Bargaining in an Ongoing Exchange with Renegotiation*

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Abstract

I develop a theory of wage determination in a novel bargaining environment with features found in the labor market. A buyer and a seller produce repeatedly in a dynamic stochastic environment with the possibility of renegotiation and with no credible commitment to a state-dependent schedule of wages. I model a bargaining game of alternating offers and show that as the time between counteroffers goes to zero, there is a single equilibrium wage in each state of the world. This wage is Nash's bargaining solution in which the surplus to be shared is a weighted combination of the surplus from exchange over an instant and the surplus from exchange over the duration of the match. As with single shot bargaining, the relative weight of each formulation of the surplus depends on the relative importance of delay versus match breakdown. The possibility of renegotiation pins down wage dynamics and is suitable for modeling wage bargaining with on the job search. I conclude by using job finding and vacancy filling rates to endogenize worker and firm bargaining power in the Nash bargaining solution.

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1 Introduction and related literature

This paper studies bargaining between two agents involved in an ongoing productive relationship in a dynamic environment without commitment to a schedule of payoffs. The analysis is intended to mirror bargaining between workers and firms over the terms of employment. The wage determines the allocation of the surplus from production. This surplus changes over time with productivity and other aspects of the productive and bargaining environment, resulting in an exchange that occurs repeatedly in a stochastic dynamic environment in which it is not possible to commit to a contingent contract of future wages that spans all eventualities.

The resulting equilibrium wage is stable, in that neither firms nor workers choose to renegotiate until the state has changed. The formulation of the wage is the familiar Nash bargaining result, in spite of the dynamic environment with no commitment. I derive a formula for endogenizing worker bargaining power, and discuss how this bargaining outcome is applicable to wage bargaining in an environment with on the job search.

Bargaining is a method of determining the division of a surplus between two (or more) parties. The seminal paper in bargaining theory is Nash (1950), which shows that there is a unique solution to the division of a convex surplus space that satisfies four axiomatic properties together: of Pareto efficiency, independence from irrelevant alternatives, symmetry, and invariance to affine transformations of the utility representations. Rubinstein (1982) relates the Nash bargaining solution to an alternating offer bargain in which declining and proposing a counteroffer reduces the value of the surplus to be split. As the time between counteroffers goes to zero, the equilibrium wage approaches the Nash bargaining outcome.

This shrinking of the surplus may be due to the cost of delay or a risk of the surplus being lost. Binmore et al. (1986) make explicit this contrast between time preference and the loss of the surplus which occurs when the match breaks down. An implication of this distinction is that the relevant outside option, the alternative to sharing the surplus, depends on which form the opportunity cost of proposing a counteroffer takes.

The bargaining environment in this paper is framed as bargaining under search and matching. The solution methodology is related to that in Trejos and Wright (1995) and Shi (1995). In their models, agents search until a compatible buyer and seller meet, bargain over the terms of their exchange, then begin searching again. The bargaining incorporates both a cost from delay and a risk of match breakdown during bargaining.

Most of the research in this field studies a one-time bargain, where there is a single moment at which the division of the surplus is determined. This is a plausible way to describe the market for many goods, particularly large, indivisible goods such as housing. However, in an ongoing relationship the parties may seek to renegotiate the terms of exchange. Workers sell their labor to firms repeatedly, sometimes for several decades, with the match surplus and the wage changing over time. Because labor laws limit the compulsion of labor, a worker and firm can not commit to a schedule of future terms of exchange.

There have been previous papers that incorporate some of the features of this analysis. Rudanko (2009) models a wage bargain between risk averse workers and risk neutral firms with limited commitment. In her model, it is efficient to commit to a constant wage, but renegotiations will occur when the surplus from the match becomes negative for either the firm or the worker. Her bargaining method does not incorporate the time cost of bargaining.

In contrast, Holden (1994) describes a scenario of wage-setting with renegotiation in which the relevant outside option is delay, not leaving the match. A wage agreement holds until conditions change sufficiently such that either firms or workers have an incentive to reopen bargaining, due to inflation or changes in demand. However, Holden does not include the risk of match breakdown during bargaining.

One other similar framing of the wage bargain is the work of Hall and Milgrom (2008). In their model, a firm and a worker make an alternating offer bargain with a delay of one business day between the arrival of counteroffers. Continuing to bargain results in both a cost from delay and an increased risk of the match breaking down. As a result, Hall and Milgrom's model incorporates aspects of both outside options, delay and leaving the match. In contrast this paper, by studying the limiting case wherein the time between counteroffers approaches zero, derives an analytical solution to the bargaining problem. This derivation allows for analysis of the relative importance of the two outside options in determining the wage and the result is compatible with renegotiation over time. This resolves a theoretical issue for considerations of wage bargaining with on the job search, the issue of changing outside options over time.

The outcome of bargaining derived in this paper is similar to that derived in Brügemann and Moscarini (2010) under complete information. Their bargaining outcome is the result of a one-time bargain in a static environment. Strikingly, their bargaining outcome has the same structural form as the outcome in this paper, in spite of the differences in the bargaining environment. Bargaining in a dynamic stochastic environment without commitment leads to a similar specification of the outcome as the environment without dynamics. Of course, the wage outcome is not static in this paper, which allows for a rich analysis of wage dynamics. I conclude with a brief analysis of the dynamics of bargaining power implied by the cyclical behavior of matching.

2 The model

This analysis uses a discrete time model. I will consider some fixed unit of time to have length one. Each period in the model will be a fraction Δ of that fixed unit of time. When a firm and worker match, they bargain over a wage to split the resulting surplus. Periods are the intervals of time over which workers (and firms) can delay and propose a counteroffer during bargaining. In moving from one period to the next, firms and workers discount at rate ρ. The discount factor is given by $\rho = e^{-r\Delta}$, which approximates discounting at rate r per fixed unit of time.

The state of the world is denoted by $x \in X$, where X is the set of all states. It is stochastic, with its evolution described by a discrete Markov process. The state includes productivity p, the separation rate s, both the worker's and firm's value of abandoning the match U and V, and both the worker's and firm's probability of match breakdown during bargaining B_W and B_F. These state variables will be more thoroughly described in the following section.

$$x = (p, s, U, V, B_W, B_F) \tag{1}$$

Productivity p is the output from the match during one fixed unit of time. When production takes place, output each period is Δp.

Matches break down exogenously at the end of each period, with probability $\psi \equiv 1 - e^{-s\Delta}$. The separation hazard, s, is the proportion of matches that break down during one fixed unit of time.

Shocks to the state arrive according to an exponential distribution with arrival rate $\lambda \equiv 1 - e^{-\ell\Delta}$. When a shock to the state occurs, a new realization is drawn according to a Markov chain process, which is independent of the value of Δ. Define π^x as the probability that, upon realization of a shock while in state x, the state remains unchanged. Further define $\pi_{x,x'}$ as the conditional probability that the new state is x' given that the current state is x and that a shock has occurred. The expected duration between shocks is $1/\ell$. The expected duration of a single state is $((1 - \pi^x)\ell)^{-1}$.

While matched, workers and firms receive a stage payoff each period. When the match breaks down, they receive a one-time state payoff, the payoff of entering an unmatched state, then the game ends. The total payoff is the discounted sum of the stage payoffs plus the discounted one-time state payoff.

When production takes place, workers derive utility from wage income according to monotone increasing period utility function $\Delta u(w)$, where w is the wage rate. For each period in which a worker is not working, they receive period utility $\Delta u(z)$, where z denotes the combined value of leisure and non-work income. If the match breaks down, workers enter a state with value U. Firms are risk neutral, and derive profit $\Delta(p - w)$ during each period where production takes place. When firms are matched but production does not take place, firms pay cost $\Delta\gamma$. If the match breaks down, firms enter a state with value V.[1]

Throughout, I make Assumption 1, which ensures that the non-trivial case, where bargaining is relevant, holds. There is a surplus to producing relative to bargaining, and there is a surplus to being matched relative to leaving the match.

[1]It is common to assume free entry of vacancy postings, so that $V = 0, \forall x$ in equilibrium, but such an assumption is not necessary for this analysis.

Assumption 1 *Existence of surplus*

1. *In all states, there is a surplus to production. For workers, I assume* $p > z, \forall p$. *For firms, I assume* $p > 0 \geq -\gamma, \forall p$.

2. *In all states, there is a surplus to remaining matched:* $\exists \tilde{w}$ *such that* $\Delta u(\tilde{w}) + \rho U' > U$ *and* $\Delta(p - \tilde{w}) + \rho V' > V, \forall x, x', \Delta$.

3. *Worker utility is continuously twice differentiable, increasing, and weakly concave with respect to the wage:*
$u'(w) > 0, u''(w) \leq 0, \forall w$.

2.1 A game of wage bargaining without credible commitment

With the payoffs and match dynamics defined, I turn to a description of the bargaining game. This section begins by laying out a game of alternating offers bargaining over ongoing production with the possibility of renegotiation. I define a subgame perfect equilibrium and characterize the limiting equilibrium as the time between counteroffers goes to zero and compare it to Nash's axiomatic bargaining solution. Throughout, I make the following assumption about information and wage determination.

Assumption 2 *Information and bargaining*

1. *There is perfect information.*

2. *Firms and workers have rational expectations about the state and the outcome of bargaining.*

3. *Neither firms nor workers can precommit to a schedule of wages for subsequent periods, state-contingent or not.*

I assume that there is no credible precommitment to wage schedules, so that the resulting (state-contingent) wage is the solution to a Bellman equation. This stands in contrast to most prior models of wage bargaining, where the outcome of the bargain is a split of the surplus, with either a fixed wage or with the wage dynamics left indeterminate.

I consider an intuitive framework for bargaining: firms and workers have alternating opportunities to propose a wage and to respond. The bargaining outcome is subject to renegotiation in subsequent periods.

When a firm and a worker match, they bargain over the wage by means of a series of offers and counteroffers, one each period. Production and wage payment do not begin until bargaining is concluded. Once an agreement is reached, it holds until either the firm or the worker decide to renegotiate the wage.

In addition to the time cost, rejecting a wage offer increases the probability of the match breaking down. If the firm proposes a wage and the worker rejects, the match breaks down with probability β_W. Similarly, when a firm rejects a wage proposed by the worker, the match breaks down with probability β_F. Together with the per-period separation rate ψ, there is probability $(1-\beta_W)(1-\psi)$ that the match remains the period after a worker rejects a wage offer. Similarly, there is probability $(1-\beta_F)(1-\psi)$ that the match remains the period after a worker rejects a wage offer. To simplify notation, I introduce terms $\delta_W = \psi + \beta_W(1-\psi)$, and $\delta_F = \psi + \beta_F(1-\psi)$, which are the probabilities of match breakdown following a rejected offer.

The match breakdown risk incurred through rejection of a wage offer varies with the duration of a period. Accordingly, $\beta_W = 1 - e^{-B_W \Delta}$ and $\beta_F = 1 - e^{-B_F \Delta}$. Parameters B_W and B_F fix the hazard rate of match breakdown in terms of fixed units of time.

The bargaining game has three subgames, depicted in Figures 1 and 2: bargaining when production took place previously at wage w, labeled $\mathbf{Prod}(w, x)$; bargaining where the firm proposes a wage, labeled $\mathbf{Barg}_F(x)$; and bargaining where the worker proposes a wage, labeled $\mathbf{Barg}_W(x)$. Every subgame starts with the components of x, that is (p, s, U, V, B_W, B_F), as state variables. The first period, when the firm and worker have no prior wage set, either the worker or the firm proposes a wage with probability $1/2$.

If the worker proposes the wage, they play subgame $\mathbf{Barg}_W(x)$; if the firm proposes the wage, they play subgame $\mathbf{Barg}_F(x)$. Let $i \in \{W, F\}$ denote the agent that proposes the wage and $-i$ denote the other agent. If agent i accepts proposed wage w', production takes place, workers receive benefit $\Delta u(w')$ and firms receive profit $\Delta(p - w')$. After production, the match breaks down with probability ψ, in which case workers and firms receive their non-match state values U' and V', discounted by ρ and allowing for a shock to change the state to a new realization of x'. If the match does not break down, there may be a shock changing to state x' and the next period the agents play subgame $\mathbf{Prod}(w', x')$, which is discounted by ρ. If the proposed wage is rejected, workers receive period payoff $\Delta u(z)$ and firms pay cost $\Delta\gamma$. The match breaks down with probability δ_{-i}, in which case there is a new realization of x', and workers and firms receive their non-match state values U' and V', discounted by ρ. If the match does not break down, there is a new realization of x' and the next period they play subgame $\mathbf{Barg}_{-i}(x')$, which is discounted by ρ.

When production has taken place in the previous period, workers and firms each have the opportunity to request a renegotiation. If both workers and firms choose not to renegotiate, production takes place, and workers receive the same wage as in the previous period. If agent $i \in \{W, F\}$ requests the renegotiation and the other agent $-i \neq i$ does not, agent $-i$ proposes a wage w'. If both the worker and the firm request a renegotiation, there is a random draw as to which agent proposes the wage, with probability $1/2$ for either agent.

This framing of the game prevents equilibria where the firm proposes a low wage, and the worker accepts then immediately requests a renegotiation (or vice versa).

One could instead consider mechanisms whereby requesting renegotiation results in foregone production or where requesting renegotiation comes with a cost that is proportional to Δ. Either of these mechanisms will result in the same equilibrium in Section 2.3, which is the limiting case where $\Delta \to 0$ characterized by condition (8). This is because the indifference conditions (4) and (5) are unaffected.

Bargaining in subgames $\mathbf{Barg}_F(x)$ and $\mathbf{Barg}_W(x)$ is the same as the bargaining in the production subgame, without the option to continue production at the previous period's wage. Consequently, a set of strategies consists of proposed wages, acceptance/rejection of proposed wages, and renegotiation, for both workers and firms.

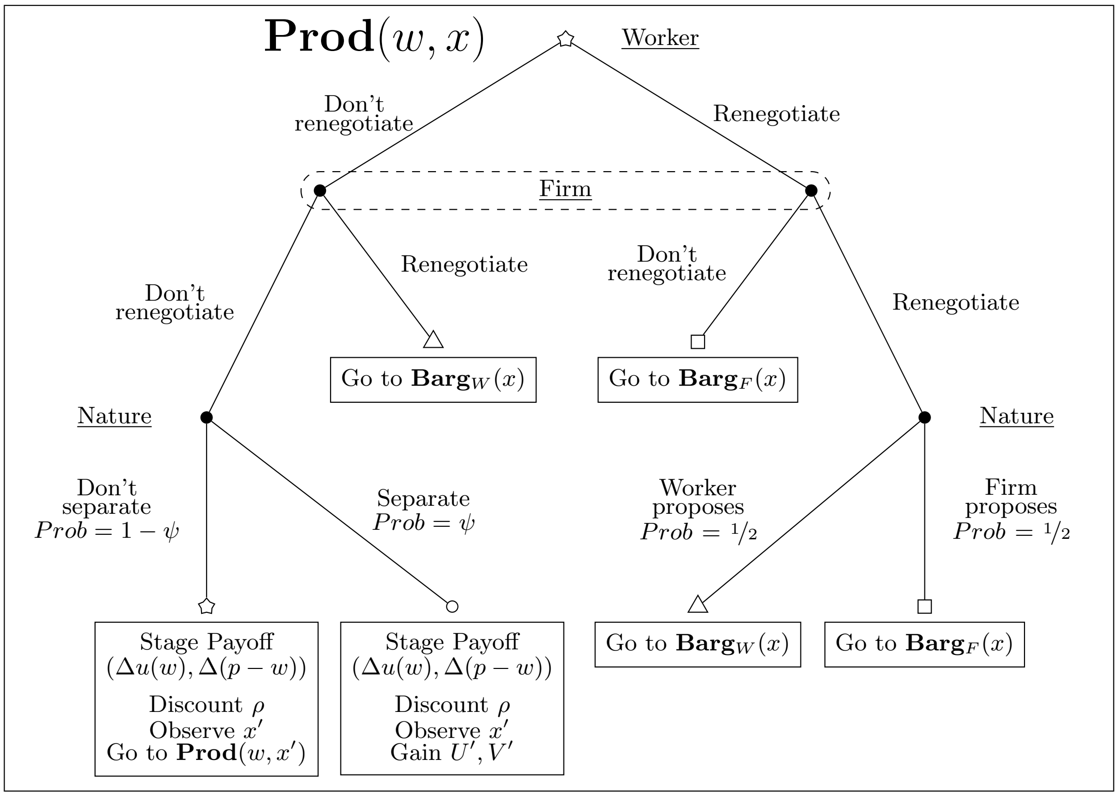
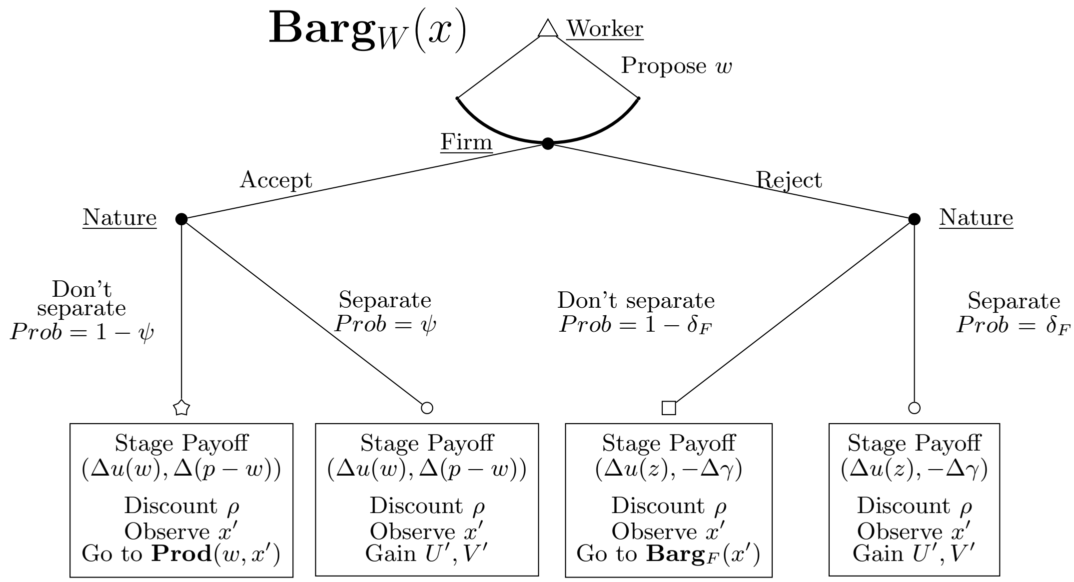
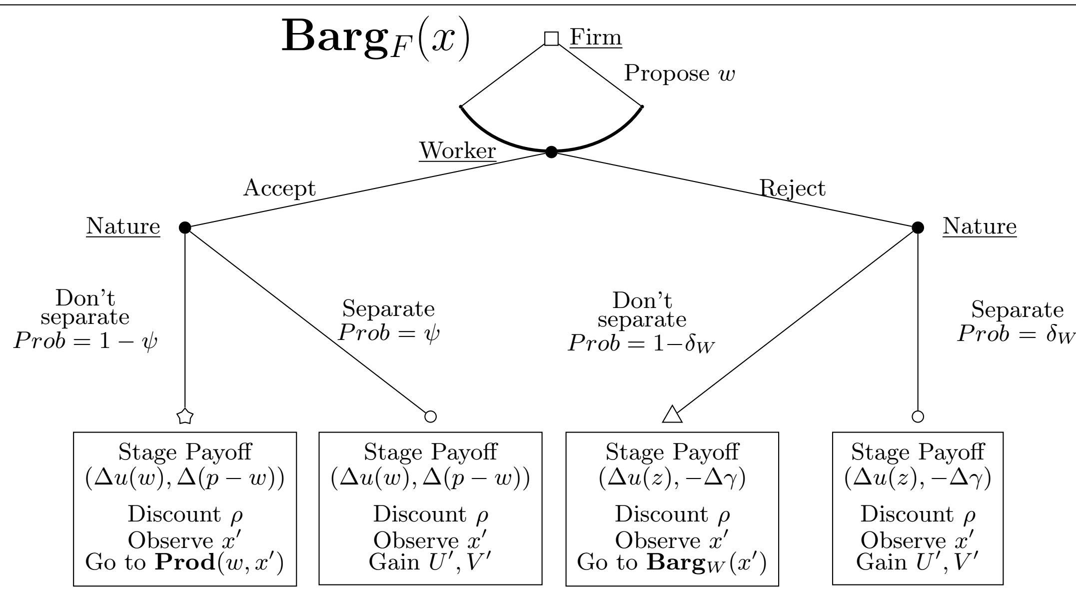


Figure 1: Production subgame of the bargaining game. Subgame in which the state is x and production took place in the previous period with wage w.



(a) Bargaining in which the state is x and the worker proposes a wage.



(b) Bargaining in which the state is x and the firm proposes a wage.

Figure 2: Non-production subgames of the bargaining game.

2.2 A characterization of equilibrium

In order to propose and characterize an equilibrium set of strategies, I first introduce a set of recursive characterizations of the value of producing under a given wage, as well as a set of conditions laying out state contingent wage strategies. First, let us label the wage strategies. When bargaining in state p, firms propose wage $\underline{w}_\Delta(x)$. Workers propose $\bar{w}_\Delta(x)$.

Strategy 1 *1. During bargaining, firms propose wage $\underline{w}_\Delta(x)$, such that the worker is indifferent between accepting the wage or rejecting and making a counteroffer. Workers propose $\bar{w}_\Delta(x)$, such that the firm is indifferent between accepting the wage or rejecting and making a counteroffer.*

2. In state x, firms accept all wage offers $w \leq \bar{w}_\Delta(x)$ and workers accept all wage offers $w \geq \underline{w}_\Delta(x)$.

3. In state x, the worker opens renegotiation when $w < \underline{w}_\Delta(x)$ and the firm opens renegotiation when $w > \bar{w}_\Delta(x)$.

Proposition 1 *Strategy 1 is a subgame perfect equilibrium.*

A proof of Proposition 1 can be found in Appendix A.

Given that the firm and the worker follow Strategy 1, consider a period in which the state is p and production takes place with wage w. Define $\pi^+_\Delta(w,x)$ as the probability that, when a shock occurs, the state in the next period is one in which $\underline{w}_\Delta(x') > w$. Define $\pi^-_\Delta(w,x)$ as the probability that, when a shock occurs, the state in the next period is one in which $\bar{w}_\Delta(x') < w$. Finally, define $\pi^\sim_\Delta(w,x)$ as the probability that, when a shock occurs, the state in the next period is one in which $\bar{w}_\Delta(x') > w > \underline{w}_\Delta(x')$.

Value function $M_\Delta(w,x)$ is the value to a worker of working in state x with wage w for a given value of Δ. $J_\Delta(w,x)$ is the equivalent value function for firms.

$$M_\Delta(w,x) = \Delta u(w) + \rho(1-\psi)E\left[M_\Delta(w',x')\right] + \rho\psi E\left[U'\right]$$
$$J_\Delta(w,x) = \Delta(p-w) + \rho(1-\psi)E\left[J_\Delta(w',x')\right] + \rho\psi E[V']$$

where the expectations operators are as follows:

$$\begin{aligned}
\rho(1-\psi)E\left[M_\Delta(w',x')\right] &= \chi^x_\Delta M_\Delta(w,x) \\
&\quad + \chi^+_\Delta(w,x)E^+_{w,x}[M_\Delta(\underline{w}_\Delta(x'),x')] \\
&\quad + \chi^-_\Delta(w,x)E^-_{w,x}[M_\Delta(\bar{w}_\Delta(x'),x')] \\
&\quad + \chi^\sim_\Delta(w,x)E^\sim_{w,x}[M_\Delta(w,x')] \\
\rho(1-\psi)E\left[J_\Delta(w',x')\right] &= \chi^x_\Delta J_\Delta(w,x) \\
&\quad + \chi^+_\Delta(w,x)E^+_{w,x}[J_\Delta(\underline{w}_\Delta(x'),x')] \\
&\quad + \chi^-_\Delta(w,x)E^-_{w,x}[J_\Delta(\bar{w}_\Delta(x'),x')] \\
&\quad + \chi^\sim_\Delta(w,x)E^\sim_{w,x}[J_\Delta(w,x')]
\end{aligned}$$

Define probability parameters $\chi_\Delta^x \equiv \rho(1-\psi)(1-\lambda(1-\pi^x))$, $\chi_\Delta^+(w,x) \equiv \rho(1-\psi)\lambda\pi_\Delta^+(w,x)$, $\chi_\Delta^-(w,x) \equiv \rho(1-\psi)\lambda\pi^-(w,x)$, and $\chi_\Delta^\sim(w,x) \equiv \rho(1-\psi)\lambda\pi^\sim(w,x)$.

Define conditional expectations operators $E_{w,x}^+ \equiv E_{x':\underline{w}_\Delta(x')>w}$, $E_{w,x}^- \equiv E_{x':\bar{w}_\Delta(x')<w}$, and $E_{w,x}^\sim \equiv E_{x'\neq x:w\in[\underline{w}_\Delta(x'),\bar{w}_\Delta(x')]}$.

The recursive value functions may be rewritten:

$$M_\Delta(w,x) = \frac{\Delta u(w) + \rho(1-\psi)E\left[M_\Delta(w',x')\right] - \chi_\Delta^x M_\Delta(w,x) + \rho\psi E[U']}{1-\chi_\Delta^x} \tag{2}$$

$$J_\Delta(w,x) = \frac{\Delta(p-w) + \rho(1-\psi)E\left[J_\Delta(w',x')\right] - \chi_\Delta^x J_\Delta(w,x) + \rho\psi E[V']}{1-\chi_\Delta^x} \tag{3}$$

The indifference conditions for acceptance versus rejection of $\underline{w}$ and $\bar{w}$ are given by conditions (4) and (5).

$$M_\Delta(\underline{w}_\Delta(x),x) = \Delta u(z) + \rho(1-\delta_W)E[M_\Delta(\bar{w}_\Delta(x'),x')] + \rho\delta_W E[U'] \tag{4}$$

$$J_\Delta(\bar{w}_\Delta(x),x) = -\Delta\gamma + \rho(1-\delta_F)E[J_\Delta(\underline{w}_\Delta(x'),x')] + \rho\delta_F E[V'] \tag{5}$$

2.3 Instantaneous counteroffers

Consider the implications of the strategy. First, I introduce simplified notation for the value functions. As I show in Proposition 2, as $\Delta \to 0$, the wage and hence the value function does not depend on which party proposes. To reflect this, the value function representations drop the Δ subscript and are a function only of the state x.

$$M(x) \equiv lim_{\Delta\to 0} M_\Delta(\bar{w}_\Delta(x),x) = lim_{\Delta\to 0} M_\Delta(\underline{w}_\Delta(x),x)$$

$$J(x) \equiv lim_{\Delta\to 0} J_\Delta(\bar{w}_\Delta(x),x) = lim_{\Delta\to 0} J_\Delta(\underline{w}_\Delta(x),x)$$

Given this outcome, the value of being matched and producing in state x can be represented as the sum of three components. The first is the discounted value of receiving period utility (or profit) $u(w(x))$ over the expected duration of being matched in state x. The second component is the expected value of being matched once the state has changed, discounted according to the expected duration until the state changes. The third is the value of being unmatched, discounted according to the expected duration until the match breaks down.

Taking the limits as $\Delta \to 0$, the value functions are defined recursively by conditions (6) and (7).

$$M(x) = \frac{u(w(x)) + \ell(1-\pi^x)E_{x'\neq x}[M(x')] + sU}{r+s+\ell(1-\pi^x)} \tag{6}$$

$$J(x) = \frac{p - w(x) + \ell(1-\pi^x)E_{x'\neq x}[J(x')] + sV}{r+s+\ell(1-\pi^x)} \tag{7}$$

Proposition 2 *The subgame perfect equilibrium in Proposition 1 exhibits the following limiting properties:*

1. *In the limit as $\Delta \to 0$, firms and workers offer the same wage $w(x) \equiv \underline{w}_\Delta(x) = \bar{w}_\Delta(x)$, for any given x.*

2. *In the limit as $\Delta \to 0$, the equilibrium wage solves condition (8).*

$$\frac{u(w(x)) - u(z) + B_W[M(x) - U]}{p - w(x) + \gamma + B_F[J(x) - V]} = u'(w(x)) \tag{8}$$

A proof of Proposition 2 can be found in Appendix B.

As $\Delta \to 0$, bargaining is instantaneous and counteroffers are never made in the equilibrium. There is a single equilibrium wage in each state and its value does not depend on which agent proposes the wage. Although firms and workers can commence bargaining at any point, they do so only when the state changes. Firms initiate bargaining when it will lower the wage, and workers initiate bargaining when it will raise the wage.

2.4 Existence and properties of the equilibrium

Having characterized the outcome of bargaining, the task remains to analyze its properties. As a first step, I show that for any given set of value functions $M(x)$ and $J(x)$, the wage is uniquely determined and is strictly positive.

Proposition 3 *Given x, for arbitrary values $M(x), J(x)$ there exists a unique, strictly positive wage that solves condition (8).*

A proof of Proposition 3 can be found in Appendix C. The properties of the wage are the result of a single crossing property of condition (8).

These properties of the wage then allow for determination of the properties of the value functions, and the wage. There is a unique equilibrium.

Proposition 4 *The subgame perfect equilibrium defined by Proposition 2 exists, and results in unique value functions $M(x)$ and $J(x), \forall x \in X$, as well as a unique wage $w(x)$.*

A proof of Proposition 4 can be found in Appendix D. The value functions $M(x)$ and $J(x)$ are shown to be the fixed point of a contraction. The properties of the equilibrium follow from this result.

2.5 Comparative statics under IID draws of the state

This model takes place in an environment with a six-dimensional state space and forward-looking dynamics. As a result, further assumptions are necessary to characterize the effect of a change in the state because of the effects of changes in the expectations about future realizations of the state. However, if we assume that the realization of the states is IID, we can draw some inferences about how the state affects the outcome. Under IID draws of the state, the equilibrium outcome has the properties summarized in Table 1.

Comparative statics with IID realizations of the state			
	$w(x)$	$M(x)$	$J(x)$
p	$\in (0,1)$	> 0	> 0
s	Ambiguous	Ambiguous	Ambiguous
U	≥ 0	> 0	≤ 0
V	≤ 0	≥ 0	> 0
B_W	< 0	< 0	> 0
B_F	> 0	> 0	< 0

Table 1: The column gives the state variable that changes, the row gives the affected outcome.

The comparative statics generally match the intuitive expectations of the outcome of bargaining. When productivity increases, all else equal, there is a larger surplus to distribute, and both firms and workers are made better off. When one party's outside option improves, they receive a larger proportion of the output from production. When one party faces a greater risk that the match will break down during bargaining, their offers become less aggressive, benefiting the other party.

The effects of a change in the separation rate are ambiguous, and the impact on the wage may even be sufficiently large to result in the worker or the firm (but not both) becoming better off after an increase in the separation rate.

3 Bargaining theory

The outcome of this bargaining game can be related to the well-known bargaining outcome described by Nash. In Nash (1950), the bargaining solution maximizes the product of the firm's and worker's surpluses. Allowing for asymmetric bargaining power, Nash's bargaining solution solves the following condition, where worker's bargaining power is $\theta \in [0,1]$.

$$w^* = argmax_w \left[\text{Worker's Surplus}(w)\right]^{\theta} \times \left[\text{Firm's Surplus}(w)\right]^{1-\theta}$$

Binmore et al. (1986) explore the links between alternating offers bargaining and Nash bargaining, and make a distinction between two kinds of outside options. In rejecting a proposed wage, firms/workers may incur some cost from delay (the foregone profit and wage), and may increase the risk that the match breaks down. In the former case, the relevant outside options that firms and workers have are the returns to not producing at any instant: $-\gamma$ and $u(z)$. The related surpluses are $p - w + \gamma$ and $u(w) - u(z)$, the net benefit to production, in terms of flows. Accordingly, Nash's bargaining solution with symmetric bargaining power solves condition (9).

$$\frac{u(w) - u(z)}{p - w + \gamma} = u'(w) \tag{9}$$

If B_W and B_F are zero, that is, if prolonging bargaining does not increase the likelihood of the match breaking down, then the bargaining game equilibrium, condition (8), is equivalent to condition (9), the characterization of Nash bargaining over flow values.

Consider instead a bargain where rejecting a wage offer brings an increased risk that the match breaks down and there is no cost from delay. The relevant outside options that firms and workers have in this situation are the values of being in the unmatched state: V and U. The related surpluses are $J(x)-V$ and $M(x)-U$, the net benefits from being matched in terms of state values. Given these surpluses, Nash's bargaining solution with worker bargaining power θ solves condition (10).

$$\frac{\theta}{1-\theta}\frac{M(x) - U}{J(x) - V} = u'(w) \tag{10}$$

Suppose that B_W and B_F are relatively large, that is, prolonging bargaining carries a significant risk that the match breaks down. Then the state values become more influential in determining the equilibrium wage and the relative effect of the cost from delay is diminished. In this situation, the bargaining game equilibrium condition (8) is approximately condition (11).

$$\frac{B_W}{B_F}\frac{M(x) - U}{J(x) - V} = u'(w) \tag{11}$$

3.1 Renegotiation and the outcome of bargaining

In condition (8), both the numerator and the denominator of the left hand side are linear combinations of the flow surpluses and the surpluses in terms of state values. The resulting bargaining outcome can be considered to be a hybrid of the two bargaining outcomes, with the relative weights determined by the discounting parameter r and the match breakdown parameters B_W and B_F.

The structure of condition (8) is similar to the equilibrium conditions derived in Trejos and Wright (1995), Shi (1995), and Brügemann and Moscarini (2010), all of which model one-time bargains. The relative weights on bargaining over flow values versus bargaining over state values are determined by the same parameters. Allowing for the possibility of renegotiation does not affect the importance of delay versus match breakdown in determining the equilibrium wage. Naturally, allowing for renegotiation does introduce wage dynamics into the outcome.

This bargaining model may prove useful for models with wage bargaining and on the job search. When an employed worker matches with a new firm, there is the question of what the constitutes the worker's outside option. The firms might engage in Bertrand competition over the worker, but if so, why would the winning firm keep paying that wage once the worker has left their other match?

The bargaining outcome in condition (8) is immune to renegotiation. The relevant outside options are delayed production and leaving the match for unemployment/vacancy, which do not change when the worker/firm is matched with another firm/worker. This is because the worker's value of leaving the match becomes relevant to the bargain when the firm leaves, and vice versa.

3.2 Endogenous worker bargaining power

If we assume that the cost of delay can be ignored, then worker bargaining power can be endogenized. Conditions (10) and (11) imply condition (12), which defines the worker bargaining power θ in terms of the increased risk of match breakdown due to bargaining. In models of wage bargaining that apply the Nash bargaining solution over state values, the worker bargaining power parameter is often treated as an exogenous parameter, either chosen somewhat arbitrarily or calibrated to match a desired equilibrium outcome. Endogenizing this parameter holds potential for improving the understanding of wage bargaining.

$$\theta = \frac{B_F}{B_W + B_F} \tag{12}$$

If the match breakdown parameters B_W and B_F can be measured empirically, it is possible to produce an estimate of the worker bargaining power parameter. As an illustrative example, consider the case where, while bargaining is ongoing, firms may encounter other workers and workers may encounter other vacancies, as in Rubinstein and Wolinsky (1985).

By making additional assumptions about matching during the bargaining process, listed in Assumption 3, it is possible to estimate an endogenous value of worker bargaining power θ.

Assumption 3 *Estimation of endogenous worker bargaining power*

1. *There is no cost due to delay during bargaining.*

2. *Workers (firms) can each only be matched with one firm (worker) at a time.*

3. *Upon encountering another match, workers (firms) abandon the current match with probability* $\alpha_1 \in [0, 1]$.

4. *Workers (firms) may not be matched with more than one firm (worker).*

5. *Bargaining workers (firms) encounter alternative vacancies (unemployed workers) at rate* α_2 *times the rate experienced by searching unemployed workers (vacancies).*

This assumption rules out multiparty wage bargains such as Bertrand competition between two workers for the same job, as in Coles and Muthoo (1998).

Let f denote the probability that an unemployed worker meets with a firm during a single fixed unit of time. Let q denote the probability that a firm with a vacancy meets an unemployed worker. For simplicity of notation, define $\alpha \equiv \alpha_1 * \alpha_2$.

Suppose that the firm has proposed a wage and the worker is deciding whether to accept the offer. If the worker rejects the wage offer, she runs the risk that the firm will find another worker and abandon the match. Similarly, if a firm rejects a proposed wage, it runs the risk that the worker will find another vacancy. These assumptions yield the following endogenizations of the match breakdown probabilities and worker bargaining power:

$$B_W = \alpha q$$
$$B_F = \alpha f$$
$$\theta = \frac{f}{q + f}$$

These series may be estimated from US data. I generate the data on the job-finding rate f according to the procedure in Shimer (2005). I estimate the vacancy-filling rate q by dividing the job-finding rate by labor market tightness, the ratio of vacancies to unemployed workers. Labor market tightness is estimated using the vacancy data generated in Barnichon (2010) matched to the data from the Job Opening and Labor Turnover Survey. The data sources are summarized in Appendix E.

Figure 3 gives the resulting estimate of worker bargaining power over time in the US. According to this result, bargaining power is procyclical, so that workers' share of the surplus from matching is larger during booms than during recessions. This is because booms are times when job offers arrive more frequently to workers and job applicants arrive less frequently to firms, with the reverse true during recessions.

In the context of bargaining over state values, this procyclicality of worker bargaining power results in wages that are more cyclical than would result if worker bargaining power was constant over the business cycle. It is worth considering how this compares to what we see in the data.

In a study of models of job search with bargained wages and endogenous vacancy posting, Shimer (2005) shows that calibrating a model with the Nash bargain over

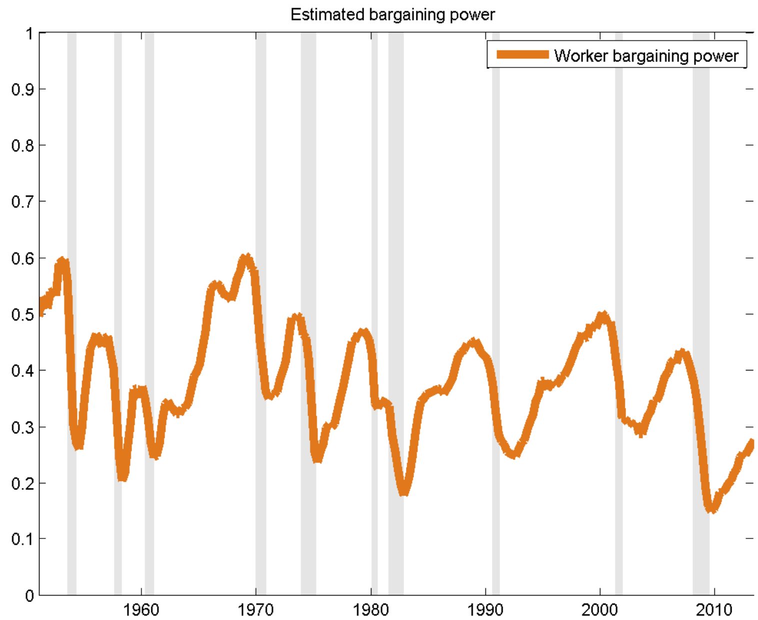


Figure 3: Values of worker bargaining power derived from match breakdown probabilities. Shaded areas indicate recessions.

state values induces, relative to US data, excessively procyclical wages and insufficient volatility in vacancies and unemployment. Shimer's calibration used constant worker bargaining power over the business cycle. The endogenous worker bargaining power derived here exacerbates this mismatch between data and model results, by increasing the volatility of wages. However, if the probability of match breakdown is small, the state values receive little weight in the surplus, and the volatility of wages is reduced. In all, this is suggestive evidence that the flow values receive most of the weight in wage bargaining.

4 Conclusion

This paper studies the outcome of bargaining in an ongoing productive relationship with renegotiation in a dynamic environment. While the bargaining environment is novel, the bargaining equilibrium is similar to outcomes in static environments. I derive a characterization of the resulting equilibrium wage which can be related to the two framings of the Nash bargain that are studied in Binmore et al. (1986). When the risk of match breakdown during bargaining is low, the bargaining outcome tends toward the Nash bargain over flow values. Alternatively, when the risk of match breakdown during bargaining is high, the bargaining outcome tends toward the Nash bargain over state values. I conclude with an analysis of the implications for worker bargaining power under the asymmetric Nash bargain over state values.

Before concluding, it is worthwhile to consider the theoretical implications of the bargaining outcome in condition (8). Similar results obtain in the monetary model of Shi (1995) and in Brügemann and Moscarini (2010). These papers study a one-time bargain, with no wage dynamics over time. The addition of renegotiation in a dynamic environment yields a similar analytic result, albeit one with wage dynamics as the state changes. This shows that the ability to renegotiate does not affect which type of Nash bargain predominates in the outcome. Rather, it is the relative importance of the costs of delay versus the risk of match breakdown that determines the relevant surplus to be split. The ability to renegotiate affects how frequently the wage will change but not the determination of the wage, conditional on renegotiation.

A Proof of subgame perfect equilibrium

Proof 1 This is a game with discounting as $\rho \in (0, 1)$ and with additively separable payoffs, so per Blackwell (1965), the one-shot deviation principle applies. A sufficient condition for subgame perfect equilibrium is to show that any single deviation from the strategy is not optimal.

Throughout, assume that workers and firms follow the strategy described in Strategy 1.

Claim 1.1 *Neither workers nor firms have an incentive to deviate from the equilibrium strategy in terms of proposed wages.*

Proof 1.1 Fix an arbitrary series $x = \{x_0, x_1, ...\}$, representing a realization of states, over consecutive periods. Define $i = argmin_{\{t\}}$ s.t. $x_t \neq x_0$ and $j = argmin_{\{t>i\}}$ s.t. $x_t \neq x_i$.

Assume that the worker and firm both follow the equilibrium strategy, and that in period 0, the worker proposes the wage. The worker proposes wage $\bar{w}_\Delta(x_0)$ and the firm accepts. The state, and consequently the wage, does not change until period i. If $\underline{w}_\Delta(x_i) > \bar{w}_\Delta(x_0)$, the worker opens renegotiation, the firm proposes $\underline{w}_\Delta(x_i)$, and the worker accepts. If $\bar{w}_\Delta(x_i) < \bar{w}_\Delta(x_0)$, the firm opens renegotiation, the worker proposes $\bar{w}_\Delta(x_i)$, and the firm accepts. If $\underline{w}_\Delta(x_i) \leq \bar{w}_\Delta(x_0) \leq \bar{w}_\Delta(x_0)$, the wage remains unchanged. Define the path of wages under the equilibrium strategy as series $\{w^*\} = \{w_0^*, w_1^*, ...\}$.

Suppose, by way of contradiction, the worker proposes $w \neq \bar{w}_\Delta(x_0)$ in period 0, then follows the equilibrium strategy. Define the path of wages under this alternative strategy as $\{\hat{w}\} = \{\hat{w}_0, \hat{w}_1, ...\}$. In the next six paragraphs, I consider all cases in which either $i > 1$ or $i = 1$, and in which the worker's proposed wage w satisfies $w > \bar{w}_\Delta(x_0)$, $w \in [\underline{w}_\Delta(x_0), \bar{w}_\Delta(x_0)]$, or $w < \underline{w}_\Delta(x_0)$.

Consider first the case where $i > 1$ and the worker proposes $w > \bar{w}_\Delta(x_0)$. The firm rejects the proposed wage, and the worker receives payoff $u(z) < u(w_0^*)$. In period 1, the firm proposes $\underline{w}_\Delta(x_0)$, and the worker accepts. $\hat{w}_t \leq w_t^*, \forall t \in \{1, 2, ..., i-1\}$, and $\hat{w}_t = w_t^*, \forall t \geq i$. The worker does not benefit from this deviation.

In the second case, $i > 1$ and the worker proposes $w \in [\underline{w}_\Delta(x_0), \bar{w}_\Delta(x_0)]$. The firm accepts the proposed wage, which remains unchanged until period i, at which point either the wage is renegotiated, or remains lower than it would have under the equilibrium strategy, as the firm would have requested renegotiation. The wage remains lower until such point as w is outside of the set of accepted wages, say at period j. From period j on, the outcome is the same as if the worker had not deviated from the strategy. In the resulting wage series, $\hat{w}_t \leq w_t^*, \forall t \in \{0, 1, ..., j-1\}$, and $\hat{w}_t = w_t^*, \forall t \geq j$. The worker does not benefit from this deviation.

In the third case, $i > 1$ and the worker proposes $w < \underline{w}_\Delta(x_0)$. The firm accepts. In period 1, the worker opens renegotiation. The firm proposes $\underline{w}_\Delta(x_0)$, and the worker accepts. The result is that $\hat{w}_t \leq w_t^*, \forall t \in \{0, 1, ..., i-1\}$, and $\hat{w}_t = w_t^*, \forall t \geq i$. The worker does not benefit from this deviation.

In the fourth case, $i = 1$ and the worker proposes $w > \bar{w}_\Delta(x_0)$. The firm rejects the proposed wage, and the worker receives payoff $u(z) < u(w_0^*)$. In period 1, the firm proposes $\underline{w}_\Delta(x_1)$, and the worker accepts. Consequently $\hat{w}_t \leq w_t^*, \forall t \in \{1, 2, ..., j-1\}$, and $\hat{w}_t = w_t^*, \forall t \geq j$. The worker does not benefit from this deviation.

In the fifth case, $i > 1$ and the worker proposes $w \in [\underline{w}_\Delta(x_0), \bar{w}_\Delta(x_0)]$. The firm accepts the proposed wage. The firm accepts the proposed wage, and then upon the change in state in period 1, either the wage is renegotiated or remains lower than it would have under the equilibrium strategy, as the firm would have requested renegotiation. The wage remains lower until such point as w is outside of the set of accepted wages, say at period j. From period j on, the outcome is the same as if the worker had not deviated from the strategy. In the resulting wage series, $\hat{w}_t \leq w_t^*, \forall t \in \{0, 1, ..., j-1\}$, and $\hat{w}_t = w_t^*, \forall t \geq j$. The worker does not benefit from this deviation.

In the sixth case, $i = 1$ and the worker proposes $w < \underline{w}_\Delta(x_0)$. The firm accepts, so that $\hat{w}_0 < w_0^*$. In the next period, the state x changes to x_1. If $\hat{w}_0 \in [\underline{w}(x_1), \bar{w}(x_1)]$, the wage is unchanged, and remains unchanged until period j, when x changes. Under the equilibrium strategy, the wage in periods 1 to j would have been $w_1^* = \bar{w}_\Delta(x_1)$. If instead $\hat{w}_0 < \underline{w}_\Delta(x_1)$, the worker requests renegotiation in period 1, and the resulting wage of $\underline{w}(x_1)$ holds until period j. Under the equilibrium strategy, the wage in periods 1 to j would have been either $\bar{w}_\Delta(x_1)$ or $\underline{w}_\Delta(x_1)$. Lastly, if $\hat{w}_0 > \bar{w}(x_1)$, the firm requests renegotiation in period 1, and the resulting wage of $\bar{w}(x_1)$ holds until period j. Under the equilibrium strategy, the wage in periods 1 to j would have been $w_1^* = \bar{w}(x_1)$. In all variations of this case, $\hat{w}_t \leq w_t^*, \forall t \in \{0, 1, ..., j-1\}$, and $\hat{w}_t = w_t^*, \forall t \geq j$. The worker does not benefit from this deviation.

Therefore, there is no deviation from the wage proposal strategy that the worker may take, in any series of draws $\{x\}$, that leaves the worker better off.

The proof for the firm's proposed wage is omitted, as is it is symmetric to the proof for the worker's proposed wages.

QED

Claim 1.2 *Neither workers nor firms have an incentive to deviate from the equilibrium strategy in terms of accepted wages.*

Proof 1.2 Suppose the state is x and the worker proposes the wage. By definition, firms are indifferent between accepting and rejecting wage offers $\bar{w}_\Delta(x)$, so it is optimal to accept any lower wage offer. Workers are indifferent between accepting and rejecting wage offers $\underline{w}_\Delta(x)$, so it is optimal to accept any higher wage offer.

QED

Claim 1.3 *Neither workers nor firms have an incentive to deviate from the equilibrium strategy in terms of opening renegotiation.*

Proof 1.3 Suppose the state is x and the wage in the previous period was w. If a firm were to open renegotiation, the worker would offer $\bar{w}_\Delta(x)$, so if $w > \bar{w}_\Delta(x)$, firms are better off if they request renegotiation. If a worker were to open renegotiation, the firm would offer $\underline{w}_\Delta(x)$, so if $w < \underline{w}_\Delta(x)$, workers are better off if they request renegotiation.

QED

B Proof of characterization of equilibrium in the limit

This is the proof of Proposition 2.

Proof 2 Throughout, assume that workers and firms follow Strategy 1. First, using conditions (2) and (3), rewrite the indifference conditions (4) and (5) as (13) and (14).

$$\begin{aligned}
\Delta\left[u(\underline{w}_\Delta(x)) - u(z)\right] = \; & \chi_\Delta^x \left[M_\Delta(\bar{w}_\Delta(x), x) - M_\Delta(\underline{w}_\Delta(x), x)\right] \\
& + \chi_\Delta^+(\underline{w}_\Delta(x), x) E^+_{\underline{w}_\Delta(x), x}\left[M_\Delta(\bar{w}_\Delta(x'), x') - M_\Delta(\underline{w}_\Delta(x'), x')\right] \\
& + \chi_\Delta^\sim(\underline{w}_\Delta(x), x) E^\sim_{\underline{w}_\Delta(x), x}\left[M_\Delta(\bar{w}_\Delta(x'), x') - M_\Delta(\underline{w}_\Delta(x), x')\right] \\
& - \rho(1-\psi)\beta_W E[M_\Delta(\bar{w}(x'), x') - U']
\end{aligned} \tag{13}$$

$$\begin{aligned}
\Delta\left[p - \bar{w}_\Delta(x) + \gamma\right] = \; & \chi_\Delta^x \left[J_\Delta(\underline{w}_\Delta(x), x) - J_\Delta(\bar{w}_\Delta(x), x)\right] \\
& + \chi_\Delta^-(\bar{w}_\Delta(x), x) E^+_{\bar{w}_\Delta(x), x}\left[J_\Delta(\underline{w}_\Delta(x'), x') - J_\Delta(\bar{w}_\Delta(x'), x')\right] \\
& + \chi_\Delta^\sim(\bar{w}_\Delta(x), x) E^+_{\bar{w}_\Delta(x), x}\left[J_\Delta(\underline{w}_\Delta(x'), x') - J_\Delta(\bar{w}_\Delta(x), x')\right] \\
& - \rho(1-\psi)\beta_F E[J_\Delta(\underline{w}_\Delta(x'), x') - V']
\end{aligned} \tag{14}$$

Claim 2.1 *In the limit as $\Delta \to 0$, both firms and workers offer the same wage $w(x) \equiv \underline{w}_\Delta(x) = \bar{w}_\Delta(x)$.*

Proof 2.1 Begin by noting that

$$\begin{aligned}
&lim_{\Delta\to 0}\chi^{+}_{\Delta}(w,x) = 0, \forall x, \forall w \in [\underline{w}_\Delta(x), \bar{w}_\Delta(x)] \\
&lim_{\Delta\to 0}\chi^{-}_{\Delta}(w,x) = 0, \forall x, \forall w \in [\underline{w}_\Delta(x), \bar{w}_\Delta(x)] \\
&lim_{\Delta\to 0}\chi^{\sim}_{\Delta}(w,x) = 0, \forall x, \forall w \in [\underline{w}_\Delta(x), \bar{w}_\Delta(x)] \\
&lim_{\Delta\to 0}\beta_W = 0 \\
&lim_{\Delta\to 0}\beta_F = 0
\end{aligned}$$

Taking the $lim_{\Delta\to 0}$ of conditions (13) and (14) gives the following:

$$\begin{aligned}
M(\bar{w}_\Delta(x), x) &= M(\underline{w}_\Delta(x), x) \\
J(\underline{w}_\Delta(x), x) &= J(\bar{w}_\Delta(x), x)
\end{aligned}$$

Applying conditions (2) and (3), I find that $\underline{w}_\Delta(x) = \bar{w}_\Delta(x), \forall x$. Define wage function $w(x)$ as this equilibrium wage as $\Delta \to 0$.

$\mathcal{QED}$

Claim 2.2 *In the limit as $\Delta \to 0$, $w(x)$ satisfies condition (8).*

Proof 2.2 Take the ratio between conditions (13) and (14).

$$\frac{\Delta\left(u(\underline{w}_\Delta(x)) - u(z)\right) + \rho(1-\psi)\beta_W E[M_\Delta(\bar{w}(x'), x') - U']}{\Delta(p - \bar{w} + \gamma) + \rho(1-\psi)\beta_F E[J_\Delta(\underline{w}_\Delta(x'), x') - V']}$$
$$=$$
$$\frac{\left\{\begin{array}{c} \chi^{x}_{\Delta}[u(\underline{w}_\Delta(x)) - u(\bar{w}_\Delta(x))] + \chi^{+}_{\Delta}(\underline{w}_\Delta(x), x)E^{+}_{\underline{w}_\Delta(x),x}[u(\underline{w}_\Delta(x')) - u(\bar{w}(x'))] \\ +\chi^{\sim}_{\Delta}(\underline{w}_\Delta(x), x)E^{\sim}_{\underline{w}_\Delta(x),x}\left[M_\Delta(\bar{w}_\Delta(x'), x') - M_\Delta(\underline{w}_\Delta(x), x')\right] \end{array}\right\}}{\left\{\begin{array}{c} \chi^{x}_{\Delta}(\bar{w}_\Delta(x) - \underline{w}_\Delta(x)) + \chi^{-}_{\Delta}(\bar{w}_\Delta(x), x)E^{-}_{\bar{w}_\Delta(x),x}[\underline{w}_\Delta(x') - \bar{w}(x')] \\ +\chi^{\sim}_{\Delta}(\bar{w}_\Delta(x), x)E^{+}_{\bar{w}_\Delta(x),x}\left[J_\Delta(\underline{w}_\Delta(x'), x') - J_\Delta(\bar{w}_\Delta(x), x')\right] \end{array}\right\}}$$

Allow $\Delta \to 0$. Applying L'Hôpital's rule to the left hand side and the limit definition of the derivative to the right hand side, we obtain the following:

$$\frac{u(w(x)) - u(z) + B_W[E[M(x')] - E[U']]}{p - w(x) + \gamma + B_F[E[J(x')] - E[V']]} = u'(w(x))$$

Note that as $\Delta \to 0$, the expected value of the state values in the next period is their current value.

$$\begin{aligned}
&lim_{\Delta\to 0}E[M_\Delta(x')] = M(x) &\qquad& lim_{\Delta\to 0}E[U'] = U \\
&lim_{\Delta\to 0}E[J_\Delta(x')] = J(x) &\qquad& lim_{\Delta\to 0}E[V'] = V
\end{aligned}$$

In the limit as $\Delta \to 0$, the equilibrium wage is characterized by condition (8).

$\mathcal{QED}$

C Proof of existence and uniqueness of a wage that solves condition (8)

This is the proof of Proposition 3.

Proof 3 Take as given value functions $M(x)$ and $J(x)$, $\forall x \in X$

Holding all else constant, the LHS of condition (8) is monotone increasing in $w(x)$. The RHS is positive and, by weak concavity of $u(\cdot)$, decreasing or of constant value.

By restricting $w(x)$ to the domain of values for which both the firm and the worker have positive surplus, we can establish that the range of the LHS of condition (8) is a subset of $[0, +\infty]$. The corresponding minimum value of $w(x)$ is $u^{-1}(u(z) - B_W[M(x) - U])$. The maximum value of $w(x)$ is $p + \gamma + B_F[J(x) - V]$. By continuity of $u(\cdot)$, we have that the range of the LHS is convex.

With the LHS convex, weakly positive, and increasing monotonically; and the RHS strictly positive and weakly decreasing by the properties of $u(\cdot)$, condition (8) exhibits a single crossing property by which, for given value functions, there is a single solution $w(x)$, for every $x \in X$.

$\mathcal{QED}$

D Proof of properties of equilibrium in the limit

This is the proof of Proposition 4.

Proof 4 Conditions (6) and (7) are mappings of M and J onto themselves. Together with the wage that solves condition (8), we may define operator $T : B^2(X) \to B^2(X)$ as follows:

$$TA(x) = \begin{pmatrix} \phi_W(w(x), A) \\ \phi_F(w(x), A) \end{pmatrix}$$

Where

- $A(x) = \begin{pmatrix} M(x) \\ J(x) \end{pmatrix}$ is a vector of the two value functions
- $w(x)$ solves (8) given M and J
- functions $\phi_W(w(x), A) = \frac{u(w(x)) + \ell(1-\pi^x)E_{x' \neq x}[M(x')] + sU}{r + s + \ell(1-\pi^x)}$ and $\phi_F(w(x), A) = \frac{p - w(x) + \ell(1-\pi^x)E_{x' \neq x}[J(x')] + sV}{r + s + \ell(1-\pi^x)}$ are the RHS of conditions (6) and (7), respectively

In order to determine the properties of the equilibrium, we will consider the related operator $\hat{T} : B^2(X) \to B^2(X)$ equilibrium can be represented as the fixed point of a contraction mapping.

$$\hat{T}A(x) = \kappa \begin{pmatrix} \phi_W(w(x), A) \\ \phi_F(w(x), A) \end{pmatrix} + (1-\kappa) \begin{pmatrix} M(x) \\ J(x) \end{pmatrix} \tag{15}$$

Where $\kappa = min_x \frac{2[r+s+\ell(1-\pi_x)]}{1+2[r+s+\ell(1-\pi_x)]} \in (0,1)$.

Any steady state for operator $\hat{T}$ will also be a steady state for operator T, and vice versa. However, operator $\hat{T}$ satisfies Blackwell's Sufficient Conditions for a contraction. This allows the proof of existence and uniqueness of the steady state solution, and characterization of its properties.

In order to prove this, we will need to know the effect on the wage of changes in the value functions $M(\cdot)$ and $J(\cdot)$. By implicit differentiation of condition (8), we have:

$$\frac{\partial w(x)}{\partial M(x)} = \frac{-B_W}{2u'(w) - [p - w(x) + \gamma + B_F[J(x) - V]]u''(w(x))} \in (-\infty, 0] \tag{16}$$

$$\frac{\partial u(w(x))}{\partial M(x)} = \frac{-B_W}{2 - [p - w(x) + \gamma + B_F[J(x) - V]]\frac{u''(w(x))}{u'(w(x))}} \in [-\frac{1}{2}, 0] \tag{17}$$

$$\frac{\partial w(x)}{\partial J(x)} = \frac{B_F}{2 - [p - w(x) + \gamma + B_F[J(x) - V]]\frac{u''(w(x))}{u'(w(x))}} \in [0, \frac{1}{2}] \tag{18}$$

$$\frac{\partial u(w(x))}{\partial J(x)} = \frac{u'(w(x))B_F}{2 - [p - w(x) + \gamma + B_F[J(x) - V]]\frac{u''(w(x))}{u'(w(x))}} \in [0, \infty) \tag{19}$$

Claim 4.1 *Condition (15) satisfies monotonicity.*

Proof 4.1 Consider first two value functions

$$A_1(x) = \begin{pmatrix} M_1(x) \\ J_1(x) \end{pmatrix} \geq \begin{pmatrix} M_2(x) \\ J_2(x) \end{pmatrix} = A_2(x), \forall x$$

Select an arbitrary x. Denote the resulting price solving condition (8) given value functions M_i and J_i with $w_i(x)$. Then condition (15) implies:

$$\hat{T}A_1(x) - \hat{T}A_2(x) = \kappa \begin{pmatrix} \phi_W(w_1(x), A_1) - \phi_W(w_2(x), A_2) \\ \phi_F(w_1(x), A_1) - \phi_F(w_2(x), A_2) \end{pmatrix} + (1-\kappa)(A_1(x) - A_2(x))$$

In order to show that monotonicity holds, we need to show that the RHS of the above condition is positive, for all $\begin{pmatrix} M_1(x) \\ J_1(x) \end{pmatrix} \geq \begin{pmatrix} M_2(x) \\ J_2(x) \end{pmatrix}, \forall x$. This can be shown by considering the effect of increasing $M(\tilde{x})$ and $J(\tilde{x})$ on $\hat{T}A(x)$, where $\tilde{x} \in X$ need not be x.

Consider <u>first</u> the effect of increasing $M(x)$ on $\hat{T}A(x)$. In order for monotonicity to hold, we need that condition (20) holds.

$$\frac{\partial TA(x)}{\partial M(x)} \geq 0 \tag{20}$$

Taking derivatives, we have that the following conditions must hold:

$$\begin{aligned} \kappa\frac{\partial \phi_W(w(x),A)}{\partial M(x)} + (1-\kappa) &\geq 0 \\ \kappa\frac{\partial \phi_F(w(x),A)}{\partial M(x)} &\geq 0 \end{aligned}$$

By the definition of functions ϕ_W and ϕ_F, we have:

$$\begin{aligned} \frac{\kappa}{r+s+\ell(1-\pi^x)}\frac{\partial u(w(x))}{\partial M(x)} + (1-\kappa) &\geq 0 \\ -\frac{\kappa}{r+s+\ell(1-\pi^x)}\frac{\partial w(x)}{\partial M(x)} &\geq 0 \end{aligned}$$

By conditions (16) and (17), we have the following, which shows that the second inequality holds.

$$\begin{aligned} -\frac{1}{2}\frac{\kappa}{r+s+\ell(1-\pi^x)} + (1-\kappa) &\geq 0 \\ 0 &\geq 0 \end{aligned}$$

By our definition of parameter κ, the following holds, so that the first inequality also holds.

$$1 - \frac{1+2[r+s+\ell(1-\pi_x)]}{2[r+s+\ell(1-\pi_x)]} * min_x \left\{ \frac{2[r+s+\ell(1-\pi_x)]}{1+2[r+s+\ell(1-\pi_x)]} \right\} \geq 0$$

In the <u>second</u> case, the effect of increasing $M(\tilde{x})$ on $\hat{T}A(x)$, where $\tilde{x} \neq x$. In order for monotonicity to hold, we need that condition (21) holds.

$$\frac{\partial TA(x)}{\partial M(\tilde{x})} \geq 0 \tag{21}$$

Taking derivatives, we have that the following conditions must hold:

$$\begin{aligned} \kappa\frac{\partial \phi_W(w(x),A)}{\partial M(\tilde{x})} &\geq 0 \\ \kappa\frac{\partial \phi_F(w(x),A)}{\partial M(\tilde{x})} &\geq 0 \end{aligned}$$

By the definition of functions ϕ_W and ϕ_F, we have the following conditions, which both hold.

$$\kappa \frac{\ell \pi_{x,\tilde{x}}}{r+s+\ell(1-\pi^x)} \geq 0$$
$$0 \geq 0$$

As the <u>third</u> case, consider the effect of increasing $J(x)$ on $\hat{T}A(x)$. In order for monotonicity to hold, we need that condition (22) holds.

$$\frac{\partial TA(x)}{\partial J(x)} \geq 0 \tag{22}$$

Taking derivatives, we have that the following conditions must hold:

$$\kappa \frac{\partial \phi_W(w(x), A)}{\partial J(x)} \geq 0$$
$$\kappa \frac{\partial \phi_F(w(x), A)}{\partial J(x)} + (1-\kappa) \geq 0$$

By the definition of functions ϕ_W and ϕ_F, we have:

$$\frac{\kappa}{r+s+\ell(1-\pi^x)} \frac{\partial u(w(x))}{\partial J(x)} \geq 0$$
$$-\frac{\kappa}{r+s+\ell(1-\pi^x)} \frac{\partial w(x)}{\partial J(x)} + (1-\kappa) \geq 0$$

By conditions (18) and (19), we have the following, which shows that the first inequality holds.

$$0 \geq 0$$
$$-\frac{1}{2} \frac{\kappa}{r+s+\ell(1-\pi^x)} + (1-\kappa) \geq 0$$

By our definition of parameter κ, the following holds, so that the second inequality also holds.

$$1 - \frac{1+2[r+s+\ell(1-\pi_x)]}{2[r+s+\ell(1-\pi_x)]} * min_x \left\{ \frac{2[r+s+\ell(1-\pi_x)]}{1+2[r+s+\ell(1-\pi_x)]} \right\} \geq 0$$

In the <u>fourth</u> case, the effect of increasing $J(\tilde{x})$ on $\hat{T}A(x)$, where $\tilde{x} \neq x$. In order for monotonicity to hold, we need that condition (23) holds.

$$\frac{\partial TA(x)}{\partial M(\tilde{x})} \geq 0 \tag{23}$$

Taking derivatives, we have that the following conditions must hold:

$$\kappa\frac{\partial\phi_W(w(x),A)}{\partial J(\tilde{x})} \geq 0$$
$$\kappa\frac{\partial\phi_F(w(x),A)}{\partial J(\tilde{x})} \geq 0$$

By the definition of functions ϕ_W and ϕ_F, we have the following conditions, which both hold.

$$0 \geq 0$$
$$\kappa\frac{\ell\pi_{x,\tilde{x}}}{r+s+\ell(1-\pi^x)} \geq 0$$

Since all four sets of conditions (20) to (23) hold, we have shown that operator $\hat{T}$ satisfies monotonicity, as any increase in the value function vector A leads to an increase in $\hat{T}A$.

$\mathcal{QED}$

Claim 4.2 *Condition (15) satisfies discounting.*

Proof 4.2 Consider arbitrary vector D comprised of two positive constants. Apply transformation T:

$$\hat{T}\left[A(x)+D\right] = \kappa\left(\begin{array}{c}\frac{u(w(x))+\ell(1-\pi^x)E_{x'\neq x}[M(x')+D_1]+sU}{r+s+\ell(1-\pi^x)}\\ \\ \frac{p-w(x)+\ell(1-\pi^x)E_{x'\neq x}[J(x')+D_2]+sV}{r+s+\ell(1-\pi^x)}\end{array}\right)+(1-\kappa)\left[A(x)+D\right]$$
$$= \hat{T}A(x) + \left[\kappa\frac{\ell(1-\pi^x)}{r+s+\ell(1-\pi^x)}+(1-\kappa)\right]D$$

Applying the definition of constant κ, we have:

$$\hat{T}\left[A(x)+D\right] = \hat{T}A(x) + \left[1-\frac{r+s}{r+s+\ell(1-\pi^x)}min_x\frac{2[r+s+\ell(1-\pi_x)]}{1+2[r+s+\ell(1-\pi_x)]}\right]D$$

The maximum value that this discount factor may take is found at the same x that solves the minimization problem for κ.

Discounting holds with discount factor $max_x\frac{1+r+s+2\ell(1-\pi_x)}{1+2[r+s+\ell(1-\pi_x)]} \in (0,1)$

$\mathcal{QED}$

Since the operator $\hat{T}$ satisfies Blackwell's Sufficient Conditions for a Contraction Mapping, it is a contraction mapping, with a unique steady state solution for value functions $M(x)$ and $J(x)$. Since any steady steady state of $\hat{T}$ is necessarily a steady state of operator T. The subgame perfect equilibrium defined by Proposition 2 therefore has unique equilibrium value functions $M(x)$ and $J(x), \forall x \in X$.

Furthermore, according to Proposition 3, there is a unique wage that solves condition (8) for each state $x \in X$.

$\mathcal{QED}$

E Data

Data sources

Data series	Source
Unemployment	BLS series LNS13000000
Unemployment (< 5 weeks)	BLS series LNS13008396
Vacancies (Conference Board)	Conference Board vacancy postings, https://sites.google.com/site/ regisbarnichon/research
Vacancies (JOLTS)	BLS series JTS00000000JOL
Employment	BLS series LNS12000000

The vacancy data is a combination of the JOLTS vacancy index and the Conference Board Help Wanted Index as compiled and adjusted in Barnichon (2010).

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