Sraffian Indeterminacy in General Equilibrium Revisited*

Naoki Yoshihara†and Se Ho Kwak
University of Massachusetts Amherst

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Abstract

Unlike Mandler's (1999; Theorem 6) impossibility result about the Sraffian indeterminacy of the steady-state equilibrium, we show that any stationary growth equilibrium with an endogenous growth rate is *indeterminate* in terms of Sraffa (1960) under the simple overlapping generation economy. Moreover, we also check that this indeterminacy is *generic*.

1 Introduction

It is well-known that Sraffa's (1960) system of equilibrium price equations contains one more unknown than equation, which leads to the indeterminacy of the steady-state equilibrium. This Sraffian indeterminacy has been regarded as a basis to argue that some non-market-competitive force is indispensable to determine the factor income distribution between capital and labor, which also set equilibrium prices of commodities. Mandler (1999) critically examined Sraffian indeterminacy by embedding the Sraffian system of price equations in a general equilibrium framework. In particular, unlike the claim of Sraffa (1960), Mandler (1999; section 6) argued that the steady-state equilibria are generically determinate.

The steady-state model in section 6 of Mandler (1999) presumes a structure of overlapping generations of agents with 2-periods lives. In this paper we reproduce the underlying narrative of the overlapping economic structure that was defined in section 6 of Mandler (1999), where a simple overlapping generation model is constructed, in which each generation is a single individual and lives

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†Corresponding author. Department of Economics, University of Massachusetts Amherst, Crotty Hall, Amherst, MA, 01002, USA; The Institute of Economic Research, Hitotsubashi University, Naka 2-1, Kunitachi, Tokyo 186-0004, Japan; and School of Management, Kochi University of Technology, Tosayamada, Kami-city, Kochi 782-8502, Japan, E-mail: nyoshihara@econs.umass.edu.

in two periods, and works only at his young age and is retired and so purchases consumption goods from the wealth due to his past saving at his old age. Then, in such a model, we respectively define the steady state equilibrium and a stationary growth equilibrium. The difference between these two notions is that, while the equilibrium investment activity consists solely of the replacements in the former notion, a positive net investment is available and its ratio to the gross investment is endogenously determined in the latter.

An interesting finding in this paper is that, even in this overlapping generation framework, the Sraffian indeterminacy is still observed: the stationary growth equilibrium is generically indeterminate though the steady state equilibrium is shown to be generically determinate. Precisely speaking, we provide a proper proof for the determinacy of the steady state equilibrium, as Mandler's (1999; section 6) own proof remains inaccurate. In contrast, we show that the stationary growth equilibrium is indeterminate in the sense of Mandler's (1999, p.699) own definition. Finally, we show that such indeterminacy occurs at almost all simple Leontief production economies.

In the rest of this paper, section 2 introduces a simple model of overlapping generation economies and defines the steady state equilibrium and a stationary growth equilibrium. Section 3 shows the indeterminate characteristic of stationary growth equilibria, and section 4 shows the genericity of the indeterminate stationary growth equilibria. As corollaries of these, the determinacy of steady-state equilibrium and its genericity are also verified in sections 3 and 4. Finally, section 5 argues the distinctive feature of the Sraffian indeterminacy in comparison with some of the neoclassical indeterminacy.

2 An overlapping generation economy at section 6 of Mandler (1999)

A simple overlapping generation model is constructed, in which each generation $t = 1, 2, \dots,$ is a single individual and lives in two periods, and works only at his young age and is retired and so purchases consumption goods from the wealth due to his past saving at his old age. Let ω_l be the labor endowment of one generation. There are n commodities which are produced in this economy and respectively used as consumption goods or capital goods. Let (A, L) be a Leontief production technique prevailed in this economy, where A is a $n \times n$ nonnegative square, productive and indecomposable matrix of reproducible input coefficients and L is a $1 \times n$ positive row vector of direct labor coefficients. Finally, let $u : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be a welfare function of lifetime consumption activities, which is common to all generations. As usual, u is assumed to be continuous and strongly monotonic. Thus, an *overlapping generation economy* is given by a profile $\langle (A, L); \omega_l; u \rangle$.

For each period t, let $p_t \in \mathbb{R}^n_+$ represent a vector of *prices* of n commodities prevailed at the end of this period; $w_t \in \mathbb{R}_+$ represent a *wage rate* prevailed at the end of this period; and $r_t \in \mathbb{R}_+$ represent an *interest rate* prevailed at the end of this period.

Each generation t at the young age is faced with the following optimization program MP^t: for a given sequence of price vectors $\{(p_t, w_t, r_t), (p_{t+1}, w_{t+1}, r_{t+1})\}$,

$$\max u\left(z_b^t, z_a^t\right)$$

subject to

$$p_t z_b^t + \frac{p_{t+1} z_a^t}{1 + r_{t+1}} \leqq w_t \omega_l,$$

where z_b^t is the consumption bundle of the generation t at the younger age; and z_a^t is the consumption bundle of this generation at the older age. Note that the monetary amount $\frac{p_{t+1} z_a^t}{1+r_{t+1}}$ is saved. In each period t, the aggregate consumption demands are

$$z^t \equiv z_b^t + z_a^{t-1}.$$

Therefore, the gross output in equilibrium is

$$y^t \geqq z^t + \omega^{t+1},$$

where ω^{t+1} is invested by the generation t for the production activity at period $t+1$. Note that the aggregate consumption demand vector z^t may contain some zero components. For such a commodity i as $z_i^t = 0$, it follows that in equilibrium, $y_i^t \geqq \omega_i^{t+1}$.

Given a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$, let $(z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}))$ be a solution of the generation $t = 1, 2, \ldots,$ to the above mentioned problem MP^t of the utility maximization under the budget constraint.

Definition 1: A *competitive equilibrium* under the overlapping generation economy $\langle (A, L); \omega_l; u \rangle$ is a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$ and sequence of gross outputs $\{y^t\}_{t \geq 0}$ satisfying the following conditions:

$$\begin{aligned}
p_t &\leqq (1 + r_t) p_{t-1} A + w_t L \ \ (\forall t); \\
y^t &\geqq z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + \omega^{t+1} \ \ (\forall t) \text{ where} \\
z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) &\equiv z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \text{ is the aggregate consumption demands at each } t; \\
A y^t &\leqq \omega^t \ \ (\forall t); \\
\frac{p_{t+1} z_a^t}{1 + r_{t+1}} &= p_t \omega^{t+1} \text{ and} \\
L y^t &\leqq \omega_l \ \ (\forall t).
\end{aligned}$$

Note that the equation $\frac{p_{t+1} z_a^t}{1+r_{t+1}} = p_t \omega^{t+1}$ represents the equilibrium condition for a gross saving and a gross investment.

In the standard static framework, the equation $\frac{p_{t+1} z_a^t}{1+r_{t+1}} = p_t \omega^{t+1}$ is not requested for the definition of competitive equilibria. However, this additional condition is necessary for the definition of competitive equilibria in the overlapping generation economy. To see it, note that $Ly^t = \omega_l$ (the excess demand condition for labor) and $Ay^t = \omega^t$ (the excess demand condition for capital goods) hold in equality for every t under an equilibrium, because of the strong monotonic property of u. Then, it follows that

$$\begin{aligned} p_t z^t + p_t \omega^{t+1} &= p_t y^t = (1 + r_t) \, p_{t-1} A y^t + w_t L y^t \\ &\Leftrightarrow \; p_t \left(z_b^t + z_a^{t-1} \right) + p_t \omega^{t+1} = (1 + r_t) \, p_{t-1} A y^t + w_t \omega_l . \end{aligned}$$

Moreover, from $Ay^t = \omega^t$, $(1 + r_t) \, p_{t-1} A y^t = (1 + r_t) \, p_{t-1} \omega^t$. By the budget constraint of each generation, it follows that

$$p_t z_a^{t-1} + p_t \omega^{t+1} = (1 + r_t) \, p_{t-1} \omega^t + \frac{p_{t+1} z_a^t}{1 + r_{t+1}} .$$

Therefore, $\frac{p_t z_a^{t-1}}{1+r_t} = p_{t-1} \omega^t$ holds if and only if $\frac{p_{t+1} z_a^t}{1+r_{t+1}} = p_t \omega^{t+1}$. Likewise, $\frac{p_t z_a^{t-1}}{1+r_t} \gtrless p_{t-1} \omega^t$ if and only if $\frac{p_{t+1} z_a^t}{1+r_{t+1}} \gtrless p_t \omega^{t+1}$. However, the latter equivalence condition is not feasible. For instance, let $\frac{p_{t+1} z_a^t}{1+r_{t+1}} > p_t \omega^{t+1}$. This implies that the generation t's gross interest revenue $(1 + r_{t+1}) \, p_t \omega^{t+1}$ at his old age is insufficient to meet his old-age-demand for consumption z_a^t. Therefore, the insufficient fund for purchasing this consumption bundle, $p_{t+1} z_a^t - (1 + r_{t+1}) \, p_t \omega^{t+1}$, must be financed by the generation $t + 1$ at the young age. However, given the strong monotonic property of u, the generation $t + 1$ at the young age does not take such a behavior as his optimal choice, which is a contradiction. Likewise, $\frac{p_{t+1} z_a^t}{1+r_{t+1}} < p_t \omega^{t+1}$ implies that the insufficient fund for the investment demands, $p_t \omega^{t+1} - \frac{p_{t+1} z_a^t}{1+r_{t+1}}$, must be financed at the young age period of generation t by the generation $t - 1$ who is at the old age. But, again given the assumption of u, the generation $t - 1$ at the old age does not take such a behavior as his optimal choice, which is again a contradiction. Therefore, in equilibrium, $\frac{p_{t+1} z_a^t}{1+r_{t+1}} = p_t \omega^{t+1}$ must hold for every generation t.

Now, consider a specific case of equilibrium where all of the investment activities are simply of the replacements. In this case, an interesting long run feature of competitive equilibrium is given as a steady-state equilibrium introduced as follows.

Definition 2 [Mandler (1999, section 6)]: A *steady-state equilibrium* under the overlapping economy $\langle (A, L) \, ; \omega_l ; u \rangle$ is a pair of a stationary price vector (p, w, r) and a gross output vector $y \geq \mathbf{0}$ such that the following conditions hold:

$$
\begin{aligned}
p &\leqq (1+r)\,pA + wL; \\
y &\geqq z\,(p,w,r) + Ay, \\
\text{where } z\,(p,w,r) &= z_b\,(p,w,r) + z_a\,(p,w,r)\,; \\
\frac{pz_a\,(p,w,r)}{1+r} &= pAy; \text{ and} \\
Ly &\leqq \omega_l.
\end{aligned}
$$

Finally, consider a case of equilibrium in which the positive net investment is observed and its ratio to the gross investment is invariant throughout the whole periods. Then, the corresponding long run feature of competitive equilibrium is given as follows.

Definition 3: A *stationary growth equilibrium* under the overlapping economy $\langle (A,L)\,;\omega_l;u\rangle$ is a profile of a stationary price vector (p,w,r), a gross output vector $y \geq \mathbf{0}$, and a common ratio of new investments to the replacements $g > -1$, such that the following conditions are satisfied:

$$
\begin{aligned}
p &\leqq (1+r)\,pA + wL; \\
y &\geqq z\,(p,w,r;g) + (1+g)\,Ay, \\
\text{where } z\,(p,w,r;g) &= z_b\,(p,w,r) + \frac{z_a\,(p,w,r)}{1+g}; \\
\frac{pz_a\,(p,w,r)}{1+r} &= (1+g)pAy; \text{ and} \\
Ly &\leqq \omega_l.
\end{aligned}
$$

3 Indeterminacy of the stationary growth equilibrium

Given the Leontief production technique (A,L), let $y^* > \mathbf{0}$ be the Frobenius eigenvector associated with the Frobenius eigenvalue $(1+R)^{-1}$ such that it is normalized to satisfy $Ly^* = 1$. This commodity bundle is called the *standard commodity* by Sraffa (1960). In this section, the standard commodity is assumed to be adopted as the *numeraire* of the price system: assume that, for any market price vector $p_t \in \mathbb{R}^n_+$, $p_t y^* = 1$ is satisfied.

Definition 4 (Mandler (1999)): Let $\langle (A,L)\,;\omega_l;u\rangle$ be an overlapping generation economy as specified above. Then, a stationary growth equilibrium $((p,w,r)\,,y,g)$ under this economy is *indeterminate* if for any $\varepsilon > 0$, there is a stationary growth equilibrium $((p',w',r')\,,y',g')$ under this economy such that $(p',w',r') \neq (p,w,r)$ and $\|(p',w',r')\,,(p,w,r)\| < \varepsilon$.

Let the profile $((p,w,r),y,g)$ be a stationary growth equilibrium. Let us take any $\varepsilon > 0$. Take r' $(\neq r)$, which is sufficiently close to r such that $\|(p',w',r'),(p,w,r)\| < \varepsilon$ holds, where $w' \equiv 1 - \frac{r'}{R}$ and $p' \equiv w'L(I-(1+r')A)^{-1}$. Then, it can be shown that there exists a steady-state equilibrium $((p',w',r'),y')$, as in Theorem A1 of Appendix. This implies that $((p,w,r),y,g)$ is indeterminate. Thus, we can summarize:

Theorem 1: Let $\langle (A,L);\omega_l;u\rangle$ be an overlapping generation economy as specified above. Then, the corresponding stationary growth equilibrium $((p,w,r),y,g)$ under this economy is *indeterminate*.

Theorem 1 can also be convinced by the standard proof. To see this point, let us reconsider the system of equations that characterizes the stationary growth equilibrium, which is given as follows:

$$\begin{aligned} p &= (1+r)pA + wL;\ (1) \\ y &= z(p,w,r;g) + (1+g)Ay;\ (2) \text{ and} \\ Ly &= \omega_l;\ (3) \text{ and} \\ (1+g)pAy &= \frac{pz_a(p,w,r)}{1+r}.\ (4) \end{aligned}$$

Note that (1) has n equations, (2) has n equations, and each of (3) and (4) has one equation. In contrast, there are n unknown variables regarding the vector y; there is one unknown variable g; and there are $(n-1)+2$ unknown variables regarding (p,w,r), assuming hereafter that commodity n is selected as the *numeraire*, instead of the standard commodity. Therefore, there are $2n+2$ unknown variables in the system of $2n+2$ equations. However, we can decrease the number of the equations based on a kind of the Walras law.

In the overlapping generation economy, the Walras law is generally given by the following equation:

$$\left[p_t\left(z_b^t + z_a^{t-1}\right) + p_tAy_{t+1} - w_t\omega_l^t\right] + (1+r_t)\left[p_{t-1}z_b^{t-1} - w_{t-1}\omega_l^{t-1}\right] = 0,\ (5)$$

which is simply derived from the value of the aggregate excess demands:

$$\left[p_t\left(z_b^t + z_a^{t-1}\right) + p_tAy_{t+1}\right] - \left[(1+r_t)p_{t-1}Ay_t + w_tLy_t\right] = 0,\ \ (5a)$$

since $w_tLy_t = w_t\omega_l^t$ and $\left[p_{t-1}z_b^{t-1} - w_{t-1}\omega_l^{t-1}\right] = -p_{t-1}Ay_t$ hold by the optimal behavior of each generation in this overlapping economy. At the steady-state price system, this equation (5) is reduced to the following form:

$$\left[p\left(z_b^t + z_a^{t-1}\right) + pAy_{t+1} - w\omega_l^t\right] + (1+r)\left[pz_b^{t-1} - w\omega_l^{t-1}\right] = 0.\ \ (5b)$$

Moreover, since $pAy_{t+1} = (1+g)pAy_t$ holds under the stationary growth equilibrium, the above (5b) can be reduced to the following form:

$$p\left(z_b^t + z_a^{t-1}\right) - \left((r-g)pAy_t + w\omega_l^t\right) = 0,$$

which is equivalent to the following equation:

$$p\left(z_b^t + z_a^{t-1}\right) = (r-g)\, pAy_t + w\omega_l^t. \text{ (5c)}$$

Then, noting that in the stationary growth equilibrium, $y = [I-(1+g)A]^{-1} z(p,w,r;g)$ should hold where $z(p,w,r;g) = z_b(p,w,r) + \frac{z_a(p,w,r)}{1+g}$, we can rewrite (5c) as follows:

$$\begin{aligned} pz(p,w,r;g) &= (r-g)\, pA(I-(1+g)A)^{-1} z(p,w,r;g) + w\omega_l \\ &\Leftrightarrow p\left[I-(r-g)A(I-(1+g)A)^{-1}\right] z(p,w,r;g) = w\omega_l. \text{ (5d)} \end{aligned}$$

By the way, let $q \equiv p\left[I-(r-g)A(I-(1+g)A)^{-1}\right]$. Then,

$$\begin{aligned} q &= p\left[I-(r-g)A(I-(1+g)A)^{-1}\right] \\ &= p-(r-g)\, pA(I-(1+g)A)^{-1} \end{aligned}$$

$$\begin{aligned} &\Leftrightarrow q(I-(1+g)A) = p(I-A) - rpA \\ &= p[I-(1+r)A] \\ &= wL. \end{aligned}$$

Thus, we have

$$q = wL(I-(1+g)A)^{-1}. \text{ (6)}$$

Note that, if $g=0$, then (6) is reduced to

$$q = wv$$

where v is the row vector of labor values:

$$v \equiv L(I-A)^{-1}.$$

By (6), the reduced form of the Walras law (5d) under the stationary growth is now reduced to

$$L(I-(1+g)A)^{-1} z(p,w,r;g) = \omega_l. \text{ (7)}$$

In particular, if $g=0$, we have:

$$v \cdot z(p,w,r;0) = \omega_l. \text{ (7a)}$$

Thus, if we have $n-1$ components of $z(p,w,r;g)$ is determined, the remaining one component of $z(p,w,r;g)$ is automatically determined by (7). Moreover, once the vector $z(p,w,r;g)$ is fixed, the unknown variables y is automatically specified by:

$$y = [I-(1+g)A]^{-1} z(p,w,r;g).$$

In summary, the system of equations (2) can be reduced to the following $n-1$ equations:

$$(I-(1+g)A)_{-n}\cdot y = z_{-n}(p,w,r;g), \text{ (2a)}$$

where $(I-(1+g)A)_{-n}$ is the $(n-1)\times n$ sub-matrix of $[I-(1+g)A]$ obtained by removing the n-th row vector of $[I-(1+g)A]$ and $z_{-n}(p,w,r;g)$ is the $(n-1)\times 1$ sub-vector of $z(p,w,r;g)$ obtained by removing the n-th component of $z(p,w,r;g)$. In other words, if the aggregate demand functions $(z_i)_{i=1,\dots,n-1}$ of $n-1$ commodities are given in the system of equations, then the aggregate demand of the remaining commodity is fixed by the demand functions of the $n-1$ commodities and the reduced form of the Walras law (7).[1] Thus, the system of $2n+1$ equations (1), (2), (3), and (4) can be reduced to the system of $2n+1$ equations (1), (2a), (3), and (4) given the reduced form of the Walras law (7). Then, since the system of $2n+1$ equations has $2n+2$ unknown variables, it has freedom of degree one. It is not difficult to see that the Jacobian matrix of the system of equations (1), (2a), (3), and (4) has rank $2n+1$. Therefore, we can show the indeterminacy of the stationary growth equilibrium by applying the implicit function theorem.

In contrast, we can see the determinacy of the steady-state equilibrium. In this case, we have the system of $2n+1$ equations (1), (2a) with $g=0$, (3), and (4) given the reduced form of the Walras law (7a). In contrast, here the system of equations has only $2n+1$ unknown variables. Again, it is not difficult to see that the Jacobian matrix of the system of equations (1), (2a) with $g=0$, (3), and (4) has rank $2n+1$. Therefore, we can show the determinacy of the steady-state equilibrium with a fixed population by applying the implicit function theorem:

Corollary 1: Let $\langle (A,L);\omega_l;u\rangle$ be an overlapping generation economy as specified above. Then, the corresponding steady-state equilibrium with a fixed population $((p,w,r),y)$ under this economy is *determinate*.

4 Openness and genericity

In this section, we examine the openness and genericity of parameter set of economies in which every stationary growth equilibrium is regular. The openness and genericity is related to the stability and coverage of indeterminacy in the perturbation of parameters to characterize the set of economies.

For the demand function of two generations z^a, z^b, labor endowment ω_ℓ^0 and for $h=(h_1,h_2,\dots,h_n)\in\mathbb{R}^n_{++}$, define a perturbed demand function with similar form in Mandler (1999) as

$$z_i^b(h) \equiv z_i^b(p,w,r)+\frac{w}{p_i}h_i$$

[1] As a result, the Walras law equation at the last line of p. 704 in Mandler (1999) should be slightly revised, as it does not reflect the condition that the investment of Ay is involved.

$$z_i^a(h) \equiv z_i^a(p,w,r) + \frac{w}{p_i} h_n$$

for each $i = 1, 2, ..., n-1$, and $\omega_l(h) \equiv \omega_\ell^0 + \sum_{i=1}^{n-1} h_i + \frac{n-1}{1+r} h_n$. Then,

$$z_i(h) \equiv z_i^b(h) + \frac{z_i^a(h)}{1+g}$$

for each $i = 1, 2, ..., n-1$. This perturbed functions satisfies Walras' law and homogeneity.

Now define a function F on the space of $n+1$ price variables $(\bar{p}, w, r)$ where $\bar{p} \equiv (p_1, ...p_{n-1})$ is normalized price, n quantity variables $(y_1, y_2, ...y_n)$, the growth rate of investments g and adding the parameter set (A, L, h) to $\mathbb{R}^{2n+1}$, *i.e.*

$$F : \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}_+ \times \mathbb{R}^{n^2}_+ \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \to \mathbb{R}^{2n+1}$$

such that

$$F(\bar{p}, w, 1+r, y, g, A, L, h) = \begin{bmatrix} z_{-n}(h) - [I - (1+g)A]_{-n}\, y \\ Ly - \omega_\ell(h) \\ p - (1+r)pA - wL \\ (1+g)pAy - \frac{pz^a(h)}{1+r} \end{bmatrix}$$

where p and L are row vectors, y is column vector, and the subscript $-n$ means the n-th row of the matrix is omitted.

Definition 5: An *economy* is a profile of (A, L, h) where (A, L) is a Leontief production technique, in which A is $n \times n$ non-negative square, productive and indecomposable matrix of reproducible input coefficients, L is $1 \times n$ positive row vector of direct labor coefficients, and $h = (h_1, h_2, ..., h_n) \in \mathbb{R}^n_{++}$.

A *regular stationary growth equilibrium* is a normalized stationary growth equilibrium vector $(\bar{p}, w, r, y, g)$ such that it satisfies $F = 0$ and the Jacobian matrix $\boldsymbol{D}F$ with respect to $p_1, ...p_{n-1}, w, r, y_1, ..., y_n, g$ is nonsingular at $(\bar{p}, w, r, y, g)$.[2] An economy (A, L, h) is *regular* if every normalized stationary

[2] The system of equation (1), (2a) (3) and (4) has $2n+1$ equations and $n+1$ price variables $(p_1, ..., p_{n-1}, w, r)$. Hence growth rate g and quantity variables $(y_1, ..., y_n)$ are to be determined simultaneously in the Jacobian. Including perturbed parameters, for any (A, L, h), $\boldsymbol{D}_{(\bar{p},y,w,r,g)}(F_{A,L,h}(\bar{p}, w, r, y, g))$ is given by:

$$\begin{bmatrix} \boldsymbol{D}_g z_{-n}(h) + A_{-n}y & [(1+g)A - I]_{-n} & \boldsymbol{D}_{\bar{p}} z_{-n}(h) & \boldsymbol{D}_w z_{-n}(h) & \boldsymbol{D}_r z_{-n}(\bar{p}, w, r, y) \\ \mathbf{0} & L & 0 & 0 & \frac{n-1}{(1+r)^2} h_n \\ \mathbf{0} & \mathbf{0} & I^*_{n-1} - (1+r)A^T_{-n} & -L^T & -pA \\ pAy & (1+g)pA & \text{(i)} & \text{(ii)} & \text{(iii)} \end{bmatrix}$$

growth equilibrium vector $(\bar{p}, w, r, y, g)$ is regular.[3] Denote the set of economies with P and the set of regular economies with P_R.

Theorem 2: P_R is open and has full measure in P.

Proof. The full measure claim of P_R is proven by the transversality theorem. Trivially, the function F defined above is smooth on the domain including all $(\bar{p}, w, r, y, g)$ and parameter (A, L, h) in $\mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}_+ \times \mathbb{R}^{n^2}_+ \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$. If $F = 0$ implies $\boldsymbol{D}_{A,L,h}F$ has full rank $2n+1$, then except a set of $(A, L, h) \in \mathbb{R}^{n^2}_+ \times \mathbb{R}^n_{++} \times \mathbb{R}^n_{++}$ of measure zero, $F_{A,L,h}(\bar{p}, w, r, y, g) : \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \times \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^{2n+1}_{++}$ has 0 as a regular value. We have $\boldsymbol{D}_{A,L,h}F$ as below:

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where

$$I^*_{n-1} = \begin{bmatrix} I_{n-1} \\ \mathbf{0} \end{bmatrix}$$

and

$$(i) = [(1+g)A_{-n}y]^T - \frac{1}{1+r}[z^a_1(p,w,r,y) + \sum_{i=1}^n p_i \frac{\partial z^a_i(p,w,r,y)}{\partial p_1}, ..., z^a_{n-1}(p,w,r,y) + \sum_{i=1}^n p_i \frac{\partial z^a_i(p,w,r,y)}{\partial p_{n-1}}],$$

$$(ii) = -\sum_1^n [\frac{p_i}{1+r} \frac{\partial z^a_i(p,w,r,y)}{\partial w}] - \frac{n}{1+r} h_n.$$

$$(iii) = \sum_1^n \frac{p_i}{1+r} [\frac{z^a_i(p,w,r,y)}{1+r} - \frac{\partial z^a_i(p,w,r,y)}{\partial r}] + \frac{nw}{(1+r)^2} h_n.$$

In addition, $\boldsymbol{D}_g z_{-n}(h)$, $\boldsymbol{D}_{\bar{p}} z_{-n}(h)$ and $\boldsymbol{D}_w z_{-n}(h)$ are calculated as:

$$\boldsymbol{D}_g z_{-n}(h) = \frac{-wh_n}{(1+g)^2}[p_1^{-1}, ..., p_{n-1}^{-1}]^T,$$

$$\boldsymbol{D}_{\bar{p}} z_{-n}(h) = \boldsymbol{D}_{\bar{p}} z_{-n}(\bar{p}, w, r, y) - \begin{bmatrix} \frac{w}{p_1^2}(h_1+h_n) & \mathbf{0} & & ... & \mathbf{0} \\ \mathbf{0} & \frac{w}{p_2^2}(h_2+h_n) & \mathbf{0} & ... & \mathbf{0} \\ & & & \ddots & \\ \mathbf{0} & ... & & \mathbf{0} & \frac{w}{p_{n-1}^2}(h_{n-1}+h_n) \end{bmatrix},$$

$$\boldsymbol{D}_w z_{-n}(h) = \boldsymbol{D}_w z_{-n}(\bar{p}, w, r, y) + [\frac{w}{p_1}(h_1+h_n), \frac{w}{p_2}(h_2+h_n), ..., \frac{w}{p_{n-1}}(h_{n-1}+h_n)]^T.$$

[3] Likewise, we can define that an economy (A, L, h) is *regular* if every normalized steady-state equilibrium vector $(\bar{p}, w, r, y)$ is regular. Then, as in the same way as the proof of Theorem 3 discussed below, it can be shown that such a regular economy is open and has full measure.

$$
\boldsymbol{D}_{A,L,h}F = \begin{bmatrix}
\frac{w}{p_1} & & & \frac{w}{(1+g)p_1} & (1+g)y^T & & & \\
& \ddots & & & & \ddots & \mathbf{0} & \mathbf{0} \\
& & \frac{w}{p_{n-1}} & \frac{w}{(1+g)p_{n-1}} & & (1+g)y^T & & \\
-1 & \dots & -1 & -\frac{n-1}{1+r} & & \mathbf{0} & & y^T \\
& & \mathbf{0} & & & \text{(iv)} & & -wI_n \\
& & \mathbf{0} & -\frac{nw}{1+r} & & \text{(v)} & & \mathbf{0}
\end{bmatrix}
$$

where the row vector y^T is the transpose of y, I_n is the $n \times n$ identity matrix, (iv)$= -(1+r)[p_1 I_n \dots p_n I_n]$ is $n{\times}n^2$ and (v)$= (1+g)[p_1 y^T \dots p_n y^T]$ is $1{\times}n^2$ matrix.[4] The first n columns are for $(h_1, ..., h_n)$, next n^2 columns are for the components of A and the last n columns are for the components of L.

To see that $\boldsymbol{D}_{A,L,h}F$ has full rank, observe that the first $(n-1) \times (n-1)$ submatrix of upper-left is nonsingular. Next, in n th row, it is observable that at least one component can remain in elementary row operations. The shape of (iv) and $-wI_n$ guarantees n nonzero rows which are linearly independent. The bottom row is also to be nonzero in elementary column operation. Therefore, $\boldsymbol{D}_{A,L,h}F$ has full rank. Since we have shown that $\boldsymbol{D}_{A,L,h}F$ has full row rank at all $(\bar{p}, w, 1+r, y, g, A, L, h)$ with $F = 0$, then by the transversality theorem, $F_{A,L,h}(\bar{p}, w, r, y, g)$ has 0 as regular value almost everywhere in P. In other words, P_R has full measure.

For the openness, to the contrary, suppose P_R is not open. Then there would be a sequence $(A, L, h)_t$ of non-regular economies converging to a regular economy $(A, L, h)_\circ \in P_R$. Correspondingly, there exists a sequence of non-regular equilibria $(\bar{p}, r, w, y, g)_t$ which converges to a regular equilibrium $(\bar{p}, r, w, y, g)_\circ$ at $(A, L, h)_\circ$. Then the corresponding Jacobian matrices $\boldsymbol{D}F(\bar{p}, w, r, y, g)_t$ of $2n+1$ rows and $2n+2$ columns exist as seen in footnote 1, which have less than full rank. For a Jacobian matrix, we can pick $2n+2$ separate square submatrices of order $2n+1$. Then determinants of square submatrices of order $2n+1$ are all zero. Now we can define a continuous function, say c from the set of Jacobian matrices to the set of $2n+2$-dimensional vectors whose components are determinants of square submatrices derived from the Jacobian $\boldsymbol{D}F$. Since $c(\boldsymbol{D}F) = (0, ..., 0) \in \mathbb{R}^{2n+2}$ for any $\boldsymbol{D}F$ of less than full rank, $c(\boldsymbol{D}F(\bar{p}, w, r, y, g)_t) = (0, ..., 0)_t \to (0, ..., 0) \in \mathbb{R}^{2n+2}$ as $t \to \infty$.

[4] Here, each $p_i I_n$ is $n \times n$ matrix:

$$
p_i I_n = \begin{bmatrix} p_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p_i \end{bmatrix}.
$$

Since $\{(0,...,0)_t\}$ converging to $(0,...,0)$ is closed in $\mathbb{R}^{2n+2}$ and c is continuous, the inverse image $c^{-1}(\{(0,...,0)_t\}) = \{\boldsymbol{D}F(\bar{p},w,r,y,g)_t\}$ whose elements are Jacobian matrices with less than full rank is closed. Hence $\boldsymbol{D}F(\bar{p},w,r,y,g)_\circ \in \{\boldsymbol{D}F(\bar{p},w,r,y,g)_t\}$ by the closedness of the set $\{\boldsymbol{D}F(\bar{p},w,r,y,g)_t\}$. Note that $c(\boldsymbol{D}F(\bar{p},w,r,y,g)_\circ) = (0,...,0) \in \mathbb{R}^{2n+2}$. This implies that the converging point of the sequence $\{\boldsymbol{D}F(\bar{p},w,r,y,g)_t\}$, each of which is correspondingly defined from $(A,L,h)_t \in P\backslash P_R$, must also have less than full rank. In other words, the convergent point of the sequence of non-regular economies must be also non-regular. This contradicts to our initial assumption. Therefore the set of regular economies P_R is open. ■

5 Concluding Remarks

In the above argument, we have shown that under the same setting of the overlapping generation economy as Mandler (1999; section 6), the Sraffian indeterminacy generically occurs in the stationary growth equilibrium. This result is a sharp contrast with the consistent claim of Mandler (1999, 2002, 2008) about the impossibility of the Sraffian indeterminacy in the steady-state equilibrium. Moreover, while indeterminacy arises in many places in neoclassical economics, such as overlapping-generations indeterminacy and factor-price indeterminacy summarized by Mandler (2002), the Sraffian indeterminacy observed in this paper has a distinctive feature in comparison with such neoclassical types.

Firstly, the overlapping-generations indeterminacy summarized by Mandler (2002) is characterized as the continuum set of equilibrium price sequences due to the arbitrariness of initial prices of commodities, but all of the equilibrium price sequences converge uniquely to the common steady-state price. In contrast, here we have focussed on the case that an equilibrium price sequence constantly consists of a steady-state equilibrium price vector from the infinite past until the infinite future, but there is a continuum set of steady-state equilibrium prices due to the continuum of factor income distribution. This suggests that the Sraffian indeterminacy and the overlapping-generations indeterminacy are quite different.

Secondly, regarding the factor-price indeterminacy, the mechanism to derive one dimension of indeterminacy in the model of three factors and two outputs discussed by Mandler (2002) is essentially the same as that discussed by Mandler (1999, section 3). That is, all of the three factors can be interpreted as that two of them are reproducible commodities, which are the same types as the output commodities, and the other is labor, but the equilibrium prices are not the steady-state ones; otherwise, another typical interpretation would be that all of the factors are primary ones. In contrast, here we have focussed on the stationary growth equilibrium of the economy where labor is the unique primary factor and capital is a bundle of multiple reproducible commodities. Moreover, Mandler's (2002) model of factor-price indeterminacy is essentially static while the stationary growth equilibrium discussed here is given in an intertemporal context. Again, the Sraffian indeterminacy and the factor-price indeterminacy are also quite different.

Finally, as Mandler's (2002) reference to Morishima (1961) indicates, it is an interesting open agenda to investigate economic implications of the Sraffian indeterminacy within the context of Turnpike theorem, which has not yet well-cultivated in modern general equilibrium theory.

6 References

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7 Appendix: On the existence of *stationary growth equilibrium*

With Definition 3, we can obtain the following existence theorem of the stationary growth equilibrium in the overlapping economy.

Theorem A1: Let $\langle (A, L); \omega_l; u \rangle$ be an overlapping generation economy as specified above. Then, for *each profit rate* $r \in [0, R)$, there exists a *stationary growth equilibrium* $((p, w, r), y, g)$ under this economy.

Proof. Let $R > 0$ be the maximal profit rate under the technique (A, L). As well-known, $\frac{1}{1+R}$ is the Frobenius eigen value of the productive and indecomposable matrix A such that there exists a unique Frobenius eigen vector $p^* > \mathbf{0}$ satisfying $p^* = (1 + R) p^* A$.

Take any $r \in [0, R)$. Then, due to the Sraffian linear distribution function, we can specify $w > 0$ as satisfying

$$r = R(1 - w).$$

Given (w, r), let

$$p \equiv wL(1 - (1 + r) A)^{-1}.$$

Then, it is well-known that in this case, $p > \mathbf{0}$ and $p = (1+r)\,pA + wL$ hold. That is, we obtain a stationary price vector (p, w, r), which is prevailed at each and every period.

Given this price information (p, w, r), consider the program MP^t of generation t. As a result, let $\left(z_b^t(p, w, r), z_a^t(p, w, r)\right)$ be a solution of generation t to the program MP^t under the stationary prices (p, w, r). But, the same solution is also optimal for generation $t-1$ as all generations have the same utility function. Therefore, without loss of generality, we get rid of the superscript "t" in the solution to each MP^t.

Thus, now, without loss of generality, let $z(p, w, r; g) \equiv z_b(p, w, r) + \frac{z_a(p,w,r)}{1+g}$, where $g > -1$ denotes a common growth rate of outputs, be the aggregate consumption demand vector. In addition, let

$$y(p, w, r; g) \equiv [I - (1+g)A]^{-1} z(p, w, r; g).$$

Note that since A is productive and indecomposable, we have $[I - (1+g)A]^{-1} > \mathbf{0}$ for any $g \in (-1, R)$, and so $y > \mathbf{0}$ holds. Moreover, it follows that

$$\begin{aligned} py(p, w, r; g) &= (1+r)\,pAy(p, w, r; g) + wLy(p, w, r; g) \\ &\Leftrightarrow pz(p, w, r) + (1+g)\,pAy(p, w, r; g) = (1+r)\,pAy(p, w, r; g) + wLy(p, w, r; g). \end{aligned}$$

Since the budget constraint of the program MP^t implies that

$$pz_b(p, w, r) + \frac{pz_a(p, w, r)}{1+r} = w\omega_l,$$

we can establish $Ly(p, w, r; g) = \omega_l$ whenever

$$\frac{pz_a(p, w, r)}{1+r} = (1+g)\,pAy(p, w, r; g)$$

is satisfied. Therefore, let us show that for any $r \in [0, R)$, there exists a unique $g(r)$ such that

$$\frac{pz_a(p, w, r)}{1+r} = (1+g(r))\,pAy(p, w, r; g(r))$$

holds.

Let $\Psi(r, g) \equiv (1+g)\,pAy(p, w, r; g) - \frac{pz_a(p,w,r)}{1+r}$. Note that if g is sufficiently close to -1, then $\Psi(r, g) \approx \Psi(r, -1) < 0$. In contrast,

$$\lim_{g \to R} \Psi(r, g) = +\infty$$

holds, as the matrix $[I - (1+g)A]^{-1}$ approaches the singular matrix. Since $\Psi(r, g)$ is continuous at every g, there exists $g(r)$ such that

$$\frac{pz_a(p, w, r)}{1+r} = (1+g(r))\,pAy(p, w, r; g(r))$$

holds. Moreover, as

$$\begin{aligned}\frac{\partial \Psi(r,g)}{\partial g} &= pAy(p,w,r;g(r)) + (1+g)\,pA\frac{\partial [I-(1+g)A]^{-1} z_b(p,w,r)}{\partial g} \\ &\quad + pA\frac{\partial [I-(1+g)A]^{-1} z_a(p,w,r)}{\partial g} \\ &> 0,\end{aligned}$$

it follows that $g(r)$ is unique. Thus, for each $r \in [0,R)$, we obtain a stationary price vector $(p(r), w(r), r)$ such that

$$\begin{aligned} p(r) &= (1+r)\,p(r)\,A + w(r)\,L; \\ y(p(r),w(r),r) &= z_b(p(r),w(r),r) + \frac{z_a(p(r),w(r),r)}{1+g(r)} + (1+g(r))\,Ay(p(r),w(r),r), \\ \frac{p(r)\,z_a(p(r),w(r),r)}{1+r} &= (1+g(r))\,p(r)\,Ay(p(r),w(r),r);\ \text{and} \\ Ly(p(r),w(r),r) &= \omega_l. \end{aligned}$$

In summary, the above-specified profile $((p(r), w(r), r), y(p(r), w(r), r), g(r))$ satisfies all of the conditions for a stationary growth equilibrium. ■