right)\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$$

where $e_k = \frac{d\log f_k}{d\log \Delta_k}$ and $\left\{\overline{\beta_k}\right\}'_k = \beta'(I-\Omega)^{-1}$ is the vector of sectors' centrality.

[20] See Section 5 for a study of propagation of firm-level shocks across sectors.

Proof *See Appendix A.5.* □

The elasticity of the "downstream" part of aggregate output with respect to the productivity of firm i in sector k is the product of three terms. The first term is the sector's centrality $\overline{\beta_k}$. As discussed earlier, the centrality measures the direct and indirect importance of a sector in the household consumption bundle. The second term captures the effect of oligopolistic competition, under monopolistic competition this term would be equal to one. Whenever the firm i in sector k is "large", i.e when $\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} > \Delta_k$, this second term is smaller than one. Because, when the productivity of a large firm increases some of the productivity gains translates in an increase of markup rather than a fall in price. Indeed, this firm already have a lot of market power and does not need to cut its price and increase its production by as much as under monopolistic competition: the pass-through is incomplete. At the sector level the price falls by less than under monopolistic competition and the effect on the "downstream" part of aggregate output is smaller. Conversely, if the productivity of a "small" firm increases, i.e when $\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} < \Delta_k$, the second term is larger than one. When the productivity of this firm increases, it decreases its price and increases its markup but also cuts the markups of larger firms. At the sector level, the price fall by more than under monopolistic competition and the effect on the equilibrium wage and the "downstream" part of aggregate output is stronger. The last term is the effect of the firm's increase in productivity on the first moment of the sector k productivity distribution. The more productive is the firm affected, the larger is this term.

The elasticity of the "downstream" part of aggregate output reflects the effect of the change in price on the real wage following an increase in $Z(k,i)$. It is easy to see this by rewriting the change in the "downstream" part of aggregate output as:[21]

$$\frac{\partial \log Y^d}{\partial \log Z(k,i)} = \overline{\beta_k}\left(\frac{1}{\varepsilon_k - 1}\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)} - \frac{\frac{\varepsilon_k}{\varepsilon_k-1} - 1}{\mu_k - 1}\frac{\partial \log \mu_k}{\partial \log Z(k,i)}\right) \tag{8}$$

The centrality $\overline{\beta_k}$ is the importance of the firm's sector good in the determination of the wage. The first term in the bracket is equal to the change in the monopolistic competition (complete pass-through) sector k's price following the increase in the productivity of firm i. The second term in the bracket is the change in sector's markup due to the change in firm i's productivity. This last term would be equal to zero under perfect competition. As described earlier, this term can be either negative or positive depending on the identity of the firm subject to the shock.

The structural importance of a firm for the "downstream" part of aggregate output is a function of the input-output network through the sector's centrality, the sector's market structure index by the Δ_k, and, the firm size through the term $\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$. The market structure and the firm size govern the change in sector's price following a change in the firm's productivity, while the input-output linkages determine the intensity of the effect of this change in sector's price on other sectors and on the equilibrium wage.

Let us now study the effect of a change in productivity of one firm in one sector on the labor share.

[21] Note that using the expression of the markup under Assumption 2 (Proposition 5) and the elasticity of Δ_k to $Z(k,i)$ in Section 3.1, it is easy to show that the elasticity of the markup is $\frac{\partial \log \mu_k}{\partial \log Z(k,i)} = -(\mu_k - 1)\frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}\right)$.

The change of the "upstream" part of aggregate output captures any change on the aggregate labor and profit income share. These changes are themselves due to changes on sectoral profit share. Changes in sectoral profit share affect the income received by the upstream sectors. To understand this, let us look at the profit share of a dollar spend on sector k's good in the simple input-output structure of Figure 4. In this simple example, the share of profit of one dollar spend on sector k's good is $1-{\mu_k}^{-1}+{\mu_k}^{-1}\omega_{kl}\left(1-{\mu_l}^{-1}\right)$. Following a shock on the productivity of firm i in sector k, the statistic Δ_k changes, let assume it increases. This increase in Δ_k increases μ_k, the markup in sector k (Proposition 5). As a consequence less income is used to pay for inputs among which sector l's good. The total share of profit/labor is affected because (i) the sector k rebates more profit to the household and (ii) the upstream sector l receives less income and therefore rebates less profit to the household. Proposition 10 generalizes the above intuition to any input-output structure.

Proposition 10 *(**Elasticity "upstream"**) Assume 2, the elasticity of the "upstream" part of aggregate output with respect to the productivity of firm i in sector k is*

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = -\frac{P^C C}{wL}\frac{P_k Y_k}{P^C C}\frac{(\mu_k-1)}{\widetilde{\mu_k}}\frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$$

where $e_k=\frac{d\log f_k}{d\log \Delta_k}$ is the elasticity of f_k, and where $\widetilde{\mu_k}$ is such that $\left\{1-\widetilde{\mu_k}^{-1}\right\}_k=(I-\widetilde{\Omega})^{-1}\left\{1-\mu_l^{-1}\right\}_l$ with $\widetilde{\Omega}=diag\left(\left\{\mu_k^{-1}\right\}_k\right)\Omega$.

Proof *See Appendix A.5.* □

The elasticity of the "upstream" part of aggregate output is the product of several terms. The first important term, $\widetilde{\mu_k}^{-1}$, is the cost share of sector k's income that is rebated as labor income to the household directly and indirectly through other sectors. The second important term is proportional to the change in sector's cost/profit share, $\frac{2e_k}{\Delta_k}\left(\Delta_k-\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}\right)\frac{\partial\log\overline{Z_k}}{\partial\log Z(k,i)}$. Under monopolistic competition, i.e when $\Delta_k\to 0$, this term is zero. Under oligopolistic competition, this term can be either positive or negative. Whenever the firm i in sector k is "large", i.e when $\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}>\Delta_k$, this term is negative and $\frac{\partial\log Y^u}{\partial\log Z(k,i)}$ becomes positive. This is very intuitive, when the productivity of a large firm increases this firms reduces its price but also uses its market power to raise its markup. At the sector level the profit share is higher which translates to a higher (resp. lower) aggregate profit share (resp. labor share) and a higher "upstream" part of aggregate output. Conversely, when the productivity of a "small" firm increases, i.e when $\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}<\Delta_k$, this term is positive and the elasticity of the "upstream" part of aggregate output is negative. This is because when the productivity of a "small" firm increases, this firm decreases its price and increases its markup but also cuts the markup of larger firms. At the sector level, the markup is reduced which translates in smaller (resp. larger) profit share (resp. labor share). The aggregate profit share (resp. labor share) is therefore reduce and so is the "upstream" part of aggregate output. The elasticity of the "upstream" part of aggregate output reflects the effect of the change in cost share on the aggregate labor share following an increase in

$Z(k,i)$. To see this, one can rewrite this elasticity as follow:

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = - \quad \frac{P_k Y_k \widetilde{\mu_k}^{-1}}{wL} \quad \frac{\partial \log(\mu_k^{-1})}{\partial \log Z(k,i)}$$

This expression shows that the effect on the aggregate labor income, the upstream part of aggregate output, is determined by sector k's direct and indirect labor share and the elasticity of sector k's cost share. Sector k's direct and indirect labor share is measured as a share of the total labor income by the sales of sector k, $P_k Y_k$, of which a share $\widetilde{\mu_k}^{-1}$ is rebated directly and indirectly as labor income. Sector k's cost share is μ_k^{-1}, because of oligopolistic competition this share is affected by changes in productivity of firms in sector k.

Both the market structure and the input-output network impact the propagation of firm level shocks on the "upstream" part of aggregate output. The markup centrality $\widetilde{\mu_k}$ is jointly determines by the input-output network and the competition intensity through the matrix $\widetilde{\Omega}$ whose elements gives the income share that flows between two sectors.

As a conclusion, the structural importance of a firm is determined by the firm size, the market structure and the input-output network. The size determines the influence of a firm on sector's price and profit share. The input-output network determines the importance of a sector in the consumption bundle and the aggregate profit/labor share. The sector's market structure interacts with both the firm size and the input-output network in shaping the structural importance of a firm. Indeed, with the firm size it governs the strength of the change in sector's price and profit share following a shock on one firm, and, with the input-output network it governs the importance of a sector for the aggregate profit/labor share.

The decomposition between the "downstream" and "upstream" part of aggregate output is valid because I assume an inelastic labor supply. In this case, any change in profit/labor share does not feed back to the wage by affecting the labor supply, while the effect of a wage increase on aggregate output is not magnified by an endogenous increase in labor supply. Relaxing the inelastic labor supply assumption won't affect the results but it affects the interpretation of the terms "downstream" and "upstream". As shown in Appendix D, with separable or GHH preferences, the output is still a function of the wage and the profit/labor share. They are themselves only a function of sectoral productivities and markups (Proposition 8) and therefore, under Assumption 2, a function of the statistics $\overline{Z_k}$ and Δ_k (Proposition 5).

5 Propagation

In this section, I show how a shock on one firm in one sector propagates to other sectors through the input-output network. The propagation to downstream sectors is due to change in price whose magnitude is governed by the competition intensity. The new propagation mechanism to upstream sectors is entirely due to the endogenous change in cost/profit share. To study the propagation of firm-level productivity shocks in the economy, I derive the elasticity of sector-level price (Proposition 11) and sales share (Proposition 12) with respect to the productivity of one firm in one sector. These results together with the elasticity of aggregate output derived in the previous section (Propositions 9 and 10) allow to derive the effect of an increase in productivity of one firm in a sector on sector-level output.

The effect of a change in productivity of one firm on other sectors' price is summarized in Proposition 11 by the elasticity of sector k's price with respect to the productivity of firm j in sector l.

Proposition 11 *(**"Downstream" Propagation**) Assume 2, the elasticity of the sector k's price with respect to the productivity of firm j in sector l is*

$$\frac{\partial \log P_k}{\partial \log Z(l,j)} = \frac{\partial \log w}{\partial \log Z(l,j)} - \frac{\psi^d_{k,l}}{\varepsilon_l - 1}\left(1 + \frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)\right)\frac{\partial \log \overline{Z_l}}{\partial \log Z(l,j)}$$

where $e_k = \frac{d\log f_k}{d\log \Delta_k}$ is the elasticity of f_k, and $\psi^d_{k,l}$ is the element (k,l) of the matrix $\psi^d = (I-\Omega)^{-1}$.

Proof *See Appendix A.6.* □

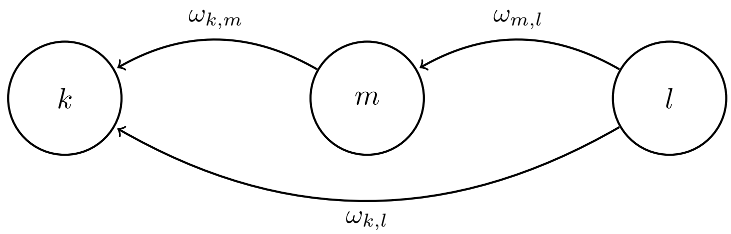
The change in sector k's price reflects the change in cost. It is the sum of the change in labor and intermediate goods cost. Following a change in productivity of firm j in sector l, the wage changes as it is described in the previous section and in Proposition 9. This is the structural importance of the firm j in sector l on the wage. This change in wage affects the cost of labor and thus sector k's price.[22]

The change in intermediate goods price is captured by the second term which is the product of (i) the direct and indirect exposure of sector k's production to sector l's good, (ii) the effect of oligopolistic competition in sector l, and, (iii) the effect of the change in productivity of firm j on sector l's. The degree of direct and indirect exposure of sector k's production to sector l's good is measured by $\psi^d_{k,l}$ the element k,l of the matrix $(I-\Omega)^{-1}$. In the simple input-output structure of Figure 4 this parameter is exactly equal to $\omega_{k,l}$. For a more general input-output structure, the number $\psi^d_{k,l}$ captures the dependence of sector k's production on sector l's good directly and through other sectors. For example, let us assume that the input-output network is as in Figure 6 i.e. sector k's is using sector l and sector m's goods to produce, and, sector m is also using sector l's good to produce. In this simple case, the direct and indirect exposure of sector k's production to sector l's good, $\psi^d_{k,l}$, is taking into account the direct consumption of sector l's good by sector k plus the indirect consumption of sector l's good through sectors m, as the latter is also using l to produce its good: $\psi^d_{k,l} = \omega_{k,l} + \omega_{k,m}\omega_{m,l}$.

The effect of oligopolistic competition is similar as the one described in the previous section, it measures the pass-through of the increase in productivity of firm j on sector l's price. Under monopolistic competition the term $1 + \frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)$ would be equal to one. Here, depending on the identity of the firm j, the response of sector l is larger or smaller than under monopolistic competition. If the productivity of a large firm increases, i.e $\frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}} > \Delta_l$, some of the increase in productivity translates in an increase in markup at the sector level and therefore the sector l's price

[22]Note that this change in wage affects all the sector in the economy and therefore it also indirectly affects sector k through its intermediate input consumption. This is the reason why there is no parameter γ_k, the labor share, in front of the term $\frac{\partial \log w}{\partial \log Z(l,j)}$.

Figure 6: An Example of Input-Output Structure



NOTE: In this simple input-output structure, firms in sector k are using labor, sector l and sector m's goods to produce their variety, firm in sector m are using labor and sector l's good, while firms in sector l are using only labor.

fall by less than under monopolistic competition. Conversely, if the increase in productivity affects a small firm, i.e $\frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}} < \Delta_l$, this firms is cutting the markup of its larger competitors and thus reduces the markup at the sector level. In this case, sector l's price falls by more than under monopolistic competition. One can see this more clearly by rewriting the elasticity of sector k's price as:

$$\frac{\partial \log P_k}{\partial \log Z(l,j)} = \frac{\partial \log w}{\partial \log Z(l,j)} - \psi^d_{k,l}\left(\frac{1}{\varepsilon_l - 1}\frac{\partial \log \overline{Z_l}}{\partial \log Z(l,j)} - \frac{\frac{\varepsilon_l}{\varepsilon_l-1} - 1}{\mu_l - 1}\frac{\partial \log \mu_l}{\partial \log Z(l,j)}\right)$$

The effect on sector k's price is stronger or smaller depending on the sign of the elasticity of sector l's markup with respect to $Z(l,j)$. As it is described above the sign of this elasticity is a function of the identity of firm j.

Without an input-output network, the term $\psi^d_{k,l}$ would be replace by one for $k = l$ and by zero otherwise i.e. a shock to one firm in a sector affects others sectors only through the effect on wage. With an input-output network but without oligopolistic competition, the sector l's markup would be constant i.e. $\frac{\partial \log \mu_l}{\partial \log Z(l,j)} = 0$. In that case, only the size of the firm would matter through $\frac{\partial \log \overline{Z_l}}{\partial \log Z(l,j)}$ and the market structure in the sector would be irrelevant.

Let us now look at the effect of a change in productivity of one firm on other sectors' sales share, this is summarized in Proposition 12 by the elasticity of sector k's sales share with respect to the productivity of firm j in sector l.

Proposition 12 ***("Upstream" Propagation)*** *Assume 2, the elasticity of the sector k's sales share with respect to the productivity of firm i in sector l is*

$$\frac{\partial \log\left(\frac{P_k Y_k}{P^C C}\right)}{\partial \log Z(l,j)} = (\psi^s_{l,k} - \mathbb{I}_{l,k})\frac{P_l Y_l}{P_k Y_k}(\mu_l - 1)\frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)\frac{\partial \log \overline{Z}_l}{\partial \log Z(l,j)}$$

where $e_k = \frac{d \log f_k}{d \log \Delta_k}$ is the elasticity of f_k, $\mathbb{I}_{l,k}$ is equal to one if $k = l$ and zero otherwise ,and, $\psi^s_{l,k}$ is the element (l,k) of the matrix $\psi^s = (I - \widetilde{\Omega})^{-1}$.

Proof *See Appendix A.7.* □

The change in sector k's sales share reflects the change in demand from sector l. The demand from sector l is determined by the total cost share of sector l and the exposure of the sector k to sector

l demand. Any change in the cost share is a change of the opposite sign of the profit share. For example, after an increase of the profit share, more income is rebated to the household as profit and less income is used to pay for inputs. Following an increase in productivity of firm j in sector l the profit share changes, and depending on the identity of firm j, it can increase or decrease. If j is large, i.e $\Delta_l < \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}$, the gain in productivity translates in an increase in markup and thus of the profit share. Conversely, if j is small, i.e $\Delta_l > \frac{Z(l,j)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}$, the increase in firm j's productivity cuts the markup of its larger competitors in sector l and the profit share at the sector level increases. The sector l's cost share is μ_l^{-1} and it is easy to rewrite the elasticity of the sector k's sales share as a function of the change in cost share:

$$\frac{\partial \log\left(\frac{P_k Y_k}{P^C C}\right)}{\partial \log Z(l,j)} = (\psi^s_{l,k} - \mathbb{I}_{l,k}) \frac{P_l Y_l}{P_k Y_k} \frac{\partial \log \mu_l^{-1}}{\partial \log Z(l,j)} \tag{9}$$

This elasticity is the product of three terms. The last two terms represent the change in total cost in sector l as a share of sector k's sales share. It is the product of sector l's income and the change in sector l's cost share following an increase in productivity of firm j in sector l. The first term represents the exposure of sector k to the direct and indirect demand of sector l. The number $\psi^s_{l,k}$, i.e the element (l, k) of the matrix $(I - \widetilde{\Omega})^{-1}$, is the share of sector l's income that goes to sector k directly or indirectly through other sectors. To understand this, let us assume that the input-output network is the simple one of Figure 6. In this case, the number $\psi^s_{k,l}$ takes into account the direct income share used by sector k to pay for the input of sector l's good. It also takes into account the indirect income share of sector k that goes to sector l through sector m since the latter is also using sector l's good to produce. In the simple case of Figure 6 we have $\psi^s_{k,l} = \mu_k^{-1}\omega_{k,l} + \mu_k^{-1}\omega_{k,m}\mu_m^{-1}\omega_{m,l}$.

Note that the sector k's sales share would be kept constant if there were no input-output network. In that case, $\psi^s = I$ and the first term of Equation 9 would be zero. Without an input-output network all the demand comes from the household who are spending a constant share of their income on each goods thanks to the Cobb-Douglas preferences. Even if the increase in productivity of firm j in sector l is affecting aggregate income (see Section 4) it is not affecting one sector more than the others. Note also that the sector k's sales share would be constant if monopolistic competition were assumed. In that case, the sector l's cost share would be fixed and the last term of Equation 9 would be zero. Without oligopolistic competition the change in productivity of firm j in sector l is not affecting the sector-level markup and the repartition of income between inputs and profit is fixed by the value of the parameter ε_k. The propagation mechanism of Proposition 12 requires both an input-output network and an endogenous market structure to operate.

Proposition 11 relates to the "downstream" propagation since sector l's shock is affecting sector k strongly for higher value of $\psi^d_{k,l}$. This is the element (k, l) of the matrix $(I - \Omega)^{-1}$ which measure the direct and indirect cost share of sector l's good in sector k production: the good is going from sector l to sector k i.e. sector k is downstream to sector l. Proposition 12 relates to the "upstream" propagation since shock to sector l is affecting sector k strongly for higher value of $\psi^s_{l,k}$. This is the element (l, k) of the matrix $(I - \widetilde{\Omega})^{-1}$ which measure the direct and indirect income share of sector

k's good in sector l production: the good is going from sector k to sector l i.e sector k is upstream to sector l. Note that here this notion of "downstream" and "upstream" are a generalization of the usual definition to take into account the indirect linkages between sectors.

6 Quantitative Results

In this economy all the aggregate uncertainty comes from the firm-level productivity stochastic process $Z_t(k, i)$. Since there is a finite number of firms in a sector, firm-level fluctuations translate into sector-level fluctuations. However, these firms play an oligopolistic competition game and take into account the impact of their decision on their sector. This results in incomplete pass-through of shocks to price and fluctuations in profit share. When the firm-level productivity process is assumed to follow random growth (Assumption 1) the two statistics $\overline{Z_{t,k}}$ and $\Delta_{t,k}$ are stochastic and follow $AR(1)$-type processes (Proposition 7). While under Assumption 2 and under oligopolistic competition, fluctuations in these statistics create fluctuations in sector-level markups and productivities according to Proposition 5. The origin of these sector-level fluctuations is "granular" (Gabaix, 2011) and they are due to the presence of large firms in a given sector. Sectors are linked through a "small world" input-output network (Figure 1) where there are a handful of hub-like sectors. Similar to Acemoglu et al. (2012) and Carvalho (2010), sector-level fluctuations do not average out and create sizable fluctuations in output as computed in Proposition 8.

In this section, I evaluate the quantitative importance of firm-level productivity shocks and oligopolistic competition in shaping the business cycle. To this end, I first calibrate the above framework to the US economy. I then simulate a path of firm-level productivity for each firms and I solve for the equilibrium allocation at each period. I then compute business cycle statics and I decompose aggregate volatility of output into fluctuations in labor income and labor share i.e. into the "downstream" and "upstream" part of aggregate output.

6.1 Calibration and Numerical Strategy

To calibrate this economy the first step is to choose preferences and deep parameters of the model. Consistently with the analysis in Section 4, I assume that labor supply is inelastic. Such assumption allows to perform the decomposition of aggregate output between its "downstream" and "upstream" part.[23] Appendix D shows how the results are affected by relaxing the inelastic labor supply assumption. The second important assumption is the choice of the parameter ε_k, here I choose this parameter to be equal to 5 in every sectors: $\forall k, \varepsilon_k = 5$. Even if this is a strong assumption, this value seems reasonable as this number has been estimated in the international trade literature to be between 3 and 9 (see Imbs and Mejean (2015) for example). Note that the estimates in the international trade literature are not necessary consistent with the above model as they are usually not assuming

[23]This assumption is also standard in the literature on the micro-origin of aggregate fluctuations and allow for comparison with Acemoglu et al. (2012), Baqaee (2016) or Baqaee and Farhi (2017a). Carvalho and Grassi (2017) assume elastic labor supply and their results should be compare with result in Appendix D.

Table 2: Baseline Calibration

Parameters	Value	Description	Target/Source
ε_k	5	substitution across firms	Monop Markup 1.25
N	389	# of sectors	BEA
N_k	578	median # firms in a sector	Census data
$\sum_k N_k$	5 576 852	total # of firms	Census data
Δ_k	0.037	median Pdty Herfindahl	sales HHI of the Census
β_k	0.027	median HH consumption share (%)	BEA
γ_k	55.90	median labor share (%)	BEA
Ω	2.19	I-O network density (when links> 1%, %)	BEA
a_k, c_k	0.34,0.30	median Firm-level pdty process	$\sigma_k = 0.1$ and Δ_k

NOTE: The first column gives the notation of the parameter in the model. The third column gives the description of the value in the second column. The fourth column is the data source or the calibration target associated.

oligopolistic competition. An elasticity of 5 across varieties within a sector implied a sectoral markup for monopolistic competition of 1.25. Finally, the sector competition is assumed to be differentiated Bertrand.

The second step consists to use concentration and input-output data to discipline the sector-level parameters. For this calibration, a sector is an industry as defined by the Bureau of Economic Analysis (BEA) in their detailed input-output classification. There are 389 sectors and the level of disaggregation is comparable for most sectors to the 5 digits NAICS classification. This is the most disaggregated level available with sectoral input-output linkages information. The U.S. Census Bureau gives information on sector-level concentration. I use the 2007 vintage of these data and especially the Herfindahl-Hirchman-Index (HHI) of sales share among the top 50 firms, namely HHI_k.[24] Under Assumption 2, it is easy to show that $\Delta_k = f_k^{-1}\left(\left(1-\varepsilon_k^{-1}\right)HHI_k + 1\right)$. I am using this relationship and the value of ε_k to back out the concentration measure of productivity at the sector level Δ_k. With these values in hand, I calibrate the Markov chain of the productivity process $\mathcal{P}^{(k)}$ to match a firm level volatility of $\sigma_k = 0.1$ and the value of Δ_k. The firm-level volatility is at the lower hand of estimate in the firm dynamics literature (see Foster et al. (2008) or Castro et al. (2015) for example). Note, however, that the literature's estimates are not necessarily consistent with the oligopolistic competition assumption in the framework presented in this paper. The matrix Ω is calibrated using the latest vintage of the detailed input-output data of the BEA for 2007. For a description of the data see the data Appendix B. The model counterpart of the input-output table provided by the BEA is $\widetilde{\Omega}$ whose elements are the share of *income* that goes from one sector to the other. Using the concentration, I compute the sector-level markup and I use the relation between Ω and $\widetilde{\Omega} = diag(\{\mu_k^{-1}\}_k)\Omega$ to recover the actual Ω whose elements are the share of *total cost* that goes from one sector to the other. As one can see in Figure 1 and as it has been shown in Acemoglu et al. (2012) or Carvalho (2014) this input-output network is a "small world" network where a handful of sector are heavily connected to the other sectors.

[24] These data are described in Appendix B.

Table 2 summarizes the parameters of the baseline calibration. In this calibration there are $N = 389$ sectors, the median number of firms in each sector is 578 while the total number of firms is equal to almost 5.6 millions firms. The median value of Δ_k across sector is 0.037 which implies a value of the HHI of 0.063. Note that merger law starts to apply in the US for a value of the HHI over 0.18. The value of the median markup is about 1.27. Under monopolistic competition this markup would be 1.25 for a value of $\varepsilon_k = 5$. In this calibration the median sector is relatively close to a sector under monopolistic competition, it reflects the conservatism of the baseline calibration. Finally, the input-output network has a density of 2.19% and is very sparse i.e. 2.19% of all the possible $N^2 = 151321$ links have a value higher than 1%.

6.2 Aggregate Volatility and Variance Decomposition

For each of the 5.6 millions firms, I simulate a path of productivity of 4000 periods.[25] To do so I use the Proposition 6 and simulate the law of motion of the productivity distribution for each sector. As in Carvalho and Grassi (2017), I follow the number of firms in each productivity bins rather than following the path of each firms. It reduces considerably the computation cost of simulations. Note that even if Assumption 2 is key for this calibration strategy by allowing the mapping between the HHI and the productivity concentration measure Δ_k, this assumption is not necessary to solve for the equilibrium allocation given the firm productivity distribution at each period. Therefore, for each period t, I solve for the full problem at the firm level from which I recover sector-level markups and productivities that I aggregate in Y_t using the Proposition 8.[26] For this 4000 periods time series, I compute aggregate volatility measured by the standard deviation of the percentage deviation of aggregate output Y_t. In Table 3, I decompose the variance of aggregate output between the contribution of the labor income and the labor share i.e the "downstream" and "upstream" part of aggregate output as in Section 4:

$$\mathbb{V}ar\Big[\log Y_t\Big] = \underbrace{\mathbb{V}ar\Big[\log Y_t^d\Big]}_{\text{variance of labor income}} + \underbrace{\mathbb{V}ar\Big[\log Y_t^u\Big]}_{\text{variance of labor share}} + \underbrace{COV}_{\text{covariance term}} \tag{10}$$

The first result is that the standard deviation of aggregate output Y_t is 0.62%. The same number in the Fernald (2014) data is 1.83%.[27] So the aggregate volatility in this model is $0.62/1.83 = 33.88\%$ of the aggregate volatility observed in the data. Note that in this model this aggregate volatility is arising purely from 5.6 millions independent firm-level shocks. The reason why this number is quantitatively non-negligible is that the central limit theorem and the "diversification argument" introduced by Lucas (1977) fail to apply. The first reason why the central limit argument fails to apply is that the diversification across firms within a sector is weak. Indeed within a sector, large firms represent a

[25] I simulate 5000 periods and drop the first 1000 periods.

[26] Rather than solving the firm-level problem for each firms, I solve this problem for each productivity bins since all the firms in a bins are perfectly homogeneous. More detailed can be found in the numerical Appendix C.

[27] This number is the standard deviation of the percentage deviation of aggregate output from an HP trend as it is computed in Carvalho and Grassi (2017).

Table 3: Aggregate Volatility

	Total	Downstream	Upstream	COV
$sd\left[\log Y_t\right]$	0.62	0.581	0.07	
$\mathbb{V}ar\left[\log Y_t\right]$	100	89.03	1.33	9.64

NOTE: First row is the standard deviation of the percentage deviation of aggregate output Y_t, of the "downstream" and the "upstream" part of aggregate output as defined in Section 4. The second row is the variance decomposition of the percentage deviation of aggregate output Y_t in Equation 10 between the contribution of labor income and labor share, i.e. the "downstream" and the "upstream" part of aggregate output. Numbers are reported in percentage points. These statistics comes from a 4000 periods simulations.

disproportionate market share as it is observed in the Census Bureau concentration data. The "granular hypothesis" introduced by Gabaix (2011) is at play: shocks to these large firms do not average out. Following a shock to one of these large firms it is unlikely that another shock of the opposite sign hits another large firm and mitigate the first one. The second reason why the central limit argument is not applying, is that the diversification across the 389 sectors is governed by the "small world" input-output network where it exists a handful of highly connected hub-like sectors. As shown by Acemoglu et al. (2012) and Carvalho (2010) diversification across sectors is weaker than without such input-output network and translates into aggregate volatility.

The second results is that 89.03% of the aggregate variance is due to the labor income i.e. the "downstream" part of aggregate output while the labor share, the "upstream" part of aggregate output, represents 1.33%, and, the covariance is about 9.64% of the aggregate variance. In an economy, without oligopolistic competition all the aggregate volatility would be due to the downstream part of aggregate output as the labor share, the "upstream" part would be constant. This illustrates the importance of the role played by the propagation of changes in profit share and competition intensity following firm-level shocks. Consistently with the Section 4's results of Propositions 9 and 10, the fact that the contribution of the "downstream" part of aggregate volatility is reduced compare to the monopolistic competition case indicates that aggregate fluctuations are led by shocks on large firms in their sectors. Following a shock on one of these large firms, the reduction in sector level price that propagates to downstream sectors is smaller as some of the gain in productivity are captured by an increase in markup. If instead the downstream propagation would have been higher than under monopolistic competition, that would have meant that fluctuations would have been led by shocks on medium size firms. Indeed, a positive shock on one of these firms translates into a higher fall in sector's price since some of this productivity gain is also reducing the sector's markup and thus strengthening the downstream propagation.

Let us further decompose the labor income i.e. the "downstream" part of aggregate output into the contribution of the wage under monopolistic competition and the contribution of the change in

Table 4: Aggregate Variance Decomposition

	"downstream"			"upstream"	COV
	wage under monop.	competition intensity	COV_w		
$\mathbb{V}ar\left[\log Y_t\right]$	89.03			1.33	9.64
$\mathbb{V}ar\left[\log Y_t\right]$	102.99	2.46	-16.42	1.33	9.64
$\mathbb{V}ar\left[\log Y_t^{Monop}\right]$	100	0	0	0	0

NOTE: First row is the variance decomposition of the percentage deviation of aggregate output Y_t between the its "downstream" and the "upstream" part. The second row include the decomposition of the "downstream" part of aggregate output between $Y_t^{d:\overline{Z_k}}$ and $Y_t^{d:\Delta_k}$. The third row is the decomposition under monopolistic competition.

competition intensity:

$$\mathbb{V}ar\left[\log Y_t^d\right] = \mathbb{V}ar\left[\log w_t L\right] = \mathbb{V}ar\Big[\underbrace{\log w_t^{monop}}_{\text{wage under monopolistic}}\Big] + \mathbb{V}ar\Big[\underbrace{\log w_t - \log w_t^{monop}}_{\text{competition intensity}}\Big] + \underbrace{COV_w}_{\text{covariance term}}$$

Under Assumption 2, Equation 5 and Proposition 5 show that under monopolistic competition the wage and the labor income, w_t^{monop}, at time t is entirely determined by $\overline{Z_{t,k}}$. Whereas the term $\log w_t - \log w_t^{monop} = -\beta'(I-\Omega)^{-1}\{\log f_k(\Delta_{t,k})^{\frac{1}{\varepsilon_k - 1}}\}_k$ is entirely determined by the competition intensity measured by the statistic $\Delta_{t,k}$. From the expression of the elasticity of the "downstream" part of aggregate output in Equation 8 the interpretation of this decomposition is even clearer: the first term is as if the sector-level markup were assumed to be fixed, while the second term is the change in the profit share. Table 4 shows the decomposition the "downstream" part of aggregate output.

The first row of Table 4 reproduces the decomposition between "downstream" and "upstream" of Table 3. In the second row the variance of aggregate output is further decomposed into the contribution of the labor income under monopolistic competition and the contribution of the change in competition intensity. The third row is the same decomposition for the case of monopolistic competition. Under monopolistic competition all the volatility of aggregate output is due to change in the sum of productivity, while change in productivity concentration would have no effect. The volatility of aggregate output under monopolistic competition is 0.63 or 34.43% of the observed aggregate volatility in the data. From an aggregate perspective, the aggregate volatility under monopolistic and oligopolistic competition look similar. However, as shown in Table 4 the propagation patterns are entirely different. As soon as oligopolistic competition is taken into account, the propagation of changes in productivity of one firm to downstream sector is dampen by the response of the competition intensity. Furthermore the latter also propagate to upstream sectors.

7 Conclusion

In this paper, I characterize the structural importance of a firm by its size, the role played by its sector in the input-output network and by its sector's market structure. I highlight the role played by the interaction between the input-output linkages and the industrial organization both theoretically and numerically. The propagation of changes in profit share turns out to be important.

This paper also relates to the important literature on the micro-origin of aggregate fluctuations by addressing its internal inconsistency and providing a new quantification. Indeed, in this paper large firms takes into account the effect of their decisions on their sector price and output. Previous papers were maintaining the atomistic behavior assumption while studying the role played by a finite number of production units. Furthermore, the quantification provides here combined the "granular" and the "network" origin of aggregate fluctuations while allowing for a flexible market structure. The aggregate volatility arising from purely idiosyncratic shocks is not much affected by the oligopolistic competition compare to the monopolistic competition benchmark, but the propagation patterns are entirely different. A new propagation channel of productivity shocks arises through the endogenous response of markups and profit shares. The downstream propagation of firm-level productivity shocks is dampen while also propagating to upstream sectors. The interaction of the oligopolistic competition and the input-output network is key for the latter.

This paper is also a starting point to study the aggregate consequence of a rise in concentration. The framework presented here allows for the aggregation of the change in concentration and trace it back to a change in concentration of firm-level productivity. Furthermore, if the concentration of firm-level productivity could be affected by policy, as merger law for example, the model would be helpful to understand the impact of such policy on the whole economy by taking into account the input-output network. I leave these subjects for future research.

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Appendix to "IO in I-O: Size, Industrial Organization and the Input-Output Network Make a Firm Structurally Important"

Basile Grassi

A Proof Appendix

A.1 Proof of Proposition 2 (Firm's Approximation)

The first step is to rewrite the system of equation of Proposition 1 in term of the following *perceived* elasticity of demand: $\varepsilon(k,i) = \frac{\mu(k,i)}{\mu(k,i)-1}$. Then, let us defined the following system of equation for a given parameter χ:

$$P(k,i) = \frac{\varepsilon(k,i)}{\varepsilon(k,i)-1}\lambda(k,i)$$

$$s(k,i) = \frac{P(k,i)y(k,i)}{P_k Y_k} = \left(\frac{P(k,i)}{P_k}\right)^{1-\varepsilon_k}$$

$$\varepsilon(k,i) = \begin{cases} \varepsilon_k & \text{Under Monopolistic Competition} \\ \varepsilon_k - \chi(\varepsilon_k - 1)s(k,i) & \text{Under Bertrand Competition} \\ \left(\frac{1}{\varepsilon_k} + \chi(1-\frac{1}{\varepsilon_k})s(k,i)\right)^{-1} & \text{Under Cournot Competition} \end{cases}$$

When $\chi = 1$, the above system is exactly the one described in Proposition 1 in term of $\varepsilon(k,i)$. When $\chi = 0$, then both the Bertrand and Cournot case reduce to the monopolistic case. I am now focusing on these two cases.

Let us reduce the above system of equation in one equation determining the sales share $s(k,i)$ of the firm i in sector k by subsituting the expression of $\varepsilon(i,k)$ and $P(k,i)$:

$$s(k,i) = \begin{cases} \left(1 - \frac{1}{\varepsilon_k - \chi(\varepsilon_k-1)s(k,i)}\right)^{\varepsilon_k - 1}\left(\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k} & \text{Under Bertrand} \\ \left(1 - \frac{1}{\varepsilon_k} - \chi(1-\frac{1}{\varepsilon_k})s(k,i)\right)^{\varepsilon_k - 1}\left(\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k} & \text{Under Cournot} \end{cases}$$

Let us rewrite the above equation in a more abstract way the unknown $X(\omega,\chi) = s(k,i)$ with $\omega = \left(\frac{\lambda(k,i)}{P_k}\right)^{1-\varepsilon_k}$ and by the function $\mathcal{H}(X,\omega,\chi)$ such that:

$$\mathcal{F}(\omega,\chi) = \mathcal{H}(X(\omega,\chi),\omega,\chi) = 0 \tag{11}$$

with

$$\mathcal{H}(X,\omega,\chi) = \begin{cases} X - \left(1 - \frac{1}{\varepsilon_k - \chi(\varepsilon_k-1)X}\right)^{(\varepsilon_k-1)}\omega & \text{Under Bertrand} \\ X - \left(1 - \frac{1}{\varepsilon_k} - \chi(1-\frac{1}{\varepsilon_k})X\right)^{(\varepsilon_k-1)}\omega & \text{Under Cournot} \end{cases}$$

As explain earlier, $X(\omega,0) = \widehat{s}(k,i)$ is the solution under monopolistic competition. The solution of this system $X(\omega,\chi)$ satisfies at the second order:

$$X(\omega,\chi) = X(\omega,0) + \chi X'(\omega,0) + \chi^2 X''(\omega,0) + o(\chi^2)$$

where $X'(\omega,\chi) := \frac{\partial X}{\partial \chi}(\omega,\chi)$ and $X''(\omega,\chi) := \frac{\partial X'}{\partial \chi}(\omega,\chi)$.

For $\chi = 1$ it yields an approximation of the solution for the Oligopolistic case:

$$X(\omega,1) \approx X(\omega,0) + X'(\omega,0) + X''(\omega,0)$$

Let us compute these derivatives by differentiating Equation 11:

$$\mathcal{F}'_\chi(\omega,\chi) = 0 = X'(\omega,\chi)\mathcal{H}'_X(X(\omega,\chi),\omega,\chi) + \mathcal{H}'_\chi(X(\omega,\chi),\omega,\chi)$$
$$\mathcal{F}''_\chi(\omega,\chi) = 0 = X''(\omega,\chi)\mathcal{H}'_X(X(\omega,\chi),\omega,\chi) + (X'(\omega,\chi))^2\mathcal{H}''_{XX}(X(\omega,\chi),\omega,\chi) + 2X'(\omega,\chi)\mathcal{H}''_{\chi X}(X(\omega,\chi),\omega,\chi)$$

From which it follows:

$$X'(\omega,\chi) = -\frac{\mathcal{H}'_{\chi}(X(\omega,\chi),\omega,\chi)}{\mathcal{H}'_{X}(X(\omega,\chi),\omega,\chi)}$$

$$X''(\omega,\chi) = -\frac{(X'(\omega,\chi))^2\mathcal{H}''_{XX}(X(\omega,\chi),\omega,\chi) + 2X'(\omega,\chi)\mathcal{H}''_{\chi X}(X(\omega,\chi),\omega,\chi)}{\mathcal{H}'_{X}(X(\omega,\chi),\omega,\chi)}$$

and evaluating this at $(\omega, 0)$:

$$X'(\omega,0) = -\frac{\mathcal{H}'_{\chi}(X(\omega,0),\omega,0)}{\mathcal{H}'_{X}(X(\omega,0),\omega,0)}$$

$$X''(\omega,0) = -\frac{(X'(\omega,0))^2\mathcal{H}''_{XX}(X(\omega,0),\omega,0) + 2X'(\omega,0)\mathcal{H}''_{\chi X}(X(\omega,0),\omega,0)}{\mathcal{H}'_{X}(X(\omega,0),\omega,0)}$$

We are left to compute the derivative of $\mathcal{H}(X,\omega,\chi)$ and substitute, which yields:

$$X'(\omega,0) = \begin{cases} -(1-\frac{1}{\varepsilon_k})X(\omega,0)^2 & \text{Under Bertrand} \\ -(\varepsilon_k - 1)X(\omega,0)^2 & \text{Under Cournot} \end{cases}$$

$$X''(\omega,0) = \begin{cases} (1-\frac{1}{\varepsilon_k})^2(1-\frac{1}{\varepsilon_k - 1})X(\omega,0)^3 & \text{Under Bertrand} \\ (\varepsilon_k - 1)^2(3-\frac{1}{\varepsilon_k-1})X(\omega,0)^3 & \text{Under Cournot} \end{cases}$$

which yields:

$$X(\omega,1) \approx \begin{cases} X(\omega,0)\left(1-(1-\frac{1}{\varepsilon_k})X(\omega,0) + (1-\frac{1}{\varepsilon_k})^2(1-\frac{1}{\varepsilon_k-1})X(\omega,0)^2\right) & \text{Under Bertrand} \\ X(\omega,0)\left(1-(\varepsilon_k-1)X(\omega,0) + (\varepsilon_k-1)^2(3-\frac{1}{\varepsilon_k-1})X(\omega,0)^2\right) & \text{Under Cournot} \end{cases}$$

By substituting $X(\omega,1) = s(k,i)$ and $X(\omega,0) = \widehat{s}(k,i)$, we get the result. □

A.2 Proof of Proposition 4 (Sector Level Markup)

To prove this proposition, I substitute the result of Proposition 1 in Equation 1 reproduced here for convenience.

$$\mu_k = \left(\sum_{i=1}^{N_k} \mu(k,i)^{-1} s(k,i)\right)^{-1} \tag{1}$$

Let us first focus on the monopolistic competition case, and then turn to the Cournot and Bertrand cases.

Monopolistic case: Let us first look at the monopolistic competition case where markups charged by firms in sector k are identical and equal to $\frac{\varepsilon_k}{\varepsilon_k-1}$. Substituting $\mu(k,i)$ in Equation 1 leads to $\mu_k = \frac{\varepsilon_k}{\varepsilon_k-1}\left(\sum_{i=1}^{N_k} s(k,i)\right)^{-1} = \frac{\varepsilon_k}{\varepsilon_k-1}$ since the sum of the sales share of firms in sector k is equal to one.

Cournot case: In this case the markup charged by firm i in sector k is equal to $\mu(k,i) = \frac{\varepsilon_k}{\varepsilon_k - 1 - (\varepsilon_k-1)s(k,i)}$, let us substitute it in Equation 1. After some simplification we have:

$$\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k}\sum_{i=1}^{N_k} s(k,i) - \frac{\varepsilon_k - 1}{\varepsilon_k}\sum_{i=1}^{N_k} s(k,i)^2 = \frac{\varepsilon_k-1}{\varepsilon_k}\left(1 - \sum_{i=1}^{N_k} s(k,i)^2\right)$$

where the last equality comes from the fact that $\sum_{i=1}^{N_k} s(k,i) = 1$.

Bertrand case: In this case the markup charged by firm i in sector k is equal to $\mu(k,i) = \frac{\varepsilon_k - (\varepsilon_k-1)s(k,i)}{\varepsilon_k - 1 - (\varepsilon_k-1)s(k,i)}$, let us substitute it in Equation 1. After some simplification we have:

$$\mu_k^{-1} = 1 - \frac{1}{\varepsilon_k}\sum_{i=1}^{N_k} s(k,i)\frac{1}{1-\frac{\varepsilon_k-1}{\varepsilon_k}s(k,i)}$$

Note that because $\varepsilon_k > 1$ and $s(k,i) < 1$ we have $0 < \frac{\varepsilon_k - 1}{\varepsilon_k}s(k,i) < 1$. We can expand the series and therefore

$\frac{1}{1-\frac{\varepsilon_k-1}{\varepsilon_k}s(k,i)} = \sum_{m=0}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^m s(k,i)^m$. After substituting in the previous equation, we get

$$\begin{aligned}
\mu_k^{-1} =& 1 - \frac{1}{\varepsilon_k}\sum_{i=1}^{N_k}\sum_{m=0}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^m s(k,i)^{m+1} \\
=& 1 - \frac{1}{\varepsilon_k}\sum_{m=0}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^m \sum_{i=1}^{N_k} s(k,i)^{m+1} \\
=& 1 - \frac{1}{\varepsilon_k} - \frac{1}{\varepsilon_k}\sum_{m=1}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^m \sum_{i=1}^{N_k} s(k,i)^{m+1} \\
=& \frac{\varepsilon_k-1}{\varepsilon_k}\left(1 - \frac{1}{\varepsilon_k-1}\sum_{m=2}^{\infty}\left(\frac{\varepsilon_k-1}{\varepsilon_k}\right)^{m-1}\sum_{i=1}^{N_k} s(k,i)^m\right)
\end{aligned}$$

where from the first to the second line, I use the fact that the sum over the firms in sector k index by i is finite, where from the second to the third line I take out the first term of the sum over the index m, and where the last line comes from rearranging terms and reindexing of the sum over m. □

A.3 Proof of Proposition 5 (Sector Allocation)

The structure of the proof is as follow. First, I found the relationship (Equation 2) between sector prices and sector level productivity and markup. Second, I show Lemmas 1 and 2 that relates the sector level productivity and markup to other sector price and the wage. Finally, I combine these results to solve for the sector allocation.

A.3.1 Proof of Equation 2:

As show earlier in Section 3.2 the sector level marginal cost and markup are such that $\lambda_k = \mu_k^{-1}P_k$. Using the fact that $\lambda_k = Z_k^{-\gamma_k} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}$, we have

$$P_k = \mu_k \left(\frac{w}{Z_k}\right)^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}} \quad \text{and} \quad \log P_k = \log \mu_k \left(\frac{w}{Z_k}\right)^{\gamma_k} + \sum_{l=1}^{N} \omega_{k,l} \log P_l \tag{12}$$

Rewriting the last equation in matrix form yields

$$\{\log P_k\}_k = \left\{\log \mu_k \left(\frac{w}{Z_k}\right)^{\gamma_k}\right\}_k + \Omega \{\log P_k\}_k$$

where the $N \times N$ matrix Ω is such that $\Omega = \{\omega_{k,l}\}_{1\le k,l\le N}$. Finally, Equation 2 comes from the pre-multiplication of the following expression by the matrix $(I-\Omega)^{-1}$

$$(I-\Omega)\{\log P_k\}_k = \left\{\log \mu_k \left(\frac{w}{Z_k}\right)^{\gamma_k}\right\}_k$$

□

A.3.2 Proof of Equation 3:

The market clearing condition for the variety i of sector k's good is such that the supply is equal to the demand from the household and from other firms in the economy:

$$P(k,i)y(k,i) = P(k,i)c(k,i) + \sum_{l=1}^{N}\sum_{j=1}^{N_l} P(k,i)x(l,j,k,i)$$

Where $c(k,i)$ is the demand of variety i of sector k's good by the household and $x(l,j,k,i)$ is the demand of variety i of sector k's good from firm j in sector l. The household's problem gives $P(k,i)c(k,i) =$

$\beta_k \left(\frac{P(k,i)}{P_k}\right)^{1-\varepsilon_k} P^C C$ while the cost minimization problem of firm j in sector l gives $P(k,i)x(l,j,k,i) = \omega_{l,k}\left(\frac{P(k,i)}{P_k}\right)^{1-\varepsilon_k} \lambda(l,j)y(l,j)$. Summing over the firms in sector k and using the fact that $P_k = \sum_{i=1}^{N_k} P(k,i)^{1-\varepsilon_k}$, we have

$$P_k Y_k = \sum_{i=1}^{N_k} p(k,i)y(k,i) = \beta_k P^C C + \sum_{l=1}^{N} \omega_{l,k} \sum_{j=1}^{N_l} \lambda(l,j)y(l,j) = \beta_k P^C C + \sum_{l=1}^{N} \omega_{l,k}\mu_l^{-1} P_l Y_l$$

where in the last equality I use the definition of the sector marginal cost λ_l and the fact that $\lambda_l = \mu_l^{-1} P_l$. Let us define the $N \times N$ matrix $\widetilde{\Omega} = \{\mu_k^{-1}\omega_{k,l}\}_{k,l}$. The above equation in vector form yields:

$$\left\{\frac{P_k Y_k}{P^C C}\right\}_k' = \beta' + \left\{\frac{P_l Y_l}{P^C C}\right\}_l' \widetilde{\Omega} \quad \Rightarrow \quad \left\{\frac{P_k Y_k}{P^C C}\right\}_k' = \beta'(I - \widetilde{\Omega})^{-1}$$

□

A.3.3 Two Lemmas:

Let us first prove two lemmas that simplified the expression of sectors' productivity and markup.

Lemma 1 ***(Productivity)*** *Under Assumption 2, the sector level productivity Z_k satisfies*

$$Z_k^{-\gamma_k} = \begin{cases} X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}} \overline{Z_k} & \text{Under Monopolistic Competition} \\ X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}} \left(\overline{Z_k} - X_k \overline{Z_k}^2 \Delta_k\right) & \text{Under Bertrand Competition} \\ X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}} \left(\overline{Z_k} - \varepsilon_k X_k \overline{Z_k}^2 \Delta_k\right) & \text{Under Cournot Competition} \end{cases}$$

where $X_k = \left(P_k^{-1} \frac{\varepsilon_k}{\varepsilon_k-1} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}\right)^{(1-\varepsilon_k)}$ and where $\overline{Z_k}$ and Δ_k are defined in Section 3.1.

Proof of Lemma 1: Let us first look at the monopolistic case before turning to the Bertrand and Cournot case.

Monopolistic case: Under monopolistic competition, firm i in sector k charges a (constant) markup $\frac{\varepsilon_k}{\varepsilon_k-1}$ over its marginal cost $\lambda(k,i)$. Note that the firm level marginal cost is equal to

$$\lambda(k,i) = Z(k,i)^{-\gamma_k} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}$$

It follows that

$$\frac{y(k,i)}{Y_k} = \left(\frac{P(k,i)}{P_k}\right)^{-\varepsilon_k} = \left(\frac{\varepsilon_k}{\varepsilon_k - 1} P_k^{-1} \lambda(k,i)\right)^{-\varepsilon_k} = Z(k,i)^{\gamma_k \varepsilon_k} \left(\frac{\varepsilon_k}{\varepsilon_k - 1} P_k^{-1} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}\right)^{-\varepsilon_k}$$

Substituting the above expression in the expression of Z_k yields

$$Z_k^{-\gamma_k} = \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} \frac{y(k,i)}{Y_k} = \left(\frac{\varepsilon_k}{\varepsilon_k - 1} P_k^{-1} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}\right)^{-\varepsilon_k} \sum_{i=1}^{N_k} Z(k,i)^{\gamma_k(\varepsilon_k - 1)}$$

which implies the result

$$Z_k^{-\gamma_k} = X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}} \sum_{i=1}^{N_k} Z(k,i)^{\gamma_k(\varepsilon_k-1)} = X_k^{\frac{\varepsilon_k}{\varepsilon_k-1}} \overline{Z_k}$$

Cournot case: Let us first note that $\frac{y(k,i)}{Y_k} = P_k p(k,i)^{-1} s(k,i) = P_k \lambda(k,i)^{-1} \mu(k,i)^{-1} s(k,i)$. The sales share under monopolistic competition are $\hat{s}(k,i) = P_k^{\varepsilon_k - 1} \left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{1-\varepsilon_k} \lambda(k,i)^{1-\varepsilon_k}$ while $\lambda(k,i) =$

$Z(k,i)^{-\gamma_k} X_k^{\frac{1}{1-\varepsilon_k}} P_k \frac{\varepsilon_k - 1}{\varepsilon_k}$. It follows that $\hat{s}(k,i) = Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k$. Note also that $\lambda(k,i)^{-1} P_k \frac{\varepsilon_k - 1}{\varepsilon_k} = Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}}$.

Under Cournot competition according to Proposition 1, we have

$$\frac{y(k,i)}{Y_k} = \lambda(k,i)^{-1} P_k \frac{\varepsilon_k - 1}{\varepsilon_k} (s(k,i) - s(k,i)^2) = Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}} (s(k,i) - s(k,i)^2)$$

Under Assumption 2,the sales share of firm i in sector k satisfies $s(k,i) - s(k,i)^2 = \hat{s}(k,i) - \varepsilon_k \hat{s}(k,i)^2$. Equipped with all these expressions, let us look at

$$\begin{aligned}
Z_k^{-\gamma_k} = \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} \frac{y(k,i)}{Y_k} &= \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}} (s(k,i) - s(k,i)^2) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \sum_{i=1}^{N_k} \left(\hat{s}(k,i) - \varepsilon_k \hat{s}(k,i)^2\right) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \sum_{i=1}^{N_k} \left(Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k - \varepsilon_k Z(k,i)^{-2\gamma_k(1-\varepsilon_k)} X_k^2\right) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \left(X_k \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k(1-\varepsilon_k)} - \varepsilon_k X_k^2 \sum_{i=1}^{N_k} Z(k,i)^{-2\gamma_k(1-\varepsilon_k)}\right) \\
&= X_k^{\frac{\varepsilon_k}{\varepsilon_k - 1}} \left(\overline{Z_k} - \varepsilon_k X_k \overline{Z_k}^2 \Delta_k\right)
\end{aligned}$$

Bertrand case:

Under Bertrand competition according to Proposition 1 $\mu(k,i) = \frac{\varepsilon_k - (\varepsilon_k - 1)s(k,i)}{\varepsilon_k - 1 - (\varepsilon_k - 1)s(k,i)}$ and therefore

$$\mu(k,i)^{-1} s(k,i) = \frac{\varepsilon_k - 1}{\varepsilon_k} \frac{1 - s(k,i)}{1 - \frac{\varepsilon_k - 1}{\varepsilon_k} s(k,i)} s(k,i) = \frac{\varepsilon_k - 1}{\varepsilon_k} s(k,i)(1 - s(k,i)) \left(1 + \frac{\varepsilon_k - 1}{\varepsilon_k} s(k,i) + (\frac{\varepsilon_k - 1}{\varepsilon_k} s(k,i))^2\right)$$

where the last equality holds for a second order approximation for $s(k,i) \longrightarrow 0$. At the second order we thus have $\mu(k,i)^{-1} s(k,i) = \frac{\varepsilon_k - 1}{\varepsilon_k} \left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2\right)$. Using the fact that $\frac{y(k,i)}{Y_k} = P_k \lambda(k,i)^{-1} \mu(k,i)^{-1} s(k,i)$ and $\lambda(k,i)^{-1} P_k \frac{\varepsilon_k - 1}{\varepsilon_k} = Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}}$, the output share of firm i in sector k is

$$\frac{y(k,i)}{Y_k} = \lambda(k,i)^{-1} P_k \frac{\varepsilon_k - 1}{\varepsilon_k} \left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2\right) = Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}} \left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2\right)$$

Under Assumption 2,the sales share of firm i in sector k satisfies $s(k,i) - \varepsilon_k^{-1} s(k,i)^2 = \hat{s}(k,i) - \hat{s}(k,i)^2$. Equipped with all these expressions, let us look at

$$\begin{aligned}
Z_k^{-\gamma_k} = \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} \frac{y(k,i)}{Y_k} &= \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k} Z(k,i)^{\gamma_k} X_k^{\frac{-1}{1-\varepsilon_k}} (s(k,i) - \varepsilon_k^{-1} s(k,i)^2) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \sum_{i=1}^{N_k} \left(\hat{s}(k,i) - \hat{s}(k,i)^2\right) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \sum_{i=1}^{N_k} \left(Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k - Z(k,i)^{-2\gamma_k(1-\varepsilon_k)} X_k^2\right) \\
&= X_k^{\frac{-1}{1-\varepsilon_k}} \left(X_k \sum_{i=1}^{N_k} Z(k,i)^{-\gamma_k(1-\varepsilon_k)} - X_k^2 \sum_{i=1}^{N_k} Z(k,i)^{-2\gamma_k(1-\varepsilon_k)}\right) \\
&= X_k^{\frac{\varepsilon_k}{\varepsilon_k - 1}} \left(\overline{Z_k} - X_k \overline{Z_k}^2 \Delta_k\right)
\end{aligned}$$

□

Lemma 2 ***(Markup)*** *Under assumption 2, the sector k's markup satisfies*

$$\mu_k^{-1} = \begin{cases} \frac{\varepsilon_k - 1}{\varepsilon_k} & \textit{Under Monopolistic Competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - \frac{1}{\varepsilon_k} X_k^2 \overline{Z_k}^2 \Delta_k\right) & \textit{Under Bertrand Competition} \\ \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - X_k^2 \overline{Z_k}^2 \Delta_k\right) & \textit{Under Cournot Competition} \end{cases}$$

where $X_k = \left(P_k^{-1} \frac{\varepsilon_k}{\varepsilon_k - 1} w^{\gamma_k} \prod_{l=1}^{N} P_l^{\omega_{k,l}}\right)^{(1-\varepsilon_k)}$ *and where* $\overline{Z_k}$ *and* Δ_k *are defined in Section 3.1.*

Proof of Lemma 2: Let us first look at the Cournot case before turning to the Bertrand case.

Cournot case: Proposition 4 shows that under Cournot competition we have $\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - \sum_{i=1}^{N_k} s(k,i)^2\right)$ while under Assumption 2, we have $s(k,i)^2 = \hat{s}(k,i)^2$. Using the fact that $\hat{s}(k,i) = Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k$

$$\mu_k^{-1} = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - X_k^2 \sum_{i=1}^{N_k} Z(k,i)^{2\gamma_k(\varepsilon_k - 1)}\right)$$

Bertrand case: As shown in the proof of Lemma 1 under Bertrand competition, the markup and the sales share of firm i in sector k satisfy up to a second order approximation:

$$\mu(k,i)^{-1} s(k,i) = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2\right)$$

It follows that the sector level markup

$$\mu_k^{-1} = \sum_{i=1}^{N_k} \frac{\varepsilon_k - 1}{\varepsilon_k}\left(s(k,i) - \varepsilon_k^{-1} s(k,i)^2\right) = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - \varepsilon_k^{-1} \sum_{i=1}^{N_k} s(k,i)^2\right)$$

since $\sum_{i=1}^{N_k} s(k,i) = 1$. Under Assumption 2, we have that $s(k,i)^2 = \hat{s}(k,i)^2$. Using the fact that $\hat{s}(k,i) = Z(k,i)^{-\gamma_k(1-\varepsilon_k)} X_k$, the result follow. □

A.3.4 Proof of the Proposition 5:

In the last step of this proof, let rewrite Equation 12 as

$$\mu_k Z_k^{-\gamma_k} = \frac{\varepsilon_k}{\varepsilon_k - 1}\left[\frac{\varepsilon_k - 1}{\varepsilon_k} P_k w^{-\gamma_k} \prod_{l=1}^{N} P_l^{-\omega_{k,l}}\right]$$

The term in the right hand side in the square bracket of the above equation is equal to $X_k^{\frac{1}{\varepsilon_k - 1}}$, we can then rewrite this equation as

$$Z_k^{-\gamma_k} = \mu_k^{-1} \frac{\varepsilon_k}{\varepsilon_k - 1} X_k^{\frac{1}{\varepsilon_k - 1}} \tag{13}$$

Bertrand case: Let us substitute in Equation 13, the expression of the productivity and the markup in Lemmas 1 and 2.

$$X_k^{\frac{\varepsilon_k}{\varepsilon_k - 1}}\left(\overline{Z_k} - X_k \overline{Z_k}^2 \Delta_k\right) = \frac{\varepsilon_k - 1}{\varepsilon_k}\left(1 - \frac{1}{\varepsilon_k} X_k^2 \overline{Z_k}^2 \Delta_k\right) \frac{\varepsilon_k}{\varepsilon_k - 1} X_k^{\frac{1}{\varepsilon_k - 1}}$$

Rearranging terms yields the following quadratic equation in unknown X_k

$$(1 - \varepsilon_k^{-1}) \overline{Z_k}^2 \Delta_k X_k^2 - \overline{Z_k} X_k + 1 = 0 \tag{14}$$

First note that the monopolistic case is nested in the above equation (see Lemmas 1 and 2). Indeed, by taking $\Delta_k = 0$ we recover the solution of the above equation under monopolistic competition: $X_k = 1/\overline{Z_k}$. Second Equation 14 admits solution only if $\overline{Z_k}^2 - 4(1 - \varepsilon_k^{-1}) \overline{Z_k}^2 \Delta_k \geq 0$ or equivalently when $\Delta_k \leq \frac{1}{4(1-\varepsilon_k^{-1})}$. In the case of strict inequality, this equation admits the following two solutions:

$$X_k^+ = \frac{1+\sqrt{1-4(1-\varepsilon_k^{-1})\Delta_k}}{2(1-\varepsilon_k^{-1})\Delta_k\overline{Z_k}} \quad \text{and} \quad X_k^- = \frac{1-\sqrt{1-4(1-\varepsilon_k^{-1})\Delta_k}}{2(1-\varepsilon_k^{-1})\Delta_k\overline{Z_k}}$$

For $\Delta_k \longrightarrow 0$, it is easy to see that $X_k^+ \longrightarrow \infty$ and $X_k^- \longrightarrow \frac{1}{\overline{Z_k}}$. To ensure continuity of the solutions with the monopolistic case, X_k^- is the only admissible solution and therefore, using the notation f_k of the Proposition 5,

$$X_k == \frac{1-\sqrt{1-4(1-\varepsilon_k^{-1})\Delta_k}}{2(1-\varepsilon_k^{-1})\Delta_k\overline{Z_k}} = \frac{f_k(\Delta_k)}{\overline{Z_k}}$$

Let us now solve for the productivity, using Lemma 1 we have $X_k^{\frac{1}{\varepsilon_k-1}}\left(\overline{Z_k}X_k - X_k^2\overline{Z_k}^2\Delta_k\right)$. Using Equation 14, to see that $X_k^2\overline{Z_k}^2\Delta_k = \frac{\overline{Z_k}X_k-1}{1-\varepsilon_k^{-1}}$, the productivity in sector k satisfies

$$Z_k^{-\gamma_k} = X_k^{\frac{1}{\varepsilon_k-1}}\frac{\varepsilon_k - \overline{Z_k}X_k}{\varepsilon_k - 1}$$

The markup expression is also found using the same reasoning. Combining Lemma 2 and Equation 14 yields that $\mu_k^{-1} = \frac{\varepsilon_k-\overline{Z_k}X_k}{\varepsilon_k}$.

Cournot case: For the Cournot case, I follow the same logic. By combining Equation 13 and Lemmas 1 and 2, X_k is the solution of the following quadratic equation:

$$(\varepsilon_k - 1)\overline{Z_k}^2\Delta_k X_k^2 - \overline{Z_k}X_k + 1 = 0 \tag{15}$$

This equation, for $\Delta_k < \frac{1}{4(\varepsilon_k-1)}$, has one admissible solution:[28]

$$X_k = \frac{1-\sqrt{1-4(\varepsilon_k-1)\Delta_k}}{2\Delta_k(\varepsilon_k-1)\overline{Z_k}} = \frac{f_k(\Delta_k)}{\overline{Z_k}}$$

Using Equation 15 and Lemmas 1 and 2, it is easy to show that under Cournot competition

$$Z_k^{-\gamma_k} = X_k^{\frac{1}{\varepsilon_k-1}}\frac{\varepsilon_k - \overline{Z_k}X_k}{\varepsilon_k - 1} \quad \text{and} \quad \mu_k^{-1} = \frac{\varepsilon_k - \overline{Z_k}X_k}{\varepsilon_k}$$

□

A.4 Proof of Proposition 8 (Equilibrium Allocation)

Wage (Equation 5): Without loss of generality, let us normalized the composite consumption good to $P^C = 1$. It implies that $0 = \log 1 = \log P^C = \sum_{k=1}^N \beta_k \log P_k = \beta' \{\log P_k\}_k$ where the last expression is an inner product of the two vectors β and $\{\log P_k\}_k$. In this last expression, let us subsitute the expression of sector-level price (Equation 2):

$$0 = \beta'(I-\Omega)^{-1}\left\{\log \mu_l \left(\frac{w}{Z_l}\right)^{\gamma_l}\right\}_l = \beta'(I-\Omega)^{-1}\{\gamma_l\}_l \log w + \beta'(I-\Omega)^{-1}\{\log \mu_l Z_l{}^{-\gamma_l}\}_l$$

Let us note that $\Omega\mathbb{I} = \left\{\sum_{l=1}^N \omega_{k,l}\right\}_k = \{1-\gamma_k\}_k = \mathbb{I} - \{\gamma_k\}_k$ where $\mathbb{I} = \{1\}_k$ is the vector of ones. It implies that $(I-\Omega)^{-1}\{\gamma_l\}_l = \mathbb{I}$. Furthermore, since $\sum_{k=1}^N \beta_k = 1$, it follows that $\beta'(I-\Omega)^{-1}\{\gamma_l\}_l = \beta'\mathbb{I} = \sum_{k=1}^N \beta_k = 1$. Using this last expression, we have the expression of the wage:

$$\log w = -\beta'(I-\Omega)^{-1}\{\log \mu_l Z_l{}^{-\gamma_l}\}_l$$

□

[28] An admissible solution such that $X_k \underset{\Delta_k\to 0}{\longrightarrow} \frac{1}{\overline{Z_k}}$.

Aggregate profit share (Equation 6): From the firm's problem it is clear that the profit $\pi(k,i)$ of firm i in sector k is such that $\pi(k,i) = P(k,i)y(k,i) - \lambda(k,i)y(k,i)$. Summing over the firms in sector k yields:

$$\pi_k = \sum_{i=1}^{N_k} P(k,i)y(k,i) - \sum_{i=1}^{N_k} \lambda(k,i)y(k,i) = P_kY_k - \lambda_kY_k = (1-\mu_k^{-1})P_kY_k$$

where I use the definition of the marginal cost in sector k and the fact that $\lambda_k = \mu_k^{-1}P_k$. Finally aggregate profit is equal to the sum of the profit in each sector:

$$\frac{Pro}{P^CC} = \sum_{k=1}^{N} \pi_k \frac{\pi_k}{P^CC} = \sum_{k=1}^{N}(1-\mu_k^{-1})\frac{P_kY_k}{P^CC} = \left\{\frac{P_kY_k}{P^CC}\right\}_k' \{1-\mu_k^{-1}\}_k$$

Substituting Equation 3 yields the result. □

Aggregate output (Equation 7): The household budget constraint is such that total expenditure is equal to the labor and profit income:

$$P^CC = wL + Pro \quad \Leftrightarrow \quad C = w + \frac{Pro}{P^CC}C$$

where I use the normalization of the price $P^C = 1$ and of the labor $L = 1$. Note that in this framework $Y = C$. Rearranging terms and taking logs give the results. □

A.5 Proof of Propositions 9 and 10 (Elasticity of Aggregate Output)

Proposition 9: Combining the expression of the wage w in Proposition 8 and the expression of the sectoral markups and productivities under Assumption 2 given by Proposition 5, we have

$$\log w = -\overline{\beta}' \left\{\log \frac{\varepsilon_k}{\varepsilon_k - 1}\left(\overline{Z_k}\right)^{\frac{-1}{\varepsilon_k-1}} f_k(\Delta_k)^{\frac{1}{\varepsilon_k-1}}\right\}_k = -\sum_{k=1}^{N}\overline{\beta}_k\left(\frac{-1}{\varepsilon_k-1}\log\overline{Z_k} + \frac{1}{\varepsilon_k-1}\log f_k(\Delta_k) + \log\frac{\varepsilon_k}{\varepsilon_k-1}\right)$$

Taking derivative of the above expression with respect to $\log \overline{Z_k}$ and $\log \Delta_k$ yields

$$\frac{\partial \log w}{\partial \log \overline{Z_k}} = \frac{\overline{\beta}_k}{\varepsilon_k - 1} \quad \text{and} \quad \frac{\partial \log w}{\partial \log \Delta_k} = -\frac{\overline{\beta}_k}{\varepsilon_k - 1}e_k$$

where e_k is the elasticity of f_k with respect to to Δ_k: $e_k = \frac{d\log f_k(\Delta_k)}{d\log\Delta_k}$. Using the fact that $\frac{\partial \log \Delta_k}{\partial \log Z(k,i)} = \frac{2}{\Delta_k}\left(\frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}} - \Delta_k\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$ and using the chain rule, we can compute the elasticity of the wage with respect to the productivity of firm i in sector k:

$$\begin{aligned}\frac{\partial \log w}{\partial \log Z(k,i)} &= \frac{\partial \log w}{\partial \log \overline{Z_k}}\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)} + \frac{\partial \log w}{\partial \log \Delta_k}\frac{\partial \log \Delta_k}{\partial \log Z(k,i)} \\ &= \frac{\overline{\beta}_k}{\varepsilon_k-1}\left(1 + \frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}\right)\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}\end{aligned}$$

□

Proposition 10: Let us first prove a technical lemma that turn out to be useful. It compute the derivative of the sector-level sales share with respect to the inverse of the sector level markups.

Lemma 3 ***(Sector-level sales share derivative)*** *Under Assumption 2, the derivative of the vector of sector-level sales share $\widetilde{\beta} = \left\{\frac{P_kY_k}{P^CC}\right\}_k$ with respect to μ_k^{-1} is*

$$\frac{\partial \widetilde{\beta}'}{\partial(\mu_k^{-1})} = \mu_k\widetilde{\beta}_k v_k'\left[(I-\widetilde{\Omega})^{-1} - I\right]$$

where v_k is the $(N \times 1)$ vector where all elements are zero except the k^{th}.

Proof of the lemma: Equation 3 in Proposition 5 tells us that $\widetilde{\beta}' \equiv \left\{\frac{P_kY_k}{P^CC}\right\}_k' = \beta'(I-\widetilde{\Omega})^{-1} = \beta'(I-S\Omega)^{-1}$

where S is the diagonal matrix with the element of the vector $\{\mu_k^{-1}\}_k$ i.e $S = diag(\mu_k^{-1})$. Thanks to the fact that for a matrix A, the derivative of its inverse is $\frac{\partial A^{-1}}{\partial x} = -A^{-1}\frac{\partial A}{\partial x}A^{-1}$, we have

$$\frac{\partial \widetilde{\beta}'}{\partial \mu_k^{-1}} = -\beta'(I-\widetilde{\Omega})^{-1}\frac{\partial (I - S\Omega)}{\partial \mu_k^{-1}}(I-\widetilde{\Omega})^{-1} = +\beta'(I-\widetilde{\Omega})^{-1}\frac{\partial S}{\partial \mu_k^{-1}}\Omega(I-\widetilde{\Omega})^{-1} = \beta'(I-\widetilde{\Omega})^{-1}\frac{\partial S}{\partial \mu_k^{-1}}S^{-1}\widetilde{\Omega}(I-\widetilde{\Omega})^{-1}$$

Note that $\frac{\partial S}{\partial \mu_k^{-1}}S^{-1} = \mu_k v_k v_k'$ with $v_k v_k'$ is the $(N \times N)$ matrix such that all elements are zeros except the k^{th} of the diagonal. Note also that $\widetilde{\Omega}(I-\widetilde{\Omega})^{-1} = (I-\widetilde{\Omega})^{-1} - I = \widetilde{\Omega} + \widetilde{\Omega}^2 + \ldots$ Using this in the above equation yields:

$$\frac{\partial \widetilde{\beta}'}{\partial (\mu_k^{-1})} = \mu_k \beta'(I-\widetilde{\Omega})^{-1} v_k v_k' \left[(I-\widetilde{\Omega})^{-1} - I\right] = \mu_k \widetilde{\beta}_k v_k' \left[(I-\widetilde{\Omega})^{-1} - I\right]$$

□

Back to the proof of Proposition 10, let us recall that $\log Y^u = -\log(1 - \frac{Pro}{P^C C})$. By using the chain rule we have

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = \frac{\partial \log Y^u}{\partial \log \overline{Z_k}}\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)} + \frac{\partial \log Y^u}{\partial \log \Delta_k}\frac{\partial \log \Delta_k}{\partial \log Z(k,i)} = \frac{\partial \log Y^u}{\partial \log \Delta_k}\frac{\partial \log \Delta_k}{\partial \log Z(k,i)}$$

where the last equality comes from the fact that $\frac{\partial \log Y^u}{\partial \log \overline{Z_k}} = 0$. Indeed, $\frac{Pro}{P^C C} = \beta'(I-\widetilde{\Omega})^{-1}\{1-\mu_k^{-1}\}_k$ is entirely determined by the parameters β and Ω and the markups μ_k and the latter are themselves entirely determined by the Δ_k. Using the expression of $\frac{\partial \log \Delta_k}{\partial \log Z(k,i)}$, we have:

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = \frac{2}{\Delta_k}\left(\frac{Z(k,i)^{(\varepsilon_k - 1)\gamma_k}}{\overline{Z_k}} - \Delta_k\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}\frac{\partial \log Y^u}{\partial \log \Delta_k} \tag{16}$$

Let us compute $\frac{\partial \log Y^u}{\partial \log \Delta_k}$:

$$\frac{\partial \log Y^u}{\partial \log \Delta_k} = \frac{\Delta_k}{1 - \frac{Pro}{P^C C}}\frac{\partial \frac{Pro}{P^C C}}{\partial \Delta_k} = \frac{\Delta_k}{\frac{wL}{P^C C}}\frac{\partial(\mu_k^{-1})}{\partial \Delta_k}\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})} = -\frac{f_k(\Delta_k)}{\varepsilon_k}\frac{\Delta_k f_k'(\Delta_k)}{f_k(\Delta_k)}\frac{P^C C}{wL}\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})}$$

where I use the chain rule in the second equality and the expression of the sector k's markup $\mu_k^{-1} = 1 - \frac{f_k(\Delta_k)}{\varepsilon_k}$ in the last equality. Note that I can use the chain rule in the second equality because the markup in sector k is entirely determined by the index Δ_k. Using the definition of the markup and the elasticity e_k of the function f_k, we have

$$\frac{\partial \log Y^u}{\partial \log \Delta_k} = -(1-\mu_k^{-1})e_k\frac{P^C C}{wL}\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})} \tag{17}$$

Let us compute $\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})}$, first thanks to Proposition 8 we have $\frac{Pro}{P^C C} = \beta'(I-\widetilde{\Omega})^{-1}\{1-\mu_k^{-1}\}_k = \widetilde{\beta}'\{1-\mu_k^{-1}\}_k$ with the notation of Lemma 3: $\widetilde{\beta} = \left\{\frac{P_k Y_k}{P^C C}\right\}_k = \beta'(I-\widetilde{\Omega})^{-1}$. Using the fact that for two vectors y and x that are function of z then $\frac{\partial(y'x)}{\partial z} = x'\frac{\partial y}{\partial z} + y'\frac{\partial x}{\partial z}$, we have

$$\begin{aligned}
\frac{\partial \frac{Pro}{P^C C}}{\partial(\mu_k^{-1})} &= \widetilde{\beta}'\frac{\partial(\{1-\mu_k^{-1}\}_k)}{\partial(\mu_k^{-1})} + \{1-\mu_k^{-1}\}_k'\frac{\partial \widetilde{\beta}}{\partial(\mu_k^{-1})} \\
&= -\widetilde{\beta}' v_k + \mu_k \widetilde{\beta}_k \{1-\mu_k^{-1}\}_k'\left[\left((I-\widetilde{\Omega})^{-1}\right)' - I\right] v_k \\
&= -\widetilde{\beta}_k + \mu_k \widetilde{\beta}_k\left[\left(\{1-\widetilde{\mu_k}^{-1}\}_k\right)' v_k - \{1-\mu_k^{-1}\}_k' v_k\right] \\
&= -\widetilde{\beta}_k + \mu_k \widetilde{\beta}_k\left[1 - \widetilde{\mu_k}^{-1} - 1 + \mu_k^{-1}\right] = -\widetilde{\beta}_k\frac{\mu_k}{\widetilde{\mu_k}} = -\frac{P_k Y_k}{P^C C}\frac{\mu_k}{\widetilde{\mu_k}}
\end{aligned} \tag{18}$$

where in the fifth line I use the definition of $\widetilde{\mu_k}^{-1}$ in Proposition 8. Substituting Equations 18 and 17 in Equation

16 yields the result:

$$\frac{\partial \log Y^u}{\partial \log Z(k,i)} = -\frac{Pro_k}{wL}\frac{\mu_k}{\widetilde{\mu_k}}\frac{2e_k}{\Delta_k}\left(\Delta_k - \frac{Z(k,i)^{(\varepsilon_k-1)\gamma_k}}{\overline{Z_k}}\right)\frac{\partial \log \overline{Z_k}}{\partial \log Z(k,i)}$$

where I use the fact that $Pro_k = P_k Y_k(1-\mu_k^{-1})$. □

A.6 Proof of Proposition 11 (Elasticity of Sector-Level Price)

Using Equation 2 and the results of Proposition 5, we have

$$\{\log P_k\}_k = (I-\Omega)^{-1}\left\{\log \overline{Z_l}^{\frac{-1}{\varepsilon_l - 1}} f_l(\Delta_l)^{\frac{1}{\varepsilon_l-1}} \frac{\varepsilon_l}{\varepsilon_l - 1}\right\}_l + \log w\mathbb{I}$$

where I used the fact that $(I-\Omega)^{-1}\{\gamma_l\}_l = \mathbb{I}$ where $\mathbb{I} = \{1\}_l$. Taking the derivative with respect to $Z(l,i)$ yields

$$\left\{\frac{\partial \log P_k}{\partial \log Z(l,i)}\right\}_k = \frac{(I-\Omega)^{-1}v_l}{\varepsilon_l - 1}\left(-\frac{\partial \log \overline{Z_l}}{\partial \log Z(l,i)} + \frac{\partial \log \Delta_l}{\partial \log Z(l,i)}e_l\right) + \log w\mathbb{I}$$

where e_l is the elasticity of f_l and v_l is the $N \times 1$ vector where the element l is one and the others are zero. Note that $\psi^d v_l = (I-\Omega)^{-1}v_l = \{\psi^d_{k,l}\}_k$. Let us substitute the expression of $\frac{\partial \log \Delta_l}{\partial \log Z(l,i)}$ to find that

$$\left\{\frac{\partial \log P_k}{\partial \log Z(l,i)}\right\}_k = -\frac{\{\psi^d_{k,l}\}_k}{\varepsilon_l - 1}\left(1 + \frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,i)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)\right)\frac{\partial \log \overline{Z_l}}{\partial \log Z(l,i)} + \log w\mathbb{I}$$

□

A.7 Proof of Proposition 12 (Elasticity of Sector-Level Sales Share)

From Equation 3 of Proposition 5, the sales share of sectors are such that

$$\left\{\frac{P_k Y_k}{P^C C}\right\}_k' = \beta'(I-\widetilde{\Omega})^{-1}$$

Using the chain rule,

$$\left\{\frac{\partial \log\left(\frac{P_k Y_k}{P^C C}\right)}{\partial \log Z(l,i)}\right\}_k' = \left\{\frac{\mu_l^{-1}}{\frac{P_k Y_k}{P^C C}}\frac{\partial \log(\mu_l^{-1})}{\partial \log Z(l,i)}\frac{\partial\left(\frac{P_k Y_k}{P^C C}\right)}{\partial(\mu_l^{-1})}\right\}_k'$$

Lemma 3 gives that $\left\{\frac{\partial\left(\frac{P_k Y_k}{P^C C}\right)}{\partial(\mu_l^{-1})}\right\}_k' = \mu_l\widetilde{\beta}_l v_l'\left[(I-\widetilde{\Omega})^{-1} - I\right]$. Since $v_l'\psi^s = \{\psi^s_{l,k}\}_k$ we have $\frac{\partial\left(\frac{P_k Y_k}{P^C C}\right)}{\partial(\mu_l^{-1})} = \mu_l\widetilde{\beta}_l(\psi^s_{l,k} - \mathbb{I}_{l,k})$ where $\mathbb{I}_{l,k} = 1$ is $l = k$ and $= 0$ otherwise. Using the expression of $\frac{\partial \log(\mu_l^{-1})}{\partial \log Z(l,i)}$, the fact that $\widetilde{\beta}_l = \frac{P_l Y_l}{P^C C}$, we have

$$\left\{\frac{\partial \log\left(\frac{P_k Y_k}{P^C C}\right)}{\partial \log Z(l,i)}\right\}_k' = \left\{\frac{P_l Y_l}{P_k Y_k}\frac{2e_l}{\Delta_l}\left(\Delta_l - \frac{Z(l,i)^{(\varepsilon_l-1)\gamma_l}}{\overline{Z_l}}\right)(\psi^s_{l,k} - \mathbb{I}_{l,k})\right\}_k'$$

□

B Data Appendix

In this paper, I use two types of data at the sector level. The first one is the I-O data of the Bureau of Economic Analysis. The second one is the concentration data of the Census Bureau.
The Bureau of Economic Analysis provide Input-Output information at different level of aggregation. I use here the detailed I-O table from 2007 which gives information on 389 sectors. They do not provide direct requirement Industry-by-Industry table but instead total Industry-by-Industry requirement table. I then use the formula $\widetilde{\Omega} = (TOT - I)TOT^{-1}$ to find the direct requirement of an industry input to produce one dollar of its

output at the steady-state. To find the value of household consumption share, I use the USE table of the Bureau of Economic Analysis, which gives for each commodity how much the household buy of this commodity. I then recover the share of income spend by the household on each industry by premultiplying these commodity spending share by the MAKE table. The MAKE table gives for each industry how much of each commodity is needed to produce one dollar of output.

The Census Bureau provides concentration measure for different level of aggregation for all sectors except for Agriculture, Forestry, Fishing and Hunting (11); Mining, Quarrying, and Oil and Gas Extraction (21); Construction (23); Management of Companies and Enterprises (55); Public Administration (92). The measure of concentration are the top 4,8,20 and 50 firms' share of total industry revenues in 2002, 2007 and 2012. For manufacturing (31-33), the census bureau also gives the Herfindahl-Hirschman Index among the 50 largest firms. I use this data for the Figures 9 and 8 in Online Appendix E. The former plots measure of sector-level concentration measure in 2002 versus 2007, the latter displays the empirical distribution of the sector-level concentration measures.

Using the correspondence table given by the Bureau of Economic Analysis between the I-O sectors classification and the NAICS 2007 classification, I matched these two data source to plot Figure 1 and to calibrate the model in section 6.

C Numerical Appendix

In this appendix, I first describe how to simulate a path of productivity for each of the 5.6 millions firms in an efficient way. Secondly, I described how to numerically solve for the equilibrium allocation without relying on Assumption 2.

Simulation of a path of Productivity Distributions

To simulate a path of productivity for a large number of firms, I follow the number of firms in each productivity bins rather than the productivity of each firms. The idea is exactly the one described in Proposition 6 and is illustrated in the discussion of the simple example of Figure 5. The key assumption is that productivity evolves on a discrete grid: the number of firms in each bins characterizes the whole distribution of productivity across firms. Since firms in a same productivity bin are exactly similar, following the number of firms in each productivity bins is equivalent to follow the productivity of each firms.

The simulation procedure follows closely the proof of Proposition 6 in Appendix F.2. For a given period t, for a given sector k, and for a given distribution of firms $g_t^{(k)}$ in this sector i.e. a vector whose elements are the number of firms in each productivity bins, we know that the number of firms in each productivity bins at time $t+1$ that were in productivity bin n at t follow a multinomial random vector with the number of trials being $g_{t,n}^{(k)}$ and the event probability given by the n^{th} row vector of the matrix $\mathcal{P}^{(k)}$. The next productivity distribution $g_{t+1}^{(k)}$ is just the sum of all these conditional distributions. This procedure makes the simulation of a path of productivity for each of the 5.6 millions firms extremely efficient. I use this procedure in all the simulation exercises in the main text of this paper.

Solving the Equilibrium Allocation

Given the distributions of productivity across firms in each sectors, I can solve for the equilibrium allocation. The first step is to solve for the Bertrand firm-level problem described in Proposition 1. Note that after substituting for the firm's marginal cost, and defining $X_k = \left(\frac{\varepsilon_k}{\varepsilon_k - 1} P_k^{-1} w^{\gamma_k} \prod_{l=1}^N P_l^{\omega_{k,l}}\right)^{1-\varepsilon_k}$ this problem is equivalent to

$$\forall i \in [|1, N_k|], \begin{cases} s(k,i) &= \left(\mu(k,i) Z(k,i)^{-\gamma_k} \frac{\varepsilon_k - 1}{\varepsilon_k}\right)^{1-\varepsilon_k} X_k \\ \mu(k,i) &= \frac{\varepsilon_k - (\varepsilon_k - 1)s(k,i)}{\varepsilon_k - 1 - (\varepsilon_k - 1)s(k,i)} \end{cases} \quad \text{and} \quad X_k = \left(\sum_{i=1}^{N_k} \mu(k,i)^{1-\varepsilon_k} Z(k,i)^{\gamma_k(\varepsilon_k - 1)} \left(\frac{\varepsilon_k}{\varepsilon_k - 1}\right)^{\varepsilon_k - 1}\right)^{-1}$$

Given firm-level productivities $Z(k,i)$, the above system of equation can be solve numerically and gives, for each sector, the firm-level sales share $s(k,i)$, the markups $\mu(k,i)$ and X_k. Note that there is one equation per firm, so when the number of firms is very large as it is in the baseline calibration the size of this system becomes too large. To save computation time, I rewrite the above system for each possible productivity level and use the number of firms in each productivity bins given by the vector $g_t^{(k)}$: the sum in the right hand side equation satisfy by X_k is now over the productivity bins rather than the firms.

$$\forall n \in [|0, M_k|], \begin{cases} s(k,n) & = \left(\mu(k,n)(\varphi_k^n)^{-\gamma_k}\frac{\varepsilon_k - 1}{\varepsilon_k}\right)^{1-\varepsilon_k} X_k \\ \mu(k,n) & = \frac{\varepsilon_k-(\varepsilon_k-1)s(k,n)}{\varepsilon_k-1-(\varepsilon_k-1)s(k,n)} \end{cases} \quad \text{and} \quad X_k = \left(\sum_{n=0}^{M_k} \mu(k,n)^{1-\varepsilon_k}(\varphi_k^n)^{\gamma_k(\varepsilon_k-1)}\left(\frac{\varepsilon_k}{\varepsilon_k-1}\right)^{\varepsilon_k-1} g_{t,n}^{(k)}\right)^{-1}$$

where $\mu(k,n)$ and $s(k,n)$ stands for the markup and the sales share of firms with productivity level φ_k^n. The number of equation of this system is the number of productivity bins M_k in sector k and is independent of the number of firms N_k in the sector k.
With the distribution of sales share and markups across firms in each sectors, I can compute sector-level productivities and markups as they are defined in Section 3.2. Given these sector level markups and productivities, the equilibrium allocation is entirely characterized by Proposition 8.

D Elastic Labor Supply

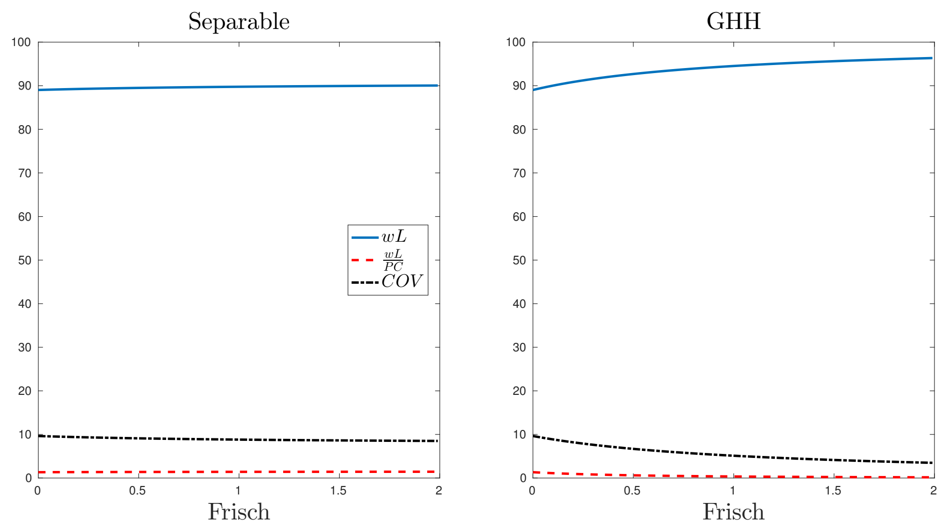
In this appendix, I show how the main results are affected by relaxing the inelastic labor supply assumption. I consider the case of separable and GHH preferences. In both cases aggregate output Y_t is a function of the equilibrium wage and the profit share as in the inelastic cases (Equation 7). With separable preferences $U(C,L) = \frac{C^{1-\eta}}{1-\eta} - \theta\frac{L^{1+1/f}}{1+1/f}$ where f is the Frisch elasticity of the labor supply and η is the coefficient of relative risk aversion, aggregate output is $\log Y_t = \left(1 + \frac{1-\eta}{1/f+\eta}\right)\log w_t - \left(1 - \frac{\eta}{1/f+\eta}\right)\log\left(1 - Pro_t/(P_t^C C_t)\right)$. With GHH preferences $U(C,L) = \frac{1}{1-\eta}\left(C - \theta\frac{L^{1+1/f}}{1+1/f}\right)^{1-\eta}$, aggregate output is $\log Y_t = (1+f)\log w_t - \log\left(1 - Pro_t/(P_t^C C_t)\right)$.
Let us define the "downstream" and "upstream" part of aggregate output as in Section 4 i.e. the (log) labor income and the (log) inverse of the labor income share respectively. With GHH preferences the elasticity in Proposition 9 is just multiply by $(1+f)$, while the result in Proposition 10 is unaffected. With these preferences the income effect does not affect the labor supply and labor income is only a function of the equilibrium wage: $w_t L_t = w_t^{1+f}\theta^{-f}$. With separable preferences, labor income is a function of aggregate output $w_t L_t = w_t^{1+f}\theta^{-f}C^{-\eta f}$ and the elasticity of the "downstream" part is a weighted sum of the elasticities in Propositions 9 and 10. The elasticity of the "upstream" part is unaffected.
The decomposition between the "downstream" and "upstream" part of the variance of aggregate output for different preferences can be found in Table 5. Each row is the decomposition of the variance of log deviation of aggregate output for different preferences. In every row the calibration is as in the baseline case of Table 2 with $\eta = 1$ and a Frisch elasticity of $f = 2$. Figure 7 plots these decompositions for a Frisch elasticity varying from 0 to 2.

Table 5: Aggregate Volatility and Elastic Labor Supply

	Total	Downstream	Upstream	COV
Inelastic	100	89.03	1.33	9.64
Separable	100	90.051	1.4452	8.5038
GHH	100	96.3625	0.16058	3.4769

NOTE: Each row is the variance decomposition of the percentage deviation of aggregate output Y_t between the contribution of labor income and labor share, i.e. the "downstream" and the "upstream" part of aggregate output. The first row is the baseline case of inelastic labor supply (as in Table 3). The second row is the case of separable preference: $U(C,L) = \frac{C^{1-\eta}}{1-\eta} - \theta\frac{L^{1+1/f}}{1+1/f}$. The third row is the case of GHH preferences: $U(C,L) = \frac{1}{1-\eta}\left(C - \theta\frac{L^{1+1/f}}{1+1/f}\right)^{1-\eta}$. The calibration is as in Table 2 with $\eta = 1$ and $f = 2$. Numbers are reported in percentage points. These statistics comes from a 4000 periods simulations.

Figure 7: Aggregate Volatility and Firsch



NOTE: Aggregate output variance decomposition between labor income and labor share as a function of the Firsch elasticity of labor supply. Left panel: for separable preferences $U(C,L) = \frac{C^{1-\eta}}{1-\eta} - \theta \frac{L^{1+1/f}}{1+1/f}$. Right panel: for GHH preferences $U(C,L) = \frac{1}{1-\eta}\left(C - \theta \frac{L^{1+1/f}}{1+1/f}\right)^{1-\eta}$. The calibration is as in Table 2 with $\eta = 1$ and $f = 2$. Numbers are reported in percentage points. These statistics comes from a 4000 periods simulations.

Online Appendix to
“IO in I-O: Size, Industrial Organization and the Input-Output Network Make a Firm Structurally Important”

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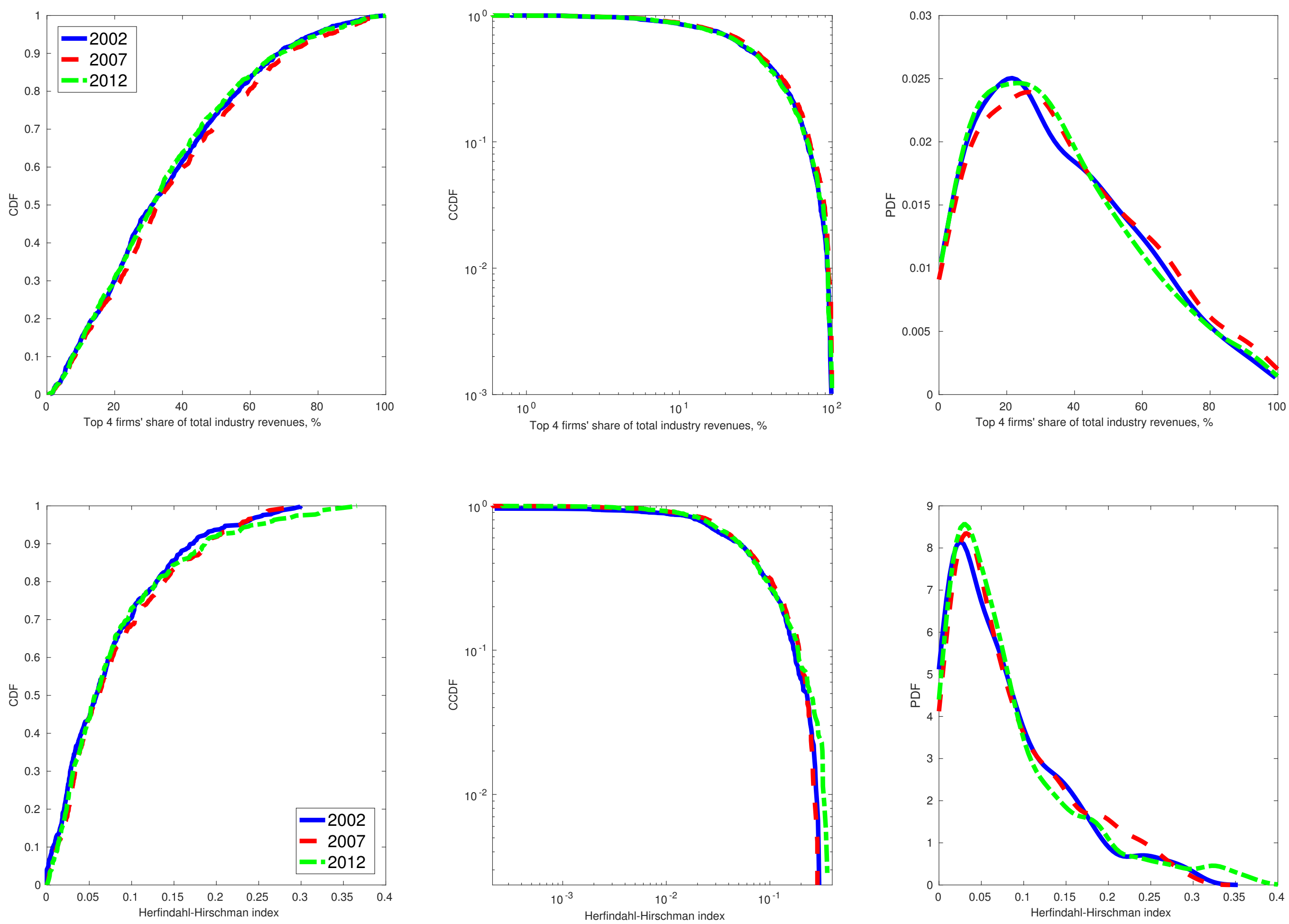
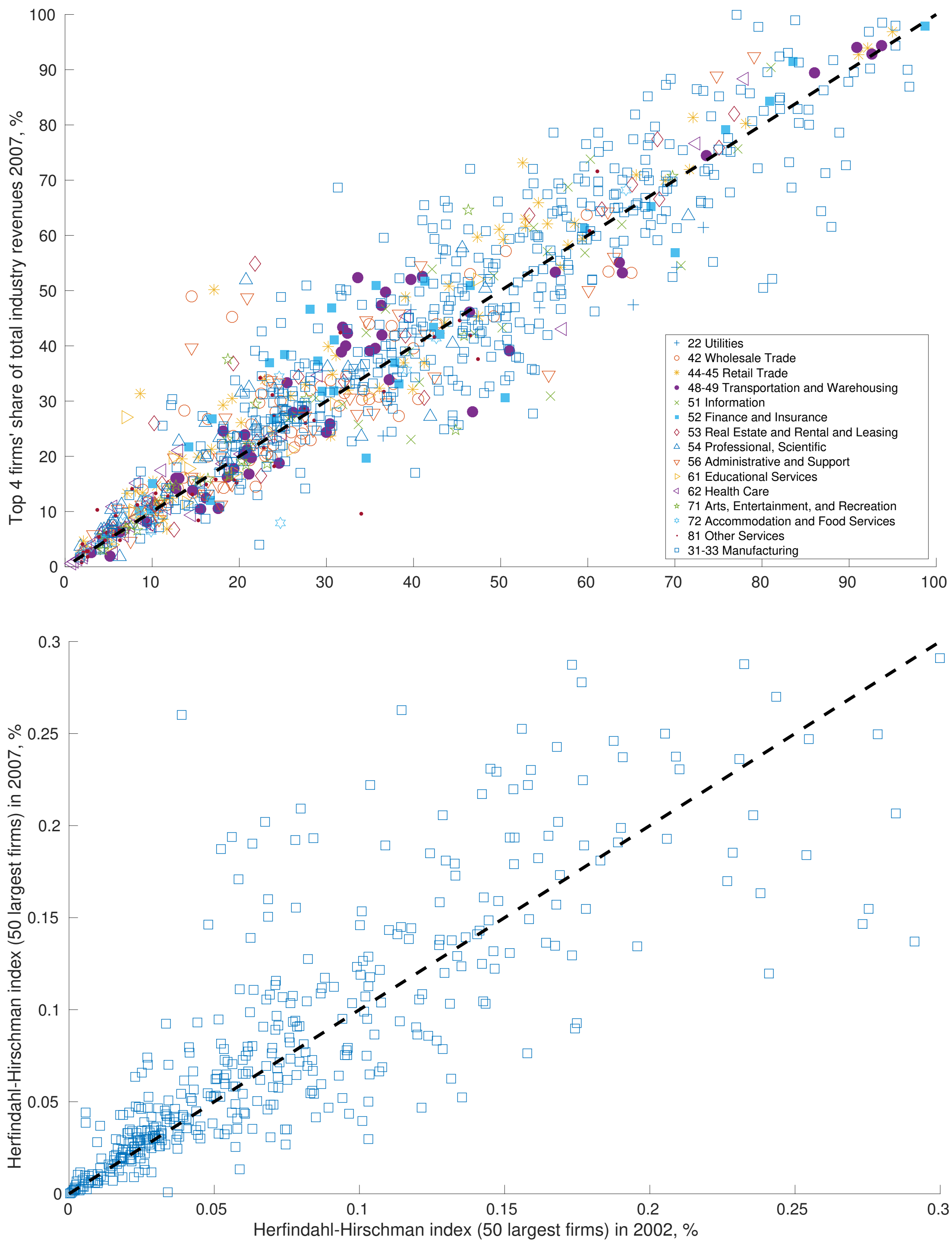
E Figures Appendix

Figure 8: Sectors’ Concentration Distribution



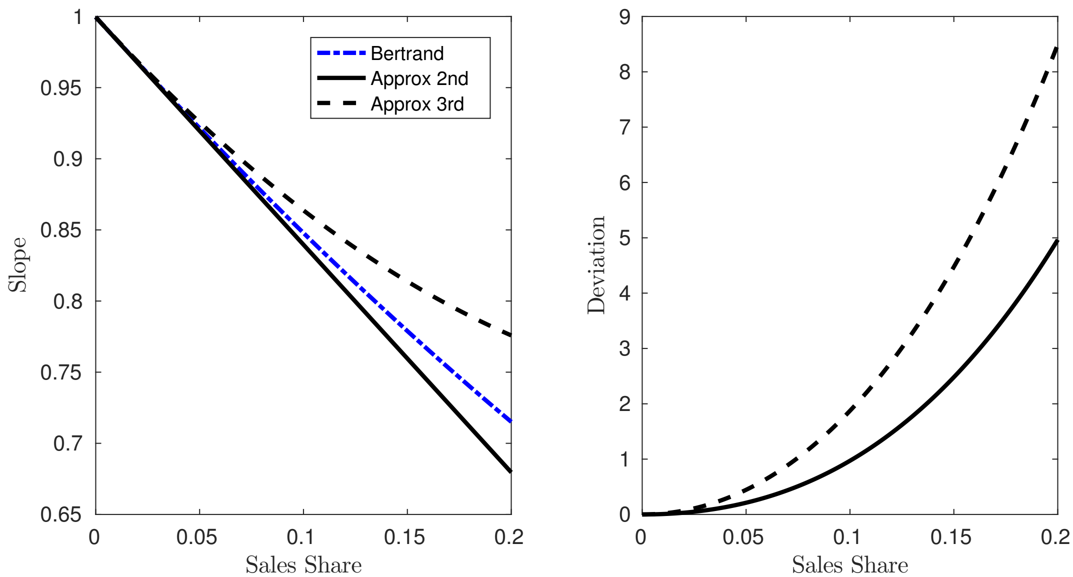
NOTE: Empirical cumulative distribution function (left), counter cumulative distribution function (center), and Kernel smoothing function estimate of the probability distribution function (right) of top four firms’ share of total revenues for 6 digits NAICS industry (top panel) and of Herfindahl-Hirschman index for the 50 largest companies for 6 digits NAICS manufacturing industries (31-33). Source: Census Bureau.

Figure 9: Sector Concentration



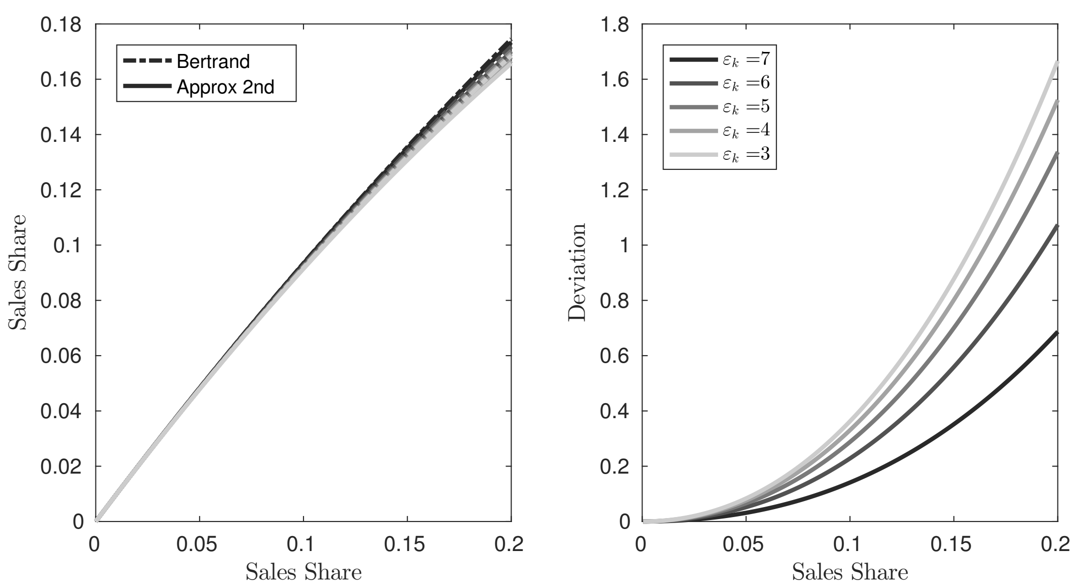
NOTE: Top Panel: Top four firms' share of total revenues in 2002 and in 2007 for 6-digit NAICS industry. The mean value is 35.37% in 2002 and 37.21% in 2007. 970 industries. Bottom Panel: Herfindahl-Hirschman-Index for the 50 largest companies in 2002 and in 2007 for 6 digits NAICS manufacturing industry (31-33). 448 industries. Source: Census Bureau.

Figure 10: Approximation of Firms' Sales Share (Slope)



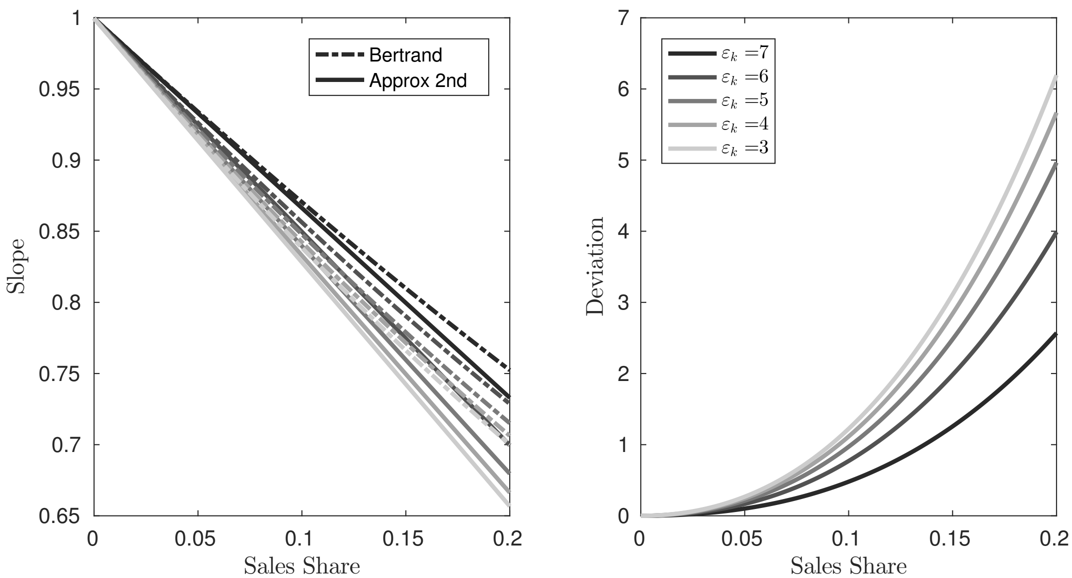
NOTE: For $\varepsilon_k = 5$. The left panel shows the slope of Bertrand sales share, the slopes of the second and the third order approximation in Proposition 2 as a function of the monopolistic sales share. The right panel shows percentage deviation of the slope of both approximations with respect to the numerical solution.

Figure 11: Approximation of Firms' Sales Share (Different ε_k)



NOTE: For different values of ε_k. The left panel shows the Bertrand sales share using a numerical solver (solid) and the second order approximation (dashed) as a function of the monopolistic sales share. The right panel shows percentage deviation of the second order approximation with respect to the numerical solution.

Figure 12: Approximation of Firms' Sales Share (Slope for Different ε_k)



NOTE: For different values of ε_k. The left panel shows the slope of Bertrand sales share (solid) and the slopes of the second order approximation (solid) in Proposition 2 as a function of the monopolistic sales share. The right panel shows percentage deviation of the slope of both approximations with respect to the numerical solution.

F Proof Appendix

F.1 Proof of Proposition 3 (Size-Volatility)

Let us first compute the variance of the growth rate of productivity. Let us call $n_{t,k,i}$ the integer such that productivity level of firm i in sector k is such that $Z_t(k,i) = \varphi_k^{n_{t,k,i}}$. Note that $Z_t(k,i)$ follows the Markovian process described in Assumption 1, therefore its growth rate satisfies:

$$\frac{Z_{t+1}(k,i) - Z_t(k,i)}{Z_t(k,i)} = \frac{\varphi_k^{n_{t+1,k,i}} - \varphi_k^{n_{t,k,i}}}{\varphi_k^{n_{t,k,i}}} = \varphi_k^{n_{t+1,k,i} - n_{t,k,i}} - 1 = \begin{cases} \varphi_k^{-1} - 1 & & a \\ 0 & \text{with proba} & b \\ \varphi_k - 1 & & c \end{cases}$$

Let us compute conditional expected growth rate of the productivity of firm i in sector k:

$$\mathbb{E}_t \left[\frac{Z_{t+1}(k,i) - Z_t(k,i)}{Z_t(k,i)} \right] = a \left(\varphi_k^{-1} - 1 \right) + c \left(\varphi_k - 1 \right) = a\varphi_k^{-1} + b + c\varphi_k - 1$$

while the conditional variance of the growth rate of $Z_t(k,i)$ is

$$\mathbb{V}ar_t \left[\frac{Z_{t+1}(k,i) - Z_t(k,i)}{Z_t(k,i)} \right] = a \left(\varphi_k^{-1} - 1 \right)^2 + c \left(\varphi_k - 1 \right)^2 - \left(a\varphi_k^{-1} + b + c\varphi_k - 1 \right)^2 = \sigma_k^2$$

These conditional moments are independent of the level $Z_t(k,i)$ at time t and they are equal to their unconditional counterpart. This complete the first part of the proof.

Let us now turn to the growth rate of the sales share $s_t(k,i)$ of firm i in sector k. To this end, I am using the approximation in Assumption 2. The first step is to find the growth rate of sales share under monopolistic competition $\hat{s}_t(k,i)$. Note that $\hat{s}_t(k,i) \propto Z_t(k,i)^{-\gamma_k(1-\varepsilon_k)}$. Keeping sectors' price and the wage constant, at the first order we have $g_{t+1}^{\hat{s}(k,i)} = -\gamma_k(1-\varepsilon_k) g_{t+1}^{Z(k,i)}$ where $g_{t+1}^x = \frac{x_{t+1} - x_t}{x_t}$.[29] Let us focus on the case of Bertrand competition. All the following calculation are very similar under Cournot. Thanks to Assumption 2, the sales share of firm i in sector k is such that $s_t(k,i) = \hat{s}_t(k,i) - (1-\varepsilon_k^{-1})\hat{s}_t(k,i)^2$ which becomes

$$g_{t+1}^{s(k,i)} = \frac{\hat{s}_t(k,i)}{s_t(k,i)} g_{t+1}^{\hat{s}(k,i)} - 2\frac{(1-\varepsilon_k^{-1})\hat{s}_t(k,i)^2}{s_t(k,i)} g_{t+1}^{\hat{s}(k,i)}$$

Using the fact that $g_{t+1}^{\hat{s}(k,i)} = -\gamma_k(1-\varepsilon_k) g_{t+1}^{Z(k,i)}$ and after some simplification, we have

$$g_{t+1}^{s(k,i)} = \gamma_k(\varepsilon_k - 1) \frac{1 - 2(1-\varepsilon_k^{-1})\hat{s}_t(k,i)}{1 - (1-\varepsilon_k^{-1})\hat{s}_t(k,i)} g_{t+1}^{Z(k,i)}$$

The conditional variance of the growth rate of firm i in sector k is

$$\mathbb{V}ar_t \left[\frac{s_{t+1}(k,i) - s_t(k,i)}{s_t(k,i)} \right] = \gamma_k^2(\varepsilon_k - 1)^2 \left(\frac{1 - 2(1-\varepsilon_k^{-1})\hat{s}_t(k,i)}{1 - (1-\varepsilon_k^{-1})\hat{s}_t(k,i)} \right)^2 \sigma_k^2$$

The above equation shows that the variance of the growth rate of the sales share of a firm is a strictly decreasing function of its level. Indeed the function $g_k : x \mapsto \gamma_k^2(\varepsilon_k - 1)^2 \frac{1-2(1-\varepsilon_k^{-1})x}{1-(1-\varepsilon_k^{-1})x}$ is strictly decreasing and the absolute value of its slope $|g_k'(x)| = \gamma_k^2(\varepsilon_k - 1)^2 \frac{(1-\varepsilon_k^{-1})}{(1-(1-\varepsilon_k^{-1})x)^2}$ is strictly increasing in ε_k. $\square$

F.2 Proof of Proposition 6 (Sector k's Productivity Distribution Dynamics)

In this section, I first derive equation 4 before solving for the stationary distribution in sector k.

[29] Equivalently, one can compute this growth rate under the stationary equilibrium, the steady-state of this economy where aggregate and sectoral quantities and prices are constant (as if they were a continuum of sectors).

Proof of Equation 4: For n such that $0 < n < M_k$, Assumption 1 implies that

$$g^{(k)}_{t+1,n} = f^{n,n-1}_{k,t+1} + f^{n,n}_{k,t+1} + f^{n,n+1}_{k,t+1}$$

where $f^{n',n}_{k,t+1}$ is the number of firms in productivity bin n' at $t+1$ that were in bin n at time t. Thanks to Assumption 1 the 3×1 vector $f^{\cdot,n}_{k,t+1} = (f^{n-1,n}_{k,t+1}, f^{n,n}_{k,t+1}, f^{n+1,n}_{k,t+1})'$ follow a multinomial distribution with number of trial $g^{(k)}_{t,n}$ and event probabilities $(a_k, b_k, c_k)'$. It follows that the 3×1 vector $f^{\cdot,n}_{k,t+1}$ has a mean $\mathbb{E}_t\left[f^{\cdot,n}_{k,t+1}\right] = g^{(k)}_{t,n}(a_k, b_k, c_k)'$ and a variance-covariance matrix equal to $g^{(k)}_{t,n}\Sigma_k$ where

$$\Sigma_k = \begin{pmatrix} a_k(1-a_k) & -a_k b_k & -a_k c_k \\ -a_k b_k & b_k(1-b_k) & -b_k c_k \\ -a_k c_k & -b_k c_k & c_k(1-c_k) \end{pmatrix}$$

Note that $f^{\cdot,n}_{k,t+1}$ are independent across n and thus

$$\mathbb{E}_t\left[g^{(k)}_{t+1,n}\right] = \mathbb{E}_t\left[f^{n,n-1}_{k,t+1}\right] + \mathbb{E}_t\left[f^{n,n}_{k,t+1}\right] + \mathbb{E}_t\left[f^{n,n+1}_{k,t+1}\right] = a_k g^{(k)}_{t,n+1} + b_k g^{(k)}_{t,n} + c_k g^{(k)}_{t,n-1}$$

$$\mathbb{V}ar_t\left[g^{(k)}_{t+1,n}\right] = \mathbb{V}ar_t\left[f^{n,n-1}_{k,t+1}\right] + \mathbb{V}ar_t\left[f^{n,n}_{k,t+1}\right] + \mathbb{V}ar_t\left[f^{n,n+1}_{k,t+1}\right] = a_k(1-a_k)g^{(k)}_{t,n+1} + b_k(1-b_k)g^{(k)}_{t,n} + c_k(1-c_k)g^{(k)}_{t,n-1}$$

For completeness, let us look at the covariance structure of the $g^{(k)}_{t+1,n}$:

$$\mathbb{C}ov_t\left[g^{(k)}_{t+1,n}; g^{(k)}_{t+1,n'}\right] = \mathbb{C}ov_t\left[f^{n,n-1}_{k,t+1} + f^{n,n}_{k,t+1} + f^{n,n+1}_{k,t+1}; f^{n',n'-1}_{k,t+1} + f^{n',n'}_{k,t+1} + f^{n',n'+1}_{k,t+1}\right] = 0 \text{ if } |n-n'| > 2$$

since the $f^{\cdot,n}_{k,t+1}$ are independent across n. For $n' = n+1$, we have:

$$\begin{aligned} \mathbb{C}ov_t\left[g^{(k)}_{t+1,n}; g^{(k)}_{t+1,n+1}\right] &= \mathbb{C}ov_t\left[f^{n,n-1}_{k,t+1} + f^{n,n}_{k,t+1} + f^{n,n+1}_{k,t+1}; f^{n+1,n}_{k,t+1} + f^{n+1,n+1}_{k,t+1} + f^{n+1,n+2}_{k,t+1}\right] \\ &= \mathbb{C}ov_t\left[f^{n,n}_{k,t+1}; f^{n+1,n}_{k,t+1}\right] + \mathbb{C}ov_t\left[f^{n,n+1}_{k,t+1}; f^{n+1,n+1}_{k,t+1}\right] \\ &= -b_k c_k g^{(k)}_{t,n} - a_k b_k g^{(k)}_{t,n+1} \end{aligned}$$

using the fact the variance-covariance matrix of $f^{\cdot,n}_{k,t+1}$ is equal to $\mu^{(k)}_{t,n}\Sigma_k$ for all $n > 0$. The same reasoning apply for $n' = n+2$.
For $n = 0$, Assumption 1 implies

$$g^{(k)}_{t+1,0} = f^{0,0}_{k,t+1} + f^{0,1}_{k,t+1}$$

and that the 2×1 vector $f^{\cdot,0}_{k,t+1} = (f^{0,0}_{k,t+1}, f^{1,0}_{k,t+1})'$ follow a multinomial distribution with number of trial $g^{(k)}_{t,0}$ and event probabilities $(a_k + b_k, c_k)'$. The same reasoning than for $n > 0$ applies.
For $n = M_k$, Assumption 1 implies

$$g^{(k)}_{t+1,M} = f^{M,M-1}_{k,t+1} + f^{M,M}_{k,t+1}$$

and that the 2×1 vector $f^{\cdot,M}_{k,t+1} = (f^{M-1,M}_{k,t+1}, f^{M,M}_{k,t+1})'$ follow a multinomial distribution with number of trial $g^{(k)}_{t,M}$ and event probabilities $(a_k, c_k + b_k)'$. The same reasoning than for $n > 0$ applies.
Gathering the above results we have in matrix form:

$$g^{(k)}_{t+1} = (\mathcal{P}^{(k)})' g^{(k)}_t + \epsilon^{(k)}_t$$

where $\epsilon^{(k)}_t$ is the $M \times 1$ vector of $\epsilon^{(k)}_{t,n}$. This complete the derivation of Equation 4.□

Stationary Distribution in Sector k: Let us drop the (k) superscript and subscript to simplify notation. The stationary distribution is a sequence that solve the following system:

$$\begin{aligned} (BC1) \qquad & g_0 = (a+b)g_0 + a g_1 \\ (BC2) \qquad & g_M = c g_{M-1} + (b+c) g_M \\ (EH) \qquad & g_n = a g_{n+1} + b g_n + c g_{n-1} \end{aligned}$$

Let us solve for the general solution of (EH). This equation is a second order linear difference equation equivalent to $0 = a g_{n+1} + (b-1) g_n + c g_{n-1} = a g_{n+1} - (a+c) g_n + c g_{n-1}$, with an associated second order polynomial

$aX^2 - (a+c)X + c = 0$ which have roots 1 and $\frac{c}{a}$. The general solution of (EH) is thus $g_n = K_1 + K_2 \left(\frac{c}{a}\right)^n$ where K_1 and K_2 are constant to solve for.
Let us substitute this general solution in the equation $(BC1)$, it yields

$$K_1 + K_2 = (a+b)(K_1+K_2) + aK_1 + aK_2\frac{c}{a} = (2a+b)K1 + (a+b+c)K2$$

since $a+b+c=1$, $(BC1)$ implies $K_1 = (2a+b)K_1$. Since $a < c$ and $a+b+c=1$, then $2a+b \neq 1$ and thus $K_1 = 0$. The general solution of this system is then $g_n = K_2\left(\frac{c}{a}\right)^n$. It is trivial to see that $(BC2)$ is satisfied by this general solution. Since $n = \frac{\log\varphi^n}{\log\varphi}$, thus $\left(\frac{c}{a}\right)^n = \exp\left(-s\log\frac{a}{c}\right) = \exp\left(-\frac{\log\varphi^n}{\log\varphi}\log\frac{a}{c}\right) = (\varphi^n)^{-\delta}$ with $\delta = \frac{\log\frac{a}{c}}{\log\varphi}$. It follows that $g_n = K_2(\varphi^n)^{-\delta}$
To solve for K_2, let us use the fact that g_n has to sum to N_k, the number of firms in sector k.

$$N_k = \sum_{n=0}^{M} g_n = K_2\sum_{n=0}^{M}\left(\varphi^{-\delta}\right)^n = K_2\frac{1-\left(\varphi^{-\delta}\right)^{M+1}}{1-\varphi^{-\delta}}$$

since $\varphi^{-\delta} < 1$. It follows that $K_2 = N_k\frac{(1-\varphi^{-\delta})}{1-(\varphi^{-\delta})^{M+1}}$ and $g_n^{(k)} = N_k\frac{(1-\varphi^{-\delta})}{1-(\varphi^{-\delta})^{M+1}}(\varphi^n)^{-\delta}$. □

F.3 Proof of Proposition 7 (Dynamics of $\overline{Z_{t,k}}$ and $\Delta_{t,k}$)

Let us define $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z_t(k,i)^\xi$ the ξth moment of the productivity distribution within sector k at time t. Note that since productivity evolves on the discrete state space $\Phi_k = \{1, \varphi_k, \cdots, \varphi_k^n, \cdots, \varphi_k^{M_k}\}$, we can rewrite $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z_t(k,i)^\xi = \sum_{i=1}^{N_k}\varphi_k^{\xi n_{t,k,i}}$ where $n_{t,k,i}$ is such that the firm i in sector k has a productivity level $\varphi^{n_{t,k,i}}$ at time t. It follows that $MZ_{t,k}(\xi) = \sum_{n=0}^{M_k}(\varphi_k^n)^\xi g_{t,n}^{(k)}$ by instead of summing over firms i, summing over productivity level φ_k^n. Below, I am showing two lemmas that totally described the dynamics of the moments $MZ_{t,k}(\xi)$ for any ξ. With these results in hand I am then characterizing the dynamics of the two moments of interest: $\overline{Z_{t,k}}$ and $\Delta_{t,k}$.

Lemma 4 ***(Dynamics of Moments of the Productivity Distribution)*** *Under Assumption 1, the ξth moment of the productivity distribution within sector k, $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z(k,i)^\xi$, satisfies*

$$MZ_{t+1,k}(\xi) = \rho_k(\xi)MZ_{t,k}(\xi) + O_{t,k}^M(\xi) + \sigma_{t,k}(\xi)\varepsilon_t$$
$$\sigma_{t,k}(\xi)^2 = \varrho_k(\xi)MZ_{t,k}(2\xi) + O_{t,k}^\sigma(\xi)$$

where ε_t is an iid (across t and k) random variable following a $\mathcal{N}(0,1)$, where $\rho_k(\xi) = a_k\varphi_k^{-\xi} + b_k + c_k\varphi_k^\xi$, and where $\varrho_k(\xi) = a_k\varphi_k^{-2\xi} + b_k + c_k\varphi_k^{2\xi} - \rho_k(\xi)^2$.

Proof of Lemma 4: Note first that

$$MZ_{t+1,k}(\xi) = \sum_{i=1}^{N_k} Z_{t+1}(k,i)^\xi = \sum_{i=1}^{N_k}\varphi_k^{\xi n_{t+1,k,i}} = \sum_{n=0}^{M_k}(\varphi_k^n)^\xi g_{t+1,n}^{(k)}$$

where $g_{t+1,n}^{(k)}$ is a stochastic as shown in Proposition 6. In the proof of this proposition we have shown that for n such that $0 < n < M_k$

$$g_{t+1,n}^{(k)} = f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1}$$

where $f_{k,t+1}^{n',n}$ is the number of firms in productivity bin n' at $t+1$ that were in bin n at time t. Given assumption 1 the 3×1 vector $f_{k,t+1}^{\cdot,n} = (f_{k,t+1}^{n-1,n}, f_{k,t+1}^{n,n}, f_{k,t+1}^{n+1,n})'$ follow a multinomial distribution with number of trial $g_{t,n}^{(k)}$ and event probabilities $(a_k, b_k, c_k)'$. In other words,

$$f_{k,t+1}^{\cdot,n} = \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n}\end{pmatrix} \rightsquigarrow \mathcal{M}ulti\left(\mu_{t,n}^{(k)}, \begin{pmatrix} a_k \\ b_k \\ c_k\end{pmatrix}\right)$$

Severini (2005) (p377 example 12.7) shows that a multinomial distribution can be approximate (i.e converge in distribution) by a multivariate normal distribution:

$$\frac{1}{\sqrt{g_{t,n}^{(k)}}}\left(f_{k,t+1}^{\cdot,n} - g_{t,n}^{(k)}\begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}\right) \underset{g_{t,n}^{(k)}\to\infty}{\xrightarrow{\mathcal{D}}} Z \rightsquigarrow \mathcal{N}(0,\Sigma_k)$$

where $\Sigma_k = \begin{pmatrix} a_k(1-a_k) & -a_kb_k & -a_kc_k \\ -a_kb_k & b_k(1-b_k) & -b_kc_k \\ -a_kc_k & -b_kc_k & c_k(1-c_k) \end{pmatrix}$.

For $n = 0$, thanks to Assumption 1 $g_{t+1,0}^{(k)} = f_{k,t+1}^{0,0} + f_{k,t+1}^{0,1}$ and the 2×1 vector $f_{k,t+1}^{\cdot,0} = (f_{k,t+1}^{0,0}, f_{k,t+1}^{1,0})'$ follow a multinomial distribution with number of trial $g_{t,0}^{(k)}$ and event probabilities $(a_k+b_k, c_k)'$. Using the same result in Severini (2005), $\frac{1}{\sqrt{g_{t,0}^{(k)}}}\left(f_{k,t+1}^{\cdot,0} - g_{t,0}^{(k)}\begin{pmatrix} a_k+b_k \\ c_k \end{pmatrix}\right) \underset{g_{t,0}^{(k)}\to\infty}{\xrightarrow{\mathcal{D}}} Z \rightsquigarrow \mathcal{N}(0,\Sigma_k^{(0)})$ where $\Sigma_k^{(0)} = \begin{pmatrix} c_k(1-c_k) & -c_k(1-c_k) \\ -c_k(1-c_k) & c_k(1-c_k) \end{pmatrix}$.

For $n = M_k$, thanks to Assumption 1 $g_{t+1,0}^{(k)} = f_{k,t+1}^{M,M} + f_{k,t+1}^{M,M-1}$ and the 2×1 vector $f_{k,t+1}^{\cdot,M} = (f_{k,t+1}^{M-1,M}, f_{k,t+1}^{M,M})'$ follow a multinomial distribution with number of trial $g_{t,M}^{(k)}$ and event probabilities $(a_k, b_k+c_k)'$. Using the same result in Severini (2005), $\frac{1}{\sqrt{g_{t,0}^{(k)}}}\left(f_{k,t+1}^{\cdot,M} - g_{t,M}^{(k)}\begin{pmatrix} a_k \\ b_k+c_k \end{pmatrix}\right) \underset{g_{t,M}^{(k)}\to\infty}{\xrightarrow{\mathcal{D}}} Z \rightsquigarrow \mathcal{N}(0,\Sigma_k^{(M)})$ where $\Sigma_k^{(M)} = \begin{pmatrix} a_k(1-a_k) & -a_k(1-a_k) \\ -a_k(1-a_k) & a_k(1-a_k) \end{pmatrix}$.

Let us keep these results in mind and let us go back to (I drop the subscript k to keep the notation parsimonious)

$$\begin{aligned}
MZ_{t+1,k}(\xi) &= \sum_{n=0}^{M}(\varphi^n)^\xi G_{t+1,n}^{(k)} = g_{t+1,0}^{(k)} + \sum_{n=1}^{M-1}(\varphi^n)^\xi g_{t+1,n}^{(k)} + (\varphi^M)^\xi g_{t+1,M}^{(k)} \\
&= f_{k,t+1}^{0,0} + f_{k,t+1}^{0,1} + \sum_{n=1}^{M-1}(\varphi^n)^\xi\left(f_{k,t+1}^{n,n-1} + f_{k,t+1}^{n,n} + f_{k,t+1}^{n,n+1}\right) + (\varphi^M)^\xi\left(f_{k,t+1}^{M,M-1} + f_{k,t+1}^{M,M}\right) \\
&= f_{k,t+1}^{0,0} + f_{k,t+1}^{0,1} + \sum_{n=1}^{M-1}(\varphi^\xi)^n f_{k,t+1}^{n,n-1} + \sum_{n=1}^{M-1}(\varphi^\xi)^n f_{k,t+1}^{n,n} + \sum_{n=1}^{M-1}(\varphi^\xi)^n f_{k,t+1}^{n,n+1} + (\varphi^M)^\xi\left(f_{k,t+1}^{M,M-1} + f_{k,t+1}^{M,M}\right) \\
&= f_{k,t+1}^{0,0} + f_{k,t+1}^{0,1} + \sum_{n=0}^{M-2}(\varphi^\xi)^{n+1} f_{k,t+1}^{n+1,n} + \sum_{n=1}^{M-1}(\varphi^\xi)^n f_{k,t+1}^{n,n} + \sum_{n=2}^{M}(\varphi^\xi)^{n-1} f_{k,t+1}^{n-1,n} + (\varphi^M)^\xi\left(f_{k,t+1}^{M,M-1} + f_{k,t+1}^{M,M}\right) \\
&= f_{k,t+1}^{0,0} + \varphi^\xi f_{k,t+1}^{1,0} + \sum_{n=1}^{M-1}(\varphi^\xi)^n\left(\varphi^\xi f_{k,t+1}^{n+1,n} + f_{k,t+1}^{n,n} + \varphi^{-\xi} + f_{k,t+1}^{n-1,n}\right) + (\varphi^\xi)^M\left(f_{k,t+1}^{M,M} + \varphi^{-\xi} f_{k,t+1}^{M-1,M}\right) \\
&= \begin{pmatrix} 1 \\ \varphi^\xi \end{pmatrix}'\begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix} + \sum_{n=1}^{M-1}(\varphi^\xi)^n\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^\xi \end{pmatrix}'\begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} + (\varphi^\xi)^M\begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}'\begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix} \\
&= \left(\rho_{k,0}g_{t,0}^{(k)} + \sqrt{\varrho_{k,0}g_{t,0}^{(k)}}\varepsilon_{t+1,0}\right) + \sum_{n=1}^{M-1}(\varphi^\xi)^n\left(\rho_k g_{t,n}^{(k)} + \sqrt{\varrho_k g_{t,n}^{(k)}}\varepsilon_{t+1,n}\right)\ldots \\
&\quad \ldots + (\varphi^\xi)^M\left(\rho_{k,M}g_{t,M}^{(k)} + \sqrt{\varrho_{k,M}g_{t,n}^{(k)}}\varepsilon_{t+1,M}\right)
\end{aligned}$$

Since $\begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} \approx Z \rightsquigarrow \mathcal{N}\left(g_{t,n}^{(k)}\begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}, g_{t,n}^{(k)}\Sigma_k\right)$ it follows that $\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^\xi \end{pmatrix}'\begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} \approx \begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^\xi \end{pmatrix}' Z \rightsquigarrow \mathcal{N}\left(g_{t,n}^{(k)}\begin{pmatrix} x^{-\xi} \\ 1 \\ x^\xi \end{pmatrix}'\begin{pmatrix} a_k \\ b_k \\ c_k \end{pmatrix}, g_{t,n}^{(k)}\begin{pmatrix} x^{-\xi} \\ 1 \\ x^\xi \end{pmatrix}'\Sigma\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^\xi \end{pmatrix}\right) = \mathcal{N}\left(g_{t,n}^{(k)}\rho_k, g_{t,n}^{(k)}\varrho_k\right)$. where $\rho_k = a_k\varphi^{-\xi} + b_k + c_k\varphi^\xi$ and $\varrho_k = a_k\varphi^{-2\xi} + b_k + c_k\varphi^{2\xi} - \rho_k^2$. The same reasoning apply for $n = M_k$ with $\rho_{k,M} = \rho_k + c(1-\varphi^\xi) := \rho_k + \widetilde{\rho}_{k,M}$ and $\varrho_{k,M} = \varrho_k - c(1-c)(1-x^{2\xi}) - 2cb(1-\varphi^\xi) - 2ca(1-\varphi^\xi) := \varrho_k + \widetilde{\varrho}_{k,M}$. The same reasoning apply for $n = 0$ with $\rho_{k,0} = \rho_k + a(1-\varphi^{-\xi}) := \rho_k + \widetilde{\rho}_{k,0}$ and $\varrho_{k,0} = \varrho_k - a(1-a)(1-x^{-2\xi}) - 2ab(1-\varphi^{-\xi}) - 2ac(1-\varphi^{-\xi}) := \varrho_k + \widetilde{\varrho}_{k,0}$.

From this it follows that

$$
\begin{aligned}
MZ_{t+1,k}(\xi) &= \left(\widetilde{\rho}_{k,0} g_{t,0}^{(k)}\right) + \rho_k \sum_{n=0}^{M} (\varphi^{\xi})^n g_{t,n}^{(k)} + (\varphi^{\xi})^M \left(\widetilde{\rho}_{k,M} g_{t,M}^{(k)}\right) + \sigma_{t,k}(\xi)\varepsilon_{t+1} \\
&= \rho_k(\xi) MZ_{t,k}(\xi) + O_{t,k}^{M}(\xi) + \sigma_{t,k}(\xi)\varepsilon_{t+1}
\end{aligned}
$$

Where $O_{t,k}^{M}(\xi) = \widetilde{\rho}_{k,0} g_{t,0}^{(k)} + (\varphi^{\xi})^M \widetilde{\rho}_{k,M} g_{t,M}^{(k)}$ .Since the $\varepsilon_{t+1,n}$ are independent across n, the variance of $\sigma_{t,k}(\xi)\varepsilon_t$ is the sum of the variances of $\sqrt{\varrho_k g_{t,n}^{(k)}}\varepsilon_{t+1,n}$ i.e

$$
\begin{aligned}
\sigma_{t,k}(\xi)^2 &= \varrho_{k,0} g_{t,0}^{(k)} + \sum_{n=1}^{M-1} (\varphi^{2\xi})^n \varrho_k g_{t,n}^{(k)} + (\varphi^{2\xi})^M \varrho_{k,M} g_{t,n}^{(k)} \\
&= \left(\varrho_k + \widetilde{\varrho}_{k,0}\right) g_{t,0}^{(k)} + \sum_{n=1}^{M-1} (\varphi^{2\xi})^n \varrho_k g_{t,n}^{(k)} + (\varphi^{2\xi})^M \left(\varrho_k + \widetilde{\varrho}_{k,M}\right) g_{t,n}^{(k)} \\
&= \widetilde{\varrho}_{k,0} g_{t,0}^{(k)} + \sum_{n=0}^{M} (\varphi^{2\xi})^n \varrho_k g_{t,n}^{(k)} + (\varphi^{2\xi})^M \widetilde{\varrho}_{k,M} g_{t,n}^{(k)} \\
&= \varrho_k(\xi) MZ_{t,k}(2\xi) + O_{t,k}^{\sigma}(\xi)
\end{aligned}
$$

where $O_{t,k}^{\sigma}(\xi) = \widetilde{\varrho}_{k,0} g_{t,0}^{(k)} + (\varphi^{2\xi})^M \widetilde{\varrho}_{k,M} g_{t,n}^{(k)}$. Moreover, ε_{t+1} follows a standard normal distribution since the $\varepsilon_{t+1,n}$ are also normally distributed. □

Lemma 5 ***(Covariance of Moments of the Productivity Distribution)*** *Under Assumption 1, the covariance between the ξth moment and the ξ'th moment of the productivity distribution within sector k is given by*

$$
\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] = \overline{\varrho}_k(\xi,\xi') MZ_{t,k}(\xi'+\xi) + O_{t,k}^{C}(\xi,\xi')
$$

where $MZ_{t,k}(\xi) = \sum_{i=1}^{N_k} Z(k,i)^{\xi}$ *and* $\overline{\varrho}_k(\xi,\xi') = a_k(1-a_k)\varphi_k^{-(\xi+\xi')} + b_k(1-b_k) + c_k(1-c_k)\varphi_k^{\xi+\xi'} - a_k b_k(\varphi_k^{-\xi} + \varphi_k^{-\xi'}) - a_k c_k(\varphi_k^{-(\xi-\xi')}\varphi_k^{\xi-\xi'}) - b_k c_k(\varphi_k^{\xi} + \varphi_k^{\xi'})$.

Proof of Lemma 5: In the proof of Lemma 4, we had

$$
MZ_{t+1,k}(\xi) = \begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix} + \sum_{n=1}^{M-1} (\varphi^{\xi})^n \begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} + (\varphi^{\xi})^M \begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}
$$

Here I drop the subscript k to keep the notation simpler. Let us compute the covariance between two moments of the productivity distribution in sector k:

$$
\begin{aligned}
&\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi); MZ_{t+1,k}(\xi')\right] \\
=&\mathbb{C}ov_t\left[\begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix} + \sum_{n=1}^{M-1} (\varphi^{\xi})^n \begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} + (\varphi^{\xi})^M \begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}; \begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix} + \sum_{n=1}^{M-1} (\varphi^{\xi'})^n \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix} + (\varphi^{\xi'})^M \left(\varphi^{-} \right. \right. \\
=&\mathbb{C}ov_t\left[\begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix}; \begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix}\right] + \sum_{n=1}^{M-1}\sum_{n'=1}^{M-1} (\varphi^{\xi})^n (\varphi^{\xi'})^{n'} \mathbb{C}ov_t\left[\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix}; \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n'-1,n'} \\ f_{k,t+1}^{n',n'} \\ f_{k,t+1}^{n'+1,n'} \end{pmatrix}\right] + \ldots \\
&\ldots + (\varphi^{\xi+\xi'})^M \mathbb{C}ov_t\left[\begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}; \begin{pmatrix} \varphi^{-\xi'} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}\right] \\
=&\mathbb{C}ov_t\left[\begin{pmatrix} 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix}; \begin{pmatrix} 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{0,0} \\ f_{k,t+1}^{1,0} \end{pmatrix}\right] + \sum_{n=1}^{M-1} (\varphi^{\xi+\xi'})^n \mathbb{C}ov_t\left[\begin{pmatrix} \varphi^{-\xi} \\ 1 \\ \varphi^{\xi} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix}; \begin{pmatrix} \varphi^{-\xi'} \\ 1 \\ \varphi^{\xi'} \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{n-1,n} \\ f_{k,t+1}^{n,n} \\ f_{k,t+1}^{n+1,n} \end{pmatrix}\right] \ldots \\
&\ldots + (\varphi^{\xi+\xi'})^M \mathbb{C}ov_t\left[\begin{pmatrix} \varphi^{-\xi} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}; \begin{pmatrix} \varphi^{-\xi'} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{k,t+1}^{M-1,M} \\ f_{k,t+1}^{M,M} \end{pmatrix}\right]
\end{aligned}
$$

where at the second line, we use the fact that $f_{k,t+1}^{\cdot,0}$ and $f_{k,t+1}^{\cdot,M}$ are independent of the $f_{k,t+1}^{\cdot,n}$ for any $0 < n < M$ and in the third line that $f_{k,t+1}^{\cdot,n}$ are independent across n. Using the fact that $\mathbb{C}ov[A'X, B'Y] = A'\mathbb{C}ov[X,Y]B$

for vectors A and B and random vectors X and Y of appropriate size, we have

$$\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi);MZ_{t+1,k}(\xi')\right]=$$
$$\begin{pmatrix}1\\\varphi^{\xi}\end{pmatrix}'\mathbb{C}ov_t\left[\begin{pmatrix}f^{0,0}_{k,t+1}\\f^{1,0}_{k,t+1}\end{pmatrix};\begin{pmatrix}f^{0,0}_{k,t+1}\\f^{1,0}_{k,t+1}\end{pmatrix}\right]\begin{pmatrix}1\\\varphi^{\xi'}\end{pmatrix}+\sum_{n=1}^{M-1}(\varphi^{\xi+\xi'})^n\begin{pmatrix}\varphi^{-\xi}\\1\\\varphi^{\xi}\end{pmatrix}'\mathbb{C}ov_t\left[\begin{pmatrix}f^{n-1,n}_{k,t+1}\\f^{n,n}_{k,t+1}\\f^{n+1,n}_{k,t+1}\end{pmatrix};\begin{pmatrix}f^{n-1,n}_{k,t+1}\\f^{n,n}_{k,t+1}\\f^{n+1,n}_{k,t+1}\end{pmatrix}\right]\begin{pmatrix}\varphi^{-\xi'}\\1\\\varphi^{\xi'}\end{pmatrix}\dots$$
$$\dots+(\varphi^{\xi+\xi'})^M\begin{pmatrix}\varphi^{-\xi}\\1\end{pmatrix}'\mathbb{C}ov_t\left[\begin{pmatrix}f^{M-1,M}_{k,t+1}\\f^{M,M}_{k,t+1}\end{pmatrix};\begin{pmatrix}f^{M-1,M}_{k,t+1}\\f^{M,M}_{k,t+1}\end{pmatrix}\right]\begin{pmatrix}\varphi^{-\xi'}\\1\end{pmatrix}$$

Using the definition of Σ, $\Sigma^{(0)}$ and $\Sigma^{(M)}$ yields

$$\mathbb{C}ov_t\left[MZ_{t+1,k}(\xi);MZ_{t+1,k}(\xi')\right]=$$
$$g^{(k)}_{t,0}\begin{pmatrix}1\\\varphi^{\xi}\end{pmatrix}'\Sigma^{(0)}\begin{pmatrix}1\\\varphi^{\xi'}\end{pmatrix}+\sum_{n=1}^{M-1}(\varphi^{\xi+\xi'})^n g^{(k)}_{t,n}\begin{pmatrix}\varphi^{-\xi}\\1\\\varphi^{\xi}\end{pmatrix}'\Sigma\begin{pmatrix}\varphi^{-\xi'}\\1\\\varphi^{\xi'}\end{pmatrix}+(\varphi^{\xi+\xi'})^M g^{(k)}_{t,M}\begin{pmatrix}\varphi^{-\xi}\\1\end{pmatrix}'\Sigma^{(M)}\begin{pmatrix}\varphi^{-\xi'}\\1\end{pmatrix}$$

To complete the proof, let us just note that

$$\begin{pmatrix}\varphi^{-\xi}\\1\\\varphi^{\xi}\end{pmatrix}'\Sigma\begin{pmatrix}\varphi^{-\xi'}\\1\\\varphi^{\xi'}\end{pmatrix}=a(1-a)\varphi^{-(\xi+\xi')}+b(1-b)+c(1-c)\varphi^{\xi+\xi'}-ab(\varphi^{-\xi}+\varphi^{-\xi'})-ac(\varphi^{-(\xi-\xi')}\varphi^{\xi-\xi'})-bc(\varphi^{\xi}+\varphi^{\xi'})$$
$$\begin{pmatrix}1\\\varphi^{\xi}\end{pmatrix}'\Sigma^{(0)}\begin{pmatrix}1\\\varphi^{\xi'}\end{pmatrix}=c(1-c)(1-\varphi^{\xi'}-\varphi^{\xi}+\varphi^{\xi+\xi'})$$
$$\begin{pmatrix}\varphi^{-\xi}\\1\end{pmatrix}'\Sigma^{(M)}\begin{pmatrix}\varphi^{-\xi'}\\1\end{pmatrix}=a(1-a)(1-\varphi^{-\xi'}-\varphi^{-\xi}+\varphi^{-(\xi+\xi')})$$

which implies that

$$\begin{aligned}&O^{C}_{t,k}(\xi,\xi')\\&=g^{(k)}_{t,0}\left(\begin{pmatrix}\varphi^{-\xi}\\1\\\varphi^{\xi}\end{pmatrix}'\Sigma\begin{pmatrix}\varphi^{-\xi'}\\1\\\varphi^{\xi'}\end{pmatrix}-\begin{pmatrix}1\\\varphi^{\xi}\end{pmatrix}'\Sigma^{(0)}\begin{pmatrix}1\\\varphi^{\xi'}\end{pmatrix}\right)+(\varphi^{\xi+\xi'})^M g^{(k)}_{t,M}\left(\begin{pmatrix}\varphi^{-\xi}\\1\\\varphi^{\xi}\end{pmatrix}'\Sigma\begin{pmatrix}\varphi^{-\xi'}\\1\\\varphi^{\xi'}\end{pmatrix}-\begin{pmatrix}\varphi^{-\xi}\\1\end{pmatrix}'\Sigma^{(M)}\begin{pmatrix}\varphi^{-\xi'}\\1\end{pmatrix}\right)\\&=g^{(k)}_{t,0}\overline{\varrho_{k,0}}+(\varphi^{\xi+\xi'})^M g^{(k)}_{t,M}\overline{\varrho_{k,M}}\end{aligned}$$

□

Proof of Proposition 7: Using Lemma 4 and the fact that $\overline{Z_{t,k}}=MZ_{t,k}\big((\varepsilon_k-1)\gamma_k\big)$ and that $\Delta_{t,k}=\overline{Z_{t,k}}^{-2}MZ_{t,k}\big(2(\varepsilon_k-1)\gamma_k\big)$, we have

$$\overline{Z_{t+1,k}}=\rho^{(Z)}_k\overline{Z_{t,k}}+o^{(Z)}_{t,k}+\sqrt{\varrho^{(Z)}_k\Delta_{t,k}+O^{(Z)}_{t,k}}\quad\overline{Z_{t,k}}\quad\varepsilon^{(Z)}_{t+1,k}$$
$$\left(\frac{\overline{Z_{t+1,k}}}{\overline{Z_{t,k}}}\right)^2\Delta_{t+1,k}=\rho^{(\Delta)}_k\Delta_{t,k}+o^{(\Delta)}_{t,k}+\sqrt{\varrho^{(\Delta)}_k\kappa_{t,k}+O^{(\Delta)}_{t,k}}\quad\Delta_{t,k}\quad\varepsilon^{(\Delta)}_{t+1,k}$$

Finally, Lemma 5 shows that the covariance $\mathbb{C}ov_t\left[\varepsilon^{(1)}_{t+1};\varepsilon^{(2)}_{t+1}\right]\neq 0$. □