Recursive Equilibrium in Krusell and Smith (1998)*

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Abstract

I combine the tools developed in two important and independent literatures - one on large economies started with Aumann (1964) and the other on dynamically incomplete markets, notably Duffie et al. (1994) - to study Krusell and Smith's incomplete markets economy with both idiosyncratic and aggregate shocks. I show the existence of generalized recursive equilibrium and characterize several important properties of the equilibrium variables. The equilibrium process admits an ergodic measure, which enables the application of the ergodic theorem for the simulation and calibration of the model. Without aggregate shocks, the existence and some characterization results carry over to economies with only idiosyncratic shocks such as Huggett (1997)'s economy.

Keywords: Neoclassical Growth Models; Incomplete Markets; Heterogeneous Agents; Aggregate and Idiosyncratic Shocks; Large Economies; Recursive Equilibrium Existence; Kakutani-Fan-Glicksberg Fixed Point Theorem; Ergodicity

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1 Introduction

Krusell and Smith (1998) provide a workhorse incomplete markets model with heterogeneous agents who are subject to both idiosyncratic and aggregate shocks. Their paper defines a recursive equilibrium and offers an algorithm to compute it .[1] Despite the increasing popularity of the model[2], little is known about the analytical properties of its equilibrium. In particular, several theoretical questions remain open: 1. Does a sequential competitive equilibrium always exist and does it admit some simple recursive form? 2. If a competitive or recursive equilibrium exists, what are the properties of the equilibrium allocation and prices? 3. Does the equilibrium process exhibit some ergodic property that is important for the simulation and calibration of the model? The present paper makes some progress toward answering these questions.

First, I define a *generalized recursive equilibrium* as a *correspondence* that maps current wealth distribution and exogenous aggregate shock to a *set* of possible prices, value function, policy function, and possible future wealth distributions. At least one element in the set satisfies short-run equilibrium conditions. I first show that any sequence of allocations and prices generated by a generalized recursive equilibrium constitutes a sequential competitive equilibrium. I then prove that a generalized recursive equilibrium always exists in Krusell and Smith's economy. In addition, if starting from any initial wealth distribution and aggregate shock, there exists no more than one sequential competitive equilibrium; then a generalized recursive equilibrium corresponds to a standard recursive equilibrium with the natural endogenous state variable - wealth distribution - as computed in Krusell and Smith (1998). In general, one can also select a recursive equilibrium from a generalized recursive equilibrium if the value function is added to the state space, which consists of the aggregate shock and wealth distribution. In the special case of the model without aggregate shocks, the existence results (and some characterization results) carry over to economies with only idiosyncratic shocks in *transitional paths* such as Huggett (1997)'s economy.

Second, I establish several properties of the agents' value and policy functions and the equilibrium process. The value function is concave, strictly increasing, and Lipschitz continuous, while the policy function is weakly increasing and Lipschitz continuous. Applying results from Duffie et al. (1994), I show that the equilibrium process has an ergodic measure. Ergodicity[3] is an important property since the existing numerical methods often involve simulating the model.

To obtain these results, I use the tools developed in two extensive theoretical literatures. The first one is the literature on large economies, i.e., economies with a *continuum* of agents, studied by Aumann (1964), Hildenbrand (1974), and many others.[4] I use the *distributional approach* developed in this literature which involves defining and character-

[1]A recursive equilibrium is a sequential competitive equilibrium that is summarized by a mapping from current wealth distribution to prices and allocations (policy function), and future wealth distribution (transition function).

[2]See Krueger et al. (2016) for a recent review of the literature using Krusell and Smith's solution method.

[3]Which is the property that the average over time of a function over the equilibrium variables is the same as the average of the function under the ergodic measure.

[4]See Khan and Yannelis (1991) for a rich collection of papers in the literature.

izing equilibria in terms of the distribution of agents' characteristics without using the explicit agent space (in the present paper the characteristics are idiosyncratic shock and wealth). However, the literature is concerned mostly with static, dynamic complete markets economies, or dynamic incomplete markets economies without aggregate shocks.[5] The literature concerns with dynamically incomplete markets with a *finite* number of infinitely-lived agents and *aggregate shocks*, including Magill and Quinzii (1994), Duffie et al. (1994) and more recently Feng et al. (2014). The idea in Duffie et al. (1994) is to construct an expectation correspondence specifying, for each possible current state, the transitions that are consistent with feasibility and satisfy short-run equilibrium conditions. The key insight in the present paper is to combine this important idea with the distributional approach in the first literature. I show that the short-run equilibrium conditions can be fruitfully represented as Bellman equations on the current and future value functions. This representation allow me to apply the techniques developed in Duffie et al. (1994) and the subsequent literature to study large dynamic economies with aggregate shocks and incomplete markets.

The present paper is related to Miao (2006) which formulates and proves the existence of a sequential competitive equilibrium in Krusell and Smith's economy. The existence proof in Miao (2006) relies on the existence and uniqueness of the value and policy functions (with arguments include individual wealth and aggregate wealth distribution) as a solution to a Bellman equation. However, the Bellman operator is not well-defined when the distribution of capital holdings is a Dirac mass at zero, which leads to an infinite marginal rate of return on capital because of the Inada condition. This problem invalidates the existence proof in Miao (2006).[6] I present the details of the problem in Appendix C, in particular Proposition 4. In the present paper, I follow a different route to establish the existence of a sequential competitive equilibrium by taking the infinite-horizon limit of finite horizon economies.[7] In order to take the limit, I derive a uniform lower bound on aggregate capital (or equivalently a upper bound on the rate of return on capital) using the agents' Euler equation and a uniform bound on the first derivative of the value function. My proof allows for unbounded utility functions, e.g., log utility as in Krusell and Smith (1998) or CRRA utility with the risk-aversion coefficient strictly greater than 1, while Miao (2006), granted that his proof might be fixed,[8] requires bounded utility functions. Lastly, the present paper offers sharp characterizations of the equilibrium variables and equilibrium process which are absent in Miao (2006).

In the next section of the paper, I present the model and the main results. Section 3 concludes with potential applications of the techniques and results developed here. The details of the proofs are presented in the appendix.

[5] In the last topic on dynamic incomplete markets economies, Acemoglu and Jensen (2015) is a recent contribution with an extensive list of references including earlier papers such as Jovanovic (1982), Bewley (1986), Hopenhayn (1992), and Huggett (1993).

[6] Another problem, pointed out in Cheridito and Sagredo (2016a), is that the equilibrium mapping in Miao (2006) might not be continuous.

[7] This is also an important technique developed in the incomplete markets literature.

[8] For example, Cheridito and Sagredo (2016b) provide a potential fix by working with the space of sequences of aggregate capital instead of the space of sequences of distributions in Miao (2006).

2 Infinite-Horizon Economy with a Continuum of Agents

The model economy is exactly the same as in Krusell and Smith (1998), which features a continuum of agents facing both aggregate and idiosyncratic shocks. Markets are incomplete since agents can only insure against the shocks by investing in productive capital.

The Environment Consider a production economy with a single final good in infinite horizon. Time runs from $t = 0$ to ∞. The economy is populated by a measure one continuum) of infinitely-lived agents (households) indexed by

$$h \in \mathcal{H} = [0, 1].$$

In each period t, there are S (finite) possible exogenous states (shocks)

$$s \in \mathcal{S} = \{1, 2, \ldots, S\}.$$

The exogenous shocks follow a first-order Markov process with the transition probabilities

$$\pi_{ss'} = \Pr(s'|s) > 0 \ \ \forall s, s' \in \mathcal{S}.$$

Let s^t denote the history of realizations of shocks up to time t:

$$s^t = (s_0, s_1, \ldots, s_t) \in \mathcal{S}^t.$$

Agents are subject to idiosyncratic shocks

$$i \in \mathcal{I} = \{1, 2, ..., I\},$$

which I assume to contain a finite number of states.[9] In addition, following Krusell and Smith (1998), I restrict the joint dynamics of aggregate and idiosyncratic shocks such that (s_t, i_t) forms a first-order Markov process with the transition matrix $\pi_{ss',ii'}$:

$$\Pr(s_{t+1} = s', i_{t+1} = i', s_t = s, i_t = i) = \pi_{ss',ii'},$$

which satisfies[10]

$$\sum_{i'} \pi_{ss',ii'} = \pi_{ss'},$$

for each $i \in \mathcal{I}$, and $s, s' \in \mathcal{S}$.

Let $(\Upsilon, \mathcal{F}, \mathbb{P})$ denote the probability space that describes the stochastic process for idiosyncratic shocks $\hat{i} = \{i_t\}_{t=0}^{\infty}$ conditional on an infinite path for the aggregate shock $s^\infty = (s_0, s_1, ...)$. I assume that the agents share the same probability space. So the family of random processes $\hat{i}^h$ can be written as

$$\hat{i}(h, \omega, s^\infty).$$

[9] The assumption that aggregate and idiosyncratic only take on only a finite number of states is consistent with Krusell and Smith (1998). Relatedly, even if one starts with a continuum of states, one would have to approximate the state space with a finite number of states to solve the model numerically.

[10] This condition means that the evolution of the aggregate state is independent of the idiosyncratic states (but not vice versa):

$$\sum_{i' \in \mathcal{I}} \Pr(s_{t+1} = s', i_{t+1} = i', s_t = s, i_t = i) = \Pr(s_{t+1} = s', s_t = s, i_t = i) = \pi_{ss'},$$

for all $i \in \mathcal{I}$.

I also endow $\mathcal{H}$ with a probability measure ϕ ($\phi(\mathcal{H}) = 1$). I make the following assumption that ensures the empirical distribution of $\hat{i}^h$ across agents to be the same as the ex-ante distribution for each $\hat{i}^h$.

Assumption 1 (Conditional No Aggregate Uncertainty). *For each $s^\infty \in \mathcal{S}^\infty$ and for almost all $\tilde{\omega} \in \Upsilon$ and $\tilde{h} \in \mathcal{H}$:*

$$\phi\left(h \in \mathcal{H} : \hat{i}(h, \tilde{\omega}, s^\infty) \in B\right) = \mathbb{P}(\omega : \hat{i}(\tilde{h}, \omega, s^\infty) \in B)$$

for each $B \in \mathcal{B}(\mathcal{I}^\infty)$.

There are at least two ways to construct $(\Upsilon, \mathcal{F}, \mathbb{P})$ and a measure ϕ such that Assumption 1 is satisfied. The first option is to choose ϕ as the standard Lesbegue measure but allow for some dependence between i^h's as in Feldman and Gilles (1985), Bergin and Bernhardt (1992), and Miao (2006). Alternative, one can choose ϕ as an extension of Lesbegue measure and keep the independence between i^h's as in Sun (2006), Sun and Zhang (2009), and Podczeck (2010). In this case, the law of large numbers for a continuum of random variables applies exactly.

I assume that, at the beginning of the economy, the fraction of agents with idiosyncratic shock i is $m_{s_0}(i)$ where $\{m_s(i)\}_{s\in\mathcal{S}, i\in\mathcal{I}}$ satisfies

$$\sum_{i\in\mathcal{I}} m_s(i)\frac{\pi_{ss',ii'}}{\pi_{ss'}} = m_{s'}(i'), \tag{1}$$

for all $s, s' \in \mathcal{S}$ and $i' \in \mathcal{I}$. Together with Assumption 1, this assumption implies that, in aggregate state s, the fraction of agents with idiosyncratic type i is $m_s(i)$, independent of the past history of aggregate shocks and agents' idiosyncratic shocks. This assumption is also assumed in Krusell and Smith (1998). This result is derived in Appendix A.

Both aggregate shock and idiosyncratic shock determine the exogenous labor supply of the agents in state (s, i): $l(s, i)$. Because the fraction of type-i agent is $m_s(i)$, the total supply of labor in aggregate state s is

$$L(s) = \sum_{i\in\mathcal{I}} m_s(i)l(s, i).$$

The idiosyncratic shock also determines the agents' discount factor in state i: $\beta(i)$. We make the following assumptions on the idiosyncratic labor supply $l(., .)$ and discount factor $\beta(.)$.

Assumption 2. *There exist $0 < \underline{l} < \bar{l}$ such that*

$$\underline{l} < l(s, i) < \bar{l}$$

for all $s \in \mathcal{S}$ and $i \in \mathcal{I}$.

There exist $0 < \underline{\beta} < \overline{\beta} < 1$ such that

$$\underline{\beta} < \beta(i) < \overline{\beta}$$

for all $i \in \mathcal{I}$.

Since $\mathcal{S}$ and $\mathcal{I}$ have finite elements, we can choose $0 < \underline{L} < \overline{L}$ such that

$$\underline{L} \le L(s) = \sum_{i\in\mathcal{I}} m_s(i)l(s, i) \le \overline{L}$$

for all $s \in \mathcal{S}$.

In each state s, there is a representative firm that produces the final output from capital and labor using an aggregate production function that employs capital and labor as input:

$$Y = F(s, K, L).$$

The aggregate state determines the productivity of the aggregate production function through the first argument.

We make the following standard assumptions on F.

Assumption 3. *F is strictly increasing, strictly concave, and has constant returns to scale in K and L.*

This assumption nests the Cobb-Douglas production function used in Krusell and Smith (1998) as a special case:

$$F(s, K, L) = A(s) K^{\alpha} L^{1-\alpha}. \tag{2}$$

I also assume that capital depreciates at rate $\delta \in (0, 1)$ in each period. The final output at time t can be transformed into future (aggregate) capital, K_{t+1}, and current (aggregate) consumption C_t according to the law of motion

$$C_t + K_{t+1} - (1-\delta) K_t = Y_t. \tag{3}$$

I further assume that:

Assumption 4. *There exists $\hat{K}$ such that*

$$F(s, K, \bar{L}) - \delta K < 0$$

for all $K \geq \hat{K}$ and for all $s \in \mathcal{S}$.

This assumption ensures that aggregate capital is always bounded above and it is satisfied by the Cobb-Douglas production function. Indeed, if the economy starts with initial aggregate capital K_0 below $\bar{K}$ where

$$\overline{K} > \max_{s \in \mathcal{S}} \max_{0 \leq K \leq \hat{K}} \left(F(s, K, \overline{L}) + (1-\delta) K \right), \tag{4}$$

then aggregate capital K_t always stays below $\bar{K}$.

Preferences and Market Arrangements In each history s^t, there are rental markets for capital and labor. Agents rent out their capital to the representative firm at competitive rental rate $r_t(s^t)$ and supply their labor endowment inelastically to the representative firm at competitive wage rate $w_t(s^t)$.

Given factor prices, agents maximize inter-temporal expected utility:

$$\mathcal{U}\left(\left\{c_t^h\left(s^t, i^{h,t}\right)\right\}\right) = \mathbb{E}_0\left[\sum_{t=0}^{\infty} \Pi_{t'=0}^{t-1} \beta(i_{t'}^h) u(c_t^h(s^t, i^{h,t}))\right]$$

where $i^{h,t} = \left(i_0^h, i_1^h, ..., i_t^h\right)$ denotes the history of idiosyncratic shocks for each agent h and the per period utility function is given by

$$u(c) = \lim_{\nu \to \sigma} \frac{c^{1-\nu} - 1}{1-\nu}.$$

I require that $\sigma \geq 1$ so that in equilibrium, consumption is bounded from below. However, it is straightforward to extend the analysis to allow for more general utility functions.

As in Krusell and Smith (1998), I assume that markets are incomplete: agents can only save in units of capital. In addition, with $\bar{K}$ being sufficiently large as in (4), I assume that the agents' choice of capital is bounded above by $\bar{k}$ sufficiently large so that:[11]

$$\bar{k} > \bar{l} + \max_{s \in \mathcal{S}} F(s, 2\overline{K}, 2\overline{L}). \tag{5}$$

As I argue below, the agents' Euler equation is crucial in deriving a lower bound for aggregate capital. But the Euler equation does not hold when the upper bound $\bar{k}$ on agents' capital holding binds. Therefore, the upper bound has to be sufficiently large to minimize its effect.

Formally, given time and history-dependent interest rates $r_t(s^t)$ and wage rates $w_t(s^t)$ agent h solves

$$\max_{\left\{c_t^h(.),k_{t+1}^h(.)\right\}} \mathcal{U}\left(\left\{c_t^h\left(s^t, i^{h,t}\right)\right\}\right) \tag{6}$$

subject to

$$c_t^h(s^t, i^{h,t}) + k_{t+1}^h(s^t, i^{h,t}) \leq (1 - \delta + r_t(s^t))k_t^h(s^{t-1}, i^{h,t-1}) + w_t(s^t)l(s_t, i_t^h), \tag{7}$$

and $c_t^h(s^t, i^{h,t}) \geq 0$ and

$$0 \leq k_{t+1}^h(s^t, i^{h,t}) \leq \bar{k}. \tag{8}$$

Given an initial condition (i_0, k_0), the optimal consumption and capital holding decisions are:

$$\left\{\left(\check{c}_t(s^t, i^t; k_0, i_0)\right), \check{k}_{t+1}\left(s^t, i^t; k_0, i_0\right)\right\}.$$

The representative firm in history s^t maximizes profit:

$$\Pi_t(s^t) = \max_{Y_t, K_t, L_t \geq 0} Y_t - r_t(s^t)K_t - w_t(s^t)L_t$$

subject to

$$Y_t \leq F(s_t, K_t, L_t).$$

Since F has constant returns to scale, in equilibrium, we must have $\Pi_t(s^t) = 0$, and

$$Y_t = F(s_t, K_t, L_t) \text{ and } r_t = F_K(s_t, K_t, L_t) \text{ and } w_t = F_L(s_t, K_t, L_t). \tag{9}$$

Equilibrium Definitions The definition of a sequential competitive equilibrium in this environment is standard.[12]

Definition 1. A sequential competitive equilibrium (SCE) - given an initial distribution of capital holdings and idiosyncratic shocks, $\{k_0^h, i_0^h\}_{h \in \mathcal{H}}$, and initial aggregate shock s_0 -

[11] A upper bound on the choice of capital holding is implicitly assumed by all numerical algorithms since capital choice is bounded by a machine's numerical upper bound.

[12] Alternatively, one can formulate the sequential competitive equilibrium using direct allocations $\left\{c_t^h, k_{t+1}^h\right\}_{t,s^t,h \in \mathcal{H}}$ which are measurable with respect to past and current aggregate shocks and each individual's idiosyncratic shocks.

consists of optimal policy functions $\left(\left\{\check{c}_t, \check{k}_{t+1}\right\}_{t,s^t}\right)$, sequences of aggregate capital and labor demands, $\{K_t, L_t\}_{t,s^t}$, and prices $\{r_t, w_t\}_{t,s^t}$ ($r_t, w_t > 0$) such that:

1. $\left\{\check{c}_t, \check{k}_{t+1}\right\}_{t,s^t}$ solves (6).

2. In each history of aggregate shocks s^t, $\{Y_t, K_t, L_t\}$ solves the representative firm's profit maximization problem, which implies (9).

3. Markets for capital, labor, and the final good clear in each history s^t:

$$\int_{\mathcal{H}} \check{k}_t(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h)\phi(dh) = K_t(s^t),$$

and

$$\int_{\mathcal{H}} l^h(s_t, i_t^h)\phi(dh) = L_t(s^t),$$

and

$$\int_{\mathcal{H}} \left(\check{c}_t(s^t, i^{h,t}; k_0^h, i_0^h) + \check{k}_{t+1}(s^t, i^{h,t}; k_0^h, i_0^h) - (1-\delta)\check{k}_t(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h)\right)\phi(dh) = Y_t(s^t).$$

Following the distributional approach in the literature on large economies, it is more convenient to work with the distributions over capital holdings (or equivalently, wealth distributions) and idiosyncratic shocks, instead of working with the allocations of capital and consumption over households $h \in \mathcal{H}$. For each distribution of asset holdings across agents $\{k^h\}_{h\in\mathcal{H}}$, consider the following probability measure μ defined by

$$\mu(A \times I) = \phi\left(h \in \mathcal{H} : \left(k^h, i^h\right) \in A \times I\right) \tag{10}$$

for each $A \times I \in \mathcal{B}\left(\left[0, \bar{k}\right]\right) \times \mathcal{B}(\mathcal{I})$, where $\mathcal{B}$ denote the Borel σ-algebras. It is immediately apparent that

$$\mu \in \Omega = \mathcal{P}\left(\left[0, \bar{k}\right] \times \mathcal{I}\right),$$

where $\mathcal{P}$ denotes the space of probability measures over $\left[0, \bar{k}\right] \times \mathcal{I}$ endowed with the weak* topology. It is well-known that Ω is compact (see for example Bogachev 2000, Theorem 8.9.3). Let $\mathcal{M}$ denote a closed subset of Ω , which I will define below. Let $\mathcal{C}$ denote the set of functions over $\left[0, \bar{k}\right] \times \mathcal{I}$, which are continuous in k. The generalized recursive equilibrium is defined over the set of distributions $\mathcal{M}$.

Definition 2. A generalized recursive equilibrium (GRE) consists of a policy correspondence

$$\mathcal{Q} : \mathcal{S} \times \mathcal{M} \rightrightarrows \mathcal{C}^2 \times \mathbb{R}_+^2$$

and a transition correspondence:

$$\mathcal{T} : \mathcal{S} \times \mathcal{M} \rightrightarrows \mathcal{M}^{\mathcal{S}}$$

and some bounds $\underline{V}, \overline{V}$, with the following property. For each $s \in \mathcal{S}$ and $\mu \in \Omega$, and $\left(\hat{V}, \hat{k}, r, w\right) \in \mathcal{Q}(s, \mu)$, we have $\underline{V} \le \hat{V} \le \overline{V}$ and there exist $(s', \mu_{s'}^+)_{s'\in\mathcal{S}} \in \mathcal{T}(s, \mu)$ and

$$\left(\hat{V}_{s'}^+, \hat{k}_{s'}^+, r_{s'}^+, w_{s'}^+\right)_{s'\in\mathcal{S}}$$

such that:

1. For each $s' \in \mathcal{S}$, $\left(\hat{V}^+_{s'}, \hat{k}^+_{s'}, r^+_{s'}, w^+_{s'}\right) \in \mathcal{Q}\left(s', \mu^+_{s'}\right)$.

2. (Market clearing) $\int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}(k,i)d\mu(k,i) + \int_{[0,\bar{k}]\times\mathcal{I}} \hat{k}(k,i)d\mu(k,i) = F(s,K,L) + (1-\delta)K$ where for $k \in [0,\bar{k}]$ and $i \in \mathcal{I}$:

$$\hat{c}(k,i) = (1-\delta+r)k + wl(s,i) - \hat{k}(k,i)$$

and

$$K = \int_{[0,\bar{k}]\times\mathcal{I}} kd\mu(k,i) \text{ and } L = \sum_{i\in\mathcal{I}} m_s(i)l(s,i).$$

3. (Firms' maximization) $r = F_K(s,K,L) > 0$ and $w = F_L(s,K,L) > 0$.

4. (Agents' maximization) For each $k \in [0,\bar{k}]$ and $i \in \mathcal{I}$, $\hat{V}$ and $\hat{V}^+$ satisfy the Bellman equation:

$$\hat{V}(k,i) = \max_{k'} u(c) + \beta(i)\sum_{i',s'} \pi_{ss',ii'}\hat{V}^+_{s'}(k',i') \tag{11}$$

s.t. $0 \le k' \le \bar{k}$ and

$$0 \le c = (1-\delta+r)k + wl(s,i) - k'.$$

In addition, $\hat{k}(k,i)$ solves (11).

5.(Distribution Consistency) For each $s' \in \mathcal{S}$,

$$\mu^+_{s'} = \hat{k} \circ_{ss'} \mu, \tag{12}$$

where the composition $\circ_{ss'}$ means that for each $i' \in \mathcal{I}$ and $A \in \mathcal{B}\left([0,\bar{k}]\right)$:

$$\mu^+_{s'}(A,i') = \sum_{i\in\mathcal{I}} \frac{\pi_{ss',ii'}}{\pi_{ss'}}\mu\left(\left(\hat{k}(.,i)\right)^{-1}(A),i\right).$$

Notice that when the correspondences $\mathcal{Q},\mathcal{T}$ are replaced by functions in the definition of GRE, we arrive at a recursive equilibrium (RE) as defined in Krusell and Smith (1998). The following lemma shows that a GRE generates a SCE starting from some initial distribution.

Lemma 1. *Let Assumption 1 hold. Starting from an initial distribution of wealth μ_0 and aggregate state s_0, sequences of policy functions and prices generated by a GRE form a SCE.*

Proof. Appendix A. □

Main Results To show the existence of a GRE, I need to impose the following properties on the production function.

Assumption 5. *For any $L > 0$ and $K > 0$:*

$$\max_{s\in\mathcal{S}} \sup_{0<\tilde{K}\le K} F_L(s,\tilde{K},L) < +\infty,$$

and

$$\max_{s\in\mathcal{S}} \sup_{0<\tilde{L}\le L} F_K(s,K,\tilde{L}) < +\infty.$$

Lastly, for any $L > 0$ and $s \in \mathcal{S}$

$$\lim_{K\to 0} F(s,K,L) = 0.$$

It is easy to verify that Assumption 5 is satisfied under the Cobb-Douglas production function (2). Together with Assumptions 2-4, this assumption guarantees the existence of a SCE in the finite horizon economy. The following lemma establishes the result.

Lemma 2. *Let Assumptions 2-5 hold. Then a SCE always exists in any finite-horizon economy.*

Proof. Appendix B. The appendix provides a precise definition of SCE in finite-horizon economies and an existence proof. Because the space of measures is an infinite-dimensional space, the proof uses an extension of the Kakutani's Fixed Point Theorem to infinite-dimensional spaces - the Fan-Glicksberg Fixed Point Theorem.

One of the difficulties in the proof is to guarantee that aggregate capital is bounded from below away from zero in equilibrium, which is equivalent to a finite relative rental rate of capital. This is the same issue that invalidates the existence proof in Miao (2006). To get around this, I make use of the agents' Euler equation:

$$u'(c_t) \geq \beta(i_t)\mathbb{E}\left[(1-\delta+r_{t+1})\,u'(c_{t+1})\right] \tag{13}$$

when $k_{t+1} < \bar{k}$. Using this equation, I show that K_{t+1} cannot be too low. Because otherwise r_{t+1} would be very high and the Euler equation implies that $u'(c_t)$ would be very high. Therefore c_t would be very low for most agents and, from the aggregate resource constraint (3), K_{t+1} would not be too low, leading to a contradiction. □

An additional assumption on the production function, Assumption 6, allows me to take the limit of equilibria in finite-horizon economies .

Assumption 6. *For any $L \geq \underline{L}$ and $s \in \mathcal{S}$,*

$$\lim_{K \to 0} F_K(s, K, L) = \infty.$$

There exists $\underline{\alpha} > 0$, such that for all $K, L > 0$:

$$\frac{LF_L(s, K, L)}{F(s, K, L)} > \underline{\alpha}.$$

For any $s, s' \in \mathcal{S}$:

$$\limsup_{K \to 0} \frac{F(s', K, \bar{L})}{F(s, K, \underline{L})} < \infty.$$

Assumption 6 is also satisfied under the Cobb-Douglas production function (2). Together with Assumption 4 and again using the agents' Euler equation (13), Assumption 6 implies that aggregate capital is bounded uniformly both above and below uniformly in finite horizon economies, i.e., the bounds are independent of the horizon . This allows me to take the limit of the equilibria in finite horizon T-period economies, as the horizon T goes to infinity, and obtain the existence of a SCE in the infinite horizon economy as well as a the existence of a GRE.

Now we arrive at the main existence theorem.

Theorem 1. *Let Assumptions 2-6 hold. Starting from an initial distribution of capital holdings and idiosyncratic shocks μ_0 with*

$$K_0 = \int_{\mathcal{H}} k_0^h \phi(dh) = \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_0(k, i) > 0,$$

there exist $0 < \underline{K} < K_0 < \overline{K}$*, such that a generalized recursive equilibrium exists over*

$$\mathcal{M} = \left\{ \mu \in \mathcal{P}\left(\left[0, \bar{k}\right] \times \mathcal{I}\right) : \underline{K} \leq \int_{\left[0,\bar{k}\right]\times\mathcal{I}} k d\mu(k,i) \leq \overline{K} \right\}.$$

Proof. I first choose $\underline{K} < K_0$ sufficiently small as in Lemma 11 and $\overline{K} > \max\left\{K_0, 2\overline{L}\right\}$ sufficiently large such that (4) is satisfied. With these bounds, the results in Appendix B, in particular Lemma 11, apply. Given these results, let Θ denote the set of

$$\left(\hat{V}(.,.), \hat{k}(.,.), r, w\right)$$

such that for each $i \in \mathcal{I}$, $\hat{k}(k,i)$ is weakly increasing in k and $0 \leq \hat{k}(k,i) \leq \bar{k}$; and $\underline{V} \leq \hat{V}(k,i) \leq \overline{V}$ for all $k \in [0, \bar{k}]$ and $\hat{V}(k,i)$ is weakly increasing and weakly concave in k, and Lipschitz continuous with a Lipschitz constant $l_V > 0$. $\underline{V}, \overline{V}, l_V$ are given in Lemma 11. In addition, $0 < \underline{r} \leq r \leq \bar{r}$ and $0 < \underline{w} \leq w \leq \bar{w}$ where $\underline{r}, \bar{r}, \underline{w}, \bar{w}$ are also given in Lemma 11. Lemma 10 shows that Θ - endowed with the topology of uniform convergence for $\hat{V}$, pointwise convergence for $\hat{k}$, and the standard topology for r and w - is sequentially compact.[13,14]

Let $g : \mathcal{S} \times \mathcal{M} \rightrightarrows \Theta \times \Theta^{\mathcal{S}}$ denote the following correspondence: for each $s \in \mathcal{S}$, $\mu \in \mathcal{M}$, $g(s,\mu)$ is the set of $\theta = \left(\left(\hat{V}(.,.), \hat{k}(.,.)\right)_{h\in\mathcal{H}}, r, w\right) \in \Theta$, and $(\theta_{s'})_{s'\in\mathcal{S}}$ with $\theta_{s'} = \left(\hat{V}^+_{s'}(.,.), \hat{k}^+_{s'}(.,.), r^+_{s'}, w^+_{s'}\right) \in \Theta$ such that

$$r = r(s,\mu) \equiv F_K(s,K,L) > 0 \quad \text{and} \quad w = w(s,\mu) \equiv F_L(s,K,L) > 0, \tag{14}$$

where

$$K = \int_{\left[0,\bar{k}\right]\times\mathcal{I}} k\mu(dk,i) > 0, \quad \text{and} \quad L = \sum_{i\in\mathcal{I}} m_s(i) l(s,i),$$

and

$$\int_{\left[0,\bar{k}\right]\times\mathcal{I}} \hat{c}(k,i) d\mu(k,i) + \int_{\left[0,\bar{k}\right]\times\mathcal{I}} \hat{k}(k,i) d\mu(k,i) = F(s,K,L) + (1-\delta)K,$$

where $\hat{c}(k,i) = (1-\delta+r)k + wl(s,i) - \hat{k}(k,i) > 0$. In addition, $\left(\hat{V}, \hat{k}\right)$ solves the functional equation (11), given $\left(\hat{V}^+_{s'}\right)_{s'\in\mathcal{S}}$.

Lemma 10 shows that g is a closed-valued correspondence.

Consider the following mapping $\mathcal{G}$ from the set of correspondences $\mathcal{V} : \mathcal{S} \times \mathcal{M} \rightrightarrows \Theta$ to itself defined as following. For each $\mathcal{V}$, $\mathcal{G}(\mathcal{V})$ is the correspondence $\mathcal{W}$ such that, for each $s \in \mathcal{S}$ and $\mu \in \mathcal{M}$, we have

$$\mathcal{W}(s,\mu) = \left\{ \begin{array}{ll} \theta = \left(\hat{V}, \hat{k}, r, w\right) \in \Theta : & \text{for each } s' \in \mathcal{S}, \exists \theta_{s'} \in \mathcal{V}\left(s', \mu^+_{s'}\right) \\ & \text{where } \mu^+_{s'} \text{ is given by (12)} \\ & \text{and } \left(\theta, (\theta_{s'})_{s'\in\mathcal{S}}\right) \in g(s,\mu) \end{array} \right\}$$

[13] In infinite dimensional spaces, compactness and sequential compactness are not equivalent. For the current theorem, we need the sequential compactness property.

[14] Notice that we do not require $\hat{k}$ to be continuous in k in the definition of Θ. Otherwise, Θ would not be sequentially compact. However, in a GRE, since $\left(\hat{V}, \hat{k}\right)$ solves (11), we can show that $\hat{k}$ is continuous.

From the definition of $\mathcal{G}$, we can establish the following properties P1-P3.

P1. If $\mathcal{V}$ is sequentially compact, in the sense that $\mathcal{V}(s,\mu)$ is sequentially compact for all $s \in \mathcal{S}$ and $\mu \in \mathcal{M}$, then $\mathcal{W} = \mathcal{G}(\mathcal{V})$ is sequentially compact.

Indeed, assume that $\{\theta^m\}_{m=0}^{\infty} \in \mathcal{W}(s,\mu)$ and $\theta^m \to \theta = \left(\hat{V}, \hat{k}, r, w\right)$. Since Θ is sequentially compact, $\theta \in \Theta$. To show that $\mathcal{W} = \mathcal{G}(\mathcal{V})$ is sequentially compact, we need to show that $\theta \in \mathcal{W}(s,\mu)$. By the definition of $\mathcal{G}$, for each $s' \in \mathcal{S}$, $\exists \theta_{s'}^m \in \mathcal{V}\left(s', \mu_{s'}^+\right)$ such that $\left(\theta^m, \left(\theta_{s'}^m\right)\right) \in g(s,\mu)$. Since $\mathcal{V}\left(s', \mu_{s'}^+\right)$ is sequentially compact, we can extract a convergent subsequence, $\theta_{s'}^{m_l} \to \theta_{s'}$ for some $\theta_{s'} \in \mathcal{V}\left(s', \mu_{s'}^+\right)$. Because g is a closed-valued correspondence, $\left(\theta, (\theta_{s'})_{s' \in \mathcal{S}}\right) \in g(s,\mu)$, which implies $\theta \in \mathcal{W}(s,\mu)$. So $\mathcal{W}$ is sequentially compact.

P2. If $\mathcal{V} \subset \mathcal{V}'$ in the sense that $\mathcal{V}(s,\mu) \subset \mathcal{V}'(s,\mu)$ for all $s \in \mathcal{S}$ and $\mu \in \mathcal{M}$, then $\mathcal{G}(\mathcal{V}) \subset \mathcal{G}(\mathcal{V}')$.

P3. Let $\mathcal{V}^0$ denote the complete correspondence: $\mathcal{V}^0(s,\mu) = \Theta$ for all $s \in \mathcal{S}$ and $\mu \in \mathcal{M}$. Then $\mathcal{G}(\mathcal{V}^0) \subset \mathcal{V}^0$.

Having defined $\mathcal{G}$ and established its properties, given $\mathcal{V}^0$, we construct the sequence of $\{\mathcal{V}^n\}_{n=0}^{\infty}$ recursively using $\mathcal{G}$: $\mathcal{V}^{n+1} = \mathcal{G}(\mathcal{V}^n)$. Then by P1, P2, and P3, we have, for all $n \geq 0$, $\mathcal{V}^{n+1} \subset \mathcal{V}^n$ and $\mathcal{V}^n$ is sequentially compact. By the existence of a SCE in (n+1)-horizon economy in Lemma 2, $\mathcal{V}^{n+1}$ is a non-empty valued correspondence.

Let $\mathcal{V}^*$ be defined by

$$\mathcal{V}^*(s,\mu) = \cap_{n=0}^{\infty} \mathcal{V}^n(s,\mu).$$

Since $\mathcal{V}^*(s,\mu)$ is the intersection of decreasing, non-empty, sequentially compact sets, $\mathcal{V}^*(s,\mu)$ is sequentially compact and non-empty. We show that $\mathcal{G}(\mathcal{V}^*) = \mathcal{V}^*$.

Indeed, by the definition of $\mathcal{V}^*$, we have $\mathcal{V}^* \subset \mathcal{V}^n$, so $\mathcal{G}(\mathcal{V}^*) \subset \mathcal{G}(\mathcal{V}^n) = \mathcal{V}^{n+1}$ for all n. This implies $\mathcal{G}(\mathcal{V}^*) \subset \mathcal{V}^*$.

Now, consider any $s \in \mathcal{S}$ and $\mu \in \Theta$ and $\theta = \left(\hat{V}, \hat{k}, r, w\right) \in \mathcal{V}^*(s,\mu)$. Since $\mathcal{V}^* \subset \mathcal{V}^{n+1} = \mathcal{G}(\mathcal{V}^n)$, there exists $\theta_{s'}^n \in \mathcal{V}^n\left(s', \mu_{s'}^+\right)$ such that $\left(\theta, \left(\theta_{s'}^n\right)_{s' \in \mathcal{S}}\right) \in g(s,\mu)$. By the sequential compactness of Θ, we can find a convergent subsequence $\{n_l\}_{l=0}^{\infty}$:

$$\left(\theta_{s'}^{n_l}\right)_{s' \in \mathcal{S}} \longrightarrow_{l \to \infty} (\theta_{s'})_{s' \in \mathcal{S}}.$$

By the sequential compactness of $\mathcal{V}^{n_l}$, we have $\theta_{s'} \in \mathcal{V}^{n_l}\left(s', \mu_{s'}^+\right)$ and since g is closed-valued, $\left(\theta, (\theta_{s'})_{s' \in \mathcal{S}}\right) \in g(s,\mu)$. Moreover, $\mathcal{V}^n\left(s', \mu_{s'}^+\right)$ is a decreasing sequence so $\theta_{s'} \in \cap_{l=0}^{\infty} \mathcal{V}^{n_l}\left(s', \mu_{s'}^+\right) = \mathcal{V}^*\left(s', \mu_{s'}^+\right)$. Therefore, by the definition of $\mathcal{G}$, we have $\theta \in \mathcal{G}(\mathcal{V}^*)$. Thus $\mathcal{V}^* \subset \mathcal{G}(\mathcal{V}^*)$.

We have shown that $\mathcal{G}(\mathcal{V}^*) \subset \mathcal{V}^* \subset \mathcal{G}(\mathcal{V}^*)$, which implies $\mathcal{G}(\mathcal{V}^*) = \mathcal{V}^*$ as desired.

Let $\mathcal{Q} = \mathcal{V}^*$. Since $\mathcal{G}(\mathcal{Q}) = \mathcal{Q}$, for each $s \in \mathcal{S}$ and each $\mu \in \mathcal{M}$, $\theta = \left(\hat{k}, \hat{V}, r, w\right) \in \mathcal{Q}(s,\mu)$, there exists $\theta_{s'} \in \mathcal{Q}\left(s', \mu_{s'}^+\right)$ for each $s' \in \mathcal{S}$ such that $\left(\theta, (\theta_{s'})_{s' \in \mathcal{S}}\right) \in g(s,\mu)$. We also define $\mathcal{T}$ as

$$\mathcal{T}(s,\mu) = \left\{ \left(\mu_{s'}^+\right)_{s' \in \mathcal{S}} : \text{given } \left(\hat{c}, \hat{k}, \hat{V}, r, w\right) \in \mathcal{Q}(s,\mu),\ \mu_{s'}^+ \text{ is determined by (12)} \right\}.$$

It follows immediately that $(\mathcal{Q}, \mathcal{T})$ defined as such forms a generalized recursive equilibrium for the economy with a continuum of agents. □

By Lemma 1, the existence of a GRE implies the existence of a SCE starting from some initial distribution. Can we go a step further and always select a RE from a GRE? The answer is no. The definition of a GRE involves policy and transition correspondences, $\mathcal{Q}$ and $\mathcal{T}$. We can show that $\mathcal{Q}$ upper-hemi continuous. Therefore there exists a measurable selection (function) $\mathcal{Q}^0$ from $\mathcal{Q}$ (Hildenbrand, 1974). Let $\mathcal{T}^0$ denote the transition function that corresponds to the selection $\mathcal{Q}^0$. However, $\left(\mathcal{Q}^0, \mathcal{T}^0\right)$ might not form a recursive equilibrium. To see this clearly, let us assume $\left(\mu_{s'}^{0,+}\right)_{s'\in\mathcal{S}} = \mathcal{T}^0\left(s,\mu\right)$ and

$$\left(\hat{V}_{s'}^{+}(.,.), \hat{k}_{s'}^{+}(.,.), r_{s'}^{+}, w_{s'}^{+}\right) \in \mathcal{Q}(s', \mu_{s'}^{+})$$

such that Conditions 2-5 in Definition 2 are satisfied. Now, it is possible that $\mu_{s'}^{0,+} = \mu$ at $s' = s$ and that Definition 2 requires

$$\left(\hat{V}_{s}^{++}(.,.), \hat{k}_{s}^{++}(.,.), r_{s}^{++}, w_{s}^{++}\right) \in \mathcal{Q}\left(s,\mu\right) \setminus \mathcal{Q}^0\left(s,\mu\right).$$

In this case, at $s' = s$, we would select the "wrong" allocation if we set

$$\left(\hat{V}_{s}^{++}(.,.), \hat{k}_{s}^{++}(.,.), r_{s}^{++}, w_{s}^{++}\right) = \mathcal{Q}^0\left(s,\mu\right).$$

The last observation indicates that, in general, we cannot always select a RE from a GRE. Therefore, we would need additional conditions to guarantee the existence of a RE. The following result provides such a sufficient condition for when a GRE gives rise to a RE.

Proposition 1. *Let Assumptions 1-6 hold. Starting from any initial distribution of capital holdings, $\mu_0(k,i)$ and exogenous aggregate state s_0, there exists a SCE. If the SCE is unique for every initial distribution of capital holdings and aggregate state, then there exists a recursive equilibrium as defined in Krusell and Smith (1998).*

Proof. Since, starting from each $s \in \mathcal{S}$ and $\mu \in \mathcal{M}$, there exists no more than one SCE, there exists a unique element

$$\left(\hat{V}, \hat{k}, r, w\right) \in \mathcal{Q}\left(s,\mu\right)$$

that satisfies Conditions 1-5 in Definition 2. Let $\mathcal{Q}^0$ denote the mapping from (s,μ) to this element, and $\mathcal{T}^0\left(s,\mu\right) = \left(\hat{k} \circ_{ss'} \mu\right)_{s'\in\mathcal{S}}$. Then $\left(\mathcal{Q}^0, \mathcal{T}^0\right)$ forms a recursive equilibrium. □

Following Duffie et al. (1994) and Miao (2006), from the generalized recursive equilibrium, for which the existence is established in Theorem 1, we can construct a recursive equilibrium if we enlarge the state space with the value function. The reason is that the short-run equilibrium condition, i.e. Condition 4 in Definition 2, only involves next-period value functions. Definition 3 and Proposition 2 below formalize the result.

Definition 3. *A recursive equilibrium (RE) with the value function as an extended state variable* consists of a subset $\mathcal{J} \subset \mathcal{S} \times \Omega \times \mathcal{C}$ and a mapping from $\xi = \left(s, \mu, \hat{V}\right) \in \mathcal{J}$ to

a. a current policy function $\hat{k}$, and current factor prices r, w ;

b. next-period wealth distributions and value functions $\left(\mu_{s'}, \hat{V}_{s'}^{+}\right)_{s'\in\mathcal{S}}$ such that

$$\xi_{s'}^{+} = \left(s', \mu_{s'}, \hat{V}_{s'}^{+}\right) \in \mathcal{J} \text{ for all } s' \in \mathcal{S},$$

and Conditions 2-5. in Definition 2 are satisfied.

Proposition 2. *Let Assumptions 2-6 hold. Then a RE with value function as an extended state variable exists.*

Proof. Given the set of distributions $\mathcal{M}$ and the correspondence $\mathcal{Q}$ in Definition 2 and Theorem 1, let

$$\mathcal{J} = \left\{ (s, \mu, \hat{V}) \in \mathcal{S} \times \mathcal{M} \times \mathcal{C} : \exists \left(\hat{V}, \hat{k}, r, w \right) \in \mathcal{Q}(s, \mu) \text{ for some } \hat{k} \in \mathcal{C} \text{ and } r, w > 0 \right\}.$$

The existence of this recursive equilibrium is a direct application of the existence of a GRE shown in Theorem 1, since any selection from the correspondence $\mathcal{Q}$ gives rise to a recursive equilibrium with the extended state variable. □

In the special case of the model without aggregate shocks (i.e., $S = 1$) the Krusell and Smith's economy becomes a neoclassical economy with only idiosyncratic shocks in a transitional path as in Huggett (1997). In particular, we have the following proposition.

Proposition 3. *Consider the case with $S = 1$ (i.e., the economy in Huggett, 1997) and let Assumptions 2-6 hold. A transitional path equilibrium exists. In addition, the value and policy functions are time-dependent and the value function is concave, increasing, and Lipschitz continuous in k; and the policy function is continuous and weakly increasing in k.*

Proof. The result stated in this proposition is a direct application of Theorem 1 to the case $S = 1$. □

Having established the existence of GRE in Theorem 1, we would like to know whether the long-run dynamics of the equilibrium exhibit some form of ergodicity. To the extent that the literature using Krusell and Smith-type models involves simulation and calibration, ergodicity is important because it allows the econometrician to calculate the moments generated by the equilibrium using simulations.

In order to apply the machinery developed in Duffie et al. (1994) to tackle this question, I use the following notation. First, let $\mathcal{Z}$ denote the extended space of exogenous and endogenous variables:

$$\mathcal{Z} = \mathcal{S} \times \mathcal{Y}$$

where

$$\mathcal{Y} = \mathcal{M} \times \mathcal{BL}_1 \times \mathcal{BL}_2.$$

In the expression for $\mathcal{Y}$, $\mathcal{BL}_1$ denotes the space of Lipschitz continuous functions bounded by $\left[\underline{V}, \overline{V}\right]$ and with Lipschitz constant l_V; and $\mathcal{BL}_2$ denotes the space of Lipschitz continuous functions bounded by $\left[0, \bar{k}\right]$ with Lipschitz constant l_k; and where l_V and l_k are given in Lemma 11.

I define the expectations correspondence $\tilde{G}$ as in Duffie et al. (1994)[15]:

$$\tilde{G} : \mathcal{Z} \rightrightarrows \mathcal{P}(\mathcal{Z})$$

by letting $\gamma \in \tilde{G}\left(s, \mu, \hat{V}, \hat{k}\right)$ iff

[15]For any Borel space X, $\mathcal{P}(X)$ denotes the space of probability measures on X, endowed with weak* topology.

(i) the marginal distribution $\gamma_{\mathcal{S}\times\mathcal{M}}$ of γ on $\mathcal{S}\times\mathcal{M}$ is $\sum_{s'\in\mathcal{S}}\pi_{ss'}\mathcal{D}_{\left(s',\mu\circ_{ss'}\hat{k}\right)}$ where $\mathcal{D}$ stands for the Dirac mass function; and

(ii) $\hat{k}(.,.)$ solves

$$\hat{V}(k,i)=\max_{k'}u(c)+\beta(i)\sum_{s',i'}\pi_{ss',ii'}\int\tilde{V}(k',i')d\gamma(.|s'),$$

subject to: $0\le k'\le\bar{k}$ and $0\le c=(1-\delta+r(s,\mu))k+w(s,\mu)l(s,i)-k'$, where $r(s,\mu)$ and $w(s,\mu)$ are defined as marginal products in (14).

Lemma 3. *$\tilde{G}$ has closed-graph and is convex-valued.*

Proof. Appendix A. □

I use the concept of ergodicity in Duffie et al. (1994). I refer readers to original paper for more detailed step-by-step explanations of the concept. In particular, with conditionally spotlessness, sunspots are used only to randomize over spotless transitions from current state to next period's state.[16] Let $\mathcal{P}_F(\mathcal{S}\times\mathcal{Y})$ denote the set of γ in $\mathcal{P}(\mathcal{S}\times\mathcal{Y})$ for which there is some $h:\mathcal{S}\to\mathcal{Y}$ with measurable graph such that $\gamma(Gr(h))=1$.

Definition 4. A Conditionally Spotless Ergodic Markov equilibrium consists of a subset $\mathcal{Z}^*\subset\mathcal{Z}$, a mapping $\Xi:\mathcal{Z}^*\to\mathcal{P}(\mathcal{Z}^*)$, and an invariant measure $\gamma^*\in\mathcal{P}(\mathcal{Z}^*)$ of Ξ such that

$$\Xi(z)\in\tilde{G}(z)$$

for all $z\in\mathcal{Z}^*$ and

(i) (ergodicity) for any invariant subset A of $\mathcal{Z}^*$, either $\gamma^*(A)=0$ or $\gamma^*(A)=1$.

(ii) (conditionally spotless) for each $z\in\mathcal{Z}^*$, there is some $M\subset\mathcal{P}_F(\mathcal{S}\times\mathcal{Y})\cap\tilde{G}$ and $\lambda\in\mathcal{P}(M)$ such that $\Xi(z)=\int\nu d\lambda(\nu)$.

With the definition, the following theorem establishes the existence of a conditionally spotless ergodic Markov equilibrium.

Theorem 2. *Let Assumptions 2-6 hold. A conditionally spotless ergodic Markov equilibrium exists.*

Proof. Given the existence of a GRE $(\mathcal{Q},\mathcal{T})$ from Theorem 1, let

$$\mathcal{Z}^*=\left\{\left(s,\mu,\hat{V},\hat{k}\right)\in\mathcal{Z}:\left(\hat{V},\hat{k},r(s,\mu),w(s,\mu)\right)\in\mathcal{Q}\left(s,\mu\right)\right\}.$$

It is easy to show that $\mathcal{Z}^*$ is compact. Because $\tilde{G}$ has closed-graph and is convex-valued, as shown in Lemma 3, the existence follows directly from Proposition 3.1 in Duffie et al. (1994). □

If a SCE starting from any initial condition is unique, as assumed in Proposition 1, then a conditionally spotless ergodic equilibrium becomes a usual (spotless) ergodic equilibrium, i.e., Ξ does not involves any sunspots. Ergodic property established in Theorem 2 is

[16] In other words, when there are multiple spotless equilibria starting from a particular state, sunspots allow the system to randomize among these equilibria (but once the sunspot random variable is realized, one cannot switch from one equilibrium to another).

important because many papers using Krusell and Smith (1998)'s solution method make use of simulations to compute their models. Ergodicity of equilibrium also allows one to apply the ergodic theorem: one can compute the model's moments using simulations. For example, for any function $\varphi \in \mathcal{L}^1(\mathcal{Z}^*, \gamma^*)$ and $\{z_t\}$ is induced by Ξ, we have:

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \varphi(z_t) = \int \varphi d\gamma^*, \ \ \gamma^*\text{-almost surely.}$$

Ergodicity is also important for the calibration or estimation of the models, for example in order to use the GMM estimators (Hansen, 1982).

3 Conclusion

In this paper, I define the concept of generalized recursive equilibrium, prove its existence, and characterize its properties in the Krusell and Smith-style neoclassical growth model with both idiosyncratic and aggregate shocks. The proof techniques apply equally well to economies with a finite number of agents and I provided the details in Appendix D. The techniques and results (existence and characterization) in the present paper should carry over to to other economies in which the agents' decision variables involve only one continuous inter-temporal choice variable such as Chang and Kim (2007), Vavra (2014), and Krueger et al. (2016). Because, in these economies one can sharply characterize the agents' value and policy functions. It is more challenging in other classes of models with more than one continuous choice variables, such as models with portfolio choices (Krusell and Smith, 1997 and Storesletten et al., 2007).

The existence proof using finite-horizon approximation also suggests a global numerical method using time iterations to solve this class of model. I implement the two-agent version of the algorithm in Appendix E and discuss its extensions for a larger number or a continuum of agents. The definitions and proofs in the present paper use the wealth distribution as a state variable and are consistent with recent algorithms approximating and discretizing the space of wealth distributions such as Reiter (2010), Gordon (2011), Childers (2015), and Sager (2016).

Lastly, I provide a rather strong and difficult to verify condition under which a generalized recursive equilibrium gives rise to a recursive equilibrium with the natural state variable - wealth distribution. In general, however, it is still an open question whether a recursive equilibrium exists in these economies. The question deserves further research given the rising importance of this class of economies.

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APPENDIX

A Supporting Results

The Derivation for the Fractions of Agents. We show by induction in t that if $\phi(h \in \mathcal{H} : i_t^h = i) = m_s(i)$ for each $i \in \mathcal{I}$ and $s \in \mathcal{S}$ the same property holds for $t+1$.

Let $B = \{\hat{i} \in \mathcal{I}^\infty : i_{t+1} = i'\}$ and $B_i = \{\hat{i} \in \mathcal{I}^\infty : i_t = i\}$. Conditioning on a path for aggregate shocks s^∞, for each $t \geq 0$ and $i \in \mathcal{I}$ we have, a.s.

$$\begin{aligned}\phi(h \in \mathcal{H} : \hat{i}(h, \tilde{\omega}, s^\infty) \in B) &= \mathbb{P}(\omega : \hat{i}(h, \omega, s^\infty) \in B) \\ &= \sum_{i \in \mathcal{I}} \mathbb{P}(\omega : \hat{i}(h, \omega, s^\infty) \in B_i \cap B),\end{aligned}$$

where the first equality holds a.s. for $\tilde{\omega} \in \Omega$ given Assumption 1. Now, a.s.

$$\begin{aligned}
\mathbb{P}(\omega : \hat{i}(h,\omega,s^\infty) \in B_i \cap B) &= \mathbb{P}(\omega : \hat{i}(h,\omega,s^\infty) \in B | \hat{i}(h,\omega,s^\infty) \in B_i) \cdot \mathbb{P}(\omega : \hat{i}(h,\omega,s^\infty) \in B_i) \\
&= \frac{\pi_{ss',ii'}}{\pi_{ss'}} \cdot \mathbb{P}(\omega : \hat{i}(h,\omega,s^\infty) \in B_i) \\
&= \frac{\pi_{ss',ii'}}{\pi_{ss'}} \cdot \phi(h \in \mathcal{H} : \hat{i}(h,\tilde{\omega},s^\infty) \in B_i) \\
&= \frac{\pi_{ss',ii'}}{\pi_{ss'}} m_s(i).
\end{aligned}$$

Therefore, for almost all $\tilde{\omega} \in \Omega$,

$$\phi(h \in \mathcal{H} : \hat{i}(h,\tilde{\omega},s^\infty) \in B) = \sum_{i \in \mathcal{I}} \frac{\pi_{ss',ii'}}{\pi_{ss'}} m_s(i) = m_{s'}(i')$$

by (1). □

Proof of Lemma 1. Consider sequences of wealth distribution μ_t, policy functions $\hat{c}_t, \hat{k}_t$ and prices r_t, w_t generated by a generalized recursive equilibrium, starting from $s_0 \in \mathcal{S}$ and a distribution $\{k_0^h, i_0^h\}_{h \in \mathcal{H}}$ such that $\mu_0(k,i)$ as defined in (10) belongs to Ω. That is, sequences of distributions $\{\mu_t(s^t)\}_{t,s^t}$ and, policy functions and value functions

$$\left\{\hat{c}_t(.,.;s^t), \hat{k}_t(.,.;s^t), \hat{V}_t(.,.;s^t)\right\}_{t,s^t},$$

and prices $\{r_t(s^t), w_t(s^t)\}_{t,s^t}$ are such that for each t, s^t, $\left(\hat{V}_t, \hat{k}_t, r_t, w_t\right) \in \mathcal{Q}(s_t, \mu_t)$, $\hat{c}_t$ is consistent with $\hat{k}_t$, and $\left(s_{t+1}, \mu_t\left(s^t, s_{t+1}\right)\right)_{s_{t+1} \in \mathcal{S}} \in \mathcal{T}(s_t, \mu_t)$ and Conditions 1-5 in Definition (2) are satisfied. For convenience, I repeat them here using the sequence notations:

1. For each $s_{t+1} \in \mathcal{S}$,

$$\begin{aligned}
&\left(\hat{V}_{t+1}\left(.,.;\left(s^t, s_{t+1}\right)\right), \hat{k}_{t+1}\left(.,.;\left(s^t, s_{t+1}\right)\right), r_{t+1}\left(s^t, s_{t+1}\right), r_{t+1}\left(s^t, s_{t+1}\right)\right) \\
&\in \mathcal{Q}\left(s_{t+1}, \mu_{t+1}\left(s^t, s_{t+1}\right)\right).
\end{aligned}$$

2. The following identity holds:

$$\begin{aligned}
&\int_{[0,\bar{k}] \times \mathcal{I}} \hat{c}_t(k,i;s^t) d\mu_t(k,i;s^t) + \int_{[0,\bar{k}] \times \mathcal{I}} \hat{k}_t(k,i;s^t) d\mu_t(k,i;s^t) \\
&= F(s_t, K_t(s^t), L(s_t)) + (1-\delta) K_t(s^t)
\end{aligned}$$

where

$$K_t(s^t) = \int_{[0,\bar{k}] \times \mathcal{I}} k d\mu_t(k,i;s^t) \text{ and } L(s_t) = \sum_{i \in \mathcal{I}} m_{s_t}(i) l(s_t, i).$$

3. $r_t(s^t) = F_K\left(s_t, K_t(s^t), L(s_t)\right) > 0$ and $w_t(s^t) = F_L\left(s_t, K_t(s^t), L(s_t)\right) > 0$.

4. For each $i \in \mathcal{I}$ and $k \in [0, \bar{k}]$, $\hat{V}_t(.,.;s^t)$ and $\left\{\hat{V}_{t+1}\left(.,.;\left(s^t, s'\right)\right)\right\}_{s' \in \mathcal{S}}$ satisfy the Bellman equation:

$$\hat{V}_t(i,k;s^t) = \max_{c,k'} u(c) + \beta(i) \sum_{i',s'} \pi_{ss',ii'} \hat{V}_{t+1}\left(i',k';\left(s^t,s'\right)\right) \tag{15}$$

s.t. $c \geq 0$ and $0 \leq k' \leq \bar{k}$ and

$$c + k' \leq (1 - \delta + r_t(s^t))k + w_t(s^t)l(s_t, i).$$

In addition, $\left(\hat{c}_t(i,k;s^t), \hat{k}_t(i,k;s^t)\right)$ solves (15).

5. For each $s_{t+1} \in \mathcal{S}$:

$$\mu_{t+1}\left(.,.;\left(s^t, s_{t+1}\right)\right) = \hat{k}_t(.,.;s^t) \circ_{s_t s_{t+1}} \mu_t(.,.;s^t),$$

where the composition operator $\circ$ is defined in (12).

I construct recursively a SCE as following:

$$\breve{k}_{t+1}(s^t, i^{h,t}; k_0^h, i_0^h) = \hat{k}_t\left(\breve{k}_t(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h), i_t^h; s^t\right)$$

$$\breve{c}_t(s^t, i^{h,t}; k_0^h, i_0^h) = \hat{c}_t\left(\breve{k}_t(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h), i_t^h; s^t\right).$$

I show that this construction is indeed a SCE as defined in Definition 1.

I first show by induction that $\mu_t(s^t)$ corresponds to the distribution (10) implied by:

$$\left\{\breve{k}_t(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h)\right\}_{h \in \mathcal{H}},$$

i.e.

$$\mu_t(A \times J; s^t) = \phi\left(h \in \mathcal{H} : \left(\breve{k}_t(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h), i_t^h\right) \in A \times J\right) \tag{16}$$

for each $A \times J \in \mathcal{B}\left([0, \bar{k}]\right) \times \mathcal{B}(\mathcal{I})$.

Identity (16) holds at $t = 0$ by definition. Now, assume that the identity holds at t, we show that it holds at $t+1$. Indeed,

$$\begin{aligned}
&\phi\left(h \in \mathcal{H} : \left(\breve{k}_{t+1}(s^t, i^{h,t}), i_{t+1}^h\right) \in A \times J\right) \\
&= \mathbb{P}\left(\hat{k}_t\left(\breve{k}_t\left(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h\right), i_{t+1}^h; s^t\right) \in A \times J\right) \\
&= \sum_{i \in \mathcal{I}} \mathbb{P}\left(\hat{k}_t\left(\breve{k}_t\left(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h\right), i_{t+1}^h; s^t\right) \in A \times J, i_t^h = i\right),
\end{aligned}$$

where the second equality is implied by Assumption 1 and the construction of $\breve{k}_{t+1}$.

The last expression can be written as:

$$\begin{aligned}
&\sum_{i \in \mathcal{I}} \mathbb{P}\left(\hat{k}_t\left(\breve{k}_t\left(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h; s_{t-1}, \mu_t\right), i_{t+1}^h; s^t\right) \in A \times J, i_t^h = i\right) \\
&= \sum_{i \in \mathcal{I}} \sum_{i' \in J} \frac{\pi_{s_t s_{t+1}, ii'}}{\pi_{s_t s_{t+1}}} \mathbb{P}\left(\breve{k}_t\left(s^{t-1}, i^{h,t-1}; k_0^h, i_0^h; s_{t-1}, \mu_t\right) \in \hat{k}_t^{-1}\left(A, i_t^h = i; s^t\right), i_t^h = i, i_{t+1}^h \in J\right) \\
&= \sum_{i \in \mathcal{I}} \sum_{i' \in J} \frac{\pi_{s_t s_{t+1}, ii'}}{\pi_{s_t s_{t+1}}} \phi\left(h \in \mathcal{H} : \breve{k}_t \in \hat{k}_t^{-1}\left(A, i'; s^t\right), i_t^h = i\right).
\end{aligned}$$

Finally, by the induction assumption at t, the last expression is equal to

$$\sum_{i \in \mathcal{I}} \sum_{i' \in J} \frac{\pi_{s_t s_{t+1}, ii'}}{\pi_{s_t s_{t+1}}} \mu_t\left((\hat{k}_t)^{-1}(A, i'; s^t), i; s^t\right),$$

which in turn equals $\mu_{t+1}\left(A \times J; s^{t+1}\right)$ from the properties of $\{\mu_t\}$ and $\left\{\hat{k}_t\right\}$. So by induction (16) holds for all t and s^t.

Consequently,

$$\int_{\mathcal{H}} \breve{k}_t(s^t, i^{h,t}; k_0^h, i_0^h)\phi(dh) = \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_t(k, i; s^t) = K_t.$$

and

$$\int_{\mathcal{H}} l^h(s_t, i_t^h)\phi(dh) = \int_{[0,\bar{k}]\times\mathcal{I}} l(s,i) d\mu_t(k, i; s^t) = \sum_{i\in\mathcal{I}} m_{s_t}(i) l(s,i) = L(s_t),$$

and

$$\int_{\mathcal{H}} \check{c}_t(s^t, i^{h,t}; k_0^h, i_0^h)\phi(dh) = \int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_t(k, i; s^t) d\mu_t(k, i; s^t)$$

and

$$\int_{\mathcal{H}} \breve{k}_{t+1}(s^t, i^{h,t}; k_0^h, i_0^h)\phi(dh) = \int_{[0,\bar{k}]\times\mathcal{I}} \hat{k}_{t+1}(k, i; s^t) d\mu_t(k, i; s^t).$$

Therefore the market clearing conditions are satisfied.

Now given the sequences of prices $\{r_t(s^t), w_t(s^t)\}$, let $\{c_t^h(s^t, i^{h,t}), k_{t+1}^h(s^t, i^{h,t})\}_{t,s^t}$ denote the allocation generated by the policy functions $\check{c}, \breve{k}$ and let $\{\tilde{c}_t^h(s^t, i^{h,t}), \tilde{k}_t^h(s^t, i^{h,t})\}_{t,s^t}$ denote a sequence that satisfies (7), and (8), we show that:

$$\hat{V}_0(k_0^h, i_0^h; s_0) = \mathbb{E}_0\left[\sum_{t=0}^{\infty} \Pi_{t'=0}^{t-1}\beta(i_{t'}^h) u(c_t^h(s^t, i^{h,t}))\right] \tag{17}$$

$$\geq \mathbb{E}_0\left[\sum_{t=0}^{\infty} \Pi_{t'=0}^{t-1}\beta(i_{t'}^h) u(\tilde{c}_t^h(s^t, i^{h,t}))\right]. \tag{18}$$

From the Bellman equation (15), we have:

$$\hat{V}_0(k_0^h, i_0^h; s_0) = \mathbb{E}_0\left[\sum_{t=0}^{T} \Pi_{t'=0}^{t-1}\beta(i_{t'}^h) u(c_t^h) + \Pi_{t'=0}^{T}\beta(i_{t'}^h)\hat{V}_{T+1}(k_t^h, i_t^h; s^t)\right]$$

Now, the second term in the right hand side is bounded (in absolute value) by

$$\bar{\beta}^{T+1}\max\left\{|\underline{V}|, |\overline{V}|\right\} \longrightarrow_{T\to\infty} 0.$$

Therefore taking $T \to \infty$, we obtain (17).

Similarly, from the Bellman equation (15), we have:

$$\hat{V}_0(k_0^h, i_0^h; s_0) \geq \mathbb{E}_0\left[\sum_{t=0}^{T} \Pi_{t'=0}^{t-1}\beta(i_{t'}^h) u(\tilde{c}_t^h) + \Pi_{t'=0}^{T}\beta(i_{t'}^h)\hat{V}_{T+1}(\tilde{k}_t^h, i_t^h; s^t)\right].$$

The second term in the right hand side is bounded below by

$$\min\{\bar{\beta}^{T+1}\underline{V}, \underline{\beta}^{T+1}\underline{V}\} \longrightarrow 0.$$

Therefore taking $T \to \infty$, we obtain (18). □

Proof of Lemma 3. First, to show that $\tilde{G}$ has closed graph, we need to show that if a sequence $\left\{\left(s_n,\mu_n,\hat{V}_n,\hat{k}_n\right)\right\}_{n=0}^{\infty}$ converges to some $\left(s^*,\mu^*,\hat{V}^*,\hat{k}^*\right)\in\mathcal{Z}$ and

$$\gamma_n\in\tilde{G}\left(s_n,\mu_n,\hat{V}_n,\hat{k}_n\right)$$

converges to $\gamma^*\in\mathcal{P}(\mathcal{Z})$ then $\gamma^*\in\tilde{G}\left(s^*,\mu^*,\hat{V}^*,\hat{k}^*\right)$. That is γ^* satisfies conditions (i) and (ii) in the definition of $\tilde{G}$.

Condition (i) requires that the marginal distributions

$$\gamma^*_{\mathcal{S}\times\mathcal{M}}=\sum_{s'\in\mathcal{S}}\pi(s,s')\mathcal{D}_{(s',\hat{k}^*\circ_{ss'}\mu^*)}.$$

This identity is equivalent to

$$\int\varphi d\gamma^*_{\mathcal{S}\times\mathcal{M}}=\sum_{s'\in\mathcal{S}}\pi(s,s')\varphi\left(s',\delta_{\hat{k}^*\circ_{ss'}\mu^*}\right),\tag{19}$$

for all bounded and continuous real-valued function $\varphi:\mathcal{S}\times\mathcal{M}\to\mathbb{R}$. Let $\hat{\varphi}$ be defined over $\mathcal{Z}$ such that

$$\hat{\varphi}(\tilde{s},\tilde{\mu},\tilde{V},\tilde{k})=\varphi(\tilde{s},\tilde{\mu})$$

for all $\tilde{s}\in\mathcal{S},\tilde{\mu}\in\mathcal{M}$ and $\tilde{V}\in\mathcal{BL}_1,\hat{k}\in\mathcal{BL}_2$. Because $\gamma_n\to\gamma^*$ in weak* topology,

$$\int\hat{\varphi}(\tilde{s},\tilde{\mu},\tilde{V},\tilde{k})d\gamma_n\to\int\hat{\varphi}(\tilde{s},\tilde{\mu},\tilde{V},\tilde{k})d\gamma^*.$$

From the definition of γ_n, we have

$$\int\hat{\varphi}(\tilde{s},\tilde{\mu},\tilde{V},\tilde{k})d\gamma_n=\int\varphi(\tilde{s},\tilde{\mu})d\gamma_{n,\mathcal{S}\times\mathcal{M}}=\sum_{s'\in\mathcal{S}}\pi(s,s')\varphi(s',\hat{k}_n\circ_{ss'}\mu_n).$$

In addition,

$$\int\hat{\varphi}(\tilde{s},\tilde{\mu},\tilde{V},\tilde{k})d\gamma^*=\int\varphi d\gamma^*_{\mathcal{S}\times\mathcal{M}}.$$

Because $\hat{k}_n\to\hat{k}^*$ uniformly and $\mu_n\to\mu^*$, $\hat{k}_n\circ_{ss'}\mu_n\to\hat{k}^*\circ_{ss'}\mu^*$ in weak* topology. Consequently, given that φ is continuous:

$$\varphi(s',\hat{k}_n\circ_{ss'}\mu_n)\to\varphi(s',\hat{k}^*\circ_{ss'}\mu^*).$$

Combining the limits and equalities, we arrive at

$$\int\varphi(\tilde{s},\tilde{\mu})d\gamma^*_{\mathcal{S}\times\mathcal{M}}=\sum_{s'\in\mathcal{S}}\pi(s,s')\varphi\left(s',\hat{k}^*\circ_{ss'}\mu^*\right).$$

Therefore, we obtain (19).

Condition (ii) requires that:

$$\hat{V}^*(k,i;s,\mu)=u(\hat{c}^*(k,i;s,\mu))+\beta(i)\sum_{s',i'}\pi_{ss',ii'}\int\tilde{V}\left(\hat{k}^*(k,i;s,\mu),i'\right)d\gamma^*(.|s')\tag{20}$$

$$\geq u(c)+\beta(i)\sum_{s',i'}\pi_{ss',ii'}\int\tilde{V}\left(k',i'\right)d\gamma^*(.|s')\tag{21}$$

for all $0\leq c=(1-\delta+r(s,\mu))k+w(s,\mu)l(s,i)-k'$ and $0\leq k'\leq\bar{k}$.

To prove (20), let

$$\varphi(s', \mu', \tilde{V}, \tilde{k}) = \tilde{V}\left(\hat{k}^*(k, i; s, \mu), i'\right)$$

which is bounded and continuous. Therefore,

$$\lim_{n\to\infty} d_n = 0$$

where

$$d_n = \int \varphi d\gamma_n(.|s') - \int \varphi d\gamma^*(.|s').$$

Consequently

$$\begin{aligned}
& u(\hat{c}_n(k, i; s, \mu)) + \beta(i) \sum_{s',i'} \pi_{ss',ii'} \int \tilde{V}\left(\hat{k}_n(k, i; s, \mu), i'\right) d\gamma_n(.|s') \\
& = u(\hat{c}^*(k, i; s, \mu)) + \beta(i) \sum_{s',i'} \pi_{ss',ii'} \int \tilde{V}\left(\hat{k}^*(k, i; s, \mu), i'\right) d\gamma^*(.|s') \\
& + u(\hat{c}_n(k, i; s, \mu)) - u(\hat{c}^*(k, i; s, \mu)) \\
& + \beta(i) \sum_{s',i'} \pi_{ss',ii'} \int \left\{\tilde{V}\left(\hat{k}_n(k, i; s, \mu), i'\right) - \tilde{V}\left(\hat{k}^*(k, i; s, \mu), i'\right)\right\} d\gamma^*(.|s') \\
& + \beta(i) d_n
\end{aligned}$$

and the terms in the last three lines goes to 0 as $n \to \infty$. Therefore, the initial expression goes to

$$u(\hat{c}^*(k, i; s, \mu)) + \beta(i) \sum_{s',i'} \pi_{ss',ii'} \int \tilde{V}\left(\hat{k}^*(k, i; s, \mu), i'; s', \mu'\right) d\gamma^*(.|s')$$

as $n \to \infty$. Following similar steps, we can prove (21).

Lastly, convex-valuedness of $\tilde{G}$ follows immediately from the fact that, for all $\gamma_1, \gamma_2 \in \mathcal{P}(\mathcal{Z})$ and $0 \le \alpha \le 1$, we have

$$\begin{aligned}
& \int \tilde{V}(k', i') d(\alpha\gamma_1(.|s') + (1-\alpha)\gamma_2(.|s')) \\
& = \alpha \int \tilde{V}(k', i') d\gamma_1(.|s') + (1-\alpha) \int \tilde{V}(k', i') d\gamma_2(.|s'),
\end{aligned}$$

for all k', i', s'. □

B Finite Horizon Economy and Proofs

We first show the existence of sequential competitive equilibrium (SCE) in a finite horizon economy. Then we show that in any SCE, prices and value and policy functions lie in compact sets. Consider the finite horizon version of the economy in Section 2 with $t = 0, 1, ..., T$. We restate the definition of a SCE using wealth distributions, i.e. the distributional approach, as following.

Given the sequence of prices

$$\left\{r_{t,T}(s^t), w_{t,T}(s^t)\right\}_{t=0,...,T; s^t \in \mathcal{S}^t},$$

the representative firm solves:

$$\max_{Y_{t,T},K_{t,T},L_{t,T}} \Pi_{t,T} = Y_{t,T} - r_{t,T}K_{t,T} - w_{t,T}L_{t,T}$$

s.t. $Y_{t,T} \leq F(s_t, K_{t,T}, L_{t,T})$. We allow for $\Pi_{t,T}$ to be potentially different from 0, but we show that in equilibrium $\Pi_t = 0$. We also assume that the profits (or losses) are divided equally across agents.

Given prices and the representative firm's profit, the value function of the agents, $\hat{V}_{t,T}(k,i;s^t)$ satisfies the Bellman equation (starting from $t = T+1$ with $\hat{V}_{T+1,T} \equiv 0$ moving backward):

$$\hat{V}_{t,T}(k,i;s^t) = \max_{c,k'} u(c) + \beta(i)\mathbb{E}_t\left[\hat{V}_{t+1,T}(k',i;s^{t+1})\right] \tag{22}$$

subject to

$$c + k' - (1-\delta)k \leq r_{t,T}(s^t)k + w_{t,T}(s^t)l(s_t,i_t) + \Pi_{t,T}(s^t), \tag{23}$$

and $c \geq 0$ and $0 \leq k \leq \bar{k}$. Let $\hat{c}_{t,T}(k,i;s^t)$ and $\hat{k}_{t,T}(k,i;s^t)$ denote the implied policy functions.

A SCE consists of prices $\{r_{t,T}(s^t), w_{t,T}(s^t)\}$, aggregate capital $K_{t,T}(s^t)$, value and policy functions $\hat{V}_{t,T}, \hat{c}_{t,T}, \hat{k}_{t,T}$ that satisfy (22) and sequences of wealth distribution $\mu_{t,T}(k,i;s^t)$ such that the following identity holds:

$$\begin{aligned}&\int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(i,k;s^t)d\mu_{t,T}(k,i;s^t) + \int_{[0,\bar{k}]\times\mathcal{I}} \hat{k}_{t,T}(i,k;s^t)d\mu_{t,T}(k,i;s^t)\\ &= F(s_t, K_{t,T}(s^{t-1}), L(s_t)) + (1-\delta)K_{t,T}(s^{t-1})\end{aligned}$$

where

$$K_{t,T}(s^{t-1}) = \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_t(k,i;s^t) \text{ and } L(s_t) = \sum_{i\in\mathcal{I}} m_{s_t}(i)l(s_t,i).$$

In addition,

$$r_{t,T}(s^t) = F_K\left(s_t, K_{t,T}(s^{t-1}), L(s_t)\right) > 0 \quad \text{and} \quad w_{t,T}(s^t) = F_L\left(s_t, K_{t,T}(s^{t-1}), L(s_t)\right) > 0.$$

Lastly, the sequences of wealth distributions are consistent with the policy functions:

$$\mu_{t+1,T}\left(.,.;(s^t,s_{t+1})\right) = \hat{k}_{t,T} \circ_{s_t s_{t+1}} \mu_{t,T}\left(.,.;s^t\right),$$

where the composition operator $\circ$ is defined in (12).

The following lemma establishes the existence of SCE in the finite horizon economy.

Lemma 4. *SCE exists in the finite horizon economy version of the model in Section 2.*

Proof. To prove the existence, following Debreu (1959), we switch to the normalization, $p_t^c + w_t + r_t = 1$, where p_t^c is the price of the consumption good, instead of normalizing p_t^c to 1.

Given a sequence $\epsilon^T = \{\epsilon_t\}_{t=0}^T$ such that $\epsilon_t > 0$ for all $t \in \{0,1,...,T\}$, let us define

$$\begin{aligned}\Delta_{\epsilon^T}^{\Sigma^T} &= \times_{(t,s^t)}\Delta_{\epsilon_t}\\ &= \left\{(p_t^c, r_t, w_t)_{t,s^t} \in \left(\mathbb{R}_+^3\right)^{\Sigma^T} : p_t^c + w_t + r_t = 1 \text{ and } p_t^c \geq \epsilon_t\right\},\end{aligned}$$

where each Δ_ϵ denotes a subset of the 3-dimensional simplex:

$$\Delta_\epsilon = \{(p^c, r, w) \gg 0 : p^c + r + w = 1 \text{ and } p^c \geq \epsilon\}.$$

Given the prices $p^T \in \Delta_{\epsilon^T}^{\Sigma_T}$. The firms maximize profit and the agents maximize the inter-temporal expected utility. In particular, in each history s^t the representative firm solves

$$\max_{Y_t, K_t, L_t \geq 0} \Pi_{t,T}(K_{t,T}, L_{t,T}) \tag{24}$$

s.t.

$$Y_{t,T} \leq F(s_t, K_{t,T}, L_{t,T})$$

and

$$\Pi_{t,T} = p^c_{t,T} Y_{t,T} - r_{t,T} K_{t,T} - w_{t,T} L_{t,T}.$$

We also impose two additional constraints:

$$0 \leq L_{t,T} \leq 2\bar{L} \quad \text{and} \quad 0 \leq K_{t,T} \leq 2\bar{K},$$

where $\bar{L}$ and $\bar{K}$ are defined in Assumption 2 and in (4).

Given $\Pi^T = \left(\Pi_{t,T}(s^t)\right)$,[17] the agents solve the dynamic programing problem: $\hat{V}_{t,T}$ is defined recursively (starting from $t = T+1$ with $\hat{V}_{T+1,T} \equiv 0$ moving backward) as

$$\hat{V}_{t,T}(k, i; s^t, p^T, \Pi^T) = \max_{c,k'} u(c) + \beta(i)\mathbb{E}_t \left[\hat{V}_{t+1,T}(k', i; s^t, p^T, \Pi^T)\right] \tag{25}$$

subject to $c \geq 0$, $0 \leq k' \leq \bar{k}$ and

$$p^c_{t,T}(s^t)\left(c + k' - (1-\delta)k\right) \leq r_{t,T}(s^t)k + w_{t,T}(s^t)l(s_t, i) + \Pi_{t,T}. \tag{26}$$

In Lemma 7, we show that the policy function for k', $\hat{k}_{t,T}(k, i; s^t)$ is uniquely-defined, continuous, and weakly increasing.

Given the policy function $\hat{k}_{t,T}$, we construct the sequence of measures $\tilde{\mu}^T = \left(\tilde{\mu}_{t,T}(.; s^t)\right)_{t,s^t}$ as following:

1. $\tilde{\mu}_{0,T} = \mu_{0,T}$
2. For $t \geq 0$,

$$\tilde{\mu}_{t+1,T}(.,.; s^{t+1}) = \hat{k}_{t,T} \circ_{s_t s_{t+1}} \mu_{t,T}(.,.; s^t),$$

where the composition operator $\circ$ is defined in (12).

We denote

$$\psi_\mu : \Delta_{\epsilon^T}^{\Sigma^T} \times \Omega^{\Sigma^T} \rightrightarrows \Omega^{\Sigma^T}$$

as the correspondence that map the sequence of prices p^T and distributions μ^T to the sequence of distributions $\tilde{\mu}^T$ as constructed above.

[17]The maximization problem (24) might have many maximizers but the maximized objective $\Pi_{t,T}$ is uniquely determined given prices $p_{t,T} \in \Delta_{\epsilon_t}$.

We form the following excess demands:

$$\text{Consumption:} x^c_{t,T}(s^t) = \int_{[0,\bar{k}]\times\mathcal{I}} \left(\hat{c}_{t,T}(k,i;s^t) + \hat{k}_{t,T}(k,i;s^t) - (1-\delta)k\right) d\mu_{t,T}(k,i;s^t) - Y_{t,T}(s^t)$$

$$\text{Capital:} x^k_{t,T}(s^t) = K_{t,T}(s^t) - \int_{[0,\bar{k}]\times\mathcal{I}} k\mu_{t,T}(k,i;s^t)$$

$$\text{Labor: } x^l_{t,T}(s^t) = L_{t,T}(s^t) - L(s_t).$$

These definitions imply exogenous bounds on the excess demands:

$$\underline{x}^c = -(1-\delta)\bar{k} - \bar{Y} < x^c_{t,T}(s^t) < \bar{x}^c = \frac{1-\underline{\epsilon}}{\underline{\epsilon}}\bar{k} + \frac{1-\underline{\epsilon}}{\underline{\epsilon}}\bar{L} + \frac{1}{\underline{\epsilon}}\bar{Y},$$

where $\underline{\epsilon} = \frac{1}{2}\min_{0\le t\le T}\epsilon_t$, and

$$\underline{x}^k = -2\bar{k} < x^k_{t,T}(s^t) < \bar{x}^k = 2\bar{K},$$

and

$$\underline{x}^l = -2\bar{L} < x^l_{t,T}(s^t) < \bar{x}^l = 2\bar{L}.$$

Let $\mathcal{K}_x$ denote the cube $\left([\underline{x}^c, \bar{x}^c] \times [\underline{x}^k, \bar{x}^k] \times [\underline{x}^l, \bar{x}^l]\right)^{\Sigma^T}$. We define the correspondence

$$\psi_x : \Delta_\epsilon^{\Sigma^T} \times \Omega^{\Sigma^T} \rightrightarrows \mathcal{K}_x$$

that maps a sequence of prices p^T and a sequence of distributions μ^T to the excess demands in every history.

Lastly,

$$\psi_p : \mathcal{K}_x \rightrightarrows \Delta_{\epsilon^T}^{\Sigma^T}$$

such that

$$p_{t,T} = \arg\max_{p\in\Delta_{\epsilon_t}} p\cdot x_{t,T}.$$

Let Φ_ϵ denote an operator (which depends on $\{\epsilon_t\}$) taking ϕ_p, ϕ_μ, ϕ_x as components:

$$\Psi_\epsilon : \Delta_{\epsilon^T}^{\Sigma^T} \times \Omega^{\Sigma^T} \times \mathcal{K}_x \rightrightarrows \Delta_{\epsilon^T}^{\Sigma^T} \times \Omega^{\Sigma^T} \times \mathcal{K}_x \tag{27}$$

$$\Psi_\epsilon = \left(\psi_p, \psi_\mu, \psi_x\right)$$

Lemma 8 shows that Ψ_ϵ is upper-hemi continuous and is non-empty, compact, and convex-valued. In addition $\Delta_\epsilon^{\Sigma^T} \times \Omega^{\Sigma^T} \times \mathcal{K}_x$ is a compact and convex subset of a locally convex Hausdorff space.[18] Therefore, by the Kakutani-Glicksberg-Fan Fixed Point Theorem, Φ_ϵ admits a fixed point. In Lemma 5 we show that, by choosing $\{\epsilon_t\}$ appropriately, this fixed point constitutes a SCE. □

The following lemma uses the fixed point in the previous lemma to establish the existence of SCE in the finite horizon economy stated in Lemma 2.

Lemma 5. *Consider the sequence $\{\epsilon_t, \underline{K}_t\}_{t=0}^T$ constructed in Lemma 6, and Ψ_ϵ as defined in (27)*

[18]These properties follow directly from the result that the space Ω of probability measures endowed with weak* topology is metrizable, shown in Bogachev (2000, Theorem 8.3.2).

given the sequence $\{\epsilon_t\}$*. Let*

$$\bar{\psi} = \left((\bar{p}_{t,T})_{t,s^t}, (\bar{\mu}_{t,T})_{t,s^t}, (\bar{x}_{t,T})_{t,s^t} \right)$$

be a fixed point of Ψ_ϵ*. Then* $\bar{\psi}$ *corresponds to a SCE.*

Proof. To show that $\bar{\psi}$ corresponds to a SCE, we need to show that $x_{t,T}(s^t) = 0$ and $r_{t,T}, w_{t,T} > 0$ for all $t \leq T$ and $s^t \in \mathcal{S}^t$. To simplify the notations, we omit the bar notation on variables. We also omit the dependence on s^t, p^T, Π^T when it is not ambiguous.

First, we notice that for all $t \leq T$ and $s^t \in \mathcal{S}^t$,

$$\begin{aligned}
p_{t,T} \cdot x_{t,T} &= p^c_{t,T} x^c_{t,T} + r_{t,T} x^k_{t,T} + w_{t,T} x^l_{t,T} \\
&= p^c_{t,T} \int_{[0,\bar{k}]\times\mathcal{I}} \left(\hat{c}_{t,T}(k,i) + \hat{k}_{t,T}(k,i) - (1-\delta)k \right) d\mu_{t,T}(k,i) - p^c_{t,T} Y_{t,T} \\
&\quad + r_{t,T} K_{t,T} - r_{t,T} \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t,T}(k,i_t) + w_{t,T} L_{t,T} - w_{t,T} L(s_t) \\
&= p^c_{t,T} \int_{[0,\bar{k}]\times\mathcal{I}} \left(\hat{c}_{t,T}(k,i) + \hat{k}_{t,T}(k,i) - (1-\delta)k \right) d\mu_{t,T}(k,i) \\
&\quad - \Pi_{t,T}(K_t, L_t) - r_{t,T} \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t,T}(k,i) - w_{t,T}(s^t) \int_{[0,\bar{k}]\times\mathcal{I}} l(s_t,i_t) d\mu_{t,T}(k,i).
\end{aligned}$$

Since $p^c_t > 0$, (26) holds with equality for each k, i. Therefore the last expression equal to 0. So

$$p_{t,T} \cdot x_{t,T} = 0 \tag{28}$$

for all t, s^t.

From the definition of a fixed point, we have

$$p_{t,T} \in \arg\max_{p \in \Delta_{\epsilon_t}} p \cdot x_{t,T}.$$

Therefore,

$$0 = p_{t,T} \cdot x_{t,T} \geq \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot x_{t,T} = x^c_{t,T},$$

or equivalently,

$$x^c_{t,T}(s^t) \leq 0 \; \forall t \leq T \text{ and } \forall s^t \in \mathcal{S}^t. \tag{29}$$

Now, we show by induction that $x_{t,T} = 0$ for all $t \leq T$ and $s^t \in \mathcal{S}^t$. In particular, we show that, $x_{0,T} = 0$ and $r_{0,T}, w_{0,T} > 0$ (Step 1) and if $x_{t-1,T} = 0$ and $r_{t-1,T}, w_{t-1,T} > 0$ for all $s^{t-1} \in \mathcal{S}^{t-1}$ then $x_{t,T} = 0$ and $r_{t,T}, w_{t,T} > 0$ and in addition $K_t > \underline{K}_t$ (Step 2).

Step 1: Starting with $t = 0$, we have just shown in (29) that $x^c_{0,T} \leq 0$.

If $x^k_{0,T} < 0$ then $r_{0,T} = 0$ (since $p_{0,T} \in \arg\max_p p \cdot x_{0,T}$ and $p_{0,T} \cdot x_{0,T} = 0$). The maximization of the representative firm, (24), at $t = 0$ implies that $K_{0,T} = 2\bar{K}$. But then $x^k_{0,T} > 0$ since we chose $2\bar{K} > K_0 = \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{0,T}(k,i;s_0)$. So $x^k_{0,T} \geq 0$.

Similarly, if $x^l_{0,T} < 0$ then $w_{0,T} = 0$. This implies that, from the maximization of the representative firm, (24), at $t = 0$, $L_{0,T} = 2\bar{L}$. But then $x^l_{0,T} > 0$ since we chose $2\bar{L} > \bar{L} > \max_{s\in\mathcal{S}} L(s)$. So $x^l_{0,T} \geq 0$.

Now, we show by contradiction that $x^c_{0,T} = 0$. Assume the contrary: $x^c_{0,T} < 0$. Then

$p^c_{0,T} = \epsilon_0$ (since $p_{0,T} \in \arg\max_p p \cdot x_{0,T}$ and $p_{0,T} \cdot x_{0,T} = 0$ and $x^k_{0,T}, x^l_{0,T} \geq 0$). Consequently,

$$r_{0,T} x^k_{0,T} + w_{0,T} x^l_{0,T} = -\epsilon_0 x^c_{0,T} > 0. \tag{30}$$

If $r_{0,T} = 0$ then $K_{0,T} = 2\bar{K}$ and

$$x^k_{0,T} \geq 2\bar{K} - K_0 > \bar{K} > 2\bar{L} > x^l_{0,T}.$$

But then, because $p_{0,T} \in \arg\max_p p \cdot x_{0,T}$, we have $r_{0,T} = 1 - \epsilon_0$ and $w_{0,T} = 0$, a contradiction with the assumption that $r_{0,T} = 0$. So we must have $r_{0,T} > 0$. And since $p_{0,T} \in \arg\max_p p \cdot x_{0,T}$, it must be that $x^k_{0,T} \geq x^l_{0,T}$. From (30), we have $x^k_{0,T} > 0$.

If $w_{0,T} = 0$ then $L_{0,T} = 2\bar{L} > 0$ and $x^l_{0,T} = L_{0,T} - L(s_0) > 0$. If $w_{0,T} > 0$ then since $p_{0,T} \in \arg\max_p p \cdot x_{0,T}$, it must be that $x^l_{0,T} \geq x^k_{0,T}$. Since we have shown above that $x^k_{0,T} \geq x^l_{0,T}$, this leads to $x^k_{0,T} = x^l_{0,T} > 0$. In either case, we have $x^l_{0,T} > 0$.

Therefore $K_{0,T} = x^k_{0,T} + K_0 > \underline{K}_0$ and $L_{0,T} = x^l_{0,T} + L(s_0) > \underline{L}$.

Now, at time $t = 0$ and s_0, (because $p_{0,T} = \epsilon_0$) $(Y_{0,T}, K_{0,T}, L_{0,T})$ solves:

$$\max_{Y,K,L} \epsilon_0 Y - r_{0,T} K - w_{0,T} L$$

s.t.

$$Y \leq F(s_0, K, L)$$

and $K \leq 2\bar{K}$, $L \leq 2\bar{L}$. Because $\epsilon_0 > 0$, $Y_{0,T} = F(s_0, K_{0,T}, L_{0,T})$ and

$$\epsilon_0 F_K(s_0, K_{0,T}, L_{0,T}) \geq r_{0,T}$$

(with equality if $K_{0,T} < 2\bar{K}$) and

$$\epsilon_0 F_L(s_0, K_{0,T}, L_{0,T}) \geq w_{0,T}$$

(with equality if $L_{0,T} < 2\bar{L}$). Therefore

$$\epsilon_0 (F_K(s_0, K_{0,T}, L_{0,T}) + F_L(s_0, K_{0,T}, L_{0,T})) \geq r_{0,T} + w_{0,T} = 1 - \epsilon_0.$$

Equivalently,

$$\epsilon_0 (1 + F_K(s_0, K_{0,T}, L_{0,T}) + F_L(s_0, K_{0,T}, L_{0,T})) \geq 1. \tag{31}$$

Because F is concave and $K_{0,T} \geq \underline{K}_0$,

$$F_K(s_0, K_{0,T}, L_{0,T}) \leq F_K(s_0, \underline{K}_0, L_{0,T}) \leq \max_{0 \leq L \leq 2\bar{L}} F_K(s_0, \underline{K}_0, L),$$

where $\max_{0 \leq L \leq 2\bar{L}} F_K(s_0, \underline{K}_0, L) < \infty$ by Assumption 5. Similarly, because $L_{0,T} \geq \underline{L}$,

$$F_L(s_0, K_{0,T}, L_{0,T}) \leq \max_{0 \leq K \leq 2\bar{K}} F_L(s_0, K, \underline{L}).$$

Therefore,

$$\begin{aligned} &\epsilon_0 (1 + F_K(s_0, K_{0,T}, L_{0,T}) + F_L(s_0, K_{0,T}, L_{0,T})) \\ &< \epsilon_0 (1 + \max_{0 \leq L \leq 2\bar{L}} F_K(s_0, \underline{K}_0, L) + \max_{0 \leq K \leq 2\bar{K}} F_L(s_0, K, \underline{L})) < 1, \end{aligned}$$

where the last inequality comes from property (41) in Lemma 6. But this contradicts the earlier inequality, (31).

So we obtain by contradiction that $x^c_{0,T} = 0$.

Now if $x_0^k > 0$ or $x_0^l > 0$ then $\max p \cdot x_{0,T} > 0$, which contradicts (28): $0 = p_{0,T} \cdot x_{0,T} = \max_p p \cdot x_{0,T}$. Therefore $x_0^k = x_0^l = 0$.

If $w_{0,T} = 0$, then $L_{0,T} = 2\bar{L}$ and $x_0^l > 0$ therefore $w_{0,T} > 0$. If $r_{0,T} = 0$ then $K_{0,T} = 2\bar{K}$ and $x_{0,T}^k = 2\bar{K} - K_0 > 0$ therefore $r_{0,T} > 0$.

Step 2: From $t-1$ to t.

Since $x_{t-1,T}^c = x_{t-1,T}^k = 0$, we have

$$\int_{[0,\bar{k}]\times\mathcal{I}} \Big(\hat{c}_{t-1,T}(k,i;s^{t-1}) + \hat{k}_{t-1,T}(k,i;s^{t-1}) - (1-\delta)k\Big)\, d\mu_{t-1,T}(k,i;s^{t-1}) - F(s_{t-1},K_{t-1},L_{t-1}) = 0$$

and

$$\int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t-1,T}(k,i;s^{t-1}) = K_{t-1}.$$

From the definition of $\mu_{t,T}, \tilde{\mu}_{t,T}$ and the fixed point property of $\bar{\psi}$,

$$\int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t,T}(k,i;s^{t}) = \int_{[0,\bar{k}]\times\mathcal{I}} \hat{k}_{t-1,T}(k,i;s^{t-1}) d\mu_{t-1,T}(k,i;s^{t-1}).$$

Therefore,

$$\int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t,T}(k,i;s^{t}) \le F(s_{t-1},K_{t-1},L_{t-1}) + (1-\delta)K_{t-1} < \bar{K},$$

where the last inequality comes from condition (4) on $\bar{K}$.

Now, we show that $x_{t,T} = 0$ and $r_{t,T}, w_{t,T} > 0$. Indeed, we show in (29) that $x_{1,T}^c(s^1) \le 0$. The following arguments are similar to the argument in Step 1.

If $x_{t,T}^k(s^t) < 0$ then $r_{t,T}(s^1) = 0$. Then $K_{t,T}(s^t) = 2\bar{K}$ but then $x_{t,T}^k(s^t) > 0$ since we have shown that $\bar{K} > \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t,T}(k,i;s^t)$. So $x_{t,T}^k(s^t) \ge 0$.

If $x_{t,T}^l(s^t) < 0$ then $w_{t,T}(s^1) = 0$. Then $L_{t,T}(s^t) = 2\bar{L}$ but then $x_{t,T}^l(s^t) > 0$ since $2\bar{L} > \max_{s\in\mathcal{S}} L(s)$. So $x_{t,T}^l(s^t) \ge 0$.

We show by contradiction that $x_{t,T}^c(s^t) = 0$. Assume to the contrary that, $x_{t,T}^c(s^t) < 0$. Then $p_{t,T}^c(s^t) = \epsilon_t$ (since $p_{t,T} \in \arg\max_{p\in\Delta_{\epsilon_t}} p \cdot x_{t,T}$ and $x_{t,T}^k, x_{t,T}^l \ge 0$). Therefore,

$$r_{t,T}(s^t)x_{t,T}^k(s^t) + w_{t,T}(s^t)x_{t,T}^l(s^t) = -\epsilon_t x_{t,T}^c(s^t) > 0. \tag{32}$$

If $r_{t,T}(s^t) = 0$ then $K_{t,T}(s^t) = 2\bar{K}$ and

$$\begin{aligned} x_{t,T}^k(s^t) &= 2\bar{K} - \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t,T}(k,i;s^t) \\ &> 2\bar{K} - \bar{K} > 2\bar{L} > x_{t,T}^l(s^t). \end{aligned}$$

But then, since $p_{t,T} \in \arg\max_p p \cdot x_{t,T}$, so $r_{t,T}(s^t) = 1 - \epsilon_t$ and $w_{t,T}(s^t) = 0$, a contradiction with the assumption that $r_{t,T}(s^t) = 0$. So we must have $r_{t,T}(s^t) > 0$. Therefore, since $p_{t,T} \in \arg\max_p p \cdot x_{t,T}$, it implies that $x_{t,T}^k(s^t) \ge x_{t,T}^l(s^t)$. From (32), we have $x_{t,T}^k(s^t) > 0$.

If $w_{t,T}(s^t) = 0$ then $L_{t,T}(s^t) = 2\bar{L} > 0$ and $x_{t,T}^l(s^t) = L_{t,T}(s^t) - L(s_t) > 0$. If $w_{t,T}(s^t) > 0$ then since $p_{t,T} \in \arg\max_p p \cdot x_{t,T}$, it must be that $x_{t,T}^l(s^t) \ge x_{0,T}^k(s^t)$. We have just shown above that $x_{t,T}^k(s^t) \ge x_{t,T}^l(s^t)$. This leads to $x_{t,T}^k(s^t) = x_{t,T}^l(s^t) > 0$. In either case, we have

$x^l_{t,T}(s^t) > 0$.

Therefore $K_{t,T}(s^t) = x^k_{t,T}(s^t) + K_{t-1,T}(s^{t-1}) > \underline{K}_{t-1}$ and $L_{t,T}(s^t) = x^l_{t,T}(s^t) + L(s_t) > \underline{L}$.

As we show in (29) $x^c_{t,T}(s^t) \leq 0$, which by the definition of $x^c_{t,T}$ implies:

$$\int_{[0,\bar{k}]\times\mathcal{I}} \left(\hat{c}_{t,T}(k,i;s^t) + \hat{k}_{t,T}(k,i;s^t) - (1-\delta)k\right) d\mu_{t,T}(k,i;s^t) - Y_{t,T}(s^t) \leq 0.$$

Therefore

$$\begin{aligned}\int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i;s^t)d\mu_{t,T}(k,i;s^t) &\leq (1-\delta)\int_{[0,\bar{k}]\times\mathcal{I}} kd\mu_{t,T}(k,i;s^t) + Y_{t,T}(s^t) \\ &\leq (1-\delta)K_{t,T}(s^t) + Y_{t,T}(s^t), \end{aligned} \tag{33}$$

where the last inequality comes from $x^k_{t,T}(s^t) \geq 0$.

From the agents' Euler equation, shown in Lemma 7,

$$p^c_{t-1,T}u'(\hat{c}_{t-1,T}) \geq \beta(i_{t-1})\mathbb{E}_{t-1}[((1-\delta)\epsilon_t + r_{t,T})u'(\hat{c}_{t,T})]$$

if $\hat{k}_{t-1,T}(k,i;s^{t-1}) < \bar{k}$. In this case, since $p^c_{t-1,T} \leq 1$,

$$\begin{aligned} u'(\hat{c}_{t-1,T}) &\geq \underline{\beta}\sum_{s_t\in\mathcal{S}}\pi_{s_{t-1}s_t}\sum_{i_t\in\mathcal{I}}\Pr(i_t|s_{t-1},s_t,i_{t-1})((1-\delta)\epsilon_t + r_{t,T})u'(\hat{c}_{1,T}) \\ &\geq \underline{\beta}\pi_{s_{t-1}s_t}\sum_{i_t}\Pr(i_t|s_{t-1},s_t,i_{t-1})((1-\delta)\epsilon_t + r_{t,T})u'(\hat{c}_{t,T}) \\ &\geq \underline{\beta}\pi_{s_{t-1}s_t}((1-\delta)\epsilon_t + r_{t,T})u'\left(\sum_{i_t}\Pr(i_t|s_{t-1},s_t,i_{t-1})\hat{c}_{t,T}\right), \end{aligned}$$

where the last inequality comes from Jensen's inequality and the convexity of u'.

Therefore

$$\left(\underline{\beta}\pi_{s_{t-1}s_t}((1-\delta)\epsilon_t + r_{t,T})\right)^{\frac{1}{\sigma}} \leq \frac{\sum_{i_t\in\mathcal{I}}\pi_{s_{t-1}s_t,i_{t-1}i_t}\hat{c}_{t,T}}{\hat{c}_{t-1,T}}.$$

Integrating over $\mu_{t-1,T}$, and by (33),[19] we obtain

$$\left(\underline{\beta}\pi_{s_{t-1}s_t}((1-\delta)\epsilon_t + r_{t,T})\right)^{\frac{1}{\sigma}} \leq \frac{Y_{t,T} + (1-\delta)K_{t,T}}{\int_{[0,\bar{k}]\times\mathcal{I}}\hat{c}_{t-1,T}(k,i)\chi_{\hat{k}_{t-1,T}<\bar{k}}d\mu_{t-1,T}(k,i)}, \tag{34}$$

where χ is the set characteristic function.

Now

$$\int_{[0,\bar{k}]\times\mathcal{I}} \bar{k}\chi_{\hat{k}_{t-1,T}\geq\bar{k}}d\mu_{t-1,T}(k,i) \leq K_t.$$

Equivalently,

$$\bar{k}\int_{[0,\bar{k}]\times\mathcal{I}} \chi_{\hat{k}_{t-1,T}\geq\bar{k}}d\mu_{t-1,T}(k,i) \leq K_t. \tag{35}$$

[19]We use the inequality that $m \leq \frac{a_j}{b_j}$ for all j implies $m \leq \frac{\int a_j}{\int b_j}$.

We also have

$$\int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i)\chi_{\hat{k}_{t,T}(i,k)<\bar{k}} d\mu_{t,T}(k,i)$$
$$= \int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i) d\mu_{t,T}(k,i) - \int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i)\chi_{\hat{k}_{t,T}\geq\bar{k}} d\mu_{t,T}(k,i). \tag{36}$$

For k such that $\hat{k}_{t-1,T}(k,i;s^t) = \bar{k}$ we have

$$\begin{aligned}&p^c_{t-1,T}(\hat{c}_{t-1,T}(k,i) + \bar{k} - (1-\delta)k)\\ &= r_{t-1,T}k + w_{t-1,T}l(s_t,i) + \Pi_{t,T}\\ &\leq (1-\epsilon_{t-1})(\bar{k}+\bar{l}) + \bar{Y},\end{aligned}$$

where $\bar{Y} = \max_{s\in\mathcal{S}} F(s, 2\bar{K}, 2\bar{L})$. Therefore, for these values of k:

$$\hat{c}_{t-1,T}(k,i) \leq \frac{(1-\epsilon_{t-1})(\bar{k}+\bar{l}) + \bar{Y}}{\epsilon_{t-1}} \leq \frac{2}{\epsilon_{t-1}}\bar{k} \tag{37}$$

since $\bar{k} > \bar{l} + \bar{Y}$, by Assumption 5.

So, (35), (36), and (37) yield

$$\begin{aligned}&\int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i)\chi_{\hat{k}_{t,T}<\bar{k}} d\mu_{t,T}(k,i)\\ &\geq \int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i) d\mu_{t,T}(k,i) - \frac{2}{\epsilon_{t-1}}\bar{k}\int_{[0,\bar{k}]\times\mathcal{I}} \chi_{\hat{k}_{t,T}\geq\bar{k}} d\mu_{t,T}(k,i)\\ &\geq \int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i) d\mu_{t,T}(k,i) - \frac{2}{\epsilon_{t-1}}K_{t,T}.\end{aligned}$$

Therefore, from (34), we get

$$\left(\underline{\beta}\pi_{s_{t-1}s_t}((1-\delta)\epsilon_t + r_{t,T})\right)^{\frac{1}{\sigma}} \leq \frac{Y_{t,T} + (1-\delta)K_{t,T}}{F(s_{t-1}, K_{t-1,T}, L_{t-1,T}) + (1-\delta)K_{t-1,T} - K_{t,T} - \frac{2}{\epsilon_{t-1}}K_{t,T}} \tag{38}$$

From the firm's problem, $(Y_{t,T}, K_{t,T}, L_{t,T})$ solves

$$\max_{Y,K,L} \epsilon_t Y - r_{t,T}K - w_{t,T}L$$

s.t.

$$Y \leq F(s_1, K, L)$$

and $0 \leq K \leq 2\bar{K}$, $0 \leq L \leq 2\bar{L}$. Since $\epsilon_t > 0$, we have $Y_t = F(s_t, K_{t,T}, L_{t,T})$ and since we established that $K_{t,T}, L_{t,T} > 0$, we have

$$\epsilon_t F_K(s_t, K_{t,T}, L_{t,T}) \geq r_{t,T}(s^t)$$

and

$$\epsilon_t F_L(s_t, K_{t,T}, L_{t,T}) \geq w_{t,T}(s^t).$$

Notice that

$$F_L(s_t, K_{t,T}, L_{t,T}) \leq F_L(s_t, K_{t,T}, \underline{L}) \leq \max_{0\leq K\leq 2\bar{K}} F_L(s_t, K, \underline{L}),$$

which is finite by Assumption 5.

Since $r_{t,T} = 1 - p^c_{t,T} - w_{t,T} = 1 - \epsilon_t - w_{t,T}$,

$$\epsilon_t F_K(s_t, K_{t,T}, L_{t,T}) \geq r_{t,T} \geq 1 - \epsilon_t(1 + \max_{0 \leq K \leq 2\bar{K}} F_L(s_t, K, \underline{L})) \tag{39}$$

Since we chose ϵ_t such that

$$\epsilon_t F_K(s_t, K_{t,T}, L_{t,T}) < 1 - \epsilon_t(1 + \max_{0 \leq K \leq 2\bar{K}} F_L(s_t, K, \underline{L}))$$

for all $K_{t,T} \geq \frac{(1-\delta)\underline{K}_{t-1}}{2\left(1+\frac{2}{\epsilon_{t-1}}\right)}$, (i.e., Property e1. in Lemma 6) inequality (39) implies that $K_{t,T} \leq \frac{(1-\delta)\underline{K}_{t-1}}{2\left(1+\frac{2}{\epsilon_{t-1}}\right)}$. So (38), together with $K_{t-1} > \underline{K}_{t-1}$, yields

$$\left(\underline{\beta}\pi_{s_{t-1}s_t}((1-\delta)\epsilon_t + 1 - \epsilon_t(1 + \max_{0 \leq K \leq 2\bar{K}} F_L(s_t, K, \underline{L})))\right)^{\frac{1}{\sigma}}$$
$$\leq \frac{Y_{t,T} + (1-\delta)K_{t,T}}{\frac{1}{2}(1-\delta)\underline{K}_{t-1}} \leq \frac{F(s_t, K_{t,T}, 2\bar{L}) + (1-\delta)K_{t,T}}{\frac{1}{2}(1-\delta)\underline{K}_{t-1}}.$$

By the choice of ϵ_t in Lemma 6 (Property e2), because $K_{t,T}$ satisfies:

$$F_K(s_t, K_{t,T}, L_{t,T}) \geq \frac{1 - \epsilon_t(1 + \max_{0 \leq K \leq 2\bar{K}} F_L(s_t, K, \underline{L}))}{\epsilon_t}$$

we have

$$\left(\underline{\beta}\pi_{s_{t-1}s_t}((1-\delta)\epsilon_t + 1 - \epsilon_t(1 + \max_{0 \leq K \leq 2\bar{K}} F_L(s_t, K, \underline{L})))\right)^{\frac{1}{\sigma}} > \frac{F(s_t, K_{t,T}, 2\bar{L}) + (1-\delta)K_{t,T}}{\frac{1}{2}(1-\delta)\underline{K}_{t-1}}.$$

This is a contradiction with the earlier inequality.

So we have shown by contraction that $x^c_{t,T} = 0$. Next, we show that $x^k_{t,T} = x^l_{t,T} = 0$ and $w_{t,T}, r_{t,T} > 0$.

Indeed, if $x^k_{t,T} > 0$ or $x^l_{t,T} > 0$ then $p_{t,T} \cdot x_{t,T} = \max p \cdot x_{t,T} > 0$, contradicting (28). Therefore $x^k_{t,T} = x^l_{t,T} = 0$.

If $w_{t,T} = 0$, then $L_{t,T} = 2\bar{L}$ and $x^l_{t,T} > 0$. Therefore $w_{t,T} > 0$. If $r_{t,T} = 0$ then $K_{t,T} = 2\bar{K}$ and $x^k_{t,T} > 0$. Therefore $r_{t,T} > 0$.

Now we show that $K_{t,T} > \underline{K}_t$. Following the derivation of (38), we obtain

$$\left(\underline{\beta}\pi_{s_{t-1}s_t}((1-\delta)p^c_{t,T} + r_{t,T})\right)^{\frac{1}{\sigma}} \leq \frac{Y_{t,T} + (1-\delta)K_{t,T}}{F(s_{t-1}, K_{t-1,T}, L_{t-1,T}) + (1-\delta)K_{t-1,T} - K_{t,T} - \frac{2}{\epsilon_{t-1}}K_{t,T}} \tag{40}$$

Therefore, if $K_{t,T} \leq \underline{K}_t$,

$$K_{t,T}\left(1 + \frac{2}{\epsilon_{t-1}}\right) \leq (1-\delta)\underline{K}_{t-1} < (1-\delta)K_{t-1,T}.$$

So, because $p^c_{t,T} \geq \epsilon_t$, (40) implies

$$\left(\underline{\beta}\pi_{s_{t-1}s_t}((1-\delta)\epsilon_t)\right)^{\frac{1}{\sigma}} < \frac{F(s_t, K_{t,T}, 2\bar{L}) + (1-\delta)K_{t,T}}{F(s_{t-1}, \underline{K}_{t-1}, \underline{L})}$$

which contradicts the definition of $\underline{K}_t$ in Lemma 6 (in particular property e2). Therefore $K_{t,T} > \underline{K}_t$. □

Lemma 6. *Given $s_0 \in \mathcal{S}$ and $K_0 > 0$, we can choose the sequence $\{\epsilon_t, \underline{K}_t\}_{t=0}^{\infty}$ recursively as follow:*

$$\underline{K}_0 = \frac{1}{2} K_0 < K_0$$

and $\epsilon_0 > 0$ such that

$$\epsilon_0 < \frac{1}{1 + \max_{0 \le L \le 2\bar{L}} F_K(s_0, \underline{K}_0, L) + \max_{0 \le K \le 2\bar{K}} F_L(s_0, K, \underline{L})}. \tag{41}$$

For $t > 0$, given ϵ_{t-1} and $\underline{K}_{t-1}$, we can choose $\epsilon_t > 0$ and $\underline{K}_t > 0$ such that the following properties e1. and e2. are satisfied:

e1. ϵ_t is sufficiently small such that

$$\epsilon_t < \min_{s \in \mathcal{S}, 0 \le L \le 2\bar{L}} \frac{1}{F_K\left(s, \frac{(1-\delta)\underline{K}_{t-1}}{2\left(1+\frac{2}{\epsilon_{t-1}}\right)}, L\right) + 1 + \max_{0 \le \tilde{K} \le 2\bar{K}} F_L(s, \tilde{K}, \underline{L})} \tag{42}$$

and

e2. for all $\hat{s}, s \in \mathcal{S}$, and for all K such that

$$F_K(s, K, L) \ge \frac{1 - \epsilon_t(1 + \max_{0 \le \tilde{K} \le 2\bar{K}} F_L(s, \tilde{K}, \underline{L}))}{\epsilon_t},$$

for some $0 \le L \le 2\bar{L}$, we have

$$\left(\underline{\beta}\pi_{\hat{s}s}((1-\delta)\epsilon_t + 1 - \epsilon_t(1 + \max_{0 \le \tilde{K} \le 2\bar{K}} F_L(s, \tilde{K}, \underline{L})))\right)^{\frac{1}{\sigma}} > \frac{F(s, K, 2\bar{L}) + (1-\delta)K}{\frac{1}{2}(1-\delta)\underline{K}_{t-1}}. \tag{43}$$

Given $\epsilon_{t-1}, \epsilon_t$, and $\underline{K}_{t-1}$, there exists $\underline{K}_t < \frac{(1-\delta)\underline{K}_{t-1}}{1+\frac{2}{\epsilon_{t-1}}}$ such that for all $\hat{s}, s \in \mathcal{S}$ and $K \le \underline{K}_t$,

$$\left(\underline{\beta}\pi_{\hat{s}s}(1-\delta)\epsilon_t\right)^{\frac{1}{\sigma}} > \frac{F(s, K, 2\bar{L}) + (1-\delta)K}{F(\hat{s}, \underline{K}_{t-1}, \underline{L})}. \tag{44}$$

Proof. The existence of $\underline{K}_0$ is obvious. By Assumption 5,

$$\max_{0 \le L \le 2\bar{L}} F_K(s_0, \underline{K}, L) < +\infty \text{ and } \max_{0 \le \tilde{K} \le 2\bar{K}} F_L(s, \tilde{K}, \underline{L}) < +\infty,$$

so there exists $\epsilon_0 > 0$ that satisfies (41). Now we construct ϵ_t and $\underline{K}_t$ recursively.

The right hand side of (42) is finite and is strictly positive. Let $\underline{\epsilon}_t > 0$ denote this value and let

$$\tilde{\epsilon}_t = \min\left\{\underline{\epsilon}_t, \frac{1}{2\left(1 + \max_{s \in \mathcal{S}, 0 \le \tilde{K} \le 2\bar{K}} F_L(s, \tilde{K}, \underline{L})\right)}\right\}.$$

By Assumption 5, there exists $\hat{K}_t$ such that for all $\hat{s}, s \in \mathcal{S}$:

$$\left(\underline{\beta}\pi_{\hat{s}s}\frac{1}{2}\right)^{\frac{1}{\sigma}} \frac{1}{2}(1-\delta)\underline{K}_{t-1} > F(s, K, \bar{L}) + (1-\delta)K, \tag{45}$$

for all $K \leq \hat{K}_t$. Again, by Assumption 5, we can find $0 < \epsilon_t < \tilde{\epsilon}_t$ such that

$$\max_{0 \leq L \leq 2\bar{L}} F_K(s, \hat{K}_t, L) \leq \frac{1 - \epsilon_t(1 + \max_{0 \leq \tilde{K} \leq 2\bar{K}} F_L(s, \tilde{K}, \underline{L}))}{\epsilon_t}. \quad (46)$$

We show that ϵ_t defined as such satisfies Properties e1. and e2.

Indeed, since $\epsilon_t < \underline{\epsilon}_t$, (42) holds, i.e., $\underline{\epsilon}_t$ satisfies e1. Now we show that $\underline{\epsilon}_t$ satisfies e2. For all $\hat{s}, s \in \mathcal{S}$, and for all K such that

$$F_K(s, K, L) \geq \frac{1 - \epsilon_t(1 + \max_{0 \leq \tilde{K} \leq 2\bar{K}} F_L(s, \tilde{K}, \underline{L}))}{\epsilon_t},$$

for some $0 \leq L \leq 2\bar{L}$, by (46), we have $K \leq \hat{K}_t$. Because

$$\epsilon_t < \frac{1}{2\left(1 + \max_{0 \leq \tilde{K} \leq 2\bar{K}} F_L(s, \tilde{K}, \underline{L})\right)}$$

we have

$$\left(\underline{\beta}\pi_{\hat{s}s}((1-\delta)\epsilon_t + 1 - \epsilon_t(1 + \max_{0 \leq \tilde{K} \leq 2\bar{K}} F_L(s, \tilde{K}, \underline{L})))\right)^{\frac{1}{\sigma}} > \left(\underline{\beta}\pi_{\hat{s}s}\frac{1}{2}\right)^{\frac{1}{\sigma}}$$

So (45) yields (43).

By Assumption 5 ($\lim_{K \to 0} F(s, K, 2\bar{L}) = 0$), there exists $\underline{K}_t$ such that

$$0 < \underline{K}_t < \frac{(1-\delta)\underline{K}_{t-1}}{1 + \frac{2}{\epsilon_{t-1}}} \quad (47)$$

and for all $\hat{s}, s \in \mathcal{S}$, and $0 < K < \underline{K}_t$,

$$\left(\underline{\beta}\pi_{\hat{s}s}(1-\delta)\epsilon_t\right)^{\frac{1}{\sigma}} F(\hat{s}, \underline{K}_{t-1}, \underline{L}) > F(s, K, 2\bar{L}) + (1-\delta)K.$$

So we obtain (44). □

Lemma 7. *Consider the value and policy functions defined recursively by the Bellman equations,* (25)*. We have the following properties:*

1. The value functions $\hat{V}_{t,T}$ are continuous, strictly increasing, strictly concave.

2. The corresponding policy correspondence, $\hat{c}_{t,T}, \hat{k}_{t,T}$ are single-valued, i.e., are functions, continuous, and $\hat{k}_{t,T}$ are weakly increasing, and the budget constraints, (26)*, hold with equality.*

3.(Euler Equation) If $k' = \hat{k}_{t,T}(k, i; s^t) < \bar{k}$ then

$$u'\left(\hat{c}_{t,T}(\left(k, i; s^{t-1}, p^T, \Pi^T\right)\right) \geq \beta(i)\mathbb{E}\left[(1 - \delta + r_{t+1,T}(s^{t+1}))u'\left(\hat{c}_{t+1,T}\left(k', i'; s^{t+1}, p^T, \Pi^T\right)\right)\right]$$

with equality if $k' > 0$.

Proof. These properties are standard. The single-valued property of the policy function comes from the fact that $u(.)$ is strictly concave. The monotonicity of $\hat{k}_{t,T}$ comes from a standard-single crossing argument using the concavity of u. □

Lemma 8. *We show that the correspondence Ψ_ϵ constructed in Lemma 4 is upper hemi-continuous, and is non-empty, compact and convex valued.*

Proof. In order to show that Ψ_ϵ is upper hemi-continuous, we need to show that given any sequence $(p^n, \mu^n, x^n) \in \Delta_{\epsilon^T}^{\Sigma^T} \times \Omega^{\Sigma^T} \times \mathcal{K}_x$ that converges to some $(p, \mu, x) \in \Delta_{\epsilon^T}^{\Sigma^T} \times \Omega^{\Sigma^T} \times \mathcal{K}_x$:

$$(p^n, \mu^n, x^n) \to (p, \mu, x)$$

and

$$(\tilde{p}^n, \tilde{\mu}^n, \tilde{x}^n) \to (\tilde{p}, \tilde{\mu}, \tilde{x})$$

and

$$(\tilde{p}^n, \tilde{\mu}^n, \tilde{x}^n) \in \Psi_\epsilon (p^n, \mu^n, x^n)$$

then we must have:

$$(\tilde{p}, \tilde{\mu}, \tilde{x}) \in \Psi_\epsilon (p, \mu, x).$$

Indeed, since

$$(\tilde{p}^n, \tilde{\mu}^n, \tilde{x}^n) \in \Psi_\epsilon (p^n, \mu^n, x^n),$$

there exists

$$\left\{Y_{t,T}^n(s^t), K_{t,T}^n(s^t), L_{t,T}^n(s^t)\right\}_{t,s^t}$$

that solves (24). Let

$$\Pi_{t,T}^n = p_{t,T}^{c,n} Y_{t,T}^n - r_{t,T}^n K_{t,T}^n - w_{t,T}^n L_{t,T}^n,$$

and $\Pi^{n,T} = \left(\Pi_{t,T}^n(s^t)\right)_{t,s^t}$.

Let $\hat{V}_{t,T}^n(k, i; s^t, p^{T,n}, \Pi^{T,n})$ denote the value functions that solves (25) given $p^{T,n}$ and $\Pi^{T,n}$ and $\hat{k}_{t,T}^n(k, i; s^t, p^{T,n}, \Pi^{T,n})$ denote the corresponding policy functions.

By choosing a convergent subsequence, we can assume that there exists

$$\left\{Y_{t,T}(s^t), K_{t,T}(s^t), L_{t,T}(s^t)\right\}_{t,s^t}$$

such that

$$\left(Y_{t,T}^n(s^t), K_{t,T}^n(s^t), L_{t,T}^n(s^t)\right) \longrightarrow_{n\to\infty} \left(Y_{t,T}(s^t), K_{t,T}(s^t), L_{t,T}(s^t)\right)$$

for all t, s^t.

First, we show that for all t and s^t, $\left(Y_{t,T}(s^t), K_{t,T}(s^t), L_{t,T}(s^t)\right)$ solves (24) given p^T.

Indeed, for any (Y, K, L) such that

$$Y \leq F(s_t, K_{t,T}, L_{t,T})$$

and

$$0 \leq L \leq 2\bar{L} \quad \text{and} \quad 0 \leq K \leq 2\bar{K},$$

since $\left(Y_{t,T}^n(s^t), K_{t,T}^n(s^t), L_{t,T}^n(s^t)\right)$ solves (24), we have

$$p_{t,T}^{n,c} Y - r_{t,T}^n K - w_{t,T}^n L \leq p_{t,T}^{n,c} Y_{t,T}^n - r_{t,T}^n K_{t,T}^n - w_{t,T}^n L_{t,T}^n.$$

Taking $n \to \infty$, we obtain

$$p_{t,T}^c Y - r_{t,T} K - w_{t,T} L \leq p_{t,T}^c Y_{t,T} - r_{t,T} K_{t,T} - w_{t,T} L_{t,T}.$$

Therefore $\left(Y_{t,T}(s^t), K_{t,T}(s^t), L_{t,T}(s^t)\right)$ solves (24) given p^T.

In addition, from the expression for $\Pi_{t,T}^n(s^t)$ and $\Pi_{t,T}(s^t)$, we also have

$$\lim_{n\to\infty} \Pi_{t,T}^n(s^t) = \Pi_{t,T}(s^t).$$

Lemma 9 then shows that

$$\hat{k}^n_{t,T}(.,i;s^t,p^{n,T},\Pi^{T,n}) \longrightarrow_{n\to\infty} \hat{k}_{t,T}(.,i;s^t,p^T,\Pi^T)$$

uniformly over $[0,\bar{k}]$.

From the definition of ψ_x, we have (to simplify the notations, we omit the dependence on $p^{n,T},\Pi^{n,T},p^T,\Pi^T$, etc.):

$$\tilde{x}^c_{t,T;n}(s^t) = \int_{[0,\bar{k}]\times\mathcal{I}} \left(\hat{c}_{t,T;n}(k,i;s^t) + \hat{k}_{t,T;n}(k;i,s^t) - (1-\delta)k\right) d\mu^n_{t,T}(k,i;s^t) - Y^n_{t,T}(s^t)$$

$$\tilde{x}^k_{t,T;n}(s^t) = K^n_{t,T}(s^t) - \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu^n_{t,T}(k,i;s^t)$$

$$\tilde{x}^l_{t,T;n}(s^t) = L^n_{t,T}(s^t) - L(s_t).$$

As shown in Lemma 7,

$$\hat{c}_{t,T;n}(k,i;s^t) + \hat{k}_{t,T;n}(k;i,s^t) - (1-\delta)k = \frac{r^n_{t,T}(s^t)k + w^n_{t,T}(s^t)l(s_t,i_t) + \Pi^n_{t,T}(K^n_{t,T},L^n_{t,T})}{p^{n,c}_{t,T}(s^t)}.$$

Therefore

$$\begin{aligned}
\tilde{x}^c_{t,T;n}(s^t) &= \int_{[0,\bar{k}]\times\mathcal{I}} \left(\hat{c}_{t,T;n}(k,i;s^t) + \hat{k}_{t,T;n}(k;i,s^t) - (1-\delta)k\right) d\mu^n_{t,T}(k,i;s^t) - Y^n_{t,T}(s^t) \\
&= \int_{[0,\bar{k}]\times\mathcal{I}} \left(\frac{r^n_{t,T}(s^t)k + w^n_{t,T}(s^t)l(s_t,i_t) + \Pi^n_{t,T}(K^n_{t,T},L^n_{t,T})}{p^{n,c}_{t,T}(s^t)}\right) d\mu^n_{t,T}(k,i;s^t) - Y^n_{t,T}(s^t) \\
&= \frac{r^n_{t,T}(s^t)}{p^{n,c}_{t,T}(s^t)} \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu^n_{t,T}(k,i;s^t) + \frac{r^n_{t,T}(s^t)}{p^{n,c}_{t,T}(s^t)} \int_{[0,\bar{k}]\times\mathcal{I}} l(s_t,i_t) d\mu^n_{t,T}(k,i;s^t) \\
&+ \frac{\Pi^n_{t,T}(K^n_{t,T},L^n_{t,T})}{p^{n,c}_{t,T}(s^t)} - Y^n_{t,T}(s^t).
\end{aligned}$$

Because $p^n \to p$, and $p^{n,c}_{t,T}, p^c_{t,T} > \epsilon_t > 0$, $\frac{r^n_{t,T}(s^t)}{p^{n,c}_{t,T}(s^t)} \to \frac{r_{t,T}(s^t)}{p^c_{t,T}(s^t)}$ and $\frac{r^n_{t,T}(s^t)}{p^{n,c}_{t,T}(s^t)} \to \frac{r_{t,T}(s^t)}{p^c_{t,T}(s^t)}$. In addition, as we show above, $\frac{\Pi^n_{t,T}(K^n_{t,T},L^n_{t,T})}{p^{n,c}_{t,T}(s^t)} \to \frac{\Pi_{t,T}(K_{t,T},L_{t,T})}{p^c_{t,T}(s^t)}$ and $Y^n_{t,T}(s^t) \to Y_{t,T}(s^t)$.

Because $\mu^n \to \mu$,

$$\int_{[0,\bar{k}]\times\mathcal{I}} k d\mu^n_{t,T}(k,i;s^t) \to \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t,T}(k,i;s^t)$$

and

$$\int_{[0,\bar{k}]\times\mathcal{I}} l(s_t,i_t) d\mu^n_{t,T}(k,i;s^t) \to \int_{[0,\bar{k}]\times\mathcal{I}} l(s_t,i_t) d\mu_{t,T}(k,i;s^t).$$

Therefore, for all t and s^t, we have:

$$\begin{aligned}\tilde{x}^c_{t,T;n}(s^t) &\to \frac{r_{t,T}(s^t)}{p^c_{t,T}(s^t)}\int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t,T}(k,i;s^t) + \frac{r_{t,T}(s^t)}{p^c_{t,T}(s^t)}\int_{[0,\bar{k}]\times\mathcal{I}} l(s_t,i_t) d\mu_{t,T}(k,i;s^t) \\ &\quad + \frac{\Pi_{t,T}(K_{t,T},L_{t,T})}{p^c_{t,T}(s^t)} - Y_{t,T}(s^t) \\ &= \int_{[0,\bar{k}]\times\mathcal{I}} \left(\hat{c}_{t,T}(k,i;s^t) + \hat{k}_{t,T}(k;i,s^t) - (1-\delta)k\right) d\mu_{t,T}(k,i;s^t) - Y_{t,T}(s^t).\end{aligned}$$

In addition, we also have $\tilde{x}^c_{t,T;n}(s^t) \to \tilde{x}^c_{t,T}(s^t)$. Therefore, for all t and s^t:

$$\tilde{x}^c_{t,T} = \int_{[0,\bar{k}]\times\mathcal{I}} \left(\hat{c}_{t,T}(k,i;s^t) + \hat{k}_{t,T}(k;i,s^t) - (1-\delta)k\right) d\mu_{t,T}(k,i;s^t) - Y_{t,T}(s^t)$$

Similarly, we can also show that, for all t and s^t:

$$\begin{aligned}\tilde{x}^k_{t,T}(s^t) &= K_{t,T}(s^t) - \int_{[0,\bar{k}]\times\mathcal{I}} k d\mu_{t,T}(k,i;s^t) \\ \tilde{x}^l_{t,T}(s^t) &= L_{t,T}(s^t) - L(s_t).\end{aligned}$$

Therefore

$$\tilde{x} \in \psi_x(p,\mu,x).$$

Following the same steps, it is also easy to show that $\tilde{p} \in \psi_p(p,\mu,x)$. Now we show that $\tilde{\mu} \in \psi_\mu(p,\mu,x)$.

Indeed, from the definition of ψ_μ, for every $A \in \mathcal{B}\left([0,\bar{k}]\right)$

$$\tilde{\mu}_{t+1,T;n}(A,i_{t+1};s^{t+1}) = \sum_{i_t\in\mathcal{I}} \Pr(i_{t+1}|i_t,s_t,s_{t+1})\mu_{t,T;n}\left(\left(\hat{k}_{t,T;n}\right)^{-1}(A),i_t;s^t\right). \quad (48)$$

We need to show that, for every $A \in \mathcal{B}\left([0,\bar{k}]\right)$

$$\tilde{\mu}_{t+1,T}(A,i_{t+1};s^{t+1}) = \sum_{i_t\in\mathcal{I}} \Pr(i_{t+1}|i_t,s_t,s_{t+1})\mu_{t,T}\left(\left(\hat{k}_{t,T;n}\right)^{-1}(A),i_t;s^t\right).$$

From the construction of the Kantorovich-Rubinstein norm for the space of measures in Bogachev (2000, Section 8.3), to show the identity, we just need to show that for all $\varphi \in Lip_1\left([0,\bar{k}]\right)$,

$$\int_0^{\bar{k}} \varphi(k)\tilde{\mu}_{t+1,T}(dk,i_{t+1};s^{t+1}) = \sum_{i_t\in\mathcal{I}} \Pr(i_{t+1}|i_t,s_t,s_{t+1}) \int_0^{\bar{k}} \varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T}(dk,i_t;s^t) \quad (49)$$

From (48), we have

$$\int_0^{\bar{k}} \varphi(k)\tilde{\mu}_{t+1,T;n}(dk,i_{t+1};s^{t+1}) = \sum_{i_t\in\mathcal{I}} \Pr(i_{t+1}|i_t,s_t,s_{t+1}) \int_0^{\bar{k}} \varphi\left(\hat{k}_{t,T;n}(k,i;s^t)\right)\mu_{t,T;n}(dk,i_t;s^t).$$

Since $\tilde{\mu}^n \to \tilde{\mu}$,

$$\lim_{n\to\infty}\int_0^{\bar{k}} \varphi(k)\tilde{\mu}_{t+1,T;n}(dk,i_{t+1};s^{t+1}) = \int_0^{\bar{k}} \varphi(k)\tilde{\mu}_{t+1,T}(dk,i_{t+1};s^{t+1}).$$

Therefore, to establish (49), we just need to show that:

$$\lim_{n\to\infty}\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T;n}(k,i;s^t)\right)\mu_{t,T;n}(dk,i_t;s^t)=\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T}(dk,i_t;s^t). \tag{50}$$

Indeed,

$$\begin{aligned}
&\left|\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T;n}(k,i;s^t)\right)\mu_{t,T;n}(dk,i_t;s^t)-\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T}(dk,i_t;s^t)\right|\\
&=\left|\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T;n}(k,i;s^t)\right)\mu_{t,T;n}(dk,i_t;s^t)-\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T;n}(dk,i_t;s^t)\right.\\
&\left.+\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T;n}(dk,i_t;s^t)-\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T}(dk,i_t;s^t)\right|\\
&\le\int_0^{\bar{k}}\left|\varphi\left(\hat{k}_{t,T;n}(k,i;s^t)\right)-\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\right|\mu_{t,T;n}(dk,i_t;s^t)\\
&+\left|\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T;n}(dk,i_t;s^t)-\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T}(dk,i_t;s^t)\right|.
\end{aligned}\tag{51}$$

We first show that

$$\lim_{n\to\infty}\int_0^{\bar{k}}\left|\varphi\left(\hat{k}_{t,T;n}(k,i;s^t)\right)-\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\right|\mu_{t,T;n}(dk,i_t;s^t)=0. \tag{52}$$

Indeed, because $\varphi\in Lip_1\left(\left[0,\bar{k}\right]\right)$,

$$\begin{aligned}
&\int_0^{\bar{k}}\left|\varphi\left(\hat{k}_{t,T;n}(k,i;s^t)\right)-\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\right|\mu_{t,T;n}(dk,i_t;s^t)\\
&\le\int_0^{\bar{k}}\left|\hat{k}_{t,T;n}(k,i;s^t)-\hat{k}_{t,T}(k,i;s^t)\right|\mu_{t,T;n}(dk.i_t;s^t)\\
&\le\sup_{0\le k\le\bar{k}}\left|\hat{k}_{t,T;n}(k,i;s^t)-\hat{k}_{t,T}(k,i;s^t)\right|\mu_{t,T;n}(\left[0,\bar{k}\right],i_t;s^t)
\end{aligned}$$

We show in Lemma 9, that $\hat{k}_{t,T;n}\to\hat{k}_{t,T}$ uniformly, therefore

$$\lim_{n\to 0}\sup_{0\le k\le\bar{k}}\left|\hat{k}_{t,T;n}(k,i;s^t)-\hat{k}_{t,T}(k,i;s^t)\right|=0,$$

In addition, since $\mu_{t,T;n}(\left[0,\bar{k}\right],i_t;s^t)\le\sum_{i\in\mathcal{I}}\mu_{t,T;n}(\left[0,\bar{k}\right],i;s^t)=1$. These two results imply (52).

Because $\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)$ is continuous, we also have:

$$\lim_{n\to\infty}\left|\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T;n}(dk,i_t;s^t)-\int_0^{\bar{k}}\varphi\left(\hat{k}_{t,T}(k,i;s^t)\right)\mu_{t,T}(dk,i_t;s^t)\right|=0.$$

Combining this limit with (52), and (51), we arrive at (50). As argued above, this implies $\tilde{\mu}\in\phi_\mu(p,\mu,x)$.

We have just established that $(\tilde{p},\tilde{\mu},\tilde{x})\in\Psi_\epsilon\left(p,\mu,x\right)$, i.e., Ψ_ϵ is upper hemi-continuous.

It is standard to show that Ψ_ϵ is compact and convex valued. The proof is facilitated by the fact that if

$$(\tilde{p}_1, \tilde{\mu}_1, \tilde{x}_1) \in \Psi_\epsilon (p, \mu, x)$$

and

$$(\tilde{p}_2, \tilde{\mu}_1, \tilde{x}_2) \in \Psi_\epsilon (p, \mu, x)$$

then $\Pi^1_{t,T} = \Pi^2_{t,T}$ for all t and s^t. Therefore by Lemma 7, $\hat{k}^1_{t,T} \equiv \hat{k}^2_{t,T}$ for all t and s^t. So $\tilde{\mu}_1 \equiv \tilde{\mu}_2$, i.e., $\psi_\mu(p, \mu, x)$ is single-valued. □

Lemma 9. *Assume that $p^{n,T} \longrightarrow_{n\to\infty} p^T$ and $\Pi^n_{t,T} \longrightarrow_{n\to\infty} \Pi_{t,T}$. In addition, $\hat{V}^n_{t,T}$ solves (25), given $p^{n,T}$ and $\Pi^{n,T}$with the corresponding policy function $\hat{k}^n_{t,T}$ and $\hat{V}_{t,T}$ solves (25) given p^T and $\Pi_{t,T}$ with the corresponding $\hat{k}_{t,T}$. Then, for all $s^t \in \mathcal{S}^t$ and $i \in \mathcal{I}$ and $k \in [0, \bar{k}]$, we have*

$$\hat{V}^n_{t,T}(., i; s^t, p^{n,T}, \Pi^{n,T}) \longrightarrow_{n\to\infty} \hat{V}_{t,T}(., i; s^t, p^T, \Pi^T)$$

pointwise, and

$$\hat{k}^n_{t,T}(., i; s^t, p^{n,T}, \Pi^{n,T}) \longrightarrow_{n\to\infty} \hat{k}_{t,T}(., i; s^t, p^T, \Pi^T)$$

uniformly over $[0, \bar{k}]$.

Proof. We show the results stated in the lemma by induction backward from $t = T + 1$.

1. At $t = T$, the result is obvious since

$$\hat{V}^n_{T,T}(k, i; s^T, p^{n,T}, \Pi^{n,T}) = u\left(\frac{r^n_{T,T}(s^T)k + w^n_{T,T}(s^T)l(s_T, i) + \Pi^n_{T,T} + (1-\delta)k}{p^{n,c}_{T,T}(s^T)}\right)$$

and

$$\hat{V}_{T,T}(k, i; s^T, p^T, \Pi^T) = u\left(\frac{r_{T,T}(s^T)k + w_{T,T}(s^T)l(s_T, i) + \Pi_{T,T} + (1-\delta)k}{p^c_{T,T}(s^T)}\right)$$

and $\hat{k}^n_{T,T} \equiv 0$ and $\hat{k}_{T,T} \equiv 0$.

2. Assume that the results in the current lemma hold for $t + 1 \leq T$, we show that they also hold for t.

Indeed, given $s^t \in \mathcal{S}$, $i \in \mathcal{I}$ and $k \geq 0$, we first show that

$$\liminf_{n\to\infty} \hat{V}^n_{t,T}(k, i; s^t, p^{n,T}, \Pi^{n,T}) \geq \hat{V}_{t,T}(k, i; s^t, p^T, \Pi^T).$$

This is immediate if the right hand side is $-\infty$, which happens if and only if

$$\Pi_{t,T}(s^t) = w_{t,T}(s^t) = k = 0.$$

Now if the right hand side is finite, for any $\nu > 0$, there exists $c \geq 0$ and $k' \in [0, \bar{k}]$ such that

$$p^c_{t,T}(s^t)\left(c + k' - (1-\delta)k\right) < r_{t,T}(s^t)k + w_{t,T}(s^t)l(s_t, i_t) + \Pi_{t,T}(s^t),$$

and

$$\hat{V}_{t,T}(k, i; s^t, p^T, \Pi^T) \leq u(c) + \beta(i)\mathbb{E}_t\left[\hat{V}^n_{t+1,T}(k', i; s^t, p^{n,T}, \Pi^{n,T})\right] + \nu$$

Because

$$\left(p^n_{t,T}, \Pi^n_{t,T}\right) \to \left(p_{t,T}, \Pi_{t,T}\right),$$

there exists N such that for all $n \geq N$

$$p^{n,c}_{t,T}(s^t)\left(c + k' - (1-\delta)k\right) \leq r^n_{t,T}(s^t)k + w^n_{t,T}(s^t)l(s_t, i_t) + \Pi^n_{t,T}(s^t).$$

Therefore,

$$\hat{V}^n_{t,T}(k, i; s^t, p^T, \Pi^{n,T}) \geq u(c) + \beta(i)\mathbb{E}_t\left[\hat{V}^n_{t+1,T}(k', i; s^t, p^{n,T}, \Pi^{n,T})\right]$$

and since $\hat{V}^n_{t+1,t}(k', i) \to \hat{V}_{t+1,T}(k', i)$ by the the induction assumption,

$$\begin{aligned}\liminf_{n\to\infty} \hat{V}^n_{t,T}(k, i; s^t, p^{n,T}, \Pi^{n,T}) &\geq u(c) + \beta(i)\mathbb{E}_t\left[\hat{V}_{t+1,T}(k', i; s^t, p^{n,T}, \Pi^{n,T})\right] \\ &\geq \hat{V}_{t,T}(k, i; s^t, p^T, \Pi^T) - \nu.\end{aligned}$$

Therefore,

$$\liminf_{n\to\infty} \hat{V}^n_{t,T}(k, i; s^t, p^{n,T}) \geq \hat{V}_{t,T}(k, i; s^t, p^T). \tag{53}$$

We show by contradiction that

$$\limsup_{n\to\infty} \hat{V}^n_{t,T}(k, i; s^t, p^{n,T}) \leq \hat{V}_{t,T}(k, i; s^t, p^T). \tag{54}$$

Case 1: $\hat{V}_{t,T}(k, i; s^t, p^T) > -\infty$. Assume to the contrary that there exists $\nu > 0$ and a subsequence $n_m \to \infty$ such that

$$\hat{V}^{n_m}_{t,T}(k, i; s^t, p^{n,T}) > \hat{V}_{t,T}(k, i; s^t, p^T) + \nu. \tag{55}$$

By the definition of V^{n_m}, there exists $c^{n_m} \geq 0$, and $k'^{n_m} \in \left[0, \bar{k}\right]$, such that

$$p^{n_m,c}_{t,T}(s^t)\left(c^{n_m} + k'^{n_m} - (1-\delta)k\right) \leq r^{n_m}_{t,T}(s^t)k + w^{n_m}_{t,T}(s^t)l(s_t, i_t) + \Pi^{n_m}_{t,T}, \tag{56}$$

and

$$\hat{V}^{n_m}_{t,T}(k, i; s^t, p^T) = u(c^{n_m}) + \beta(i)\mathbb{E}_t\left[\hat{V}^{n_m}_{t+1,T}(k'^{n_m}, i; s^t, p^{n_m,T})\right].$$

By choosing subsequences, we can assume that $c^{n_m} \to c^*$ and $k'^{n_m} \to k^*$ for some $c^* \geq 0$ and $k^* \in \left[0, \bar{k}\right]$. From (56), and because

$$\left(p^n_{t,T}, \Pi^n_{t,T}\right) \to \left(p_{t,T}, \Pi_{t,T}\right),$$

we have

$$p^c_{t,T}(s^t)\left(c^* + k^* - (1-\delta)k\right) \leq r_{t,T}(s^t)k + w_{t,T}(s^t)l(s_t, i_t) + \Pi_{t,T}.$$

Since $\hat{V}^{n_m}_{t+1,T} \to \hat{V}_{t+1,T}$ pointwise and $\hat{V}_{t+1,T}$ is continuous and increasing in k, we obtain[20]

$$\limsup_{m\to\infty} \hat{V}^{n_m}_{t+1,T}(k'^{n_m}, i; s^t, p^{n_m,T}) \leq \hat{V}_{t+1,T}(k^*, i; s^t, p^T).$$

Consequently,

$$\limsup_{m\to\infty} \hat{V}^{n_m}_{t,T}(k, i; s^t, p^T) \leq u(c^*) + \beta(i)\mathbb{E}_t\left[\hat{V}_{t+1,T}(k^*, i; s^t, p^T)\right] \leq \hat{V}_{t,T}\left(k, i; s^t, p^T\right).$$

[20]For any $0 \leq k^* < \tilde{k}$,

$$\begin{aligned}\limsup_{m\to\infty} \hat{V}^{n_m}_{t+1,T}(k'^{n_m}, i; s^t, p^{n_m,T}) &\leq \lim_{m\to\infty} \hat{V}^{n_m}_{t+1,T}(\tilde{k}, i; s^t, p^{n_m,T}) \\ &= \hat{V}_{t+1,T}(\tilde{k}, i; s^t, p^T).\end{aligned}$$

Taking the limit $\tilde{k}$ to k^*, we obtain the desired inequality.

This contradicts (55). So we obtain (54) by contradiction.

Case 2: $\hat{V}_{t,T}(k, i; s^t, p^T) = -\infty$. Then

$$\Pi_{t,T}(s^t) = w_{t,T}(s^t) = k = 0.$$

From the budget's constraint for $\hat{V}^n_{t,T}$, we have

$$\hat{V}^n_{t,T}(0, i; s^t, p^T, \Pi^{n,T}) \leq u\left(\frac{w^n_{t,T}(s^t)l(s_t, i_t) + \Pi^n_{t,T}(s^t)}{p^{n,c}_{t,T}(s^t)}\right) + \beta(i)\mathbb{E}_t\left[\hat{V}^n_{t+1,T}(\bar{k}, i; s^t, p^{n,T}, \Pi^{n,T})\right].$$

Now,

$$\lim_{n\to\infty} w^n_{t,T}(s^t) = w_{t,T}(s^t) = 0$$
$$\lim_{n\to\infty} \Pi^n_{t,T}(s^t) = \Pi_{t,T}(s^t) = 0,$$

and $p^{n,c}_{t,T}(s^t) > \epsilon_t > 0$, and $u(0) = -\infty$. Therefore,

$$\lim_{n\to\infty} u\left(\frac{w^n_{t,T}(s^t)l(s_t, i_t) + \Pi^n_{t,T}(s^t)}{p^{n,c}_{t,T}(s^t)}\right) = -\infty.$$

In addition, $\hat{V}^n_{t+1,T}(\bar{k}, i; s^t, p^{n,T}, \Pi^{n,T})$ is finite. So

$$\limsup_{n\to\infty} \hat{V}^n_{t,T}(0, i; s^t, p^T, \Pi^{n,T}) = -\infty = \hat{V}_{t,T}(k, i; s^t, p^T).$$

We have shown that, in either case, we obtain (54). Combining this inequality, with (53), we finally get the desired limit

$$\lim_{n\to\infty} \hat{V}^n_{t,T}(k, i; s^t, p^{n,T}, \Pi^{n,T}) = \hat{V}_{t,T}(k, i; s^t, p^T, \Pi^T).$$

Given $k \in [0, \bar{k}]$, we also show by contradiction that

$$\lim_{n\to\infty} \hat{k}^n_{t,T}(k, i; s^t, p^{n,T}, \Pi^{n,T}) = \hat{k}_{t,T}(k, i; s^t, p^T, \Pi^T).$$

Assume to the contrary. Then, there exists a subsequence $\{n_m\}$ such that

$$k'^{n_m} = \hat{k}^n_{t,T}(k, i; s^t, p^{n,T}, \Pi^{n,T}) \to k^*$$

for some $k^* \in [0, \bar{k}]$ and $k^* \neq \hat{k}_{t,T}(k, i; s^t, p^T, \Pi^T)$. Let c^{n_m} be defined such that the budget constraint, (56) holds with equality. Taking, further subsequence if necessary, we can assume that $c^{n_m} \to c^*$ for some c^*. As shown above (c^*, k^*) must satisfy the budget constraint at p^T, Π^T, and

$$\hat{V}_{t,T}\left(k, i; s^t, p^T, \Pi^T\right) = \lim_{m\to\infty} \hat{V}^{n_m}_{t,T}(k, i; s^t, p^T, \Pi^{n,T}) = u(c^*) + \beta(i)\mathbb{E}_t\left[\hat{V}_{t+1,T}(k^*, i; s^t, p^T, \Pi^T)\right].$$

Therefore, $k^* = \hat{k}_{t,T}(k, i; s^t, p^T, \Pi^T)$ (by Lemma 7 the maximizer is unique). This is a contradiction.

So we have established the pointwise convergence of $\hat{k}^n_{t,T}$ to $\hat{k}_{t,T}$. Because $\hat{k}_{t,T}$ and $\hat{k}^n_{t,T}$ are increasing and continuous (by Lemma 7), the convergence is uniform. □

Lemma 10. *Θ, g defined in Theorem 1 satisfy:*

1. *Θ is sequentially compact.*

2. *g is a closed-valued correspondence.*

Proof. Proof of Part 1: We endow the space of increasing function with pointwise convergence topology and we endow the space of V function with the sup norm topology.

$\mathcal{MF}$ denote the space of monotone functions from $[0, \bar{k}]$ to $[0, \bar{k}]$. We endow $\mathcal{MF}$ with the topology of pointwise-convergence. Then, by Helly's selection theorem, $\mathcal{MF}$ is sequentially compact.[21]

$\mathcal{BL}_1$ denote the space of Lipschitz continuous function with the Lipschitz constant l_V, defined in (61), and bounded below by $\underline{V}$ and bounded above by $\overline{V}$. We endow $\mathcal{BL}_1$ with the topology of convergence in sup norm. Then, by Ascoli-Arzela theorem, $\mathcal{BL}_1$ is sequentially compact.

Proof of Part 2: We need to show that $g(s, \mu)$ is closed for all $s \in \mathcal{S}$ and $\mu \in \Omega$. That is, for any sequence $\left(\theta^n, (\theta^n_{s'})_{s' \in \mathcal{S}}\right)_{n=0}^{\infty} \in g(s, \mu)$ such that

$$\left(\theta^n, (\theta^n_{s'})_{s' \in \mathcal{S}}\right)_{n=0}^{\infty} \longrightarrow_{n \to \infty} \left(\theta, (\theta_{s'})_{s' \in \mathcal{S}}\right)$$

then $\left(\theta, (\theta_{s'})_{s' \in \mathcal{S}}\right) \in g(s, \mu)$.

By the definition of convergence (topology) in different spaces, we have $\hat{k}^n \to \hat{k}$ (pointwise convergence) and $\hat{V}^n \to \hat{V}$ and $\hat{V}^{+n}_{s'} \to \hat{V}$ (convergence in sup norm).

First, since $\left(\theta^n, (\theta^n_{s'})_{s' \in \mathcal{S}}\right)_{n=0}^{\infty} \in g(s, \mu)$, we have

$$\hat{V}^n(k, i) \geq u\left((1 - \delta + r^n)k + w^n l(s, i) - k'\right) + \beta(i) \sum_{i', s'} \pi_{ss', ii'} \hat{V}^{+n}_{s'}(k', i')$$

for each $k' \in [0, \bar{k}]$. Taking the limit $n \to \infty$, we have

$$\hat{V}(k, i) \geq u\left((1 - \delta + r)k + wl(s, i) - k'\right) + \beta(i) \sum_{i', s'} \pi_{ss', ii'} \hat{V}^{+}_{s'}(k', i')$$

for all $k' \in [0, k]$. Therefore

$$\hat{V}(k, i) \geq \max_{k' \in [0, \bar{k}]} u\left((1 - \delta + r)k + wl(s, i) - k'\right) + \beta(i) \sum_{i', s'} \pi_{ss', ii'} \hat{V}^{+}_{s'}(k', i').$$

Now, since $\left(\theta^n, (\theta^n_{s'})_{s' \in \mathcal{S}}\right)_{n=0}^{\infty} \in g(s, \mu)$:

$$\hat{V}^n(k, i) = u\left((1 - \delta + r^n)k + w^n l(s, i) - \hat{k}^n(k, i)\right) + \beta(i) \sum_{i', s'} \pi_{ss', ii'} \hat{V}^{+n}_{s'}(\hat{k}^n(k, i), i').$$

We show that

$$\lim_{n \to \infty} \hat{V}^{+n}_{s'}(\hat{k}^n(k, i), i') = \hat{V}^{+}_{s'}\left(\hat{k}(k, i), i'\right).$$

Indeed

$$\begin{aligned} \left|\hat{V}^{+n}_{s'}(\hat{k}^n(k, i), i') - \hat{V}^{+}_{s'}\left(\hat{k}(k, i), i'\right)\right| &\leq \left|\hat{V}^{+n}_{s'}(\hat{k}^n(k, i), i') - \hat{V}^{+n}_{s'}(\hat{k}(k, i), i')\right| \\ &+ \left|\hat{V}^{+n}_{s'}(\hat{k}(k, i), i') - \hat{V}^{+}_{s'}\left(\hat{k}(k, i), i'\right)\right|. \end{aligned}$$

[21] See Exercise 7.13 in Rudin (1976) for an elementary proof.

The first term goes to zero because $\hat{V}_{s'}^{+n}$ is Lipschitz continuous and $\hat{k}^n$ converges pointwise to $\hat{k}$ and the second term goes to 0 because of the pointwise convergence of $\hat{V}_{s'}^{+}$ to $\hat{V}_{s'}^{+}$.

Therefore

$$\hat{V}(k,i) = u\left((1-\delta+r)k + wl(s,i) - \hat{k}(k,i)\right) + \beta(i)\sum_{i',s'}\pi_{ss',ii'}\hat{V}_{s'}^{+}(\hat{k}(k,i),i').$$

So

$$\hat{V}(k,i) = \max_{k'\in[0,\bar{k}]} u\left((1-\delta+r)k + wl(s,i) - k'\right) + \beta(i)\sum_{i',s'}\pi_{ss',ii'}\hat{V}_{s'}^{+}(k',i'),$$

and $\hat{k}(k,i)$ is a maximizer. □

Now we find bounds for the endogenous variables.

Lemma 11. *There exist $0 < \underline{K} < K_0$, $0 < \underline{r} < \bar{r}$ and $0 < \underline{w} < \bar{w}$, and $\underline{V} < \overline{V}$ and $l_V, l_k > 0$, such that in competitive equilibrium in the finite horizon economy, starting with an initial wealth distribution $\mu_0(k,i)$ and*

$$K_0 = \int_{[0,\bar{k}]\times\mathcal{I}} d\mu_0(k,i)$$

we have, for all $t \le T$ and $s^t \in \mathcal{S}^t$:

1. *$K_{t,T}(s^t) \ge \underline{K}$*
2. *$r_{t,T}(s^t) \in [\underline{r},\bar{r}]$ and $w_{t,T}(s^t) \in [\underline{w},\bar{w}]$*
3. *$\hat{V}_{t,T}(k,i;s^t) \in [\underline{V},\bar{V}]$*
4. *$0 \le \hat{V}'_{t,T}(k,i;s^t) \le l_V$ and $0 \le \hat{k}'_{t,T}(k,i;s^t) \le l_k$.*[22]

Proof. By Assumption 6, there exists $\underline{K} < \min\left\{K_0, \frac{L}{2}\right\}$ such that:

1.There exists $\gamma > 0$, such that, for all $K \le \underline{K}$,

$$\frac{F(s',K,L(s'))}{F(s,K,L(s))} < \gamma,$$

for all $s, s' \in \mathcal{S}$.

2. For all $K \le \underline{K}$,

$$F_K(s,K,L(s)) > \max\left\{1, \frac{\left(\gamma^{\frac{2(2-\delta)}{\underline{\alpha}}}\right)^{\sigma}}{\underline{\beta}\min_{s,s'}\pi_{ss'}}\right\} \tag{57}$$

for all $s \in \mathcal{S}$.

We show that if for some t and $s^t \in \mathcal{S}^t$, $K_{t,T}(s^t) \ge \underline{K}$ then $K_{t+1,T}(s^t,s) \ge \underline{K}$ for all $s \in \mathcal{S}$.

Assume to the contrary that $K_{t+1,T}\left(s^{t+1}\right) < \underline{K}$. We will show that this leads to a contradiction.

To simplify the exposition, we use the notation $z_{t,T}(k,i)$ as shorthand for $z_{t,T}(k,i;s^t)$, where $z_{t,T}$ can be the value, policy, or pricing functions, $\hat{V}_{t,T}$ or $\hat{c}_{t,T}, \hat{k}_{t,T}$ or $r_{t,T}, w_{t,T}$. In a

[22] $\hat{V}$ and $\hat{k}$ might not be differentiable everywhere because of the borrowing constraint, $0 \le k' \le \bar{k}$. In this case we can use the concept of generalized derivatives and the associated Envelope Theorems in Milgrom and Segal (2002).

competitive equilibrium, $L_{t,T}(s^t) = L(s_t) = \sum_{i\in\mathcal{I}} m(i,s_t)l(i,s_t)$, so we write L_t instead of $L_{t,T}$.

From the first order condition, if $\hat{k}_{t,T}(k,i) < \bar{k}$ then

$$u'(\hat{c}_{t,T}(i,k)) \geq \beta(i)\mathbb{E}_t\left[(1-\delta+F_K(s_{t+1},K_{t+1,T},L_{t+1}))u'(\hat{c}_{t+1,T}(i,\hat{k}_{t,T}(i,k)))\right].$$

Therefore, since $K_{t+1,T} < \underline{K}$, $F_K(s_{t+1},K_{t+1,T},L_{t+1}) > F_K(s_{t+1},\underline{K},L_{t+1})$ and the last inequality implies:

$$\begin{aligned} u'(\hat{c}_{t,T}(i,k)) &\geq \left\{\min_{s\in\mathcal{S}}(1-\delta+F_K(\underline{K},L(s),s))\right\}\beta(i)\pi_{s_ts_{t+1}}\sum_{i_{t+1}}\frac{\pi_{s_ts_{t+1},ii_{t+1}}}{\pi_{s_ts_{t+1}}}u'\left(\hat{c}_{t+1,T}(\hat{k}_{t,T}(i,k),i_{t+1})\right) \\ &\geq \left\{\min_{s\in\mathcal{S}}(1-\delta+F_K(\underline{K},L(s),s))\right\}\underline{\beta}\pi_{s_ts_{t+1}}u'\left(\sum_{i_{t+1}}\frac{\pi_{s_ts_{t+1},ii_{t+1}}}{\pi_{s_ts_{t+1}}}\hat{c}_{t+1,T}(\hat{k}_{t,T}(k,i),i_{t+1})\right) \end{aligned}$$

where the last inequality comes from the fact that $u'(c) = c^{-\sigma}$ is strictly convex.

Consequently,

$$\left(\left\{\min_{s\in\mathcal{S}}(1-\delta+F_K(s,\underline{K},L(s)))\right\}\underline{\beta}\pi_{s_ts_{t+1}}\right)^{\frac{1}{\sigma}} \leq \frac{\sum_{i_{t+1}}\frac{\pi_{s_ts_{t+1},ii_{t+1}}}{\pi_{s_ts_{t+1}}}\hat{c}_{t+1,T}(\hat{k}_{t,T}(k,i),i_{t+1})}{\hat{c}_{t,T}(k,i_t)}.$$

Therefore, using the basic result in footnote 19,

$$\begin{aligned} &\left(\left\{\min_{s\in\mathcal{S}}(1-\delta+F_K(\underline{K},L(s),s))\right\}\underline{\beta}\pi_{s_ts_{t+1}}\right)^{\frac{1}{\sigma}} \\ &\leq \frac{\sum_{i_t,i_{t+1}}\frac{\pi_{s_ts_{t+1},i_ti_{t+1}}}{\pi_{s_ts_{t+1}}}\int_{\hat{k}_{t,T}<\bar{k}}\hat{c}_{t+1,T}(\hat{k}_{t+1,T}(k,i_{t+1}),i_{t+1})\mu_{t,T}(dk,i_t)}{\sum_{i_t}\int_{\hat{k}_{t,T}<\bar{k}}\hat{c}_{t,T}(k,i_t)\mu_{t,T}(dk,i_t)} \end{aligned} \tag{58}$$

Now, we show that this would lead to a contradiction.

Indeed, for k such that $\hat{k}_{t,T}(k,i;s^t) = \bar{k}$ we have

$$\begin{aligned} &\hat{c}_{t,T} + \bar{k} - (1-\delta)k \\ &= r_{t,T}k + w_{t,T}l(s_t,i) \\ &\leq F_K(s_t,K_{t,T},L_t)\bar{k} + F_L(s_t,K_{t,T},L_t)\bar{l}. \end{aligned}$$

or

$$\hat{c}_{t,T} \leq (-\delta+F_K(s_t,K_{t,T},L_t))\bar{k} + F_L(s_t,K_{t,T},L_t)\bar{l}. \tag{59}$$

In addition,

$$\underline{K} > K_{t+1,T} \geq \sum_i\int_{\hat{k}_{t,T}=\bar{k}}\hat{k}_{t,T}(k,i)\mu_{t,T}(dk,i)$$

Therefore

$$\frac{K_{t+t,T}}{\bar{k}} > \sum_i\int_{\hat{k}_{t,T}=\bar{k}}\mu_{t,T}(dk,i).$$

Combining this inequality with (59), we obtain

$$
\begin{aligned}
&\int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i)d\mu_{t,T}(k,i) \\
&< K_{t+1,T}(-\delta + F_K(s_t, K_{t,T}, L_{t,T})) + K_{t+1,T}F_L(s_t, K_{t,T}, L_{t,T})\frac{\bar{l}}{\bar{k}} \\
&< K_{t+1,T}(-\delta + F_K(s_t, K_{t,T}, L_{t,T})) + K_{t+1,T}F_L(s_t, K_{t,T}, L_{t,T}) \\
&< K_{t,T}F_K(s_t, K_{t,T}, L_{t,T}) - \delta K_{t+1,T} + \frac{1}{2}L_{t,T}F_L(s_t, K_{t,T}, L_{t,T})) \qquad (60)
\end{aligned}
$$

where the second inequality comes from (5), which implies $\bar{k} > \bar{l}$ and the last inequality comes from $K_{t+1,T} < \underline{K} < K_{t,T}$ and $\underline{K} < \frac{L}{2}$

Therefore,

$$
\begin{aligned}
&\int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i)d\mu_{t,T}(k,i) \\
&= \int_{[0,\bar{k}]\times\mathcal{I}} \hat{c}_{t,T}(k,i)d\mu_{t,T}(k,i) - \int_{[0,\bar{k}]\times\mathcal{I},\hat{k}_{t,T}=\bar{k}} \hat{c}_{t,T}(k,i)d\mu_{t,T}(k,i) \\
&= Y_{t,T} + (1-\delta)K_{t,T} - K_{t+1,T} - \int_{[0,\bar{k}]\times\mathcal{I},\hat{k}_{t,T}=\bar{k}} \hat{c}_{t,T}(k,i)d\mu_{t,T}(k,i) \\
&= K_{t,T}F_K(s_t, K_{t,T}, L_{t,T}) + L_{t,T}F_L(s_t, K_{t,T}, L_{t,T}) + (1-\delta)K_{t,T} - K_{t+1,T} \\
&- \int_{[0,\bar{k}]\times\mathcal{I},\hat{k}_{t,T}=\bar{k}} \hat{c}_{t,T}(k,i)d\mu_{t,T}(k,i).
\end{aligned}
$$

Replacing the last item with (60), we have

$$
\begin{aligned}
\int_{[0,\bar{k}]\times\mathcal{I},\hat{k}_{t,T}<\bar{k}} \hat{c}_{t,T}(k,i)d\mu_{t,T}(k,i) &> \frac{1}{2}L_{t,T}F_K(s_t, K_{t,T}, L_{t,T}) + (1-\delta)\left(K_{t,T} - K_{t+1,T}\right) \\
&> L_{t,T}F_L(s_t, K_{t,T}, L_{t,T}).
\end{aligned}
$$

Assumption 6 implies that

$$
L_{t,T}F_L(s_t, K_{t,T}, L_{t,T}) > \underline{\alpha}F(s_t, K_{t,T}, L_{t,T}) > \underline{\alpha}F(s_t, K_{t+1,T}, L_{t,T}).
$$

In addition

$$
\begin{aligned}
&\sum_{i_t} \frac{\pi_{s_t s_{t+1}, i_t i_{t+1}}}{\pi_{s_t s_{t+1}}} \int_{\hat{k}_{t,T}<\bar{k}} \hat{c}_{t+1,T}(\hat{k}_{t,T}(k,i_t), i_{t+1})\mu_{t,T}(dk, i_t) \\
&= \sum_{i_{t+1}} \int_{k<\bar{k}} \hat{c}_{t+1,T}(k, i_{t+1})\mu_{t+1,T}(dk, i_{t+1}) \\
&< \sum_{i_{t+1}} \int_0^{\bar{k}} \hat{c}_{t+1,T}(i_{t+1}, k)\mu_{t+1,T}(dk, i_{t+1}) \\
&= Y_{t+1,T} + (1-\delta)K_{t+1,T} - K_{t+2,T} \\
&< F(s_{t+1}, K_{t+1,T}, L_{t+1}) + (1-\delta)K_{t+1,T}.
\end{aligned}
$$

Therefore (58) becomes

$$\begin{aligned}
&\left(\left\{\min_s(1-\delta+F_K(s,\underline{K},L(s)))\right\}\underline{\beta}\pi_{s_ts_{t+1}}\right)^{\frac{1}{\sigma}}\\
&<\frac{F(s_{t+1},K_{t+1,T},L_{t+1})+(1-\delta)K_{t+1,T}}{\frac{\underline{\alpha}}{2}F(s_t,K_{t,T},L_{t,T})}.\\
&<\frac{2F(s_{t+1},K_{t+1,T},L_{t+1})}{\underline{\alpha}F(s_t,K_{t+1,T},L_t)}\left(1+(1-\delta)\frac{K_{t+1,T}}{F(s_{t+1},K_{t+1,T},L_{t+1})}\right)\equiv\mathcal{E}.
\end{aligned}$$

Because

$$\frac{K_{t+1,T}}{F(s_{t+1},K_{t+1,T},L_{t+1})}<\frac{1}{F_K(s_{t+1},K_{t+1},L_{t+1})}<\frac{1}{F_K(s_{t+1},\underline{K},L_{t+1})}<1,$$

and by (57), we have

$$\mathcal{E}<\frac{F(s_{t+1},K_{t+1,T},L_{t+1})}{F(s_t,K_{t+1,T},L_{t,T})}\frac{2(2-\delta)}{\underline{\alpha}}<\gamma\frac{2(2-\delta)}{\underline{\alpha}}.$$

However, this contradicts the definition of $\underline{K}$, which satisfies (57). We obtain the desired contradiction.

Therefore, by contradiction, $K_{t,T}\geq\underline{K}$ for all t and s^t. Now, for each t and s^t, we have

$$F_K(s_t,\bar{K},L(s_t))\leq r_{t,T}=F_K(s_t,K_{t,T},L_t)\leq F_K(s_t,\underline{K},L(s_t)).$$

Hence $r_{t,T}\in[\underline{r},\bar{r}]$, where

$$0<\underline{r}=\min_{s\in\mathcal{S}}F_K(s,\bar{K},L(s))\quad\text{and}\quad\bar{r}=\max_{s\in\mathcal{S}}F_K(s,\underline{K},L(s)).$$

Similarly, there exist $0<\underline{w}<\bar{w}$, such that $w_{t,T}\in[\underline{w},\bar{w}]$ for all t and s^t.

From the (22), for all $k\in\left[0,\bar{k}\right]$, $i\in\mathcal{I}$, and $t\leq T$ and $s^t\in\mathcal{S}^t$, we have

$$\frac{1-(\bar{\beta})^{T-t+1}}{1-\bar{\beta}}u\left((1-\delta+\bar{r})\bar{k}+\bar{w}\bar{l}\right)\geq\hat{V}_{t,T}(k,i)\geq\frac{1-(\underline{\beta})^{T-t+1}}{1-\underline{\beta}}u\left(\underline{wl}\right).$$

Let

$$\overline{V}=\sup_{t,T:0\leq t\leq T}\frac{1-(\bar{\beta})^{T-t+1}}{1-\overline{\beta}}u\left((1-\delta+\bar{r})\bar{k}+\bar{w}\bar{l}\right)\quad\text{and}\quad\underline{V}=\inf_{t,T:0\leq t\leq T}\frac{1-(\underline{\beta})^{T-t+1}}{1-\underline{\beta}}u\left(\underline{wl}\right).$$

Then

$$\underline{V}\leq\hat{V}_{t,T}(k,i;s^t)\leq\overline{V}$$

for all $k\in\left[0,\bar{k}\right]$, $i\in\mathcal{I}$, and $t\leq T$ and $s^t\in\mathcal{S}^t$.

Now

$$\hat{V}_{t,T}(k,i)=u\left(\hat{c}_{t,T}(k,i)\right)+\beta(i)\mathbb{E}_t\left[\hat{V}_{t+1,T}(k',i)\right].$$

Therefore

$$u\left(\hat{c}_{t,T}(k,i)\right)>\underline{V}-\bar{\beta}\max\{\overline{V},0\}.$$

Since $\lim_{c\to0}u(c)=-\infty$, there exists $\underline{c}>0$ such that

$$\hat{c}_{t,T}(k,i)>\underline{c}$$

for all $k\in\left[0,\bar{k}\right]$, $i\in\mathcal{I}$, and $t\leq T$ and $s^t\in\mathcal{S}^t$.

From the envelope condition

$$\hat{V}'_{t,T}(k,i) = \left(1-\delta + r_{t,T}(s^t)\right) u'\left(\hat{c}_{t,T}\left(k,i\right)\right),$$

Therefore

$$\hat{V}'_{t,T}(k,i) \leq \left(1-\delta+\bar{r}\right) u'(\underline{c}) \equiv l_V \tag{61}$$

for all $k \in \left[0,\bar{k}\right]$, $i \in \mathcal{I}$, and $t \leq T$ and $s^t \in \mathcal{S}^t$.

If $\hat{k} = 0$ or $\bar{k}$ then $\hat{k}' = 0$. Now if $\hat{k} \in \left(0,\bar{k}\right)$, the first-order on k' holds with equality

$$u'\left((1-\delta) + r_{t,T}(s^t)k - \hat{k}\right) = \beta(i)\mathbb{E}_t\left[\hat{V}'_{t,+1T}(\hat{k},i')\right].$$

Differentiate both sides with respect to , we obtain

$$u''\left(\left(1-\delta+r_{t,T}(s^t)\right)k - \hat{k}\right)\left(1-\delta+r_{t,T}(s^t)-\hat{k}'(k,i)\right) = \beta(i)\mathbb{E}_t\left[\hat{V}''_{t,+1T}(\hat{k},i')\right]\hat{k}'(k,i) \leq 0.$$

Therefore

$$\hat{k}'(k,i) \leq 1-\delta+r_{t,T}(s^t) \leq 1-\delta+\bar{r} = l_k. \tag{62}$$

□

C Relation to Miao (2006)

Consider the economy with a continuum of agents in Section 2. Let $\hat{\mathcal{P}}$ denote the set of probability measures μ over $[0,\infty) \times \mathcal{I}$ such that

$$\int_{\mathbb{R}_+ \times \mathcal{I}} k d\mu(k,i) \leq \bar{K},$$

where $\bar{K}$ is defined in (4) and

$$\hat{\mathcal{P}}^{\infty} = \times_{t=0}^{\infty} \hat{\mathcal{P}}^{\mathcal{S}^t}.$$

The existence proof in Miao (2006) relies on the fixed point of the following operator:

$$\boldsymbol{T} : \mathcal{C}\left([0,\infty), \mathcal{I}, \mathcal{S}, \hat{\mathcal{P}}^{\infty}\right) \to \mathcal{C}\left([0,\infty), \mathcal{I}, \mathcal{S}, \hat{\mathcal{P}}^{\infty}\right)$$

define for each

$$\tilde{\mu} = \left(\left\{\mu_t(s^t)\right\}_{t\geq 0, s^t \in \mathcal{S}^t}\right) \in \hat{\mathcal{P}}^{\infty}$$

as

$$\begin{aligned} \boldsymbol{T}V\left(k,i;s_0,\tilde{\mu}\right) = & \max_{k' \in \Gamma(k,i,s,\mu_0)} u\left(\left(1+r\left(s_0,\mu_0\right)-\delta\right)k + w(s_0,\mu_0)l(i) - k'\right) \\ & + \beta(i) \sum_{s_1 \in \mathcal{S}, i' \in \mathcal{I}} \pi_{s_0 s_1;ii'} V\left(k,i;s_1,\left\{\mu_{t+1}(s^{t+1})\right\}_{t\geq 0}\right) \end{aligned} \tag{63}$$

where

$$\Gamma\left(k,i,s,\mu_0\right) = \left\{k' : 0 \leq k' \leq \left(1+r\left(s_0,\mu_0\right)-\delta\right)k + w(s_0,\mu_0)l(i)\right\}.$$

Miao shows that operator $\boldsymbol{T}$ is a contraction mapping (as an application of the Blackwell Theorem). Therefore, $\boldsymbol{T}$ admits a unique fixed point $\hat{V}$ and the corresponding policy function is

$$\hat{k}\left(k, i; s_0, \tilde{\mu}\right)$$

We define

$$\Lambda\tilde{\mu} = \tilde{\tilde{\mu}}$$

where $\tilde{\tilde{\mu}}_0 = \tilde{\mu}_0$ and

$$\tilde{\tilde{\mu}}_{t+1}(s^{t+1})\left(A, i'\right) = \sum_{i\in\mathcal{I}} \frac{\pi_{s_t s_{t+1}, ii'}}{\pi_{s_t s_{t+1}}} \tilde{\mu}_t\left(\hat{k}^{-1}\left(A, i; s_t, \{\mu_\tau(s^\tau)\}_{\tau\geq t}\right)\right),$$

for all $A \in \mathcal{B}(\mathbb{R}^+)$. The mapping Λ is continuous in $\hat{\mathcal{P}}^\infty$. Therefore by the Brouwer-Schauder-Tychonoff Fixed Point Theorem, Λ admits a fixed point, which corresponds to a sequentially competitive equilibrium.

However this proof does not directly apply to Krusell and Smith (1998)'s model in which the production function satisfies the Inada condition at zero aggregate capital. Most importantly because of the following two reasons.

First of all, because of the Inada condition on the production function, the operator $\boldsymbol{T}$ is not well-defined when $\mu_0 = \mathcal{D}(0)$, where $\mathcal{D}(x)$ is the Dirac mass at x because

$$r(s_0, \mathcal{D}(0)) = +\infty.$$

Therefore V must be defined over $\hat{\mathcal{P}}_*^\infty$ where $\mathcal{P}_* = \hat{\mathcal{P}}\backslash\mathcal{D}(0)$.

However, the following proposition shows that Λ does not preserve $\hat{\mathcal{P}}_*^\infty$, i.e., there exists $\tilde{\mu} \in \hat{\mathcal{P}}_*^\infty$ such that, $\Lambda\tilde{\mu} \notin \hat{\mathcal{P}}_*^\infty$. The intuition is that if aggregate capital in $\tilde{\mu}$ is very high, the implied marginal rate of return on capital (interest rate) is very low. Together with a sufficiently low discount factor, the agents will not want to save, leading to zero aggregate capital in $\tilde{\mu}$.[23] The following proposition formalizes this intuition.

Proposition 4. *Assume that $\beta(i) = \beta$ for all $i \in \mathcal{I}$ and let $\beta \in (0,1)$ sufficiently small such that*

$$u'\left(\bar{l}F_L(s, \bar{K}, L(s))\right) > \beta u'\left(\underline{l}F_L(s', \bar{K}, L(s'))\right) \tag{64}$$

for all $s, s' \in \mathcal{S}$. Then there exists K^* such that, for all $K \leq K^*$, and

$$\tilde{\mu} = \left(\mathcal{D}\left(K\right), \{\mathcal{D}\left(\bar{K}\right)\}_{t>0, s^t\in\mathcal{S}^t}\right)$$

we have

$$\Lambda\tilde{\mu} = \left(\mathcal{D}(K), \{\mathcal{D}(0)\}_{t>0, s^t\in\mathcal{S}^t}\right) \notin \hat{\mathcal{P}}_*^\infty.$$

Proof. Because $\lim_{c\downarrow 0} u'(c) = +\infty$ and $\lim_{K\downarrow 0} F(s_0, K, L(s_0)) - \delta K = 0$, there exists K^* such that for all $K \leq K^*$, we have

$$u'\left(F(s_0, K, L(s_0)) - \delta K\right) \geq \beta u'\left(\underline{l}F_L(s', \bar{K}, L(s'))\right) \tag{65}$$

for all $K \leq K^*$. Let $\tilde{\mu}$ be defined above. Using the agents' Euler equation, we have, at $k = K$, the solution to (63) involves:

$$k_t = 0 \text{ for all } t > 0. \tag{66}$$

[23] The discount factor has to be sufficiently low to dominate the precautionary saving motive coming from uncertain labor income.

Indeed, we just have to verify that:

$$u'\left(l(i_t)F_L(s_t,\bar{K},L(s_t))\right) \geq \beta\mathbb{E}_t\left[\left(1-\delta+F_K\left(s_{t+1},\bar{K},L(s_{t+1})\right)\right)u'\left(l(i_{t+1})F_L(s_{t+1},\bar{K},L(s_{t+1}))\right)\right] \tag{67}$$

for all $s_t, s_{t+1} \in \mathcal{S}$ and $i_t, i_{t+1} \in \mathcal{I}$ and:

$$u'\left(F(s_0,K,L(s_0))-(1-\delta)K\right) \geq \beta\mathbb{E}_0\left[\left(1-\delta+F_K\left(s_1,\bar{K},L(s_1)\right)\right)u'\left(l(i_1)F_L(s_1,\bar{K},L(s_1))\right)\right]. \tag{68}$$

Because for all $s \in \mathcal{S}$

$$F\left(s,\bar{K},L(s)\right)-\delta\bar{K} < 0$$

we have

$$F_K\left(s,\bar{K},L(s)\right) < \delta$$

or

$$1-\delta+F_K\left(s_1,\bar{K},L(s_1)\right) < 1.$$

So (67) and (68) follow directly from (64) and (65).

From (66), we obtain

$$\Lambda\tilde{\mu} = \left(\mathcal{D}(K), \{\mathcal{D}(0)\}_{t>0,s^t\in\mathcal{S}^t}\right).$$

□

Second of all, $\hat{\mathcal{P}}_*^{\infty}$, endowed with the product topology (of weak* topology in $\hat{\mathcal{P}}_*$) is not a a compact set. Therefore one cannot apply the Brouwer-Schauder-Tychonoff Fixed Point Theorem for continuous functions defined on this set.

In the present paper, I follow a different route to establish the existence of a competitive equilibrium by taking the limit of finite horizon economies as in Appendix B. I derive a lower bound on aggregate capital using the agents' Euler equation, hence indirectly rule out $\mathcal{D}(0)$. Another way to put it is that $\mathcal{D}(0)$ implies an infinite marginal rate of return r on capital but in Lemma 4, by restricting prices on Δ_ϵ, we impose an upper bound on r. I show that this bound does not bind in equilibrium.

D Finite Agents Economy

In this Appendix, I present the finite agent version of the Krusell and Smith (1998)'s model. The proof for the existence of a generalized recursive equilibrium in this model is similar but simpler than the one for the economy with a continuum of agents in Section 2.

D.1 Infinite-Horizon Economy

The Environment Consider an endowment, a single consumption (final) good economy in infinite horizon. Time runs from $t = 0$ to ∞. The economy is populated by H representative, infinitely-lived agents (households) indexed by:

$$h \in \mathcal{H} = \{1, 2, \ldots, H\}$$

Each representative agent represents a continuum of measure $\frac{1}{H}$ of identical agents. The preferences over the streams of consumption of agent h is given by

$$\mathcal{U}\left(\left\{c_t^h(s^t)\right\}_{t\geq 0, s^t\in\mathcal{S}^t}\right) = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(\Pi_{t'=0}^{t}\beta^h(s_{t'})\right)u(c_t^h)\right] \tag{69}$$

where

$$u(c) = \lim_{\nu\to\sigma}\frac{c^{1-\nu}-1}{1-\nu}$$

and the discount factor β_t^h depends on the aggregate state s_t. We require $\sigma \geq 1$ so that in equilibrium, consumption is bounded from below.

In each period t, there are S (finite) possible exogenous states (shocks)

$$s \in \mathcal{S} = \{1, 2, \ldots, S\}.$$

The shocks capture both idiosyncratic uncertainties (or more precisely household level uncertainties) and aggregate uncertainties. For example, state s can be a vector:

$$s = (A, i_1, ..., i_H),$$

where A is the aggregate productivity and i_h's are idiosyncratic shocks capturing agents' labor productivity and/or discount rate. As pointed out in Den Haan (2001), one caveat with a finite number of agents is that each idiosyncratic shock is by construction an aggregate shock because it changes the aggregates, for example, aggregate labor supply when i_h determines idiosyncratic labor supply. However, when the number of agents is very large, the effects of each idiosyncratic shock on the aggregates become negligible. In the limit with a continuum of agents considered in the next section, by the law of large numbers, an idiosyncratic shock does not have direct aggregate effects.

The exogenous shock follows a first-order Markov process with the transition probabilities $\pi(s, s')$. Let s^t denote the history of realizations of shocks up to time t:

$$s^t = (s_0, s_1, \ldots, s_t) \in \mathcal{S}^t.$$

At time t, state s_t determines the agents' endowments, $l^h(s_t) > 0$ units of labor for $h \in \mathcal{H}$. We assume that, there exist $\underline{L}, \bar{L} > 0$ such that:

$$\underline{L} \leq \frac{1}{H}\sum_{h\in\mathcal{H}} l^h(s_t) \leq \bar{L}$$

for all $s \in \mathcal{S}$. State s_t also determines the agents' discount factor, $\beta^h(s_t)$. In addition, there exist $0 < \underline{\beta}, \bar{\beta} < 1$ such that:

$$\underline{\beta} < \beta^h(s_t) < \bar{\beta}.$$

In each state s, there is a representative firm that produces the final output from capital and labor using an aggregate production function that employs capital and labor as input:

$$Y = F(s, K, L).$$

The aggregate state determines the productivity of the aggregate production function through the first argument. We assume that the production function F satisfies Assumptions 3, 4, and 5 in Section 2.

Market Arrangements In each history s^t, there are rental markets for capital and labor market. Agents of type h rent out their capital to the representative firm at competitive rental rate $r_t(s^t)$ and supply their labor endowment inelastically to the representative firm at competitive wage rate $w_t(s^t)$.

We assume that markets are incomplete inter-temporally, i.e., the agents can only hold capital to insure against idiosyncratic and aggregate shocks. Therefore they face the sequential budget constraints:

$$c_t^h(s^t) + k_{t+1}^h(s^t) - (1-\delta)k_t^h(s^{t-1}) \leq r_t(s^t)k_t^h(s^{t-1}) + w_t(s^t)l^h(s_t) \tag{70}$$

and the borrowing constraints:

$$k_{t+1}^h(s^t) \geq 0. \tag{71}$$

Agent h solves

$$\max_{c^h,k^h} \mathcal{U}\left(\left\{c_t^h(s^t)\right\}_{t,s^t}\right) \tag{72}$$

subject to (70) and (71).

The representative firm in history s^t maximizes profit:

$$\Pi_t(s^t) = \max_{Y_t,K_t,L_t \geq 0} Y_t - r_t K_t - w_t L_t$$

subject to

$$Y_t \leq F(s_t, K_t, L_t).$$

Since F has constant returns to scale, in equilibrium, we must have $\Pi_t(s^t) = 0$ and

$$Y_t = F(s_t, K_t, L_t) \text{ and } r_t = F_K(s_t, K_t, L_t) \text{ and } w_t = F_L(s_t, K_t, L_t). \tag{73}$$

The definition of a SCE in this environment is standard.

Definition 5. A SCE given an initial distribution of capital holdings $\{k_0^h\}_{h\in\mathcal{H}}$ consists of an allocation $\left(\{c_t^h, k_{t+1}^h\}_{t,s^t}\right)_{h\in\mathcal{H}}$ and $\{K_t, L_t\}_{t,s^t}$ and prices $\{r_t, w_t\}_{t,s^t}$ $(r_t, w_t > 0)$ such that:

1. For each agent $h \in \mathcal{H}$, $\{c_t^h, k_{t+1}^h\}_{t,s^t}$ maximizes the intertemporal expected utility (69) subject to the sequential budget constraints, (70) and borrowing constraints, (71).

2. In each history s^t, $\{Y_t, K_t, L_t\}$ solves the representative firm's profit maximization problem, i.e., (73) is satisfied.

3. Markets for capital, labor, and final good clear in each history s^t:

$$\frac{1}{H}\sum_{h\in\mathcal{H}} k_t^h(s^t) = K_t(s^t)$$

and

$$\frac{1}{H}\sum_{h\in\mathcal{H}} l^h(s^t) = L_t(s^t)$$

and

$$\frac{1}{H}\sum_{h\in\mathcal{H}} \left(c_t^h(s^t) + k_{t+1}^h(s^t) - (1-\delta)k_t^h(s^{t-1})\right) = Y_t(s^t).$$

Let Ω denote a set of wealth distributions, or equivalently of the distributions of capital holdings, and is a *compact* subset of $\mathbb{R}^H_+$:

$$\Omega = \left\{ \left(k^h_t \right)_{h \in \mathcal{H}} \right\} \subset \mathbb{R}^H_+.$$

Following Krusell and Smith (1998), we define a generalized recursive equilibrium as following.

Definition 6. A GRE is a policy correspondence and a transition correspondence:

$$\mathcal{Q} : \mathcal{S} \times \Omega \rightrightarrows \mathbb{R}^{3H+2}_+$$

and

$$\mathcal{T} : \mathcal{S} \times \Omega \rightrightarrows \Omega$$

and $\underline{c} > 0$ such that for all $s \in \mathcal{S}$ and $\{k^h\}_{h \in \mathcal{H}} \in \Omega$ and $\left(\{c^h, k^h_+, v^h\}_{h \in \mathcal{H}}, r, w \right) \in \mathcal{Q}\left(s, \{k^h\}_{h \in \mathcal{H}}\right)$, we have $c^h \geq \underline{c}$ and there exists

$$\left(\left\{ c^h_+(s'), k^h_{++}(s'), v^h_+(s') \right\}_{h \in \mathcal{H}}, r_+(s'), w_+(s') \right)_{s' \in \mathcal{S}}$$

that satisfies the following properties:

1. $\{k^h_+\}_{h \in \mathcal{H}} \in \mathcal{T}\left(s, \{k^h\}_{h \in \mathcal{H}}\right)$
2. $\left(\{c^h_+, k^h_{++}, v^h_+\}_{h \in \mathcal{H}}, r_+, w_+ \right) \in \mathcal{Q}\left(s', \{k^h_+\}_{h \in \mathcal{H}}\right)$
3. (Market clearing) $\frac{1}{H} \sum_{h \in \mathcal{H}} c^h + \frac{1}{H} \sum_{h \in \mathcal{H}} k^h_+ = F(s, K, L) + (1 - \delta)K$ where

$$K = \frac{1}{H} \sum_{h \in \mathcal{H}} k^h > 0 \text{ and } L = \frac{1}{H} \sum_{h \in \mathcal{H}} l^h(s) > 0.$$

4. (Firms' maximization) $r = F_K(s, K, L) > 0$ and $w = F_L(s, K, L) > 0$.
5. (Agents' maximization) For each $h \in \mathcal{H}$

$$u'(c^h) \geq \beta^h(s) \sum_{s' \in \mathcal{S}} \pi_{ss'} \left(1 - \delta + r_+(s')\right) u'(c^h_+(s')) \tag{74}$$

with equality if $k^h_+ > 0$ and

$$c^h + k^h_+ = (1 - \delta + r)k^h + wl^h \tag{75}$$

and

$$v^h = u(c^h) + \beta^h(s) \sum_{s' \in \mathcal{S}} \pi_{ss'} v^h_+(s'). \tag{76}$$

A recursive equilibrium is a generalized recursive equilibrium in Definition 6 with the correspondences $\mathcal{Q}, \mathcal{T}$ being single-valued.

The following lemma shows the connection between a generalized recursive equilibrium and competitive equilibrium.

Lemma 12. *A sequence of allocations and prices generated by a GRE forms a SCE.*

Proof. Appendix D.2. □

To show the existence of a GRE, we need the following properties on the production function.

Assumption 7. *There exists K^* such that for any $0 < K < K^*$ and $\underline{L} \leq L \leq \bar{L}$, and $s, s' \in \mathcal{S}$:*

$$\left(\underline{\beta} \min_{s'' \in \mathcal{S}} (1 - \delta + F_K(s'', K, L))\right)^{\frac{1}{\sigma}} > \frac{F(s', K, \underline{L}) + (1 - \delta)K}{F(s, K, \bar{L}) - \delta K}.$$

This assumption requires that the marginal rate of return on capital is very high when capital is low. Together with the agents' Euler equation, it implies a lower bound on aggregate capital in any competitive equilibrium.

It is easy to verify that the last two assumptions hold for the Cobb-Douglas production function in (2) since $F_K(s'', K, L) \to \infty$ as $K \to 0$ and $\frac{F(s', K, L)}{F(s, K, L)}$ is bounded above as $K \to 0$.

Armed with the assumptions above, we arrive at the first existence result.

Theorem 3. *Assume that Assumptions 3-5, and 7 hold. Given any initial distribution of capital, $\frac{1}{H} \sum_{h \in \mathcal{H}} k_0^h = K_0 > 0$, there exist $0 < \underline{K} < K_0 < \bar{K}$ such that a GRE exists with $\Omega = \left\{ \left(k^h\right)_{h \in \mathcal{H}} \in \mathbb{R}_+^H : \underline{K} \leq \frac{1}{H} \sum_{h \in \mathcal{H}} k^h \leq \bar{K} \right\}$.*

Proof. We choose $\bar{K}$ sufficiently large:

$$\bar{K} > \max \left\{ K_0, \hat{K}, \max_{s \in \mathcal{S}} \max_{0 \leq K \leq \hat{K}} \left(F(s, K, \bar{L}) + (1 - \delta) K \right), 2\bar{L} \right\}, \tag{77}$$

where $\hat{K}$ is defined in Assumption 4, and $\underline{K}$ sufficiently small:

$$\underline{K} < \min \left\{ K_0, K^* \right\}, \tag{78}$$

where K^* is defined in Assumption 7.

The proofs follow closely the steps in Cao (2010). I first show that a SCE exists for any finite horizon economy. In addition, the equilibrium variables in a finite horizon economy always lie in a compact set. Then I take the limit of the horizon to infinity and construct appropriate correspondences to show the existence of a GRE.

However, Cao (2010) assumes that each agent receives an strictly positive amount of final good endowment in every period and history of shocks. In this paper, we relax this assumption. We only require that each agent receives an strictly positive amount of labor endowment in every period and history of shocks. We show that aggregate capital is always bounded from below:

$$K_{t,T}(s^t) \geq \underline{K}$$

for all t and s^t, where $K_{t,T}(s^t)$ is the aggregate capital at time t and in history s^t in the T-period economy. Therefore wage rate is bounded from below:

$$w_{t,T}(s^t) = F_L(s_t, K_{t,T}(s^t), L_{t,T}(s^t)) \geq \underline{w}$$

for some $\underline{w} > 0$. Together with a strictly positive labor endowment, the lower bound on wage rate implies a strictly positive labor income, which plays a similar role to a strictly positive final good endowment in Cao (2010).

To show that aggregate capital is bounded from below, we use the agents' Euler equation, (74):

$$u'(c_t^h) \geq \beta_t^h \mathbb{E}_t \left[(1 - \delta + r_{t+1}) \, u'(c_{t+1}^h) \right].$$

This equation implies that if K_{t+1} is too small, the rate of return on capital r_{T+1} is every high, driving up saving from time t, and in turn, increasing K_{t+1}. Assumption 7 then leads to a contradiction.

The details of the proof are given in Appendix D.2. □

Generalized Recursive Equilibrium and Recursive Equilibrium As discussed in Section 2, in general, we cannot always select a recursive equilibrium from a GRE. Therefore, we would need additional conditions to guarantee the existence of a recursive equilibrium. The following result provides such a sufficient condition for when a GRE gives rise to a recursive equilibrium.

Corollary 1. *Assume that the conditions in Theorem 3 are satisfied. We have:*

1. Starting from any wealth distribution $\{k_0^h\}_{h\in\mathcal{H}} \in \mathbb{R}_+^H$ and exogenous state $s_0 \in \mathcal{S}$, there exists a SCE.

2. In addition if the SCE is unique for every initial wealth distribution and exogenous state, there exists a recursive equilibrium.

Proof. 1. By Lemma 12, starting from any distribution of capital holdings $\{k_0^h\}_{h\in\mathcal{H}}$ and aggregate state s, the sequences of allocation and prices generated by a GRE is a SCE. Theorem 3 guarantees the existence of a GRE. Hence, a SCE exists.

2. Because starting from each $s \in \mathcal{S}$ and $\{k^h\}_{h\in\mathcal{H}} \in \Omega$, there exists no more than one SCE, there exists a unique element

$$\left(\left\{c^h, k_+^h, v^h\right\}_{h\in\mathcal{H}}, r, w\right) \in \mathcal{Q}\left(s_t, \left\{k^h\right\}_{h\in\mathcal{H}}\right)$$

that satisfies Conditions 1.-5. in Definition 6. Let $\mathcal{Q}^0$ denote the mapping from $\left(s, \{k^h\}_{h\in\mathcal{H}}\right)$ to this element, and $\mathcal{T}^0\left(s, \{k^h\}_{h\in\mathcal{H}}\right) = \{k_+^h\}_{h\in\mathcal{H}}$. Then $\left(\mathcal{Q}^0, \mathcal{T}^0\right)$ forms a recursive equilibrium. □

A GRE also gives rise to a recursive equilibrium if we allow for more (endogenous) state variables in addition to the agents' capital holdings. This point is emphasized more generally in Duffie et al. (1994).

Corollary 2 (Recursive Equilibrium with Extended State Variables)**.** *Given the set of distributions Ω and the correspondence $\mathcal{Q}$ in Definition 6 and Theorem 3, let*

$$\Xi = \left\{ \left(s, \left\{k^h\right\}, \left\{c^h, v^h\right\}\right) \in \mathcal{S} \times \Omega \times \mathbb{R}^{2H} \;\middle|\; \begin{array}{l} \left(\left\{c^h, k_+^h, v^h\right\}, r, w\right) \in \mathcal{Q}\left(s_t, \left\{k^h\right\}\right) \\ \text{for some } \left(k_+^h\right) \text{ and } r, w > 0 \end{array} \right\}$$

where $\{x^h\}$ is a short-cut for $\{x^h\}_{h\in\mathcal{H}}$.

A recursive equilibrium with an extended state variable can be constructed over Ξ as a mapping from from $\xi = \left(s, \{k^h\}, \{c^h, v^h\}\right)$ to

a. a current capital choices and values $\{k_+^h\}$, and current factor prices r, w ;

b. next period capital holdings and consumption $\left(s', \{k_+^h\}, \{c_+^h, v_+^h\}\right)_{s'\in\mathcal{S}}$ such that

$$\xi_{s'}^+ = \left(s', \left\{k_+^h\right\}, \left\{c_+^h, v_+^h\right\}\right) \in \Xi \textit{ for all } s' \in \mathcal{S},$$

and Conditions 3. − 5. *in Definition* 2 *are satisfied.*

We notice that from the firms' maximization problem (Condition 4), r and w are pinned down by the current states s and $K = \frac{1}{H}\sum_h k^h$. *Therefore, from the agents' budget constraint,* (75), c^h *pins down* k_+^h. *Consequently, the values of* $\{c^h, v^h\}$ *uniquely select the element in* $\mathcal{Q}(s, \{k^h\})$ *and* $\mathcal{T}(s, \{k^h\})$, *i.e. , for any* $\xi = \left(s, \{k^h\}, \{c^h, v^h\}\right) \in \Xi$ *there exists a unique tuple* $\left(k_+^h\right), r, w$ *such that* $\left(\{c^h, k_+^h, v^h\}, r, w\right) \in \mathcal{Q}\left(s_t, \{k^h\}\right)$. *The selection gives rise to a recursive equilibrium in the extended state space.*

In the following section, we extends the analysis above in this environment with a finite number of (representative) agents to the environment with a continuum of agents.

D.2 Finite-Horizon Economy and Proofs

To prove Theorem 3, first we show the existence of a SCE in Lemma 13. The proof of this lemma uses Kakutani's Fixed Point Theorem.

We consider a finite horizon economy that lasts for $T+1$ periods, $t = 0, 1, ..., T$. Given prices

$$\{r_{t,T}(s^t), w_{t,T}(s^t)\}_{t\le T, s^t\in\mathcal{S}^t}$$

the representative firm solves

$$\max_{Y_t, K_t, L_t} \Pi_t = Y_t - r_t K_t - w_t L_t$$

s.t. $Y_t \le F(s_t, K_t, L_t)$. We allow for Π_t potentially be different from 0, but we show that in equilibrium $\Pi_t = 0$. We also assume that the profits (or losses) are divided equally across agents.

Given prices and the representative firm's profit, agents solve

$$\max_{c_{t,T}^h, k_{t+1,T}^h} \mathbb{E}_0 \left[\sum_{t=0}^{T} \left(\Pi_{t'=0}^{t} \beta^h(s_t)\right) u(c_{t,T}^h)\right] \tag{79}$$

s.t.

$$c_{t,T}^h + k_{t+1,T}^h \le (1-\delta) k_{t,T}^h + r_{t,T} k_{t,T}^h + w_{t,T} l_t^h + \frac{1}{H}\Pi_t$$

and

$$c_{t,T}^h, k_{t+1,T}^h \ge 0.$$

A SCE is defined similarly as in Definition 5. Lemma 13 show that a SCE exists. Lemma 14 shows that

$$\underline{c} \le c_{t,T}^h \le \bar{c} \qquad 0 \le k_{t,T}^h \le \bar{k} \qquad \underline{v} \le v_{t,T}^h \le \bar{v}$$
$$\underline{K} \le K_t \le \bar{K} \qquad \underline{r} \le r_{t,T} \le \bar{r} \qquad \underline{w} \le w_{t,T} \le \bar{w}$$

with the bounds appropriately defined.

Lemma 13. *Given an initial distribution of capital holding* $\{k_0^h\}_{h\in\mathcal{H}}$ *such that*

$$K_0 = \frac{1}{H}\sum_{h\in\mathcal{H}} k_0^h > 0,$$

a SCE exists in the finite horizon economy.

Proof. The proof uses Kakutani's Fixed Point Theorem as in Cao (2010), which builds upon Debreu (1959).

To simplify the proof, we switch from choosing the final good as numeraire to the following normalization:

$$p^c_{t,T}(s^t) + w_{t,T}(s^t) + r_{t,T}(s^t) = 1.$$

The sequential budget constraint of the consumers become:

$$p^c_{t,T}(c^h_{t,T} + k^h_{t+1,T} - (1-\delta)k^h_{t,T}) \leq r_{t,T}k^h_{t,T} + w_{t,T}l^h_t + \frac{1}{H}\Pi_t.$$

The objective function of the representative firms

$$\Pi_{t,T} = p^c_{t,T}Y_{t,T} - r_{t,T}K_{t,T} - w_{t,T}L_{t,T}. \tag{80}$$

Given a sequence $\tilde{\epsilon} = \{\epsilon_t\}_{t=0}^T$ such that $\epsilon_t > 0$ for $t = 0, 1, ..., T$, we impose an additional restriction on the set of normalized prices:

$$p^c_{t,T}(s^t) \geq \epsilon_t > 0. \tag{81}$$

This restriction effectively puts an upper bound on marginal rate of returns on capital:

$$\frac{r_{t,T}}{p^c_{t,T}} \leq \frac{1-\epsilon_t}{\epsilon_t}$$

therefore a lower bound on aggregate capital.

For $\epsilon > 0$, let Δ_ϵ denote the subset of $\mathbb{R}^3_+$:

$$\Delta_\epsilon = \left\{(p^c, w, r) \in \mathbb{R}^3_+ : p^c + w + r = 1 \text{ and } p^c \geq \epsilon > 0\right\}. \tag{82}$$

For each history s^t , given our normalization and the additional restriction (81):

$$\left(p^c_{t,T}(s^t), w_{t,T}(s^t), r_{t,T}(s^t)\right) \in \Delta_{\epsilon_t}.$$

We also denote

$$\Delta^{\Sigma^T}_{\tilde{\epsilon}} = \left\{\left(p^c_{t,T}, r_{t,T}, w_{t,T}\right)_{t,s^t} : \left(p^c_{t,T}, r_{t,T}, w_{t,T}\right) \in \Delta_{\epsilon_t}\right\}.$$

Given the prices, the representative firm maximizes (80) subject to

$$0 \leq Y_{t,T}, K_{t,T}, L_{t,T} \quad \text{and} \quad Y_{t,T} \leq F(s_t, K_t, L_t).$$

To ensure the compactness of the maximization problem, we impose additional restrictions:

$$K_{t,T} \leq 2\bar{K} \quad \text{and} \quad L_{t,T} \leq 2\bar{L},$$

for all t and s^t, where $\bar{K}$ is defined in (4).

Similarly, each consumers maximize (79) subject to

$$p^c_{t,T}c^h_{t,T} + p^c_{t,T}\left(k^h_{t+1,T} - (1-\delta)k^h_{t,T}\right) \leq r_{t,T}k^h_{t,T} + w_{t,T}l^h_t + \frac{1}{H}\Pi_{t,T} \tag{83}$$

and

$$0 \leq c^h_{t,T}, k^h_{t+1,T}$$

for all t and s^t.

Because the representative firms' choices are restricted on a compact set. Their profits Π_t are bounded above:

$$\Pi_{t,T}(s^t) < \bar{\Pi}$$

for all t, s^t. Given the initial distribution of capital holding, the budget constraints (83) and the exogenous lower bound on $p^c_{t,T}$, (81), it is easy to show that there exist $\hat{c}, \hat{k} > 0$ such that

$$c^h_{t,T} < \hat{c} \quad \text{and} \quad k^h_{t+1,T} < \hat{k}$$

for all h, t, s^t.

Let ψ_x denote the correspondence that maps each set of prices

$$\left\{ \left(p_{t,T}(s^t), r_{t,T}(s^t), w_{t,T}(s^t) \right) \right\}_{s^T \in \Sigma^T}$$

to the excess demand in each market in each history:

$$\begin{aligned} \psi_x : \Delta_{\tilde{\epsilon}}^{\Sigma^T} & \rightrightarrows \mathbb{R}^{3\|\Sigma^T\|} \\ p^T \in \Delta_{\tilde{\epsilon}}^{\Sigma^T} & \mapsto x^T = (\text{excess demands}) \end{aligned}$$

The component of the excess demand in each market corresponds to the component of the price system in that market:

$$\begin{aligned} \text{Consumption:} x^c_{t,T}(s^t) &= \frac{1}{H} \sum_{h \in \mathcal{H}} \left(c^h_{t,T}(s^t) + k^h_{t+1,T}(s^t) - (1-\delta) k^h_t(s^{t-1}) \right) - Y_{t,T}(s^t) \\ \text{Capital:} x^k_{t,T}(s^t) &= K_{t,T}(s^t) - \frac{1}{H} \sum_h k^h_{t,T}(s^{t-1}) \\ \text{Labor: } x^l_{t,T}(s^t) &= L_{t,T}(s^t) - L(s_t). \end{aligned}$$

It is standard to show that ψ_x is upper hemi-continuous and compact, convex-valued.[24] Given that each individual choices $c^h_{t,T}, k^h_{t+1,T}$ are bounded, ψ_x is bounded by a closed cube $\mathcal{K}_x \subset \mathbb{R}^{\Sigma^T}$. For example,

$$-\hat{k} \leq x^k_{t,T}(s^t) \leq 2\bar{K}$$

for all t, s^t.

Consider the following correspondence:

$$\begin{aligned} \Psi : \Delta_{\tilde{\epsilon}}^{\Sigma^T} \times \mathcal{K}_x & \rightrightarrows \Delta_{\tilde{\epsilon}}^{\Sigma^T} \times \mathcal{K}_x \\ \left\{ p^T \in \Delta_{\tilde{\epsilon}}^{\Sigma^T}, x^T \in \mathcal{K}_x \right\} & \mapsto \left\{ \arg\max_{\tilde{p} \in \Delta_{\tilde{\epsilon}}^{\Sigma^T}} \tilde{p} \cdot x^T \right\} \times \psi_x(p^T). \end{aligned}$$

It is also standard to show that Ψ is a upper hemi-continuous, non-empty, compact, and convex valued correspondence. Kakutani's Fixed Point Theorem then guarantees that Ψ has a fixed point $\left(\bar{p}^T, \bar{x}^T\right)$. By choosing $\tilde{\epsilon}$ appropriately, we can show that $\left(\bar{p}^T, \bar{x}^T\right)$ corresponds to a competitive equilibrium. The proof is similar to the one in Lemma 5 and Lemma 6 below so we omit the details here. For example, $\tilde{\epsilon}$ and $\{\underline{K}_t\}_{t=0}^T$ are chosen

[24] The additional restriction (81) is crucial for the upper hemi-continuity of ψ_x. Without the restriction ψ_x is not upper hemi-continuous.

recursively using Assumption 5 and the agents' Euler equation (84):

1. ϵ_0 and K_0 are chosen as in Lemma 6.
2. Given $\underline{K}_{t-1}$, ϵ_t is chosen sufficiently small such that, for any K such that

$$F_K(s,K,L) \geq \frac{1-\epsilon_t\left(1+\max_{0\leq K\leq 2\bar{K}} F_L(s,K,\underline{L})\right)}{\epsilon_t},$$

for some $L \in [0, 2\bar{L}]$ we have

$$\left(\pi_{s_-s}\left((1-\delta)\,\epsilon_t + 1 - \epsilon_t\left(1+\max_{0\leq\tilde{K}\leq 2\bar{K}} F_L(s,\tilde{K},\underline{L})\right)\right)\right)^{\frac{1}{\sigma}} > \frac{F(s,K,\bar{L}) + (1-\delta)K}{F(s_-,\underline{K}_{t-1},\underline{L}) + (1-\delta)\underline{K}_{t-1} - K}.$$

for all $s, s_- \in \mathcal{S}$.

$\underline{K}_t$ is chosen such that for all $s, s_- \in \mathcal{S}$, and $K_t \leq \underline{K}_t$:

$$\left(\pi_{s_-s}\,(1-\delta)\,\epsilon\right)^{\frac{1}{\sigma}} > \frac{F(s,K,\bar{L}) + (1-\delta)K}{F(s_-,\underline{K}_{t-1},\underline{L}) + (1-\delta)\underline{K}_{t-1} - K}.$$

□

Lemma 14. *Consider a SCE with the initial aggregate capital $K_0 > 0$ and let $\underline{K}, \bar{K}$ be defined as in (78) and (4). Then for all $t \in \{0, ..., T\}$ and $s^t \in \mathcal{S}^t$, we have $\underline{K} \leq K_{t,T}(s^t) \leq \bar{K}$ and, for all $h \in \mathcal{H}$:*

$$0 \leq c^h_{t,T}, k^h_{t,T} \leq H \max_{s\in\mathcal{S}} \{F(s,\bar{K},L(s)) + (1-\delta)\bar{K}\} = \bar{c} = \bar{k},$$

and

$$\underline{r} = \min_{s\in\mathcal{S}} \min_{\underline{L}\leq L\leq\bar{L}} F_K(s,\bar{K},L(s)) \leq r_{t,T} \leq \bar{r} = \max_{s\in\mathcal{S}} \max_{\underline{L}\leq L\leq\bar{L}} F_K(s,\underline{K},L(s))$$

and

$$\underline{w} = \min_{s\in\mathcal{S}} \min_{\underline{K}\leq K\leq\bar{K}} F_L(s,K,\bar{L}) \leq w_{t,T} \leq \bar{w} = \max_{s\in\mathcal{S}} \max_{\underline{K}\leq K\leq\bar{K}} F_L(s,K,\underline{L}).$$

In addition, there exists $\underline{c} > 0$ such that $c^h_{t,T} \geq \underline{c}$ for all t and s^t and $h \in \mathcal{H}$.

Proof. First we show by induction that $K_{t,T}(s^{t-1}) \leq \bar{K}$ for all t and s^t. At $t = 0$, this property is satisfied by the definition of $\bar{K}$. Assume that the property holds for t, and all $s^t \in \mathcal{S}^t$, we show that it holds for $t+1$ and $s^{t+1} \in \mathcal{S}^{t+1}$.

Indeed, from the market clearing conditions, we have

$$\begin{aligned}
K_{t+1}(s^t) &= \frac{1}{H}\sum_{h\in\mathcal{H}} k^h_{t+1,T}(s^t) \\
&= Y_{t,T}(s^t) + (1-\delta)K_t(s^{t-1}) - \frac{1}{H}\sum_{h\in\mathcal{H}} c^h_{t,T}(s^t) \\
&\leq Y_{t,T}(s^t) + (1-\delta)K_t(s^{t-1}) \\
&= F\left(s_t, K_t(s^{t-1}), L(s_t)\right) + (1-\delta)K_t(s^{t-1}).
\end{aligned}$$

If $K_t(s^{t-1}) \geq \hat{K}$ then

$$\begin{aligned} K_{t+1}(s^t) &= K_t(s^{t-1}) + F\left(s_t, K_t(s^{t-1}), L(s_t)\right) - \delta K_t(s^{t-1}) \\ &\leq K_t(s^{t-1}) \leq \bar{K}. \end{aligned}$$

If $K_t(s^{t-1}) \leq \hat{K}$ then

$$\begin{aligned} K_{t+1}(s^t) &= F\left(s_t, K_t(s^{t-1}), L(s_t)\right) + (1-\delta)K_t(s^{t-1}) \\ &\leq \max_{s \in \mathcal{S}} \max_{0 \leq K \leq \hat{K}} F(K, L(s), s) + (1-\delta)K \leq \bar{K} \end{aligned}$$

So in either case we have $K_{t+1}(s^t) \leq \bar{K}$.

Therefore, by induction, we have $K_{t,T}(s^{t-1}) \leq \bar{K}$ for all t and s^t.

Now we show by induction that

$$K_{t,T}(s^{t-1}) \geq \underline{K}$$

for all t and s^t.

By the definition of $\underline{K}$, $K_0 \geq \underline{K}$. Now assume that $K_t(s^{t-1}) \geq \underline{K}$ for all $s^t \in \mathcal{S}^t$, we show by contradiction that $K_{t+1}(s^t) \geq \underline{K}$. Assume to the contrary, i.e. $K_{t+1}(s^t) < \underline{K}$ for some $s^t \in \mathcal{S}^t$.

From the first order condition of the agents, we have

$$u'(c^h_{t,T}) \geq \underline{\beta}\mathbb{E}_t\left[(1-\delta+F_K(s_{t+1}, K_{t+1}, L_{t+1}))u'(c^h_{t+1,T})\right] \tag{84}$$

for all $h \in \mathcal{H}$. Therefore

$$\begin{aligned} u'(c^h_{t,T}) &\geq \min_{s_{t+1}\in\mathcal{S}}(1-\delta+F_K(s_{t+1}, \underline{K}, L_{t+1}))\underline{\beta}\mathbb{E}_t\left[u'(c^h_{t+1,T})\right] \\ &\geq \min_{s_{t+1}\in\mathcal{S}}(1-\delta+F_K(s_{t+1}, \underline{K}, L_{t+1}))\underline{\beta}u'\left(\mathbb{E}_t[c^h_{t+1,T}]\right) \end{aligned}$$

where the last inequality comes from the fact that $u'(c) = c^{-\sigma}$ is strictly convex.

Consequently,

$$\frac{\mathbb{E}_t\left[c^h_{t+1,T}\right]}{c^h_{t,T}} \geq \left(\underline{\beta}\min_{s_{t+1}\in\mathcal{S}}(1-\delta+F_K(s_{t+1}, \underline{K}, L_{t+1}))\right)^{\frac{1}{\sigma}}$$

and[25]

$$\frac{\mathbb{E}_t\left[\frac{1}{H}\sum_{h\in\mathcal{H}} c^h_{t+1,T}\right]}{\frac{1}{H}\sum_{h\in\mathcal{H}} c^h_{t,T}} \geq \left(\underline{\beta}\min_{s_{t+1}\in\mathcal{S}}(1-\delta+F_K(s_{t+1}, \underline{K}, L_{t+1}))\right)^{\frac{1}{\sigma}}.$$

From the market clearing conditions, we have

$$\frac{1}{H}\sum_{h\in\mathcal{H}} c^h_{t+1,T} \leq F(s_{t+1}, K_{t+1}, L_{t+1}) + (1-\delta)K_{t+1} \leq \max_{s\in\mathcal{S}} F(s, \underline{K}, \bar{L}) + (1-\delta)\underline{K}$$

[25]We use the inequality that $m \leq \frac{a_j}{b_j}$ for all j implies $m \leq \frac{\sum a_j}{\sum b_j}$.

and

$$\frac{1}{H}\sum_{h\in\mathcal{H}} c^h_{t,T} = F(s_t, K_t, L_t) + (1-\delta)K_t - K_{t+1} \geq F(s_t, \underline{K}, \bar{L}) - \delta\underline{K}.$$

Therefore,

$$\frac{\mathbb{E}_t\left[\frac{1}{H}\sum_{h\in\mathcal{H}} c^h_{t+1,T}\right]}{\frac{1}{H}\sum_{h\in\mathcal{H}} c^h_{t,T}} \leq \frac{\max_{s\in\mathcal{S}} F(s, \underline{K}, \bar{L}) + (1-\delta)\underline{K}}{F(s_t, \underline{K}, \bar{L}) - \delta\underline{K}}.$$

So finally, we obtain

$$\frac{\max_{s\in\mathcal{S}} F(s, \underline{K}, \bar{L}) + (1-\delta)\underline{K}}{F(s_t, \underline{K}, \bar{L}) - \delta\underline{K}} \geq \left(\underline{\beta} \min_{s_{t+1}\in\mathcal{S}} (1-\delta + F_K(s_{t+1}, \underline{K}, L_{t+1}))\right)^{\frac{1}{\sigma}}.$$

However, this contradicts the inequality in Assumption 7. Therefore we must have $K_{t+1}(s^t) \geq \underline{K}$. So by contradiction, $K_t(s^{t-1}) \geq \underline{K}$ for all t and s^t.

The other inequalities for c^h_t, k^h_t, r_t, w_t follow immediately.

Now we show that there exists $\underline{c} > 0$ such that $c^h_t \geq \underline{c}$ for all t, s^t, and h. Indeed, from the agents' maximization problem, since starting from any history s^t, an agent can always consumes her labor endowment, we have

$$u(c^h_t(s^t)) + \mathbb{E}_t\left[\sum_{t+1}^{\infty}\prod_{t''=t'}^{t'} \beta(s_{t''})u(c^h_{t'})\right] \geq u(\underline{wl}) + \mathbb{E}_t\left[\sum_{t+1}^{\infty}\prod_{t''=t'}^{t'} \beta(s_{t''})u(\underline{wl})\right]$$
$$\geq \frac{1}{1-\bar{\beta}}u(\underline{wl}).$$

In addition, $c^h_{t'} \leq \bar{c}$ for all t'. Therefore

$$u(c^h_t(s^t)) + \frac{\beta}{1-\underline{\beta}}u(\bar{c}) \geq \frac{1}{1-\bar{\beta}}u(\underline{wl}).$$

So

$$c^h_t(s^t) \geq \underline{c} = u^{-1}\left(\frac{1}{1-\bar{\beta}}u(\underline{wl}) - \frac{\bar{\beta}}{1-\underline{\beta}}u(\bar{c})\right) > 0.$$

□

Proof of Theorem 3. The steps of the proof are similar to the ones for Theorem 1 presented in the main paper. With some abuse, we re-use several notations in the proof such as Θ, g etc.

Given the bounds determined in Lemma 14, let Θ denote the set of

$$\left(\left(c^h, k^h_+, v^h\right)_{h\in\mathcal{H}}, r, w\right)$$

such that $\underline{c} \leq c^h \leq \bar{c}$, $0 \leq k^h_+ \leq \bar{k}$, $\underline{v} \leq v^h \leq \bar{v}$, $\underline{r} \leq r \leq \bar{r}$, and $\underline{w} \leq w \leq \bar{w}$.

Let $g : \mathcal{S}\times\Omega \rightrightarrows \Theta\times\Theta^{\mathcal{S}}$ denote the following correspondence: for each $s \in \mathcal{S}$, $\omega = \left(k^h\right)_{h\in\mathcal{H}} \in \Omega$, $g(s,\omega)$ is the set of $\theta = \left(\left(c^h, k^h_+, v^h\right)_{h\in\mathcal{H}}, r, w\right) \in \Theta$ and $(\theta_{s'})_{s'\in\mathcal{S}} \in \Theta^{\mathcal{S}}$

with $\theta_{s'} = \left(\left\{ c_+^h(s'), k_{++}^h(s'), v_+^h(s') \right\}_{h \in \mathcal{H}}, r_+(s'), w_+(s') \right)_{s' \in \mathcal{S}}$ such that:

$$\frac{1}{H} \sum_{h \in \mathcal{H}} c^h + \frac{1}{H} \sum_{h \in \mathcal{H}} k_+^h = F(s, K, L) + (1 - \delta) K$$

where $K = \frac{1}{H} \sum_{h \in \mathcal{H}} k^h > 0$ and $L = \frac{1}{H} \sum_{h \in \mathcal{H}} l^h(s) > 0$, and $r = F_K(s, K, L) > 0$ and $w = F_L(s, K, L) > 0$. In addition, (13), (75), and (76) are satisfied.

It is easy to show that g is a closed-valued correspondence.

Consider the following mapping $\mathcal{G}$ from the set of correspondences $\mathcal{V} : \mathcal{S} \times \Omega \rightrightarrows \Theta \subset \mathbb{R}^{3H+2}$ to itself as following. For each $\mathcal{V}$, $\mathcal{G}(\mathcal{V})$ is the correspondence $\mathcal{W}$ such that, for each $s \in \mathcal{S}$ and $\omega = (k^h)_{h \in \mathcal{H}} \in \Omega$, we have

$$\mathcal{W}(s, \omega) = \left\{ \begin{array}{ll} \theta = \left((c^h, k_+^h, v^h)_{h \in \mathcal{H}}, r, w \right) \in \Theta : & \text{for each } s' \in \mathcal{S}, \exists \theta_{s'} \in \mathcal{V}\left(s', (k_+^h)_{h \in \mathcal{H}}\right) \\ & \text{and } \left(\theta, (\theta_{s'})_{s' \in \mathcal{S}}\right) \in g(s, \omega) \end{array} \right\}$$

From the definition of $\mathcal{G}$, we have the following properties P1-P3:

P1. If $\mathcal{V}$ is compact in the sense that $\mathcal{V}(s, \omega)$ is compact for all $s \in \mathcal{S}$ and $\omega \in \Omega$, then $\mathcal{W} = \mathcal{G}(\mathcal{V})$ is compact.

Indeed, assume $\{\theta^m\}_{m=0}^\infty \in \mathcal{W}(s, \omega)$, and $\theta^m \to \theta = \left((c^h, k_+^h, v^h)_{h \in \mathcal{H}}, r, w \right)$. Since Θ is compact, $\theta \in \Theta$. To show that $\mathcal{G}(\mathcal{V})$ is compact, we need to show that $\theta \in \mathcal{W}(s, \omega)$. By the definition of $\mathcal{G}$, for each $s' \in \mathcal{S}$, $\exists \theta_{s'}^m \in \mathcal{V}\left(s', (k_+^h)_{h \in \mathcal{H}}\right)$ and $\left(\theta^m, (\theta_{s'}^m)\right) \in g(s, \omega)$. Since $\mathcal{V}\left(s', (k_+^h)_{h \in \mathcal{H}}\right)$ is compact, we can extract a converging subsequence, $\theta_{s'}^{m_l} \to \theta_{s'}$ for some $\theta_{s'} \in \mathcal{V}\left(s', (k_+^h)_{h \in \mathcal{H}}\right)$. Because g is a closed valued correspondence, $\left(\theta, (\theta_{s'})_{s' \in \mathcal{S}}\right) \in g(s, \omega)$, which implies $\theta \in \mathcal{W}(s, \omega)$.

P2. If $\mathcal{V} \subset \mathcal{V}'$ in the sense that $\mathcal{V}(s, \omega) \subset \mathcal{V}'(s, \omega)$ for all $s \in \mathcal{S}$ and $\omega \in \Omega$ then $\mathcal{G}(\mathcal{V}) \subset \mathcal{G}(\mathcal{V}')$.

P3. Let $\mathcal{V}^0$ denote the complete correspondence: $\mathcal{V}^0(s, \omega) = \Theta$ for all $s \in \mathcal{S}$ and $\omega \in \Omega$. Then $\mathcal{G}(\mathcal{V}^0) \subset \mathcal{V}^0$.

Given $\mathcal{V}^0$, we construct the sequence of $\{\mathcal{V}^n\}_{n=0}^\infty$ recursively using $\mathcal{G}$: $\mathcal{V}^{n+1} = \mathcal{G}(\mathcal{V}^n)$. Then by P1, P2, and P3, we have $\mathcal{V}^{n+1} \subset \mathcal{V}^n$ and is non-empty, compact valued. Non-empty-ness comes from the existence of competitive equilibrium in the $n+1$-horizon economy proved in Lemma 13.

Let $\mathcal{V}^*$ be defined by

$$\mathcal{V}^*(s, \omega) = \cap_{n=0}^\infty \mathcal{V}^n(s, \omega).$$

Since $\mathcal{V}^*(s, \omega)$ is the intersection of decreasing compact sets, $\mathcal{V}^*(s, \omega)$ is compact and is non-empty. We show that $\mathcal{G}(\mathcal{V}^*) = \mathcal{V}^*$.

Indeed, by the definition of $\mathcal{V}^*$, we also have $\mathcal{V}^* \subset \mathcal{V}^n$, to $\mathcal{G}(\mathcal{V}^*) \subset \mathcal{G}(\mathcal{V}^n) = \mathcal{V}^{n+1}$, so $\mathcal{G}(\mathcal{V}^*) \subset \mathcal{V}^*$.

Now, for each $s \in \mathcal{S}$ and $\omega \in \Theta$ and $\theta = \left(\left\{ c^h, k_+^h, v^h \right\}_{h \in \mathcal{H}}, r, w \right) \in \mathcal{V}^*(s, \omega)$. Since $\mathcal{V}^* \subset \mathcal{V}^n$, there exists $\theta_{s'}^n \in \mathcal{V}^n\left(s', (k_+^h)_{h \in \mathcal{H}}\right)$ and $\left(\theta, (\theta_{s'}^n)_{s' \in \mathcal{S}}\right) \in g(s, \omega)$. By the compactness of Θ, we can find a converging subsequence $\{n_l\}_{l=0}^\infty$, $\left(\theta_{s'}^{n_l}\right)_{s' \in \mathcal{S}} \longrightarrow_{l \to \infty} (\theta_{s'})_{s' \in \mathcal{S}}$. By the compactness of $\mathcal{V}^{n_l}$, we have $\theta_{s'} \in \mathcal{V}^{n_l}\left(s', (k_+^h)_{h \in \mathcal{H}}\right)$ and since g has closed graph,

$(\theta, (\theta_{s'})_{s' \in \mathcal{S}}) \in g(s, \omega)$. Moreover, $\mathcal{V}^n \left(s', (k_+^h)_{h \in \mathcal{H}}\right)$ is a decreasing sequence so $\theta_{s'} \in \cap_{l=0}^{\infty} \mathcal{V}^{n_l} \left(s', (k_+^h)_{h \in \mathcal{H}}\right) = \mathcal{V}^* \left(s', (k_+^h)_{h \in \mathcal{H}}\right)$. So by the definition of $\mathcal{G}$, we have $\theta \in \mathcal{G}(\mathcal{V}^*)$. Therefore $\mathcal{V}^* \subset \mathcal{G}(\mathcal{V}^*)$.

Since $\mathcal{G}(\mathcal{V}^*) \subset \mathcal{V}^* \subset \mathcal{G}(\mathcal{V}^*)$, it implies that $\mathcal{G}(\mathcal{V}^*) = \mathcal{V}^*$.

Let $\mathcal{Q} = \mathcal{V}^*$. Since $\mathcal{G}(\mathcal{Q}) = \mathcal{Q}$, for each $s \in \mathcal{S}$ and each $\omega = (k^h)_{h \in \mathcal{H}} \in \Omega$, $\theta = \left((c^h, k_+^h, v^h)_{h \in \mathcal{H}}, r, w\right) \in \mathcal{Q}(s, \omega)$, there exists $\theta_{s'} \in \mathcal{Q}\left(s', (k_+^h)_{h \in \mathcal{H}}\right)$ for each $s' \in \mathcal{S}$ and $(\theta, (\theta_{s'})_{s' \in \mathcal{S}}) \in g(s, \omega)$. We also define $\mathcal{T}$ as

$$\mathcal{T}(s, \omega) = \left\{ \left(k_+^h\right)_{h \in \mathcal{S}} : \left(\left(c^h, k_+^h, v^h\right)_{h \in \mathcal{H}}, r, w\right) \in \mathcal{Q}(s, \omega) \text{ for some } \left(c^h, v^h\right)_{h \in \mathcal{H}} \text{ and some } r, w \right\}.$$

It is immediate that $(\mathcal{P}, \mathcal{T})$ defined as such forms a GRE for the economy with a finite number of types. □

Proof of Lemma 12. Consider sequences of allocation and prices generated by a GRE, starting from $s_0 \in \mathcal{S}$ and $\{k_0^h\}_{h \in \mathcal{H}} \in \Omega$. That is, sequences $\{c_t^h(s^t), k_{t+1}^h(s^t), v_t^h(s^t)\}_{t, s^t, h}$ and $\{r_t(s^t), w_t(s^t)\}_{t, s^t}$ such that for each t, s^t

$$\left(\left\{c_t^h(s^t), k_{t+1}^h(s^t), v_t^h(s^t)\right\}_{h \in \mathcal{H}}, r_t(s^t), w_t(s^t)\right) \in \mathcal{Q}\left(s_t, \left\{k_t^h(s^t)\right\}_{h \in \mathcal{H}}\right),$$

and

$$\left(\left\{c_{t+1}^h(s^{t+1}), k_{t+2}^h(s^{t+1}), v_{t+1}^h(s^{t+1})\right\}_{h \in \mathcal{H}}, r_{t+1}(s^{t+1}), w_t(s^{t+1})\right) \in \mathcal{Q}\left(s_{t+1}, \left\{k_{t+1}^h(s^t)\right\}_{h \in \mathcal{H}}\right),$$

and Conditions 3-4 in Definition 6 are satisfied (with the variable without subscript stands for the variable at time t, the variables with subscript $+$ stands for the variables at time $t+1$ and the variables with subscript $++$ stands for the variables at time $t+2$, for example c^h stands for c_t^h, c_+^h stands for c_{t+1} and c_{++}^h stands for c_{t+2}^h, etc.).

The market clearing conditions are satisfied obviously. We just need to verify that given $\{r_t(s^t), w_t(s^t)\}$, the allocation $\{c_t^h(s^t), k_{t+1}^h(s^t)\}_{t, s^t}$ solves agent h's maximization problem, (72). That is for any alternative allocation $\{\tilde{c}_t^h(s^t), \tilde{k}_{t+1}^h(s^t)\}_{t, s^t}$ that satisfies (70) and (71), we have

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \left(\Pi_{t'=0}^{t-1} \beta^h(s_{t'})\right) u(c_t^h)\right] \geq \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \left(\Pi_{t'=0}^{t-1} \beta^h(s_{t'})\right) u(\tilde{c}_t^h)\right]. \tag{85}$$

The proof of this inequality follows closely Duffie et al. (1994). First, we show by induction that for all $T \geq 0$:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \left(\Pi_{t'=0}^{t-1} \beta^h(s_{t'})\right) u(c_t^h)\right] \geq \mathbb{E}_0 \left[\sum_{t=0}^{T} \left(\Pi_{t'=0}^{t-1} \beta^h(s_{t'})\right) u(\tilde{c}_t^h)\right] + \mathbb{E}_0 \left[\sum_{T+1}^{\infty} \left(\Pi_{t'=0}^{t-1} \beta^h(s_{t'})\right) u(c_t^h)\right]$$
$$+ \mathbb{E}_0 \left[\left(\Pi_{t'=0}^{T-1} \beta^h(s_{t'})\right) u'(c_T^h) \left(\tilde{k}_{T+1}^h - k_{T+1}^h\right)\right]. \tag{86}$$

For $T = 0$, inequality (86) is

$$u(c_0^h) \geq u(\tilde{c}_0^h) + u'(c_0^h) \left(\tilde{k}_1^h - k_1^h\right),$$

which is true because

$$u(c_0^h) \geq u(\tilde{c}_0^h) + u'(c_0^h)\left(c_0^h - \tilde{c}_0^h\right),$$

from the concavity of $u(.)$ and from

$$\tilde{c}_0^h + \tilde{k}_1^h \leq (1-\delta+r_0)k_0^h = c_0^h + k_1^h,$$

which implies $c_0^h - \tilde{c}_0^h \geq \tilde{k}_1^h - k_1^h$.

Now, assume that (86) holds for T, we need to show that it also holds for $T+1$, i.e.

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(\Pi_{t'=0}^{t-1}\beta^h(s_{t'})\right)u(c_t^h)\right] \geq \mathbb{E}_0\left[\sum_{t=0}^{T+1}\left(\Pi_{t'=0}^{t-1}\beta^h(s_{t'})\right)u(\tilde{c}_t^h)\right] + \mathbb{E}_0\left[\sum_{T+2}^{\infty}\left(\Pi_{t'=0}^{t-1}\beta^h(s_{t'})\right)u(c_t^h)\right]$$
$$+\mathbb{E}_0\left[\left(\Pi_{t'=0}^{T}\beta^h(s_{t'})\right)u'(c_{T+1}^h)\left(\tilde{k}_{T+2}^h - k_{T+2}^h\right)\right]. \tag{87}$$

Given (86) holds for T, to show (87), we just need to show:

$$\mathbb{E}_0\left[\left(\Pi_{t'=0}^{T}\beta^h(s_{t'})\right)u(c_{T+1}^h)\right] + \mathbb{E}_0\left[\left(\Pi_{t'=0}^{T-1}\beta^h(s_{t'})\right)u'(c_T^h)\left(\tilde{k}_{T+1}^h - k_{T+1}^h\right)\right]$$
$$\geq \mathbb{E}_0\left[\left(\Pi_{t'=0}^{T+1}\beta^h(s_{t'})\right)u(\tilde{c}_{T+1}^h)\right] + \mathbb{E}_0\left[\left(\Pi_{t'=0}^{T}\beta^h(s_{t'})\right)u'(c_{T+1}^h)\left(\tilde{k}_{T+2}^h - k_{T+2}^h\right)\right].$$

Equivalently,

$$\mathbb{E}_0\left[\beta^h(s_T)u(c_{T+1}^h)\right] + \mathbb{E}_0\left[u'(c_T^h)\left(\tilde{k}_{T+1}^h - k_{T+1}^h\right)\right]$$
$$\geq \mathbb{E}_0\left[\beta^h(s_T)u(\tilde{c}_{T+1}^h)\right] + \mathbb{E}_0\left[\beta^h(s_T)u'(c_{T+1}^h)\left(\tilde{k}_{T+2}^h - k_{T+2}^h\right)\right]. \tag{88}$$

Because of Condition 5. in Definition 6,

$$u'(c_T^h) = \beta_T^h(s_T)\mathbb{E}_T\left[(1-\delta+r_{T+1})\,u'(c_{T+1}^h)\right],$$

if $k_{T+1}^h > 0$, and

$$u'(c_T^h) \geq \beta_T^h(s_T)\mathbb{E}_T\left[(1-\delta+r_{T+1})\,u'(c_{T+1}^h)\right],$$

if $k_{T+1}^h = 0$, which implies $\tilde{k}_{T+1}^h - k_{T+1}^h \geq 0$. Therefore

$$\mathbb{E}_0\left[u'(c_T^h)\left(\tilde{k}_{T+1}^h - k_{T+1}^h\right)\right]$$
$$\geq \mathbb{E}_0\left[\beta_T^h(s_T)\,(1-\delta+r_{T+1})\,u'(c_{T+1}^h)\left(\tilde{k}_{T+1}^h - k_{T+1}^h\right)\right].$$

From this inequality, we obtain (88) if

$$u(c_{T+1}^h) + (1-\delta+r_{T+1})\,u'(c_{T+1}^h)\left(\tilde{k}_{T+1}^h - k_{T+1}^h\right)$$
$$\geq u(\tilde{c}_{T+1}^h) + u'(c_{T+1}^h)\left(\tilde{k}_{T+2}^h - k_{T+2}^h\right). \tag{89}$$

Since

$$c_{T+1}^h + k_{T+2}^h = (1-\delta+r_{T+1})\,k_{T+1}^h$$
$$\tilde{c}_{T+1}^h + \tilde{k}_{T+2}^h \leq (1-\delta+r_{T+1})\,\tilde{k}_{T+1}^h,$$

we have

$$(1-\delta+r_{T+1})\left(\tilde{k}^h_{T+1}-k^h_{T+1}\right)\geq \tilde{k}^h_{T+2}-k^h_{T+2}+\tilde{c}^h_{T+1}-c^h_{T+1}.$$

Plugging this into (89), we obtain the desired inequality if

$$u(c^h_{T+1})+u'(c^h_{T+1})\left(\tilde{c}^h_{T+1}-c^h_{T+1}\right)\geq u(\tilde{c}^h_{T+1}),$$

which is true because $u(.)$ is concave.

Having established (86), we are ready to show (85). First we observe that, because Ω is compact, there exists $\bar{k}>0$ such that $k^h_t(s^t)\leq \bar{k}$ for all h,t,s^t. Now, from (86), taking $T\to\infty$ and noticing that

$$\mathbb{E}_0\left[\sum_{T+1}^{\infty}\left(\Pi_{t'=0}^{t-1}\beta^h(s_{t'})\right)u(c^h_t)\right]\geq \bar{\beta}^{T+1}\frac{1}{1-\bar{\beta}}u(\underline{c})\to_{T\to\infty}0$$

and

$$\begin{aligned}&\mathbb{E}_0\left[\left(\Pi_{t'=0}^{T-1}\beta^h(s_{t'})\right)u'(c^h_T)\left(\tilde{k}^h_{T+1}-k^h_{T+1}\right)\right]\\ &\geq -\mathbb{E}_0\left[\left(\Pi_{t'=0}^{T-1}\beta^h(s_{t'})\right)u'(c^h_T)k^h_{T+1}\right]\\ &\geq -\bar{\beta}^T u'(\underline{c})\bar{k}\longrightarrow_{T\to\infty}0,\end{aligned}$$

we obtain (85). □

E Numerical Algorithm and Examples

The existence proofs in Section 2 and Appendix D also suggest an algorithm to compute recursive equilibria, alternative to the one put forth in Krusell and Smith (1998), using the equilibria in finite horizon economies. The next subsection presents the algorithm and the one following presents two numerical examples for two-agent economies.

E.1 Numerical Algorithm

I propose an algorithm to compute the GRE as defined in Definition 6 for finite agent economies, assuming that the equilibrium is indeed a recursive equilibrium. That is, we seek to compute the functions (instead of correspondences) $\mathcal{Q}$ and $\mathcal{T}$ defined over $\mathcal{S}\times\Omega$.

Notice that Ω can be re-parametrized as:

$$\Omega=[\underline{K},\bar{K}]\times\Delta^H=\left\{\left(K,\left(\omega^h\right)_{h\in\mathcal{H}}\right):\underline{K}\leq K\leq\bar{K}\text{ and }0\leq\omega^h\leq 1,\sum_{h\in\mathcal{H}}\omega^h=1\right\},$$

where $\omega^h_t=\frac{k^h_t}{HK_t}$. We calculate recursively, for each $T\geq 0$, the function φ_T from Ω, the set of current wealth distributions, to current prices and allocations, and to future wealth distributions. Function φ_T corresponds to the equilibrium mapping, for the $(T+1)-$horizon economy presented in Appendix D.2, from the initial distribution of capital holdings and aggregate shock in period 0 to allocation and prices in the period. Indeed,

$$\varphi_T : \quad \mathcal{S} \times [\underline{K}, \bar{K}] \times \Delta^H \rightrightarrows \quad \mathbb{R}_+^{4H+2}$$
$$\left\{ s \in \mathcal{S}, K \in [\underline{K}, \bar{K}], \omega \in \Delta^H \right\} \mapsto \left\{ \left(c^h, k_+^h, \lambda^h, v^h \right)_{h \in \mathcal{H}}, r, w \right\}$$

defined as follows.

1. For $T = 0$:

$$\varphi_0 : \left\{ s \in \mathcal{S}, K \in [\underline{K}, \bar{K}], \omega \in \Delta^H \right\} \mapsto \left\{ \left(c^h, k_+^h, \lambda^h, v^h \right)_{h \in \mathcal{H}}, r, w \right\}$$

where

$$r = F_K(s, K, L(s)) \quad \text{and} \quad w = F_L(s, K, L(s))$$

and for all $h \in \mathcal{H}$

$$c^h = (1 - \delta + r)\, \omega^h K + w l^h(s) \quad \text{and} \quad v^h = u(c^h),$$

and

$$k_+^h = 0 \quad \text{and} \quad \lambda^h = 0.$$

2. For $T > 0$, assuming that we have calculated φ_{T-1}, φ_T is calculated as:

$$\varphi_T : \left\{ s \in \mathcal{S}, K \in [\underline{K}, \bar{K}], \omega \in \Delta^H \right\} \mapsto \left\{ \left(c^h, k_+^h, \lambda^h, v^h \right)_{h \in \mathcal{H}}, r, w \right\}$$

such that, for each $s' \in \mathcal{S}$,

$$\left(c_+^h, k_{++}^h, \lambda_+^h, v_+^h \right) = \varphi_{T-1} \left(s', K_+, \left(\omega_+^h \right)_{h \in \mathcal{H}} \right)$$

where

$$K_+ = \frac{1}{H} \sum_{h \in \mathcal{H}} k_+^h$$

and for each $h \in \mathcal{H}$:

$$\omega_+^h = \frac{k_+^h}{\sum k_+^h},$$

and Conditions 2.-5. in Definition 6 are satisfied:

A1. $\frac{1}{H} \sum_{h \in \mathcal{H}} c^h + \frac{1}{H} \sum_{h \in \mathcal{H}} k_+^h = F(s, K, L(s)) + (1 - \delta) K.$

A2. $r = F_K(s, K, L(s)) > 0$ and $w = F_L(s, K, L(s)) > 0$

A3. For each $h \in \mathcal{H}$

$$u'(c^h) = \beta^h(s) \sum_{s' \in \mathcal{S}} \pi_{ss'} \left(1 - \delta + r_+(s') \right) u'(c_+^h(s')) + \lambda^h \tag{90}$$

with $\lambda^h \geq 0$ and

$$\lambda^h k_+^h = 0$$

and

$$c^h + k_+^h = (1 - \delta + r) k^h + w l^h$$

and

$$v^h = u(c^h) + \beta^h(s) \sum_{s' \in \mathcal{S}} \pi_{ss'} v_+^h(s').$$

Condition A3 is a reformulation of Condition 5 in Definition 6 using the multipliers λ^h's and the complementary-slackness condition. Notice also that for each $\left(K, \left(\omega^h\right)_{h\in\mathcal{H}}\right) \in [\underline{K}, \bar{K}] \times \Delta^H$, the conditions in A2. and A3. gives us $4H+2$ equations for $4H+2$ unknowns (including λ^h's). The market clearing, condition A1., is satisfied by summing up the budget constraint of each agent.

Discretization and Approximation We discretize Ω using Ω^d:

$$\Omega^d = \left\{\underline{K} = K_1^d < K_2^d < ... < K_N^d = \bar{K}\right\} \times \left\{\omega_m^d\right\}_{m=1}^M \tag{91}$$

where $\omega_m^d \in \Delta^H$ for $m = 1, ..., M$.

Let φ_T^d denote the discrete approximation of φ_T over $\mathcal{S} \times \Omega^d$. For each $s \in \mathcal{S}$ and at each $\left(K_n^d, \omega_m^d\right)$, we solve for

$$\varphi_T^d\left(s, K_n^d, \omega_m^d\right) = \left(c^h, k_+^h, \lambda^h, v^h\right)$$

such that Conditions A2 and A3 are satisfied. In Conditions A2 and A3, the future values $\hat{c}_+^h, \hat{v}_+^h$ are computed using multi-dimensional cubic splines approximation:

$$\left(c_+^h, v_+^h, \lambda_+^h, k_{++}^h\right) = \varphi_{T-1}^d\left(s', \frac{1}{H}\sum k_+^h, \frac{k_+^h}{\sum k_+^h}\right). \tag{92}$$

Fixing a precision ν, the algorithm converges when

$$\left\|\varphi_T^d - \varphi_{T-1}^d\right\|_{\mathcal{S}\times\Omega^d} \leq \nu.$$

E.2 Numerical Results

We present two numerical examples in economies with two agents. When $H = 2$, we just need to keep track of the wealth share of agent 1 because $\omega^2 = 1 - \omega^1$. Therefore, in (91),

$$\left\{\omega_m^d\right\}_{m=1}^M = \left\{\tilde{\omega}_m^1\right\}_{m=1}^M$$

where

$$0 = \tilde{\omega}_1^1 < \tilde{\omega}_2^1 < ... < \tilde{\omega}_M^1 = 1.$$

In the first example, Subsection E.2.1, the agents differ in labor productivity but have the same discount factor. In the second example, Subsection E.2.2, the agents have the same labor productivity but differ in their discount factor.

E.2.1 Heterogeneous Income

There are two representative agents $h \in \{1, 2\}$ in the economy of mass $\frac{1}{2}$ each. The agents share the same intertemporal expected utility

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \log c_t^h\right].$$

In each period, the exogenous aggregate state of the economy is a pair of states (s, i) where $s \in \{b, g\}$ and $i \in \{0, 1\}$. State s determines the aggregate productivity $A(s)$ and aggregate labor supply $L(s)$. The aggregate production function is Cobb-Douglas, (2).

State i determines which agent is employed. If $i = 0$ then agent 1 is unemployed and agent 2 is employed, and vice versa for $i = 1$.[26] The employed agent has $2(1 - v)L(s)$ units of labor and the unemployed agent has $2vL(s)$ units of labor. v stands for unemployment transfers by the government and is set at 7%.

The parameters are taken from Krusell and Smith (1998, Section 2) in particular, the discount rate and the production parameters are:

$$\beta = 0.99 \quad \delta = 0.025 \quad \alpha = 0.36.$$

The aggregate productivity and aggregate labor supply are:

$$\begin{bmatrix} A(b) & A(g) \end{bmatrix} = \begin{bmatrix} 0.99 & 1.01 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} L(b) & L(g) \end{bmatrix} = \begin{bmatrix} 0.2944 & 0.3140 \end{bmatrix}, \tag{93}$$

with the transition matrix $\pi = [\pi_{ss'ii'}]$, directly taken from Krusell and Smith (1998, Section 2):[27]

$$\pi = \begin{bmatrix} 0.5250 & 0.3500 & 0.0312 & 0.0938 \\ 0.0389 & 0.8361 & 0.0021 & 0.1229 \\ 0.0938 & 0.0312 & 0.2917 & 0.5833 \\ 0.0091 & 0.1159 & 0.0243 & 0.8507 \end{bmatrix}$$

Figure 1 shows next period aggregate capital, K_+ as a function of current period aggregate capital, K, in state $(b, 0)$, for two different values of ω: $\omega = 0$ and $\omega = 1$. The figure shows that future aggregate capital depends on not only current aggregate capital but also on current wealth share ω of agent 1.

Given the global nonlinear solution for φ_∞, we can also simulate forward and carry out a regression exercise as in Krusell and Smith (1998). From 10,000-period simulation (with the first 1000 periods dropped), we obtain the following regression results:

$$\log K' = 0.0438 + 0.9832 \log K; \qquad R^2 = 0.999223$$

in good times and

$$\log K' = 0.0167 + 0.9923 \log K; \qquad R^2 = 0.997372$$

in bad times. These regression results tell us that, in the simulated paths of the economy, current aggregate capital seems to be a sufficient state variable to forecast future aggregate capital, which Krusell and Smith call an "approximate aggregation" property. However, Figure 1 tells us that this property does not hold globally.

As a comparison, we also solve the Krusell and Smith's model, with the exact parameters above, but in which idiosyncratic shocks are truly idiosyncratic. We obtain the following regression results:

$$\log K' = 0.0906 + 0.9631 \log K; \qquad R^2 = 0.999999$$

[26] This approximation of a fully idiosyncratic income process using a two agent income process is similar to the approximation in Heaton and Lucas (1995).

[27] In the transition matrix, we use the convention $\{1, 2, 3, 4\}$ correspond to $\{(b, 0), (b, 1), (g, 0), (g, 1)\}$ respectively.

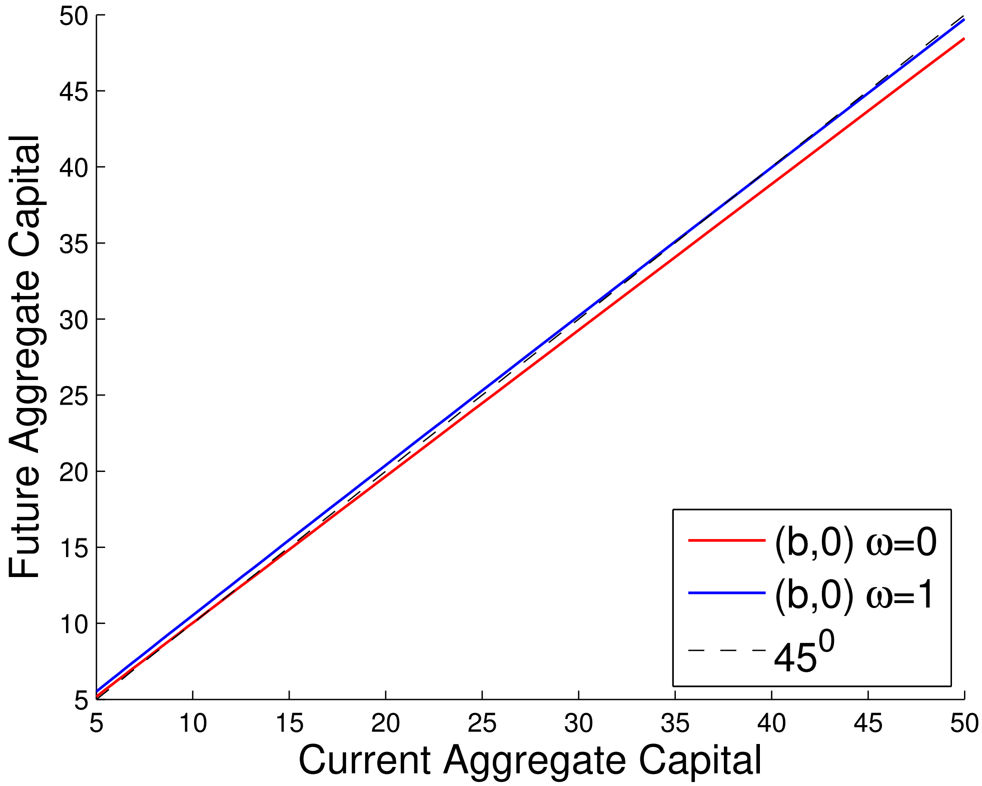

Figure 1: Evolution of Aggregate Capital in Bad Times

in good times

$$\log K' = 0.0807 + 0.9651 \log K; \qquad R^2 = 0.999999$$

and in bad times.

The approximate evolution of aggregate capital is not too different in the two-agent economy compared to the Krusell and Smith's economy. But we observe that the auto-correlation coefficients for log aggregate capital are lower than those in the two-agent economy. The R^2 are also slightly higher than in the two-agent economy.

E.2.2 Heterogenous Betas

In this example, we assume that the agents face idiosyncratic shocks that determine their discount rate. The discount factor can be low ($\underline{\beta}$) or high ($\bar{\beta}$), where:

$$\underline{\beta} = 0.9858 \quad \text{and} \quad \bar{\beta} = 0.9930,$$

taken from Krusell and Smith (1998, Section 3). As in their paper, the transition from one to the other is determined such that the average duration for individual β is 50 years, which corresponds to agents' lifetime. To simplify the exercise, we assume that the two agents have the same labor productivity, which varies with the aggregate state, s. The aggregate productivity and aggregate labor supply are given in (93). The evolution of the aggregate state is the same as in the previous example. The other aggregate state i determines the agents' discount factor ($i = 0$ agent 1 has low discount factor and agent 2 has high discount factor and vice versa for $i = 1$). The evolution of aggregate state i is independent of the evolution of aggregate state s.

Figure 2 shows next period aggregate capital, K_+ as a function of current period aggregate capital, K, in state $(b, 0)$, for two different values of ω: $\omega = 0$ and $\omega = 1$. The figure shows that future aggregate capital depends mostly on current aggregate capital and does not vary visibly with the current wealth share ω of agent 1.

As in the previous example, from 10,000-period simulation (with the first 1000 periods dropped), we obtain the following regression results:

$$\log K' = 0.0916 + 0.9633 \log K; \qquad R^2 = 0.999999$$

in good times

$$\log K' = 0.0789 + 0.9662 \log K; \qquad R^2 = 0.999999$$

and in bad times. Because future aggregate capital depends mostly on current aggregate capital, the fitness of the linear regressions are very high.

As in the previous example, these regression results are comparable to the ones in Krusell and Smith (1998)'s model in which the discount rates are truly idiosyncratic:

$$\log K' = 0.0871 + 0.9662 \log K; \qquad R^2 = 0.999981$$

in good times and

$$\log K' = 0.0836 + 0.9670 \log K; \qquad R^2 = 0.999976$$

in bad times.

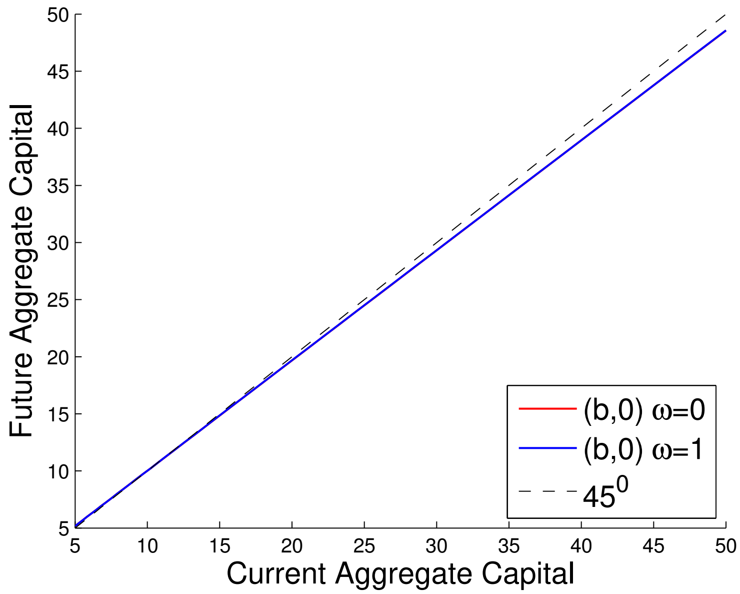

Figure 2: Evolution of Aggregate Capital in Bad Times

E.3 Discussion of the Algorithm for Many Agents and for Continuum of Agents

The discretization and approximation method laid out in Subsection E.1 applies to general model in Section D with many agents. However, when the number of agents is larger than 2, the algorithm suffers from the curse of dimensionality, i.e., it takes many points to discretize Ω using Ω^d (the number of points is approximately $N^{\dim(\Omega)}$ where N is the number of points used to discretize each dimension). There are two ways to get around this problem.

First, notice that from Conditions A1-A3, in Subsection E.1, we just need to solve for $\left(c^h\right)_{h\in\mathcal{H}}$ as functions of the exogenous and endogenous states $(s, K, \omega) \in \Omega$. The idea is to approximate numerically c^h's using some basis functions

$$\{\xi_1, \xi_2, ..., \xi_m\}$$

that is

$$c^h\left(s, K, \omega\right) = \sum_{i=1}^{m} \hat{c}_i^h \xi(s, K, \omega).$$

We can then solve for the approximation coefficients $\left(\hat{c}_i^h\right)_{i=1,2,...,,m}^{h\in\mathcal{H}}$. The advantage of this algorithm is that the number of basis functions can be significantly smaller than the number of points to discretize Ω and does not increase fast with the dimension of Ω. The basis functions can be polynomials as in Judd (1992) and Gaspar and Judd (1997). Second, we can use Smolyak (1963)'s sparse-grid collocation method. The method only requires knowing the value of c^h's at a few number of collocation points in Ω to approximate the whole functions. A comprehensive exposition of the method and applications can be found in Maliar and Maliar (2014).

Using these ideas, the algorithm in Subsection E.1 can potentially be applied to the model in Section 2 with a continuum of agents. However, there are two major difficulties. First, the endogenous state space Ω of probability measures is infinite-dimensional. Therefore, we need to approximate Ω with a finite-dimensional space. For example, one can approximate Ω with the set of convex combinations of Dirac masses, i.e., for each $\mu \in \Omega$ we approximate μ by

$$\mu \approx \sum_{i \in \mathcal{I}} \sum_{j=1}^{M} \hat{\mu}_{i,j} \mathcal{D}(k_j)$$

where $0 \leq k_1 < k_2 < ... < k_M \leq \bar{k}$, and $\sum_{i \in \mathcal{I}} \sum_{j=1}^{M} \hat{\mu}_{i,j} = 1$.[28] Second, following Definition 2, for each $\mu \in \Omega$, we need to solve for the value and policy functions, $\hat{V}$ and $\hat{k}$ (in contrast to the case with a finite number of agents, we just need to solve for a vector of current consumptions and future capital holdings). In other words, we need to solve for the value and policy functions that depend both on capital holding and wealth distribution, $\hat{V}(k, i; s, \mu)$ and $\hat{k}(k, i; s, \mu)$. Having approximated Ω with a finite $n-$dimensional space, $\hat{V}$ and $\hat{k}$ become functions over $(n+1)$ dimensions. They can then be approximated using basis functions or Smolyak's sparse grid method.

While these are viable paths to implement the algorithm for many agents or for a continuum of agents, they would require a significant amount of engineering and thus lie outside the scope of the present paper.

[28] Approximating distributions using Dirac masses is similar the histogram technique used in Reiter (2010) and Young (2010).