THE LATTICE OF ENVY-FREE MATCHINGS

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Abstract. In a many-to-one matching model, we show that the set of envy-free matchings is a lattice. A Tarski operator on this lattice, which can be interpreted as modeling vacancy chains, has the set of stable matchings as its fixed points. (JEL C78)

1. Introduction

This paper studies envy-free matchings in a many-to-one matching model (e.g. between hospitals and doctors). The set of envy-free matchings turns out to be a lattice, and the set of stable matchings equals the set of fixed points of a Tarski operator on this lattice. Informally speaking, envy-freeness is a relaxation of stability that allows blocking pairs involving a doctor and an empty position of a hospital. Envy-free matchings emerge as natural states in some markets. For example, in senior level labor markets (Blum Roth and Rothblum 1997), positions typically become available when current incumbents retire, and the resulting empty positions are often filled with candidates who are incumbents elsewhere. If the market is stable before the retirements, then it will be envy-free after. This paper establishes some structure on the set of envy-free matchings, and the Tarski operator may serve as a model of re-equilibration in such a setting. It will become clear that if a market does not move

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quickly to re-equilibration–i.e. if it is often observed before the Tarski operator has converged to a stable matching–then envy-free matchings will be observed.

This work resembles, in many ways, the earlier matching literature that studies stability through Tarski's fixed point theorem (Tarski 1955), with the main difference being that we start from what we show to be a lattice of *matchings*. Beginning with Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004), Hatfield and Milgrom (2005), Ostrovsky (2008) and others analyzed stable allocations using a function from a lattice to itself whose fixed points are stable. These earlier papers all start with a lattice that is larger than the set of matchings, for instance, many of them build their operators on the set of "pre-matchings", in which A being matched to B does not imply B is matched to A. In this paper, we show that the lattice fixed point method also works if our base lattice is the set of envy-free matchings.

One way to compare these new results with the previous literature is to consider what they tell us about how stable matches may be reached. The previous literature can be interpreted as modeling how a centralized process like the deferred acceptance algorithm converges through a series of offers and rejections to a set of offers and acceptances that coincide with a matching, and that matching is stable. The present analysis can be interpreted as modeling how a decentralized sequence of offers, in which firms fill vacant positions, and cause vacancies in other firms, produces a sequence of envy-free matchings if it begins from one, and eventually converges to a stable matching. If the process stops short of convergence, the stopping point will itself be an envy-free matching from which further vacancy chains can arise.

When thinking about lattices in two-sided matching models, economists sometimes focus on lattices like the one originally introduced by Conway (see Knuth 1976, Roth and Sotomayor 1990): the join of two matchings will match all agents on one side of the market to their more preferred partners, and, dually, the meet will match these agents to their less preferred partners. However we will show that the set of envy-free matchings forms a lattice that does not exhibit this kind of duality between the join and the meet. It is worth mentioning that, in Blum et al (1997), even though they did not study the underlying lattice structure, they obtained some lattice-theoretic results on the one-to-one marriage model through the method of modifying preferences, some of which are extended to many-to-one settings in Cantala (2011). We further investigate and generalize some of their conclusions in many-to-one markets with the help of a Tarski operator and provide some general algebraic insights. The envy-free matchings considered here generalize the "simple" matchings studied by Sotomayor (1996) and the "firm-quasi-stable" matchings studied by Blum et al (1997).[1]

In the next section we describe the model. Results on the lattice structure are presented in section 3. Section 4 explores the implications of these results together with the interaction between envy-free matchings and traditional stable matchings. Section 5 studies possible generalizations and counterexamples. In section 6, we discuss applications of envy-free matchings: section 6.1 is devoted to vacancy chains in senior level labor markets, and section 6.2 introduces the role of envy-freeness in the constrained matching models. Section 7 concludes.

2. Model and Preliminaries

We work with a standard two-sided many-to-one matching model with strict preferences. There is a finite set of hospitals $\mathbf{H}$ and a finite set of doctors $\mathbf{D}$. Each doctor d has strict preferences $\succ_d$ over the set of hospitals and being unmatched, denoted by $\emptyset$. Each hospital h: (1) has a capacity q_h; (2) has strict preferences $\succ_h$ over subsets of doctors and being unmatched, again denoted by $\emptyset$; (3) furthermore its preference

[1]See also Sotomayor (1999, 2000, 2006, 2016) for applications and generalizations of simple matchings in other settings.

over groups of doctors is **responsive**: for any $D' \subseteq \mathbf{D}$, if $|D'| > q_h$ then $\emptyset \succ_h D'$; if $|D'| \leq q_h$ then for any $d_1 \in \mathbf{D} \backslash D' \cup \{\emptyset\}$, $d_2 \in D'$: $D' \succ_h D' \cup \{d_1\} \setminus \{d_2\}$ if and only if $d_2 \succ_h d_1$. In words, h considers D' as **unacceptable** (worse than being unmatched) if D' contains more doctors than h's capacity; the ranking of each doctor is independent of his colleagues, and any two groups of doctors that differ in a single doctor are preference ordered by the preference for individual doctors.

A **matching** μ will specify which doctors are matched to each hospital; more formally, it is a mapping such that: (1) $\forall h \in \mathbf{H}$, $\mu(h) \subseteq \mathbf{D}$, $|\mu(h)| \leq q_h$, and $\forall d \in \mathbf{D}$, $\mu(d) \in \mathbf{H} \cup \{\emptyset\}$; (2) $\mu(d) = h \Leftrightarrow d \in \mu(h)$. We say a matching μ is **individually rational** if: (1) $\forall d \in \mathbf{D}$, $\mu(d) \succsim_d \emptyset$; and (2) $\forall h \in \mathbf{H}$, $d \in \mu(h)$ implies $d \succsim_h \emptyset$. A doctor-hospital pair (d, h) **blocks** μ if $h \succ_d \mu(d)$ and at least one of the following situations happen:

(1) $\exists d' \in \mu(h)$ such that $d \succ_h d'$;

(2) $|\mu(h)| < q_h$ and $d \succ_h \emptyset$.

A matching μ is **stable** if and only if it is individually rational and there is no blocking pair. In such a framework, Gale and Shapley (1962) and Roth (1985a) showed that a stable matching always exists, furthermore there is a **doctor optimal stable matching** that all doctors weakly prefer to any other stable matching, and similarly there is a **hospital optimal stable matching** that all hospitals weakly prefer to any other stable matching.

Now we are ready to introduce envy-free matchings. In the definition of stability there are two kinds of blocking pairs; in envy-free matchings, we allow type (2) blocking pairs involving a vacant position, while disallowing type (1) blocking pairs involving a filled position:[2]

[2]A matching that disallows type (2) blocking pairs while allowing type (1) blocking pairs is often called a non-wasteful matching. And a matching is stable if and only if it is both envy-free and

Definition 2.1. Given a matching μ, a doctor d has **justified envy** toward d' who is assigned to hospital h, if (i) $h \succ_d \mu(d)$ and (ii) $d \succ_h d'$. We say μ is **envy-free** if it is individually rational and no doctor has justified envy.

Notice that the absence of justified envy can be regarded as reflecting a kind of fairness of the hiring process to doctors: if one doctor has a position that another would have preferred, it is because this was the choice of the employer. If (d, h) forms a blocking pair in an envy-free matching, then h must have at least one empty position. Let $\mathcal{L}$ denote the set of envy-free matchings.

Here are some examples of envy-free matchings: all stable matchings are envy-free; the empty matching in which everyone is unmatched is envy-free; a matching in which one hospital h is matched to its most preferred k $(1 \leq k \leq q_h)$ doctors that list h as acceptable (everyone else is unmatched) is envy-free; and after each round of a hospital proposing deferred acceptance algorithm, the temporary matching is envy-free. Compared to stability, envy-freeness preserves fairness but sacrifices efficiency.

In the one-to-one matching market, where $q_h = 1$ for all hospitals h, envy-free matchings correspond to Sotomayor's (1996) simple matchings: A matching μ is **simple** if it is individually rational and whenever a blocking pair (d, h) exists we have $\mu(h) = \emptyset$. She used simple matchings to provide a non-constructive proof for the existence of stable matchings in the one-to-one marriage model. In this paper we investigate the structure of both envy-free matchings and stable matchings in the many-to-one setting.

When studying stable matchings in many-to-one markets with responsive preferences, a common proof technique is to represent each hospital by multiple copies each

non-wasteful. That is, an envy-free matching may be wasteful, i.e. it may leave unfilled positions that could have been filled.

having a quota of one, to make the matching one-to-one. More precisely, for each hospital h with quota q_h, divide it into q_h copies: $h_1, h_2, ..., h_{q_h}$. Consider a new market ("the related marriage market"), in which the agents are doctors and copies of hospitals. Each hospital copy h_i has the same preference over doctors as h; and each doctor follows his original preference when comparing two copies of different hospitals h and h'; and strictly prefers h_p over h_q if and only if $p < q$. Then there is a natural bijection between matchings in the two markets: a matching μ in the many-to-one market, that matches hospital h to $\mu(h)$ corresponds to a matching μ' in the related marriage market where h_i is matched to h's i-th most preferred doctor among $\mu(h)$. It is well-known that μ is stable in the many-to-one market if and only if μ' is stable in the related marriage market (Roth and Sotomayor 1990). We have an analogous result for envy-free matchings:

Proposition 2.2. *A matching μ is envy-free if and only if the corresponding matching μ' in the related marriage market is simple.*

Proof of Proposition 2.2:

($\Rightarrow$) Let μ be an envy-free matching, for the sake of contradiction, assume μ' is not simple, then there exists a blocking pair (h_i, d) such that $\mu'(h_i) = d' \in \mathbf{D}$. If $\mu'(d) = h'_j$ or $\emptyset$, then d has justified envy towards d' in μ, contradiction. Otherwise, $\mu'(d) = h_j$ for some $j \neq i$, but this is impossible, since if $i < j$ then $d' \succ_{h_i} d$, and if $i > j$ then $h_j \succ_d h_i$, either way (h_i, d) can not be a blocking pair in μ' to begin with.

($\Leftarrow$) Let μ' be a simple matching. Assume μ is not envy-free, then there exists some d and d' such that d has justified envy towards d', which means $(d, \mu'(d'))$ blocks μ', contradiction. $\square$

3. Lattice Structure

We first prove that the set of envy-free matchings forms a lattice, under the partial order $\succsim_D$ of the **common preference of doctors** (i.e. $\mu \succsim_D \mu'$ if and only if $\forall d \in D$, $\mu(d) \succsim_d \mu'(d)$). Recall that a partially ordered set L is called a **join-semilattice** if any two elements in L have a least upper bound (called join, denoted by $\vee$); and a **meet-semilattice** if any two elements in L have a greatest lower bound (called meet, denoted by $\wedge$). A partially ordered set L is a **lattice** if it is both a join-semilattice and meet-semilattice. Before we present the proof, here is a simple example that illustrates the result:

Example 3.1. Consider the following preference profile with **D**$= \{d_1, d_2\}$ and **H**$= \{h_1, h_2, h_3\}$, each hospital has capacity 1 and unlisted means unacceptable.

d_1: $h_1 \succ_{d_1} h_2$

d_2: $h_3 \succ_{d_2} h_2$

h_1: d_1

h_2: $d_1 \succ_{h_2} d_2$

h_3: d_2

There are seven envy-free (simple) matchings. We document them with a pair (x, y), where x is the index of d_1's hospital and y is the index of d_2's hospital. If x or y is empty, then the corresponding doctor is unmatched. For example, matching (b) below means $\mu(d_1) = h_2$ and $\mu(d_2) = \emptyset$.

(a): $(,)$

(b): $(2,)$

(c): $(1, 3)$

(d): $(1,)$

(e): $(1, 2)$

(f): $(, 3)$

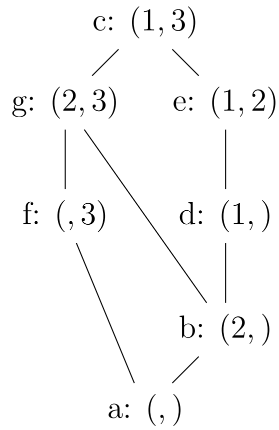

FIGURE 1. Hasse Diagram For Example 3.1

(g): $(2,3)$

Under the common preference of doctors $\succsim_D$, these matchings form a lattice, whose Hasse diagram is shown in figure 1. Notice unlike in stable matchings, the common preference of hospitals is no longer a dual order of $\succsim_D$. For example, matching (a) is the worst outcome for all doctors, and it is also the worst for all hospitals.

To show that the set of envy-free matchings forms a lattice, we need to find the join and meet for any two envy-free matchings μ and μ'. Just like in Conway's lattice theorem of stable matchings (see Knuth 1976), a natural candidate for $\mu \vee \mu'$ is a "matching" λ that matches each doctor d to his more preferred hospital between $\mu(d)$ and $\mu'(d)$ (and each hospital h to the corresponding matched doctors); similarly a "matching" ν that matches each doctor d to his less preferred hospital between $\mu(d)$ and $\mu'(d)$ is a candidate for $\mu \wedge \mu'$. From the example above, we see that ν is not necessarily an envy-free matching: look at (e) and (g), both d_1 and d_2 like h_2 less, therefore ν is not individually rational because $\{d_1, d_2\}$ violates the quota of h_2 and hence is unacceptable. On the other hand, λ defines an envy-free matching. We have the following lemma:

Lemma 3.2. *The matching λ defined above is always envy-free and therefore the set of envy-free matchings $\mathcal{L}$ is a join-semilattice under $\succsim_D$.*

Proof of Lemma 3.2: First, we show λ is an individually rational matching. Since each doctor d matched to h in λ is matched to h either in μ or μ', hence $d \succ_h \emptyset$; similarly $h \succ_d \emptyset$. Then the only thing left to prove is $|\lambda(h)| \leq q_h$. Assume otherwise, then we have $|\lambda(h)| > q_h \geq |\mu(h)|$ and $|\lambda(h)| > q_h \geq |\mu'(h)|$. From $|\lambda(h)| > |\mu'(h)|$ we know that there must exist a doctor d such that $\lambda(d) = h$ but $\mu'(d) \neq h$, this implies $\mu(d) = h$. Similarly there is a doctor d' such that $\mu(d') \neq h$ and $\mu'(d') = h$. Without loss of generality, say $d \succ_h d'$. Then d envies d''s position in matching μ' (by definition of λ, $h \succ_d \mu'(d)$), contradicting the fact that μ' is envy-free.

Now we show λ is envy-free. Assume otherwise, say d has justified envy towards d' for hospital h in λ, i.e. $d \succ_h d'$, $h \succ_d \lambda(d)$ and $\lambda(d') = h$. Then d' is matched to h in μ or μ'. Without loss of generality, assume $\mu(d') = h$. By definition of λ, we have $h \succ_d \lambda(d) \succsim_d \mu(d)$. But this means d has justified envy towards d' in matching μ, contradiction.

Therefore λ is an envy-free matching. It is clear that λ is the least upper bound of μ and μ', and so the set of envy-free matchings is a join-semilattice under $\succsim_D$. □

With a technical lemma we can show that $\mathcal{L}$ is a lattice, even though the economic interpretation of meet is unclear. (We further discuss the meet in Remark 4.7 and identify situations in which a Conway-like meet is well-defined.)

Lemma 3.3. *A finite join-semilattice with a minimum is a lattice.*

Readers are referred to Stanley (2011) for a complete proof of this lemma.[3]

[3]The idea is that, for any two elements x and y in a finite join-semilattice L, define a set $M_{x,y} = \{z|\ z \leq x, z \leq y\}$. This set is always non-empty because of the minimum element. Then one can check that $x \wedge y = \vee M_{x,y}$.

Theorem 3.4. *The set of envy-free matchings $\mathcal{L}$ is a lattice under $\succsim_D$.*

Proof of Theorem 3.4: by Lemma 3.2, $\mathcal{L}$ is a join-semilattice, and the matching in which everyone is unmatched is the minimum of $\mathcal{L}$, then by Lemma 3.3, $\mathcal{L}$ is a lattice under $\succsim_D$. □

Remark 3.5. The set $\mathcal{L}$ is a lattice but it is not distributive. One easy way to check this is: in Example 3.1, a-f-g-c and a-b-d-e-c are two maximal chains with different lengths, but every maximal chain in a finite distributive lattice has the same length. One can also directly check $(g \wedge e) \vee d = d$ while $(g \vee d) \wedge (e \vee d) = e$. It is clear that the minimum of $\mathcal{L}$ is the matching at which everyone is unmatched, but what is the maximum? Sotomayor (1996) showed that in one-to-one matching markets, when preferences are strict, the doctor optimal stable matching is the only weakly Pareto optimal matching for doctors among all simple matchings. It is straightforward to extend her proof (or use Proposition 2.2) to our many-to-one model with strict responsive preferences. Combine this with the fact that the maximum element of $\mathcal{L}$ must be weakly Pareto optimal for doctors among all envy-free matchings, we conclude that the maximum element of $\mathcal{L}$ is exactly the doctor optimal stable matching. One should also note that the responsiveness assumption on preferences is crucial to this kind of structure; we will show in section 5 that when preferences are substitutable, the lattice property no longer holds.

Now we apply Tarski's fixed point theorem to the lattice of envy-free matchings and give an alternative proof of the existence and lattice property of stable matchings. Note that a lattice is called complete if every subset (and not just every pair of elements) has a meet and a join, and that every finite lattice is complete. Below is the statement of Tarski's famous result:

Theorem 3.6. *(Tarski 1955) Let $(\mathcal{L}, \leq)$ be a complete lattice and $T : \mathcal{L} \to \mathcal{L}$ be isotone, i.e. $\forall x, y \in \mathcal{L}$, $x \leq y \Rightarrow T(x) \leq T(y)$, then the set of fixed points of T is nonempty and forms a complete lattice with respect to $\leq$.*

We next define a suitable Tarski operator T. (In Section 6 we will explain how T can be interpreted as one step in a vacancy chain process, in which hospitals with empty positions fill them, possibly by hiring incumbents at other hospitals and hence creating new vacant positions.)

Definition 3.7. Given an envy-free matching μ, let B_μ denote the set of all blocking pairs in μ. Define:
$B_\mu^1 = \{(d,h) \in B_\mu | \ \forall (d',h) \in B_\mu, d \succsim_h d'\}$, i.e. B_μ^1 consists of each hospital's favorite blocking pair in B_μ (if any);
$B_\mu^2 = \{(d,h) \in B_\mu^1 | \ \forall (d,h') \in B_\mu^1, h \succsim_d h'\}$, which consists of each doctor's favorite blocking pair in B_μ^1 (if any).
Because preferences are strict, B_μ^1 identifies at most one blocking pair for each hospital h. Similarly B_μ^2 picks out at most one blocking pair for each doctor d. Now we can define an operator T that maps $\mathcal{L}$ (the lattice of envy-free matchings) to itself as follows:

for all d such that there is some h that $(d,h) \in B_\mu^2$, define $T(\mu)(d) = h$ (h is unique by definition of B_μ^2);
for all other d such that there is no h that $(d,h) \in B_\mu^2$, define $T(\mu)(d) = \mu(d)$;
define $T(\mu)(h) = \{d| \ T(\mu)(d) = h\}$, so that $T(\mu)$ remains a matching.

In words, T satisfies all blocking pairs (d,h) in B_μ^2 by transferring each doctor d to the corresponding hospital h. Below are two immediate but important implications of the definitions:

Proposition 3.8. *For any envy-free matching μ:*
(I). $T(\mu) \succsim_D \mu$;
(II). $T(\mu) = \mu$ if and only if μ is stable.

To check that Theorem 3.6 applies, we need to show two things: (1) $T(\mu)$ is an envy-free matching; and (2) T is isotone. For (1), first note that each d and h shows up at most once in B^2_μ, so T is well-defined and will not cause a capacity violation. (Recall if (d, h) is a blocking pair then h must have at least one unfilled position.) By construction of B^2_μ, whenever we have a blocking pair $(d, h) \in B^2_\mu$, d must be h's most preferred doctor who forms a blocking pair with h, thus after transferring d to h, no doctor will have justified envy towards d. Also, after we satisfy all the blocking pairs in B^2_μ, d has no justified envy towards any other doctor. Therefore $T(\mu)$ must be an envy-free matching.

Before proving the monotonicity of T and invoking Tarski's fixed point theorem, we should point out that, (1) and Proposition 3.8 combined already imply the existence of stable matchings: start with an arbitrary envy-free matching, repeatedly applying T is a Pareto improving process for all doctors until it reaches a fixed point. Since doctors' welfare is bounded above, the Pareto improving phase must terminate. In other words, eventually this process leads to a fixed point of T, namely a stable matching. This is more of a special property of envy-free matchings than of our carefully crafted operator. Since all blocking pairs in envy-free matchings only involve empty positions, any process of satisfying blocking pairs that moves from one envy-free matching to another will be Pareto improving for all doctors. Thus any such procedure will stop in finite time, and achieve a stable allocation.[4] This is why we do not need a careful algorithm like Roth and Vande Vate (1990) to make sure the process of fixing blocking pairs does not cycle (as in Knuth 1976) when dealing with envy-free matchings. For instance, another simple operator that will do the job is to select an arbitrary blocking pair from B^1_μ and satisfy it. (But this operator is not isotone.) The main benefit of Tarski's fixed point theorem is the lattice structure of fixed points, and the monotonicity of T also has interesting implications in our case. So we return to point (2): its proof is non-trivial and deserves to be a separate lemma:

[4]See Sotomayor (2006) for more discussion on such processes, under a variety of different settings.

Lemma 3.9. *If μ and μ' are two envy-free matchings such that $\mu \succsim_D \mu'$, then for any doctor d, $T(\mu)(d) \succsim_d T(\mu')(d)$, which in turn implies $T(\mu) \succsim_D T(\mu')$.*

Proof of Lemma 3.9: If d is not involved in any blocking pair in $B^2_{\mu'}$, then $T(\mu)(d) \succsim_d \mu(d) \succsim_d \mu'(d) = T(\mu')(d)$. Hereafter assume there is some hospital h such that $(d,h) \in B^2_{\mu'}$, then by definition $T(\mu')(d) = h$. We discuss 4 cases: (I). Hospital h fills its quota in μ. Recall that $(d,h) \in B^2_{\mu'}$ implies h does not fill its quota in μ'. Then there must exist a doctor d', who is matched to h in μ but not in μ'. Since $\mu \succsim_D \mu'$, then $h \succ_{d'} \mu'(d')$, therefore (d',h) must be a blocking pair in μ' (h has unfilled positions in μ'). Since (d,h) is h's favorite blocking pair in μ', then $d \succ_h d'$. Now look at μ, since d can not have justified envy towards d', we must have $\mu(d) \succsim_d h$. Thus $T(\mu)(d) \succsim_d \mu(d) \succsim_d h = T(\mu')(d)$. (II). Hospital h does not fill its quota in μ, but (d,h) is not a blocking pair in μ. Then again $\mu(d) \succsim_d h$ and $T(\mu)(d) \succsim_d \mu(d) \succsim_d h = T(\mu')(d)$. (III). Hospital h does not fill its quota in μ, and (d,h) is a blocking pair in μ; furthermore $(d,h) \in B^2_{\mu}$. Then $T(\mu)(d) = h = T(\mu')(d)$. (IV). Hospital h does not fill its quota in μ, and (d,h) is a blocking pair in μ but $(d,h) \notin B^2_{\mu}$. Then (d,h) is absent from B^2_{μ}, either because h likes another blocking pair (d',h) better, or (d,h) is h's favorite blocking pair but d likes another blocking pair (d,h') better. The former situation can not happen, since (d',h) will also be a blocking pair in μ', contradicting the fact that $(d,h) \in B^2_{\mu'}$. For the latter case, $T(\mu)(d) = h' \succ_d h = T(\mu')(d)$. The above four cases are exhaustive and therefore $T(\mu)(d) \succsim_d T(\mu')(d)$. □

Theorem 3.10. *The set of stable matchings is non-empty and forms a lattice with respect to $\succsim_D$.*

Proof of Theorem 3.10: The lattice of envy-free matchings is finite and therefore complete, and the operator T defined above maps $\mathcal{L}$ to itself. Moreover, T is isotone, thus by Theorem 3.6, the set of fixed points of T, namely the stable matchings

(Proposition 3.8 (II)), is non-empty and forms a lattice under $\succsim_D$. □

Furthermore, we can pin down the closed form expression of the fixed point of T, starting from any envy-free matching μ. First, we present a lemma, which is interesting in its own right.

Lemma 3.11. *Let μ_1 be an envy-free matching and μ_2 be a stable matching, then $\mu_1 \vee \mu_2$ is stable.*

Proof of Lemma 3.11: Let μ'_1 and μ'_2 be the corresponding matchings in the related marriage market. By Proposition 2.2, μ'_1 is simple and μ'_2 is stable. Theorem A.3 in Blum et al (1997) states that, the join of a simple matching and a stable matching is stable. Therefore $\mu'_1 \vee \mu'_2$ is stable in the related marriage market, which in turn implies that the corresponding $\mu_1 \vee \mu_2$ in the many-to-one market is stable.[5] □

One might find Lemma 3.11 a little surprising.[6] We will offer some intuition in Remark 4.3. Also, there will be a dual version of this result in Lemma 4.6.

Theorem 3.12. *Let μ be an envy-free matching, denote the fixed point of T starting from μ by $F(\mu)$. Then $F(\mu) = \mu \vee \mu_H$, where μ_H is the hospital optimal stable matching.*

In particular, starting with the empty matching, repeatedly applying T will converge to μ_H, i.e. T performs (a version of) the hospital proposing deferred acceptance algorithm.

[5]Remember the join operator $\vee$ matches each doctor to his more preferred hospital, therefore $\mu'_1 \vee \mu'_2$ in the related marriage market is the corresponding matching of $\mu_1 \vee \mu_2$ in the many-to-one market.

[6]The proof of Theorem A.3 in Blum et al (1997) is non-trivial: it uses a version of the decomposition lemma (listed as Lemma 4.5 in our paper) together with discussing three cases. See also Cantala (2011).

Proof of Theorem 3.12: By Lemma 3.11, $\mu \vee \mu_H$ is stable. By monotonicity of T, $\mu \vee \mu_H \succsim_D \mu$ implies $\mu \vee \mu_H \succsim_D F(\mu)$ (notice we always achieve $F(\mu)$ in finite time). On the other hand, T is a weakly Pareto improving operator for all the doctors, i.e. $T(\mu) \succsim_D \mu$, then $F(\mu) \succsim_D \mu$. Also, since μ_H is doctor pessimal among all stable matchings, we have $F(\mu) \succsim_D \mu_H$, thus $F(\mu) \succsim_D \mu \vee \mu_H$. Therefore the antisymmetry of $\succsim_D$ implies $F(\mu) = \mu \vee \mu_H$. □

4. Further Relations To Stable Matchings

Recall the "Rural Hospital Theorem" (Roth 1986): if a hospital does not fill its quota in one stable matching, then it is matched to the same set of doctors in all stable matchings. This implies that each hospital fills the same number of positions at every stable matching. Envy-freeness is a relaxation of stability, so we can ask, will we be able to fill more positions in an envy-free matching than in a stable matching? The negative answer lies in the lattice structure:

Corollary 4.1. *The total number of hospital positions filled in any envy-free matching will not exceed the total number of positions filled in any stable matching.*

Proof of Corollary 4.1: Let μ be any envy-free matching and μ_D be the doctor optimal stable matching. Then $\mu_D \succsim_D \mu$, since μ_D is the maximum element in the lattice. That means each doctor weakly likes his assignment in μ_D more than in μ, then weakly more doctors are assigned to a hospital in μ_D than in μ. Hence weakly more positions are filled in μ_D (and, by the Rural Hospital Theorem, in any stable matching) than in μ.[7] □

[7] One colleague suggested the following extension of Corollary 4.1: if a hospital does not fill its quota in one stable matching, is the set of doctors it is matched with in any envy-free matching a subset of the set of doctors it is matched with in all stable matchings? Unfortunately the answer is negative. Here is a simple example: 2 hospitals h_1 and h_2, 1 doctor d, d prefers h_1 to h_2 and all agents

What can we say about an envy-free matching and the hospital optimal stable matching μ_H?

Corollary 4.2. *Let μ be any envy-free matching. If $\mu \succsim_D \mu_H$, then μ is stable.*

Proof of Corollary 4.2: If $\mu \succsim_D \mu_H$, then $\mu \vee \mu_H = \mu$. By Theorem 3.12, $F(\mu) = \mu$, which means μ is stable. □

Remark 4.3.
(1). The contraposition of Corollary 4.2 says that if μ is envy-free but not stable, then at least one doctor is strictly worse off in μ than in μ_H.
(2). Note that Corollary 4.2 does not rule out the case that μ is envy-free but not stable, and μ is incomparable to μ_H. This can happen, so the envy-free but not stable matchings need not lie completely below the stable matchings.
(3). Corollary 4.2 "supports" Lemma 3.11. If one can prove Corollary 4.2 directly, then Lemma 3.11 follows as a consequence, since $\mu_1 \vee \mu_2 \succsim_D \mu_2 \succsim_D \mu_H$ implies $\mu_1 \vee \mu_2$ is stable.

The set of envy-free matchings is generally much larger than the set of stable matchings. Given an envy-free matching μ, there can potentially be some blocking pairs (d, h); if for each h involved in some blocking pair, we reduce the quota of h to the number of doctors it is currently matched to, then we obtain a stable matching in the new market. And vice versa, given a stable matching, if we increase the quotas of some hospitals, then this matching is envy-free in the new market. With this idea in mind, let's inspect a theorem proved by Gale and Sotomayor in the one-to-one matching market:

Theorem 4.4. *(Gale and Sotomayor 1985) In a one-to-one matching market, suppose H is contained in H' and μ_D and μ_H are the doctor, hospital optimal stable*

are acceptable to each other. Then the only stable outcome is to match d with h_1. However another matching assigning d to h_2 is envy-free. Therefore from the point of view of h_2, this conjecture does not hold. We thank him for sharing his idea.

matching respectively in market (D, H, P) *(*P *represents the preferences). Let* μ'_D, μ'_H *be the doctor and hospital optimal stable matching respectively in market* (D, H', P')*, where preferences* P' *agrees with* P *on* D *and* H*, then the following statements hold:*
$\mu_H \succsim_H \mu'_H$ *under* P*;* $\mu'_H \succsim_D \mu_H$ *under* P'*;*
$\mu'_D \succsim_D \mu_D$ *under* P'*;* $\mu_D \succsim_H \mu'_D$ *under* P*.*

In words, if some hospitals are removed from the market, then all doctors are weakly worse off and all remaining hospitals are weakly better off in both doctor optimal and hospital optimal stable matchings.

The fact that all doctors are weakly worse off in the doctor optimal stable matching is implied by the lattice structure, since the doctor optimal stable matching in a smaller market resulting when some hospitals are removed from the larger market is a simple matching in the larger market, which is weakly worse (for all doctors) than the maximum element in the lattice, namely the doctor optimal stable matching in the larger market. On the other hand, we can give a lower bound on the number of simple matchings, by using the statement that all hospitals remaining in the smaller market are weakly better off in the doctor optimal stable matching. Let H^* be the set of hospitals that are matched in the doctor optimal stable matching (in the larger market). We claim there are at least $2^{|H^*|}$ many simple matchings in the market: for each $H^{**} \subseteq H^*$, consider the doctor optimal stable matching in a market consists of the original set of doctors and H^{**}. It is a simple matching in the larger market and since all hospitals in H^{**} are weakly better off, they must remain matched in this simple matching. Therefore different H^{**} gives us a different simple matching in the larger market (since H^{**} is exactly the set of hospitals that are matched in the simple matching). Consequently there are at least $2^{|H^*|}$ many simple matchings in such a market.

Now we are ready to revisit the meet of two envy-free matchings. In Example 3.1, we showed that, in general, the Conway style meet ν that matches each doctor d to his less preferred hospital between $\mu(d)$ and $\mu'(d)$ (and each hospital h to the corresponding matched doctors) may not be a matching. However, for a stable matching μ and an envy-free matching μ', ν is well-defined. Furthermore, it is envy-free.

To prove this, we need a version of decomposition lemma proved in Blum et al (1997).[8] Readers are referred to the appendix of their paper for a proof.

Lemma 4.5. *(Blum et al 1997) In a one-to-one matching model, let μ be a stable matching and μ' be a simple matching. Define $H(\mu, \mu')$ to be the set of hospitals h such that $\mu(h) \succ_h \mu'(h)$ and h does not belong to any blocking pair in μ', and define $D(\mu', \mu)$ to be the set of doctors d such that $\mu'(d) \succ_d \mu(d)$. Then both μ and μ' are isomorphisms between $H(\mu, \mu')$ and $D(\mu', \mu)$.*

With this lemma, we obtain a dual result of Lemma 3.11:

Lemma 4.6. *If μ is a stable matching and μ' is an envy-free matching, then ν that matches each doctor d to his less preferred hospital between $\mu(d)$ and $\mu'(d)$ is also an envy-free matching. Moreover, it is the meet of μ and μ' in the lattice of envy-free matchings.*

Proof of Lemma 4.6: We only prove the case where each hospital has a quota of 1. The general case then follows from Proposition 2.2. (Just like the argument in Lemma 3.11.)

First, ν is a matching, i.e. no two doctors d_1 and d_2, both like a hospital h less, when comparing their assignments in μ and μ'. Assume otherwise, say d_1 and d_2 both choose h in ν. Without loss of generality, assume $h = \mu(d_1)$ and $h = \mu'(d_2)$, then by definition of ν, $\mu'(d_1) \succ_{d_1} h = \mu(d_1)$. Notice $\mu'(d_1) \succ_{d_1} \mu(d_1)$ implies

[8] The proof of Lemma 3.11 also depends on this result.

$d_1 \in D(\mu', \mu)$, then by Lemma 4.5, $h \in H(\mu, \mu')$. But applying Lemma 4.5 again, we have $d_2 \in D(\mu', \mu)$, which means $\mu'(d_2) \succ_{d_2} \mu(d_2)$, so h isn't d_2's less preferred hospital.

Now we show ν is simple. Assume this is not true, then there exists some hospital h such that $\nu(h) = d \in \mathbf{D}$ and (d', h) blocks ν, for some doctor d'. Notice by definition of ν, $h = \nu(d)$ is either $\mu(d)$ or $\mu'(d)$, similarly $\nu(d')$ is either $\mu(d')$ or $\mu'(d')$. If $h = \nu(d) = \mu(d) = \mu'(d)$, then (d', h) would block either μ or μ' which would contradict that they are simple. So we take $\mu(d)$ and $\mu'(d)$ to be distinct, and consider four cases: (I). $\nu(d) = \mu(d)$ and $\nu(d') = \mu(d')$. In this case (d', h) blocks μ, contradiction. (II). $\nu(d) = \mu'(d)$ and $\nu(d') = \mu'(d')$. In this case (d', h) blocks μ', contradiction. (III) $\nu(d) = \mu(d)$ and $\nu(d') = \mu'(d')$. By definition of ν, $\nu(d) = \mu(d) \Rightarrow \mu'(d) \succ_d \mu(d)$, i.e. $d \in D(\mu', \mu)$. Then by Lemma 4.5, $h = \mu(d) \in H(\mu, \mu')$, which means: (a). $d = \mu(h) \succ_h \mu'(h)$ and (b). h does not belong to any blocking pair in μ'. We show that (a) and (b) contradict each other. Since (d', h) forms a blocking pair in ν, we have $d' \succ_h d$. Then by transitivity, $d' \succ_h d \succ_h \mu'(h)$. Now look at μ', $h \succ_{d'} \nu(d') = \mu'(d')$ and $d' \succ_h \mu'(h)$, then (d', h) blocks μ', contradicts statement (b). (IV). $\nu(d) = \mu'(d)$ and $\nu(d') = \mu(d')$. In this case, $h = \mu'(d)$, then μ' is simple implies that h does not belong to any blocking pair in μ'. Notice $h \succ_{d'} \nu(d') = \mu(d')$, since (d', h) blocks ν. Then (d', h) is not a blocking pair in μ implies $\mu(h) \succ_h d'$. Also, we have $d' \succ_h d$, then $\mu(h) \succ_h d = \mu'(h)$. Therefore we have $h \in H(\mu, \mu')$, then by Lemma 4.5, $d = \mu'(h) \in D(\mu', \mu)$, i.e. $\mu'(d) \succ_d \mu(d)$. This contradicts the fact that $\nu(d) = \mu'(d)$. The above four cases are exhaustive, thus no matter what, ν is a simple matching.

It is clear that, when ν is well-defined, it will be the meet of μ and μ', in the lattice of envy-free matchings. □

Remark 4.7. (On the meet of two envy-free matchings:)

Given two envy-free matchings μ_1 and μ_2, let H_1 be the set of positions that has to be removed from $\mathbf{H}$ so that μ_1 is stable in the new market. Similarly define H_2 to be the set of positions that has to be removed from $\mathbf{H}$ so that μ_2 is stable in the new market. If one of H_1 and H_2 is contained in the other, say $H_1 \subseteq H_2$, then μ_2 is envy-free in the market created by removing H_1 from $\mathbf{H}$. By Lemma 4.6, ν is well-defined for μ_1 and μ_2, in the market with positions $\mathbf{H}\backslash H_1$, then it will also be envy-free in the original market consisting of positions in $\mathbf{H}$. Therefore for such μ_1 and μ_2, a Conway-like meet is well-defined. (Lemma 4.6 is the special case when $H_1 = \emptyset$.)

5. Generalization and Counterexamples

In this section we discuss some generalization attempts. As we mentioned in section 2, envy-freeness is one of the possible generalizations of simpleness defined by Sotomayor. The key idea of simple matching is that if there exists a blocking pair, then one of the agents involved must be unmatched. In envy-free matchings, we ensure any blocking hospital has at least one empty position in such a spirit. Another natural way of generalizing simple matchings to a many-to-one model is by requiring the doctor involved in a blocking pair to be unmatched, i.e. if (h, d) forms a blocking pair in μ, then $\mu(d) = \emptyset$. Let's call this kind of matching **Doctor-quasi-stable**. With such a notion of quasi-stability, we should focus on the common preference of hospitals, since we more or less ignored doctors' welfare by allowing matchings to persist even if some unmatched doctors form blocking pairs. Unfortunately, we lose the lattice structure in Doctor-quasi-stable matchings. Example 3.1 is still a valid counterexample for defining meet in the sense of Conway (switching roles of h and d of course). The following simple example will illustrate that a Conway-like join for hospitals is also problematic in this setting.

Example 5.1. Let there be one hospital h with quota $q_h = 3$ and four doctors d_1, d_2, d_3, d_4. The preference of h is: $d_1 \succ_h d_2 \succ_h d_3 \succ_h d_4 \succ_h \emptyset$, and each doctor prefers h than being unmatched. It is easy to see that both $\mu(h) = \{d_1, d_4\}$ and $\mu'(h) = \{d_2, d_3\}$

are Doctor-quasi-stable, but then defining the join of these two matchings needs more information than that preferences are responsive.

Therefore we cannot conduct a similar lattice theoretic analysis on Doctor-quasi-stable matchings. Now, let's turn to another type of generalization. Originally we worked with responsive preferences, which are a special case of substitutable preferences (for hospitals). In matching theory, substitutability is usually described by a collection of choice functions C_h (for each hospital h), which picks out the most preferred group of doctors for hospital h, given any subset of doctors. The preferences of hospitals satisfy **substitutability** if: $\forall h \in \mathbf{H}$, $D \subseteq \mathbf{D}$, d and $d' \in D$, we have $d \in C_h(D) \Rightarrow d \in C_h(D \setminus \{d'\})$. It is clear that the choice functions associated with responsive preferences satisfy the substitutability condition. Stability is also well-defined in this setting, a matching μ is stable if the following two conditions hold: (1) individual rationality: $\mu(d) \succsim_d \emptyset$ and $\mu(h) = C_h(\mu(h))$ for all doctor d and hospital h; (2) no blocking pair: there is no doctor hospital pair (d, h) such that $h \succ_d \mu(d)$ and $d \in C_h(\mu(h) \cup \{d\})$. One might wonder whether we can generalize our results to substitutable preferences.

The following example will rule out some naive attempts. The preferences in this example are taken from Roth and Sotomayor (1990).

Example 5.2. Let there be two hospitals h_1, h_2 with $q_{h_1} = 2$ and $q_{h_2} = 1$, and three doctors d_1, d_2, d_3. The preferences go as follows:

$h_1 : \{d_1, d_2\} \succ_{h_1} \{d_1, d_3\} \succ_{h_1} \{d_2, d_3\} \succ_{h_1} \{d_3\} \succ_{h_1} \{d_2\} \succ_{h_1} \{d_1\} \succ_{h_1} \emptyset$

$h_2 : \{d_3\} \succ_{h_2} \emptyset$

$d_1 : h_1 \succ_{d_1} h_2 \succ_{d_1} \emptyset$

$d_2 : h_1 \succ_{d_2} h_2 \succ_{d_2} \emptyset$

$d_3 : h_1 \succ_{d_3} h_2 \succ_{d_3} \emptyset$

One can easily verify that this preference profile is substitutable but not responsive, and there is a unique stable matching μ:

$\mu(d_1) = h_1$, $\mu(d_2) = h_1$ and $\mu(d_3) = h_2$.

We would have to be careful about how we define "envy-freeness" in the case of substitutable preferences.[9] If we do not modify envy-freeness at all, in the above stable matching μ, d_3 has justified envy towards d_2 since $\mu(d_2) = h_1$, $d_3 \succ_{h_1} d_2$ and $h_1 \succ_{d_3} h_2$. This is unreasonable; clearly we do not want envy-freeness to conflict with stability. One way to resolve this issue is by extending envy-freeness in the following way:[10] a doctor d has justified envy towards d' who is assigned to hospital h in matching μ, if (i). $h \succ_d \mu(d)$ and (ii). $C_h(\{d\} \cup \mu(h)) = \{d\} \cup (\mu(h) \setminus \{d'\})$. In other words, h is willing to kick d' out in order to admit d. Notice that under this definition of envy-freeness, a stable matching is always envy-free. Another possible modification is to make sure whenever (h, d) forms a blocking pair, h has at least one empty position. But with the above two definitions we lose the lattice structure: a matching μ' that $\mu'(d_3) = h_1$, and everyone else is unmatched, satisfies both quasi-stability concepts, but if we let each doctor choose their more preferred hospital between μ and μ', all of them will choose h_1, which violates the quota of h_1. So a Conway-like join would fail with these quasi-stability notions. (And again the meet fails by Example 3.1.) Of course we do not rule out the possibility that these matchings still form a lattice in the mathematical sense, but if both join and meet are not Conway-like, their interpretation would be unclear.

One of the reasons that we are unable to generalize our results to substitutable preferences is that our analysis relies heavily on the hospitals' preferences on individual doctors. Especially when proving the monotonicity of the Tarski operator T, we used

[9]See Sotomayor (1999) for a generalized definition of "envy-free" matchings in many-to-many matching markets with substitutable (but not necessarily strict) preferences.

[10]We thank an anonymous referee for suggesting this extension.

a full elementary combinatorial argument that focuses on the rankings of each individuals; while most other papers (e.g. every paper we mentioned in the introduction section) in the literature obtain this property through (usually trivial) set inclusions, which is naturally group oriented. But perhaps here is another reason that is more fundamental: the set of stable matchings under substitutable preferences, does not behave very nicely. Roth (1985b) showed that, in a many-to-one matching market, assuming substitutability on hospitals' preferences, there are counterexamples in which both of the Conway style join and meet (for doctors) of two stable matchings fail to be stable.[11] Therefore the set of stable matchings in such settings lacks necessary structure, let along envy-free matchings that are "less organized".

6. Applications

6.1. **Vacancy Chains.**

This section explores the role of envy-free matchings in vacancy chains that appear in senior level labor markets. Firms and workers in labor markets correspond to hospitals and doctors respectively in our model. In a senior level labor market where an initial matching is present, a small change in one of the firms (e.g. a worker retires or a new position is created) may trigger a chain reaction. For instance, imagine a professor at Harvard retires, his position may be filled by a professor from MIT. Now MIT has a vacant position, which may in turn attract a Stanford professor. Then Stanford would want to hire a new professor, and so on. This is a "vacancy chain". If the initial matching is stable, then after some workers retire (and are removed from the market), or/and some new positions are created, this matching may no longer be

[11]Li (2013) showed that the Conway style join and meet for hospitals are not well-defined among stable matchings in such markets either. However, with additional "cardinal monotonicity" conditions on hospitals' preferences, Alkan (2002) showed that the set of stable matchings forms a Conway-like lattice.

stable, but it remains envy-free. Another type of change that may result in a vacancy chain is some new workers joining the market. When this happens, the matching may not be envy-free any more. But if firms always prefer experienced workers to new ones, then envy-freeness is still preserved. Therefore envy-free matchings can be a good tool for studying vacancy chains. This is in fact the motivating example of Blum et al (1997), who constructed vacancy chain models with simple matchings. Clearly envy-freeness is a more general and suitable description of the market if some workers from the same firm retire at the same time or some firms create multiple positions at once.

Our Tarski operator describes a possible re-stabilization process following retirements or the creation of new positions. When a stable matching becomes an envy-free matching due to market changes, firms may start to make offers to workers. If each firm involved in a blocking pair makes an offer to its most preferred worker who forms a blocking pair with it (B^1_μ in section 3), and each worker accepts his most preferred offer received (B^2_μ), then a new envy-free matching is formed and the first round of vacancy chains is completed. If this process repeats until all vacancy chains have ended, then our discussion in section 3 shows that it will stop at a stable matching.[12] Moreover, if a centralized clearing-house exists, then it can expedite this firm-proposing like procedure by matching each worker to his partner in $\mu \vee \mu_H$, by invoking Theorem 3.12. Note that this process is not strategy-proof for the workers (in either the decentralized or centralized versions), since μ_H is not.

[12]This process often takes time to converge, and sometimes it ends before reaching the final stable matching. For instance, in the process of hiring junior faculty members, after an offer is rejected, academic departments sometimes decline to review additional potentially acceptable candidates, and delay hiring until the next academic year. As another example, envy-free but unstable (wasteful) outcomes are observed by Andersson et al. (2017 in preparation) in some sequential procedures for assigning students to public and private schools in Turkey and Sweden.

6.2. **Constrained Matching Models.**

The implications of the Rural Hospital Theorem sometimes give a reason for labor market administrators to avoid stable matchings, at which some unpopular "rural hospitals" will not be able to recruit enough doctors to meet their needs. To avoid such a situation, sometimes distributional constraints are introduced. For instance, starting from 2009, the Japanese government imposed regional caps in its national medical matching program to make sure enough doctors are assigned to rural hospitals. In response, several papers such as Kamada and Kojima (2010) and Goto et al (2014) develop modified matching models, seek relaxed stability notions, and propose new strategy-proof algorithms. Envy-freeness is one of the concepts adopted. Of course, a desirable matching should satisfy not only envy-freeness, but also some weakened efficiency constraints: since e.g. the empty matching is envy-free but inefficient. Kamada and Kojima (2017 in preparation) discover that when the constraints satisfy a what they call "generalized upper bound" assumption, there always exists a feasible envy-free matching that is most preferred by all doctors, comparing to any other feasible envy-free matching. And they develop an algorithm for constructing such a matching with the help of Tarski's fixed point theorem. A similar situation arises in school choice problems (see Abdulkadiroglu and Sönmez 2003, Ehlers et al 2014, for example). To avoid socioeconomic segregation, many school choice programs in the United States seek to impose constraints on enrollment at public schools. Under these constraints, stability with respect to original preferences may be unachievable. Often some version of envy-freeness (also called fairness[13] in the controlled school choice literature) becomes a goal of market designers.

[13] See for example Balinski and Sönmez (1999).

7. Conclusion

In a many-to-one matching model with responsive preferences, the set of envy-free matchings is a lattice, and the set of stable matchings equals the set of fixed points of a Tarski operator that can be interpreted as satisfying blocking pairs of the sort that arise in vacancy chains. The set of envy-free matchings seems likely to be of independent interest, since these matchings can arise as the intermediate matchings on the way to restabilizing a previously stable market following retirements or the creation of new positions. In decentralized markets, in which the re-stabilization process takes time, it may be that these intermediate states, in which some firms are actively seeking to fill recent vacancies, will be the typical state of the market.

References

[1] Abdulkadiroglu, Atila, and Tayfun Sönmez (2003). "School choice: A mechanism design approach." The American Economic Review 93.3: 729-747.

[2] Adachi, Hiroyuki (2000). "On a characterization of stable matchings." Economics Letters 68.1: 43-49.

[3] Alkan, Ahmet (2002). "A class of multipartner matching markets with a strong lattice structure." Economic Theory 19.4: 737-746.

[4] Andersson, Ola, Tommy Andersson, Umut Dur, Sinan Ertemel and Onur Kesten (2017). "Sequential Student Admissions: A Tale of Two Countries." In preparation.

[5] Balinski, Michel, and Tayfun Snmez (1999). "A tale of two mechanisms: student placement." Journal of Economic theory 84.1: 73-94.

[6] Blum, Yosef, Alvin E. Roth, and Uriel G. Rothblum (1997). "Vacancy chains and equilibration in senior-level labor markets." Journal of Economic Theory 76.2: 362-411.

[7] Cantala, David (2011). "Agreement toward stability in matching markets." Review of Economic Design 15.4: 293-316.

[8] Echenique, Federico, and Jorge Oviedo (2004). "Core many-to-one matchings by fixed-point methods." Journal of Economic Theory 115.2: 358-376.

[9] Ehlers, Lars, Isa E. Hafalir, M. Bumin Yenmez and Muhammed A. Yildirim (2014). "School choice with controlled choice constraints: Hard bounds versus soft bounds." Journal of Economic Theory 153: 648-683.

[10] Fleiner, Tamás (2003). "A fixed-point approach to stable matchings and some applications." Mathematics of Operations Research 28.1: 103-126.

[11] Gale, David, and Lloyd S. Shapley (1962). "College admissions and the stability of marriage." American mathematical monthly: 9-15.

[12] Gale, David, and Marilda Sotomayor (1985). "Some remarks on the stable matching problem." Discrete Applied Mathematics 11.3: 223-232.

[13] Goto, Masahiro, Naoyuki Hashimoto, Atshushi Iwasaki, Yujiro Kawasaki, Suguru Ueda, Yosuke Yasuda and Makoto Yokoo (2014). "Strategy-proof matching with regional minimum quotas." Proceedings of the 2014 international conference on Autonomous agents and multi-agent systems.

[14] Gusfield, Dan, and Robert W. Irving (1989). The stable marriage problem: structure and algorithms. Vol. 54. Cambridge: MIT press.

[15] Hatfield, John William, and Paul R. Milgrom (2005). "Matching with contracts." American Economic Review: 913-935.

[16] Irving, Robert W., and Paul Leather (1986). "The complexity of counting stable marriages." SIAM Journal on Computing 15.3: 655-667.

[17] Kamada, Yuichiro, and Fuhito Kojima (2010). "Improving efficiency in matching markets with regional caps: The case of the Japan residency matching program." Discussion Papers, Stanford Institute for Economic Policy Research.

[18] Kamada, Yuichiro, and Fuhito Kojima (2017). "Fair Matching under Constraints." In preparation.

[19] Knuth, Donald E. (1976). Marriages Stables. Montréal: Les Presses de l'Université de Montréal.

[20] Li, Jianrong (2013). "A note on Roth's consensus property of many-to-one matching." Mathematics of Operations Research 38.2: 389-392.

[21] Ostrovsky, Michael (2008). "Stability in supply chain networks." The American Economic Review: 897-923.

[22] Roth, Alvin E. (1985a). "The college admissions problem is not equivalent to the marriage problem." Journal of Economic Theory 36.2: 277-288.

[23] Roth, Alvin E. (1985b). "Conflict and coincidence of interest in job matching: some new results and open questions." Mathematics of Operations Research 10.3: 379-389.

[24] Roth, Alvin E. (1986). "On the allocation of residents to rural hospitals: a general property of two-sided matching markets." Econometrica: Journal of the Econometric Society: 425-427.

[25] Roth, Alvin E., and Marilda A. Oliveira Sotomayor (1990). Two-sided matching: A study in game-theoretic modeling and analysis. No. 18. Cambridge University Press.

[26] Roth, Alvin E., and John H. Vande Vate (1990). "Random paths to stability in two-sided matching." Econometrica: Journal of the Econometric Society: 1475-1480.

[27] Sotomayor, Marilda (1996). "A non-constructive elementary proof of the existence of stable marriages." Games and Economic Behavior 13.1: 135-137.

[28] Sotomayor, Marilda (1999). "Three remarks on the many-to-many stable matching problem." Mathematical social sciences 38.1: 55-70.

[29] Sotomayor, Marilda (2000). "Existence of stable outcomes and the lattice property for a unified matching market." Mathematical Social Sciences 39.2: 119-132.

[30] Sotomayor, Marilda (2006). "The role played by simple outcomes in the pairwise coalition formation process in matching markets." (working paper)

[31] Sotomayor, Marilda (2016). "The role played by the well-behaved matchings in the coalition formation process of the stable matchings for the roommate market." (working paper)

[32] Stanley, Richard P. (2011). Enumerative combinatorics. Vol. 1. Cambridge university press.

[33] Tarski, Alfred (1955). "A lattice-theoretical fixpoint theorem and its applications." Pacific journal of Mathematics 5.2: 285-309.