Multiple Treatments with Strategic Interaction[*]

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Abstract

We develop an empirical framework in which we identify and estimate the effects of treatments on outcomes of interest when the treatments are results of strategic interaction (e.g., bargaining, oligopolistic entry, decisions in the presence of peer effects). We consider a model where agents play a discrete game with complete information whose equilibrium actions (i.e., binary treatments) determine a post-game outcome in a non-separable model with endogeneity. Due to the simultaneity in the first stage, the model as a whole is incomplete and the selection process fails to exhibit the conventional monotonicity. Without imposing parametric restrictions or large support assumptions, this poses challenges in recovering treatment parameters. To address these challenges, we first analytically characterize regions that predict equilibria in the first-stage game with possibly more than two players, whereby we find a certain monotonic pattern of these regions. Based on this finding, we derive bounds on the average treatment effects (ATE's) under nonparametric shape restrictions and the existence of excluded variables. We also introduce and point identify a multi-treatment version of local average treatment effects (LATE's).

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1 Introduction

We develop an empirical framework in which we identify and estimate the heterogeneous effects of treatments on outcomes of interest where the treatments are results of strategic

interaction (e.g., bargaining, oligopolistic entry, decisions in the presence of peer effects or strategic effects). Treatments are determined as an equilibrium of a game and these strategic decisions of players endogenously affect common or player-specific outcomes. For example, one may be interested in the effects of newspaper entry on local political behaviors, the effects of entry of carbon-emitting companies on local air pollution and health outcomes, the effects of the presence of potential entrants in nearby markets on pricing or investment decisions of incumbents, the effects of large supermarkets' exit decisions on local health outcomes, and the effects of provision of limited resources where individuals make participation decisions under peer effects as well as based on their own gains from the treatment. As reflected in some of these examples, our framework allows us to study *externalities of strategic decisions*, such as societal outcomes resulting from firm behaviors. Ignoring strategic interaction in treatment selection processes may lead to biased, or at least less informative, conclusions about the effects of interest.

We consider a model where agents play a discrete game with complete information, whose equilibrium actions (i.e., a profile of binary endogenous treatments) determine a post-game outcome in a nonseparable model with endogeneity. We are interested in various treatment parameters in this model. In recovering these parameters, the setting of this paper poses several challenges. First, the first-stage game posits a structure in which binary dependent variables are simultaneously determined, thereby making the model as a whole *incomplete.* Second, due to this simultaneity, the selection process does not exhibit the conventional monotonic property á la Imbens and Angrist (1994). Furthermore, we want to remain flexible with other components of the model. That is, we make no assumptions on the joint distributions of the unobservables nor parametric restrictions on the player's payoff function and on how treatments affect the outcome. Also, we do not impose any arbitrary equilibrium selection mechanism as a way of solving the multiplicity of equilibria. In nonparametric models with multiplicity or/and endogeneity, identification may be achieved with excluded instruments of large support. Even though such a requirement is met in practice, estimation and inference can still be problematic (Andrews and Schafgans (1998), Khan and Tamer (2010)). We thus allow instruments and other exogenous variables to be discrete and have small supports.

The first contribution of this paper is that, as an important initial step to address the challenges described above, we analytically characterize regions that predict equilibria in the first-stage game. Complete analytical characterization of the equilibrium regions for more than two players has not been studied in the literature.[1] Under symmetry and strategic substitutability restrictions on the payoff functions, we fully characterize the geometric properties of the regions in the space of unobservables, which describe the properties of equilibria in the game. More importantly, we show that these regions exhibit a monotonic pattern in terms of the number of players who choose to take the action—e.g., the number of entrants in an entry game.

The second contribution of this paper is that, after restoring a generalized version of monotonicity in the selection process, we show how the model structure and the data can be informative about treatment parameters, such as the average treatment effects (ATE's)

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[1]To estimate payoff parameters, Berry (1992) partly characterizes equilibrium regions. To calculate the bounds on these parameters, Ciliberto and Tamer (2009) simulate their moment inequalities model that are implied by the shape of these regions, especially the regions for multiple equilibria. While their approaches are enough for the purpose of their analyses, full analytical results are critical for the identification analysis of the current paper.

and the local ATE (LATE's). We first establish the bounds on the ATE and other related parameters with possibly discrete instruments of small support. We also show that tighter bounds on the ATE can be obtained by introducing (possibly discrete) exogenous variables excluded from the first-stage game. This is especially motivated when the outcome variable is generated from externalities incurred by the players. We can derive sharp bounds as long as the outcome variable is binary. Further, with continuous instruments of large supports, we show that multiplicity and endogeneity become irrelevant and the ATE is point identified. To derive informative bounds, we impose nonparametric shape restrictions on the outcome function, such as conditional symmetry and uniformity. The symmetry assumption is eventually relaxed by using instruments that vary enough to offset the effect of strategic substitutability. We provide a simple testable implication for the existence of such instruments variation provided that the payoff unobservables are mutually independent. The symmetry assumption may alternatively be relaxed by assuming that strategic interaction occurs only within subgroups of players, thus allowing for partial symmetry. Next, we introduce and point identify a multi-treatment version of the LATE. The simultaneity in the selection process does not permit the usual equivalence result by Vytlacil (2002) between the specification of a threshold-crossing selection rule and Imbens and Angrist (1994)'s monotonicity assumption. A monotonic pattern restored for the equilibrium regions, however, enables us to recover the LATE for a treatment of "dichotomous states." A marked feature of our analyses is that for the sharp bounds on the ATE and the identification of the LATE, player-specific instruments are not necessary.

Partial identification in single-agent nonparametric triangular models with binary endogenous variables has been studied in Shaikh and Vytlacil (2011) and Chesher (2005), among others. Shaikh and Vytlacil (2011) provide bounds on the ATE in this setting. In a slightly more general model, Vytlacil and Yildiz (2007) achieve point identification with an exogenous variable that is excluded from the selection equation and has a large support. Our bound analysis builds on these papers, but we study a multi-agent model with strategic interaction as a key component of the model. A few existing studies have extended a single-treatment model to a multiple-treatment setting (e.g., Heckman et al. (2006), Jun et al. (2011)), but their models maintain monotonicity in the selection process and none of them allow simultaneity among the multiple treatments resulting from agents' interaction as we do in this paper.

In interesting recent work, Pinto (2015), Heckman and Pinto (2015), and Lee and Salanié (2016) extend the monotonicity of the selection process in multi-valued treatments settings, but they generally consider different types of treatment selection mechanisms than ours. Pinto (2015) and Heckman and Pinto (2015) introduce unordered monotonicity, and Lee and Salanié (2016) consider more general non-monotonicity. The latter paper does mention entry games as one example of treatment selection processes they allow, but they assume known payoffs and bypass the multiplicity of equilibria, which is one of the emphases of our paper. Also, Lee and Salanié (2016)'s main focus is on identification of marginal treatment effects with continuous instruments. In another important work, Chesher and Rosen (2017) consider a wide class of generalized instrumental variable models in which our model falls and propose a systematic method of characterizing sharp identified sets for admissible structures. The focus of the present paper is to point and partially identify particular structural features (i.e., treatment parameters) analytically, and to investigate how the identification is related to the

exogenous sources of variation in the model and to the equilibrium characterization in the treatment selection process. Calculating the sharp bounds on these treatment parameters using their general approach involves projections of identified sets that may require additional parametric restrictions. Lastly, Han (2018) considers nonparametric identification of optimal treatment regimes and dynamic treatment effects, where generalized monotonicity in the determination of each period's outcome and treatment is restored under some nonparametric shape restrictions.

Without triangular structures, Manski (1997), Manski and Pepper (2000) and Manski (2013) also propose bounds on the ATE with multiple treatments under various monotonicity assumptions, including an assumption on the sign of treatment response. We take an alternative approach that is more explicit about treatments interaction while remaining agnostic about the direction of treatment response. Our results suggest that, provided that there exist exogenous variation excluded from the selection process, the bounds calculated from this approach can be more informative than those from their approach. Among these papers, Manski (2013) is the closest to ours in that it considers multiple treatments and multiple agents with simultaneous interaction, but with an important difference from our approach. The interaction in his setting is through individuals which are the unit of observation. On the other hand, our setting features the interaction through the treatment/player unit, and the unit of observation is i.i.d. markets or regions in which the first-stage game is played and from which the outcome variable may emerge.

Identification in models for binary games with complete information has been studied in Tamer (2003) and Ciliberto and Tamer (2009), Bajari et al. (2010), among others. The present paper contributes to this literature by considering post-game outcomes in the model, especially those that are not the game players' direct concerns. As related work that considers post-game outcomes, Ciliberto et al. (2016) introduce a model where firms make simultaneous decisions of entry and pricing upon entry. As a result, their model can be seen as a multi-agent extension of a sample selection model. The model considered in this paper, on the other hand, is a multi-agent extension of a model for endogenous treatments. Ciliberto and Tamer (2009) and Ciliberto et al. (2016) recover model primitives as their parameters of interest and they impose parametric assumptions to facilitate their analyses. In contrast, our parameters of interest are functionals of the primitives (but excluding the game parameters) and thus allow our model to remain essentially nonparametric. Also a different approach to partial identification under multiplicity is employed, as their approach is not applicable to the particular setting of this paper even if the distribution of the unobserved payoff types is assumed to be known.

The paper is organized as follows. Section 2 introduces the model, the parameters of interest, and motivating examples. As the first main result of this paper, Section 3 presents the analytical characterization of equilibrium regions for many players. Section 4 delivers the partial identification results of this paper. We start by conducting the bound analysis on the ATE's for a two-player case and a binary dependent variable as an illustration. Then we extend the results to a many-player case with a more general dependent variable. Section 5 relaxes the symmetry assumption introduced in the previous section, and Section 6 discusses an extension of the model, point identification under large support, and relationship to Manski (2013). The LATE parameter is introduced and identified in Section 7. Section 8 presents a numerical illustration. Unless noted, the proofs of theorems and lemmas are collected in

Appendix D.

For a generic $\tilde{S}$-vector $\boldsymbol{v} \equiv (v_1, ...v_{\tilde{S}})$, let $\boldsymbol{v}_{-s}$ denote an $(\tilde{S}-1)$-vector where s-th element is dropped from v, i.e., $\boldsymbol{v}_{-s} \equiv (v_1, ..., v_{s-1}, v_{s+1}, ..., v_{\tilde{S}})$. When no confusion arises, we sometimes change the order of entry and write $\boldsymbol{v} = (v_s, \boldsymbol{v}_{-s})$ for convenience. For a multivariate function $f(\boldsymbol{v})$, the integral $\int_A f(\boldsymbol{v})d\boldsymbol{v}$ is understood as a multi-dimensional integral over a set A contained in the space of $\boldsymbol{v}$. Vectors in this paper are row vectors.

2 Setup and Motivating Examples

Let $\boldsymbol{D} \equiv (D_1, ..., D_S) \in \mathcal{D} \subseteq \{0,1\}^S$ be a S-vector of observed binary treatments and $\boldsymbol{d} \equiv (d_1, ..., d_S)$ be its realization, where S is fixed. We assume that $\boldsymbol{D}$ is predicted as a pure strategy Nash equilibrium of a complete information game with S players who make entry decisions or individuals who choose to receive treatments.[2] Let Y be an observed post-game outcome that results from profile $\boldsymbol{D}$ of endogenous treatments. It can be an outcome common to all players or an outcome specific to each player. Let $(X, Z_1, ..., Z_S)$ be observed exogenous covariates. We consider a model of a semi-triangular system:

$$Y = \theta(\boldsymbol{D}, X, \epsilon_{\boldsymbol{D}}), \tag{2.1}$$

$$D_s = 1\left[\nu^s(\boldsymbol{D}_{-s}, Z_s) \geq U_s\right], \qquad s \in \{1, ..., S\}, \tag{2.2}$$

where s is an index for players or interchangeably for treatments. Without loss of generality we normalize the scalar U_s to be distributed as $Unif(0,1)$, and $\nu^s : \mathbb{R}^{S-1+d_{z_s}} \to (0,1]$ and $\theta : \mathbb{R}^{S+d_x+1} \to \mathbb{R}$ are unknown functions that are nonseparable in their arguments. We allow the unobservables $(\epsilon_{\boldsymbol{D}}, U_1, ..., U_S)$ to be arbitrarily dependent to one another. Although the notation suggests that the instruments Z_s's are player/treatment-specific they are not necessarily required to be so for the analyses of this paper; see Appendix C for this discussion. The exogenous variables X are variables excluded from all the equations for D_s. The existence of X is not necessary but useful for the bound analysis of the ATE, and it can be motivated when Y is generated from externalities incurred by the players and thus does not enter the players' first-stage payoff functions. There may be covariates W common to all the equations for Y and D_s, which is suppressed for succinctness. Implied from the complete information game, player s's decision D_s depends on the decisions of all others $\boldsymbol{D}_{-s}$ in $\mathcal{D}_{-s}$, and thus $\boldsymbol{D}$ is determined by a simultaneous system. The model (2.1)–(2.2) is *incomplete*, i.e., the model primitives and the covariates do not uniquely predict $(Y, \boldsymbol{D})$ due to the possible existence of multiple equilibria in the first-stage game of treatment selection. Moreover, the conventional monotonicity in the sense of Imbens and Angrist (1994) is not exhibited in the selection process due to simultaneity. The unit of observation, indexed by market or geographical region i, is suppressed in all the expressions.

The potential outcome of receiving $\boldsymbol{D} = \boldsymbol{d}$ can be written as

$$Y_{\boldsymbol{d}} = \theta(\boldsymbol{d}, X, \epsilon_{\boldsymbol{d}}), \qquad \boldsymbol{d} \in \mathcal{D},$$

and $\epsilon_{\boldsymbol{D}} = \sum_{\boldsymbol{d} \in \mathcal{D}} 1[\boldsymbol{D} = \boldsymbol{d}]\epsilon_{\boldsymbol{d}}$. We are interested in the ATE and related parameters. With

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[2]While mixed strategy equilibria are not considered in this paper, it may be possible to extend the setup to incorporate mixed strategies following the argument in Ciliberto and Tamer (2009).

the average structural function (ASF) $E[Y_{\boldsymbol{d}}|X = x]$ for vector $\boldsymbol{d} \in \mathcal{D}$, the ATE can be written as

$$E[Y_{\boldsymbol{d}} - Y_{\boldsymbol{d}'}|X = x] = E[\theta(\boldsymbol{d}, x, \epsilon_{\boldsymbol{d}}) - \theta(\boldsymbol{d}', x, \epsilon_{\boldsymbol{d}'})], \tag{2.3}$$

for $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{D}$. Another parameter of interest is the average treatment effect on the treated (ATT): $E[Y_{\boldsymbol{d}} - Y_{\boldsymbol{d}'}|D = \boldsymbol{d}'', Z = z, X = x]$ for $\boldsymbol{d}, \boldsymbol{d}', \boldsymbol{d}'' \in \mathcal{D}$. Unlike the ATT or the treatment of the untreated in the single-treatment case, $\boldsymbol{d}''$ does not necessarily equal $\boldsymbol{d}$ or $\boldsymbol{d}'$ here. One might also be interested in the sign of the ATE, which in this multi-treatment case is essentially establishing an ordering among the ASF's. Lastly, we are interested in the LATE, which will be considered later after necessary concepts are introduced.

As an example of the ATE, we may choose $\boldsymbol{d} = (1, ..., 1)$ and $\boldsymbol{d}' = (0, ..., 0)$ to measure some cancelling-out effect, or we may be interested in more general nonlinear effects. Another example would be choosing $\boldsymbol{d} = (1, \boldsymbol{d}_{-s})$ and $\boldsymbol{d}' = (0, \boldsymbol{d}_{-s})$ for given $\boldsymbol{d}_{-s}$. In the latter example, we can learn interaction effects of treatments, i.e., how much the average gain (ATE) from treatment s is affected by other treatments: suppressing the conditioning on $X = x$,

$$E\left[Y_{1,\boldsymbol{d}_{-s}} - Y_{0,\boldsymbol{d}_{-s}}\right] - E\left[Y_{1,\boldsymbol{d}'_{-s}} - Y_{0,\boldsymbol{d}'_{-s}}\right],$$

where $Y_{\boldsymbol{d}}$ is interchangeably written as $Y_{\boldsymbol{d}_s,\boldsymbol{d}_{-s}}$ here. For example with $\boldsymbol{d}_{-s} = (1, ..., 1)$ and $\boldsymbol{d}'_{-s} = (0, ..., 0)$, complementarity between treatment s and all the other treatments can be represented as $E\left[Y_{1,\boldsymbol{d}_{-s}} - Y_{0,\boldsymbol{d}_{-s}}\right] - E\left[Y_{1,\boldsymbol{d}'_{-s}} - Y_{0,\boldsymbol{d}'_{-s}}\right] > 0$. Sometimes, we instead want to focus on learning about complementarity between two treatments, while averaging over the other $S - 2$ treatments. This can be dealt with a more general framework of defining the ASF and ATE by introducing a partial potential outcome; this is discussed in Appendix A.

In identifying these treatment parameters, suppose we attempt to recover the effect of a single treatment with $\boldsymbol{D}^1$ being a scalar in model (2.1)–(2.2) *conditional on* $\boldsymbol{D}^2 = \boldsymbol{D}_{-s} = \boldsymbol{d}_{-s}$, and then recover the effects of multiple treatments by transitively using these effects of single treatments. This strategy is not valid since D^2 is a function of D^1 and also due to multiplicity. Therefore, the approaches in the literature with single-treatment, single-agent triangular models are not directly applicable and a new theory is demanded in this more general setting.

We provide two examples to which model (2.1)–(2.2) may apply; other examples mentioned in the introduction are discussed in Appendix B.

Example 1 (Externality of airline entry). *In this example, we are interested in the effects of airline competition on local air quality and health. Consider multiple airline companies making entry decisions in local market i defined as a route that connects a pair of cities. Let Y_i denote the air pollution levels or average health outcomes of this local market. Let $D_{s,i}$ denote airline s's decision to enter market i, which is correlated with some unobserved characteristics of the local market that affect Y_i. The parameter $E[Y_{\boldsymbol{d},i} - Y_{\boldsymbol{d}',i}]$ captures the effects of a market structure on pollution or health. One interesting question would be whether the ATE is nonlinear in the number of airlines as companies may share the market and operate more efficiently when facing more competition. As related work, Schlenker and Walker (2015) document how sensitively local health outcomes, such as acute respiratory*

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diseases, are affected by the change in airline schedules. Economic activity variables, such as population and income, can be included in W_i, since they not only affect the outcomes but also the entry decisions. The excluded variable X_i can be characteristics of the local market that directly affect pollution or health levels, such as weather shocks or the share of pollution-related industries in the local economy. We assume that, conditional on W_i, these factors affect the outcome but do not enter the payoff functions of the airlines. The instruments $Z_{s,i}$ are cost shifters that affect entry decisions. When Y_i is a health outcome, pollution levels can be included in X_i.

Example 2 (Media and political behavior). *In this example, the question is how media affects political participation or electoral competitiveness. In county or market i, either $Y_i \in [0,1]$ can denote voter turnout, or $Y_i \in \{0,1\}$ can denote whether an incumbent is re-elected or not. Let $D_{s,i}$ denote a market entry decision by local newspaper type s, which is correlated with unobserved characteristics of the county. In this example, $Z_{s,i}$ can be the neighborhood counties' population size and income, which is common to all players $(Z_{1,i} = \cdots = Z_{S,i})$. Lastly, X_i can include changes in voter ID regulations. Using a linear panel data model, Gentzkow et al. (2011) show that the number of newspapers in the market significantly affects the voter turnout but find no evidence whether it affects the re-election of incumbents. More explicit modeling of the strategic interaction among newspaper companies can be important to capture competition effects on political behavior of the readers.*

3 Geometric Characterization of Equilibrium Regions

As an important step for the analyses of this paper, we formally characterize the regions in the space of the unobservables that predict equilibria of the treatment selection process in the first-stage game. The analytical characterization of the equilibrium regions when there are more than two players $(S > 2)$ can generally be complicated (Ciliberto and Tamer (2009, p. 1800)) and has not been fully studied in the literature. We make the following assumptions on the first-stage nonparametric payoff function for each $s \in \{1, ..., S\}$. Let $\mathcal{Z}_s$ be the support of Z_s.

Assumption SS. *For every $z_s \in \mathcal{Z}_s$, $\nu^s(\boldsymbol{d}_{-s}, z_s)$ is strictly decreasing in each element of $\boldsymbol{d}_{-s}$.*

Assumption SY1. *For every $z_s \in \mathcal{Z}_s$, $\nu^s(\boldsymbol{d}_{-s}, z_s) = \nu^s(\tilde{\boldsymbol{d}}_{-s}, z_s)$ for any permutation $\tilde{\boldsymbol{d}}_{-s}$ of $\boldsymbol{d}_{-s}$.*

Assumption SS asserts that the agents' treatment decisions are produced in a game with strategic substitutability. The strictness of the monotonicity is not important for our purpose but convenient in making statements about the regions. Assumption SY1 imposes symmetry (conditional on $Z_s = z_s$) in terms of opponents' decisions, which trivially holds in the two-player case and becomes crucial with many players in the characterization by simplifying the regions of multiple equilibria. This assumption is related to the exchangeability assumption in classical entry games (e.g., Berry (1992), Kline and Tamer (2012)), which imposes that the payoff of a player is a function of the number of other entrants, or the anonymity assumption in large games (e.g., Kalai (2004), Menzel (2016)).[3] In the language of Ciliberto and Tamer

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[3]This assumption is imposed as part of a monotonicity assumption (Assumption 3.2) in Kline and Tamer (2012). The "symmetry of payoffs" has a different meaning in their paper.

([2009](#)), although SY1 restricts heterogeneity in the *fixed competitive effects* (i.e., how each of other entrants affects one's payoff), the nonseparability between $\boldsymbol{d}_{-s}$ and z_s in $\nu^s(\boldsymbol{d}_{-s}, z_s)$ allows heterogeneity how each player is affected by other entrants; this heterogeneity is related to the *variable competitive effects*.

We begin by introducing some notations for equilibrium profiles. For $k = 1, ..., S$, let $\boldsymbol{e}_k$ be an S-vector of all zeros except the k-th element being a unity, and let $\boldsymbol{e}_0 \equiv (0, ..., 0)$. For $j = 0, ..., S$, define $\boldsymbol{e}^j \equiv \sum_{k=0}^{j} \boldsymbol{e}_k$, which is an S-vector where the first j elements are unity and the rest are zero. For some positive integers n_s, define a permutation function $\sigma : \{n_1, ..., n_S\} \to \{n_1, ..., n_S\}$, which has to be a one-to-one function. For example,

$$\left(\begin{array}{ccccc} n_1 & n_2 & n_3 & n_4 & n_5 \\ \sigma(n_1) & \sigma(n_2) & \sigma(n_3) & \sigma(n_4) & \sigma(n_5) \end{array} \right) = \left(\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{array} \right).$$

Let Σ be a set of all possible permutations. Define a set of all possible permutations of $\boldsymbol{e}^j = (e_1^j, ..., e_S^j)$ as

$$M_j \equiv \left\{ \boldsymbol{d}^j : \boldsymbol{d}^j = (\sigma(e_1^j), ..., \sigma(e_S^j)) \text{ for } \sigma(\cdot) \in \Sigma \right\} \tag{3.1}$$

for $j = 0, ..., S$. Note M_j is constructed to be a set of all equilibrium profiles with j treatments selected or j entrants, and it partitions $\mathcal{D} = \bigcup_{j=0}^{S} M_j$. There are $S!/j!(S-j)!$ distinct $\boldsymbol{d}^j$'s in M_j. For example with $S = 3$, $\boldsymbol{d}^2 \in M_2 = \{(1,1,0), (1,0,1), (0,1,1)\}$ and $\boldsymbol{d}^0 \in M_0 = \{(0,0,0)\}$. Note $\boldsymbol{d}^0 = \boldsymbol{e}^0 = (0, ..., 0)$ and $\boldsymbol{d}^S = \boldsymbol{e}^S = (1, ..., 1)$.

Let $\boldsymbol{D}(\boldsymbol{z}) \equiv (D_1(z_1), ..., D_S(z_S))$ where $\boldsymbol{z} \equiv (z_1, ..., z_S)$ and $D_s(z_s)$ is the potential treatment decision had the player s been assigned $Z_s = z_s$. We are interested in characterizing a region R of $\boldsymbol{U} \equiv (U_1, ..., U_S)$ in $\mathcal{U} \equiv (0, 1]^S$ that satisfies $\boldsymbol{U} \in R \Leftrightarrow \boldsymbol{D}(\boldsymbol{z}) \in M_j$ for some j. Let $\tilde{\boldsymbol{e}}^j$ be a $(S-1)$-vector where the first j elements are unity and the rest are zero for $j = 0, ..., S-1$. By Assumption SY1, $\nu^s(\tilde{\boldsymbol{e}}^j, z_s)$ is the only relevant payoff function to define the equilibrium regions. For notational simplicity, let $\nu_j^s(z_s) \equiv \nu_{\tilde{\boldsymbol{e}}^j}^s(z_s) \equiv \nu^s(\tilde{\boldsymbol{e}}^j, z_s)$. Now, for each equilibrium profile, we define regions of $\boldsymbol{U}$ that are Cartesian products in $\mathcal{U}$:

$$R_{\boldsymbol{d}^0}(\boldsymbol{z}) \equiv \prod_{s=1}^{S} (\nu_0^s(z_s), 1], \qquad R_{\boldsymbol{d}^S}(\boldsymbol{z}) \equiv \prod_{s=1}^{S} \left(0, \nu_{S-1}^s(z_s)\right]$$

and, given $\boldsymbol{d}^j = (\sigma(e_1^j), ..., \sigma(e_S^j))$ for some $\sigma(\cdot) \in \Sigma$[4] and $j = 1, ..., S-1$,

$$R_{\boldsymbol{d}^j}(\boldsymbol{z}) = \left\{ \boldsymbol{U} : (U_{\sigma(1)}, ..., U_{\sigma(S)}) \in \left\{ \prod_{s=1}^{j} \left(0, \nu_{j-1}^{\sigma(s)}(z_{\sigma(s)})\right] \right\} \times \left\{ \prod_{s=j+1}^{S} \left(\nu_j^{\sigma(s)}(z_{\sigma(s)}), 1\right] \right\} \right\}. \tag{3.2}$$

For example, for $\sigma(\cdot)$ such that $\boldsymbol{d}^1 = (\sigma(1), \sigma(0), \sigma(0)) = (0, 1, 0)$,

$$R_{010}(\boldsymbol{z}) = \left(\nu_1^1(z_1), 1\right] \times \left(0, \nu_0^2(z_2)\right] \times \left(\nu_1^3(z_3), 1\right].$$

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[4]Sometime we use the notation $\boldsymbol{d}_\sigma^j$ to emphasize the permutation function $\sigma(\cdot)$ from which $\boldsymbol{d}^j$ is generated.
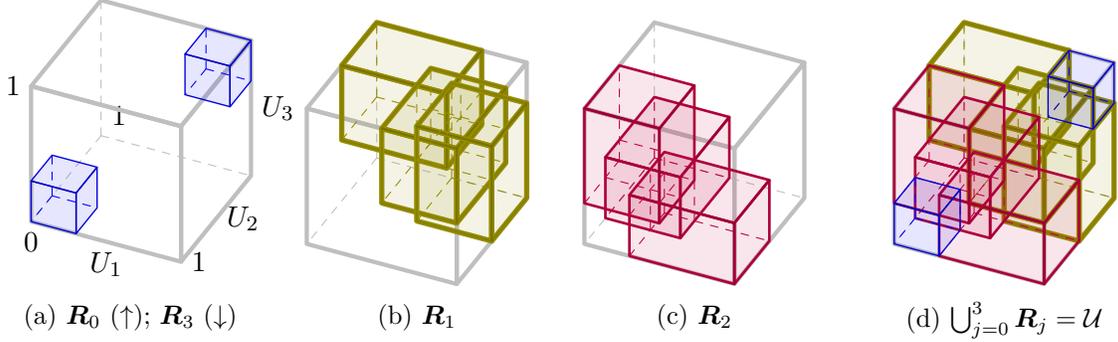
Figure 1: Illustration of equilibrium regions in treatment selection process (Proposition 3.1) for three players ($S = 3$).

Lastly, define the region of all equilibria with j treatments selected or j entrants as

$$\boldsymbol{R}_j(\boldsymbol{z}) \equiv \bigcup_{\boldsymbol{d} \in M_j} R_{\boldsymbol{d}}(\boldsymbol{z}). \tag{3.3}$$

In what follows, we establish the geometric properties of these regions.

Definition 3.1. *Sets A and B are neighboring sets when there exists a point in one set whose open ε-ball has nonempty intersection with the other set for any $\varepsilon > 0$.*

Two sets with a nonempty intersection are trivially neighboring sets. Two disjoint sets can possibly be neighboring sets when they share a "border". Let $\mathcal{Z}$ be the supports of $\boldsymbol{Z} \equiv (Z_1, ..., Z_S)$.

Proposition 3.1. *Consider the first-stage game (2.2). Under Assumptions SS and SY1, the following holds: For every $\boldsymbol{z} \in \mathcal{Z}$ (which is suppressed),*
(i) $\boldsymbol{R}_j \cap \boldsymbol{R}_{j'} = \emptyset$ for $j, j' = 0, ..., S$ with $j \neq j'$;
(ii) $\boldsymbol{R}_j$ and $\boldsymbol{R}_{j-1}$ are neighboring sets for $j = 1, ..., S$;
(iii) $\boldsymbol{R}_j$ and $\boldsymbol{R}_{j-t}$ are not neighboring sets for $j = t, ..., S$ and $t \geq 2$;
(iv) $\bigcup_{j=0}^{S} \boldsymbol{R}_j = \mathcal{U}$.

This proposition fully characterizes the equilibrium regions. Figure 1 illustrates the results of Proposition 3.1 for $S = 3$ with $\boldsymbol{R}_0 = R_{000}$, $\boldsymbol{R}_1 = R_{100} \cup R_{010} \cup R_{001}$, $\boldsymbol{R}_2 = R_{110} \cup R_{101} \cup R_{011}$ and $\boldsymbol{R}_3 = R_{111}$; also see Figures 4 and 5 for illustrations of individual $R_{\boldsymbol{d}^j}$'s and regions of multiple equilibria for this case.

For concreteness, we henceforth discuss Proposition 3.1 in terms of an entry game. By (i) and the fact that M_S and M_0 are singleton, one can conclude that $R_{\boldsymbol{d}^S}$ and $R_{\boldsymbol{d}^0}$ are regions of unique equilibrium. For $j = 1, ..., S - 1$, however, $R_{\boldsymbol{d}^j} \cap R_{\tilde{\boldsymbol{d}}^j}$ is not necessarily empty for $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$. In particular, $R_{\boldsymbol{d}^j} \cap R_{\tilde{\boldsymbol{d}}^j}$ are regions of multiple equilibria. By (i), there is no multiple equilibria where one equilibrium has j entrants and another has j' entrants for $j' \neq j$. This is reminiscent of Berry (1992) and Bresnahan and Reiss (1990, 1991) in that the equilibrium is unique in terms of the number of entrants. In other words, $\boldsymbol{D}(\boldsymbol{z}) \in M_j$ is uniquely predicted by $\boldsymbol{U} \in \boldsymbol{R}_j(\boldsymbol{z})$. In the present paper, this result is obtained under substantially weaker conditions on the payoff function than those in Berry (1992).

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Player s	1	2	3	4	5
Decision d_s^j	1	1	0	1	0
Decision d_s^{j-1}	1	0	0	0	1

Table 1: An example of equilibria that differ by one entrant with $S = 5$ and $j = 3$.

Proposition 3.1(ii)–(iii) assert that regions are neighboring sets when the number of entrants differs by one, but not when the number of entrants differs by more than one. By (i), neighboring sets in (ii) are disjoint neighboring sets. Let $A \sim B$ denote that A and B are neighboring sets. Note that $A \sim B$ implies $B \sim A$ and vice versa. Then (i)–(iii) immediately imply that $\boldsymbol{R}_j$'s are disjoint regions that lie in $\mathcal{U}$ in a monotonic fashion, where all possible neighboring relationships are expressed as

$$\boldsymbol{R}_1 \sim \boldsymbol{R}_2 \sim \cdots \sim \boldsymbol{R}_{S-1} \sim \boldsymbol{R}_S. \tag{3.4}$$

Proposition 3.1(iv) implies that an equilibrium always exists in a discrete game with strategic substitutes, regardless of the number of players or the shape of the distribution of unobservables. That is, an econometric model for this game is coherent (Tamer (2003); Chesher and Rosen (2012)), which extends the finding with a two-player game in the literature. Proposition 3.1(i) and (iv) imply that $\boldsymbol{R}_j$ for $j = 1, ..., S$ partition the entire $\mathcal{U}$. Note that, reversion (or crossing) of the "border" of the partition does not occur, otherwise it violates (iii).

Proposition 3.1(i)–(iii) can be shown by utilizing the properties of sets defined as Cartesian products (Proposition D.1 in Appendix D) and by observing that the pairs of equilibrium profiles in question obey certain rules. For example for $\boldsymbol{d}^j$ and $\boldsymbol{d}^{j-1}$ in (ii), there always exists a player s^* such that $d_{s^*}^j = 1$ and $d_{s^*}^{j-1} = 0$ by contradiction. For all other players, each equilibrium decision must be one of the following four pairs: $(d_s^j, d_s^{j-1}) \in \{(1,1), (0,0), (1,0), (0,1)\}$ $\forall s \neq s^*$. One possibility of $\boldsymbol{d}^j$ and $\boldsymbol{d}^{j-1}$ is where all the four pairs occur (although not necessary) as displayed in Table 1 with $S = 5$, $j = 3$ and $s^* = 2$ (or 4). Now to prove (ii), we show that $R_{\boldsymbol{d}^j} \sim R_{\boldsymbol{d}^{j-1}}$ $\forall d^j \in M_j$ and $\forall d^{j-1} \in M_{j-1}$. For any Cartesian products $R = \prod_{s=1}^S r_s$ and $Q = \prod_{s=1}^S q_s$, it satisfies that $R \sim Q$ if and only if $r_s \sim q_s$ $\forall s$. But it can be shown that for each of (d_s^j, d_s^{j-1}) pairs above $\forall s$, U_s falls into respective intervals r_s and q_s that satisfy $r_s \sim q_s$. This is formally shown as part of the proof of Proposition 3.1 in Appendix D.

Lastly, we introduce a uniformity assumption that is required in this multi-agent setting.

Assumption M1. *For any $z_s, z_s' \in \mathcal{Z}_s$, either $\nu^s(\boldsymbol{d}_{-s}, z_s) \geq \nu^s(\boldsymbol{d}_{-s}, z_s')$ $\forall \boldsymbol{d}_{-s} \in \mathcal{D}_{-s}$ and $\forall s \in \{1, ..., S\}$, or $\nu^s(\boldsymbol{d}_{-s}, z_s) \leq \nu^s(\boldsymbol{d}_{-s}, z_s')$ $\forall \boldsymbol{d}_{-s} \in \mathcal{D}_{-s}$ and $\forall s \in \{1, ..., S\}$.*

The uniformity is across $\boldsymbol{d}_{-s}$ and s. Note that this assumption is weaker than a conventional monotonicity that $\nu^s(\boldsymbol{d}_{-s}, z_s)$ is either non-decreasing or non-increasing in z_s for all $\boldsymbol{d}_{-s}$ and s. Assumption M1 is justifiable especially when z_s is chosen to be of the same kind for all players. For example in an entry game, if z_s is chosen to be each player's cost shifters, then the payoffs would decrease in their costs for all players.

Now we are ready to state the first main result of this paper. For $j = 0, ..., S$, define the

region of all equilibria with *at most j* entrants as

$$\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \equiv \bigcup_{k=0}^{j} \boldsymbol{R}_k(\boldsymbol{z}).$$

Although this region is hard to express explicitly in general, it has a simple feature that serves our purpose:

Theorem 3.1. *Under Assumptions SS, SY1 and M1 and for any given $\boldsymbol{z}, \boldsymbol{z}' \in \mathcal{Z}$, either*

$$\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \subseteq \boldsymbol{R}^{\leq j}(\boldsymbol{z}') \ \forall j, \quad or \quad \boldsymbol{R}^{\leq j}(\boldsymbol{z}) \supseteq \boldsymbol{R}^{\leq j}(\boldsymbol{z}') \ \forall j. \tag{3.5}$$

Theorem 3.1 establishes a generalized version of monotonicity in the treatment selection process.[5] This theorem plays a crucial role in calculating the bounds on the treatment parameters, in showing sharpness of the bounds, and in introducing the LATE. In showing Theorem 3.1, since deriving the explicit expression of $\boldsymbol{R}^{\leq j}$ can be cumbersome, we infer its form by focusing on the "border" of $\boldsymbol{R}^{\leq j}$ and using the results of Proposition 3.1; see the proof in Appendix D.[6]

4 Partial Identification of the ATE

4.1 Preliminaries

To characterize the bounds on the treatment parameters, we make the following assumptions. Unless otherwise noted, the assumptions hold for each $s \in \{1, ..., S\}$.

Assumption IN. $(X, \boldsymbol{Z}) \perp (\epsilon_{\boldsymbol{d}}, \boldsymbol{U}) \ \forall \boldsymbol{d} \in \mathcal{D}$.

Assumption E. $(\epsilon_{\boldsymbol{d}}, \boldsymbol{U})$ *are continuously distributed* $\forall \boldsymbol{d} \in \mathcal{D}$.

Assumption EX. *For each* $\boldsymbol{d}_{-s} \in \mathcal{D}_{-s}$, $\nu^s(\boldsymbol{d}_{-s}, Z_s)|X$ *is nondegenerate.*

Assumptions IN, EX and all the analyses below can be understood as "conditional on W," the common covariates in X and $\boldsymbol{Z} = (Z_1, ..., Z_S)$. Assumption EX is related to the exclusion restriction and the relevance condition of the instruments Z_s.

We now impose two shape restrictions on the outcome function $\theta(\boldsymbol{d}, x, \epsilon_{\boldsymbol{d}})$ via restrictions on

$$\vartheta(\boldsymbol{d}, x; \boldsymbol{u}) \equiv E[\theta(\boldsymbol{d}, x, \epsilon_{\boldsymbol{d}})|\boldsymbol{U} = \boldsymbol{u}]$$

a.e. $\boldsymbol{u}$. These restrictions on the conditional mean are weaker than those that are directly imposed on $\theta(\boldsymbol{d}, x, \epsilon_{\boldsymbol{d}})$. Let $\mathcal{X}$ be the supports of X.

Assumption M. *For every* $x \in \mathcal{X}$, *either* $\vartheta(1, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) \geq \vartheta(0, \boldsymbol{d}_{-s}, x; \boldsymbol{u})$ *a.e.* $\boldsymbol{u} \ \forall \boldsymbol{d}_{-s} \in \mathcal{D}_{-s}$ *or* $\vartheta(1, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) \leq \vartheta(0, \boldsymbol{d}_{-s}, x; \boldsymbol{u})$ *a.e.* $\boldsymbol{u} \ \forall \boldsymbol{d}_{-s} \in \mathcal{D}_{-s}$

---

[5]This notion of generalized monotonicity is also used in Han (2018) for the determination of both the outcome and the treatment of each time period in a dynamic context.

[6]Berry (1992) derives the probability of an event that the number of entrants is less than a certain value, which can be written as $\Pr[\boldsymbol{U} \in \boldsymbol{R}^{\leq j}(\boldsymbol{z})]$ using our notation. This result is not sufficient for the purpose of our paper.

Assumption M holds in a leading case of binary Y with a threshold crossing model that satisfies uniformity:

Assumption M*. *(i)* $\theta(\boldsymbol{d}, x, \epsilon_{\boldsymbol{d}}) = 1[\mu(\boldsymbol{d}, x) \geq \epsilon_{\boldsymbol{d}}]$ *where $\epsilon_{\boldsymbol{d}}$ is scalar and $F_{\epsilon_{\boldsymbol{d}}|\boldsymbol{U}} = F_{\epsilon_{\boldsymbol{d}'}|\boldsymbol{U}}$ for any $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{D}$; (ii) for every $x \in \mathcal{X}$, either $\mu(1, \boldsymbol{d}_{-s}, x) \geq \mu(0, \boldsymbol{d}_{-s}, x)$ $\forall \boldsymbol{d}_{-s} \in \mathcal{D}_{-s}$ or $\mu(1, \boldsymbol{d}_{-s}, x) \leq \mu(0, \boldsymbol{d}_{-s}, x)$ $\forall \boldsymbol{d}_{-s} \in \mathcal{D}_{-s}$.*

Assumption M* implies Assumption M. Assumption M can be stated in twofold, corresponding to (i) and (ii) of Assumption M*: (a) for every x and $\boldsymbol{d}_{-s}$, either $\vartheta(1, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) \geq \vartheta(0, \boldsymbol{d}_{-s}, x; \boldsymbol{u})$ a.e. $\boldsymbol{u}$, or $\vartheta(1, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) \leq \vartheta(0, \boldsymbol{d}_{-s}, x; \boldsymbol{u})$ a.e. $\boldsymbol{u}$; (b) for every x, each inequality statement in (a) holds for all $\boldsymbol{d}_{-s}$. For an outcome function with a scalar index, $\theta(\boldsymbol{d}, x, \epsilon_{\boldsymbol{d}}) = \tilde{\theta}(\mu(\boldsymbol{d}, x), \epsilon_{\boldsymbol{d}})$, part (a) is implied by $\epsilon_{\boldsymbol{d}} = \epsilon_{\boldsymbol{d}'} = \epsilon$ (or more generally $F_{\epsilon_{\boldsymbol{d}}|\boldsymbol{U}} = F_{\epsilon_{\boldsymbol{d}'}|\boldsymbol{U}}$) for any $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{D}$ and $E[\tilde{\theta}(t, \epsilon)|\boldsymbol{U} = \boldsymbol{u}]$ being strictly increasing (decreasing) in t a.e. $\boldsymbol{u}$.[7] Functions that satisfy the latter assumption include: strictly monotonic functions such as transformation models $\tilde{\theta}(t, \epsilon) = r(t + \epsilon)$ where unknown $r(\cdot)$ is a strictly increasing; and functions that are not strictly monotonic such as limited dependent variables models $\tilde{\theta}(t, \epsilon) = 1[t \geq \epsilon]$ or $\tilde{\theta}(t, \epsilon) = 1[t \geq \epsilon](t - \epsilon)$. There can be, however, functions that violate the latter assumption but satisfy part (a). For example, consider a threshold crossing model with a random coefficient: $\theta(\boldsymbol{d}, x, \epsilon) = 1[\phi(\epsilon)\boldsymbol{d}\beta^\top \geq x\gamma^\top]$ where $\phi(\epsilon)$ is nondegenerate. When $\beta_s \geq 0$, then $E[\theta(1, \boldsymbol{d}_{-s}, x, \epsilon) - \theta(0, \boldsymbol{d}_{-s}, x, \epsilon)|\boldsymbol{U} = \boldsymbol{u}] = \Pr\left[\frac{x\gamma^\top}{\beta_s + \boldsymbol{d}_{-s}\beta_{-s}^\top} \leq \phi(\epsilon) \leq \frac{x\gamma^\top}{\boldsymbol{d}_{-s}\beta_{-s}^\top}|\boldsymbol{U} = \boldsymbol{u}\right]$ and thus nonnegative a.e. $\boldsymbol{u}$, and vice versa. Part (a) also does not impose any monotonicity of θ in $\epsilon_{\boldsymbol{d}}$ (i.e., $\epsilon_{\boldsymbol{d}}$ can be a vector).

Part (b) of Assumption M imposes mild uniformity as we deal with more than one treatment. Uniformity is required across different values of $\boldsymbol{d}_{-s}$ but not across s, which means that different treatments can have different directions of monotonicity. More importantly, knowledge on the direction of the monotonicity is not necessary, unlike Manski (1997) or Manski (2013) where the semi-monotone treatment response is assumed for possible multiple treatments.

Assumption SY. *For every $x \in \mathcal{X}$, $\vartheta(\boldsymbol{d}, x; \boldsymbol{u}) = \vartheta(\tilde{\boldsymbol{d}}, x; \boldsymbol{u})$ a.e. $\boldsymbol{u}$ for any permutation $\tilde{\boldsymbol{d}}$ of $\boldsymbol{d}$.*

For a benchmark analysis, we first maintain this conditional symmetry since it is convenient to simplify the analysis given our incomplete model. Assumption SY imposes symmetry in the functions as long as the observed characteristics X remain the same. This assumption is relaxed in Section 5 by either using instruments that offset strategic substitutability or allowing partial symmetry. An assumption related to SY is also found in Manski (2013).

Heuristically, the following is the idea of the bound analysis. For given $\boldsymbol{d} \in \mathcal{D}$, consider

$$E[Y_{\boldsymbol{d}}|X] = E[Y_{\boldsymbol{d}}|\boldsymbol{Z}, X] = E[Y|\boldsymbol{D} = \boldsymbol{d}, \boldsymbol{Z}, X] \Pr[\boldsymbol{D} = \boldsymbol{d}|\boldsymbol{Z}]$$
$$+ \sum_{\boldsymbol{d}' \neq \boldsymbol{d}} E[Y_{\boldsymbol{d}}|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z}, X] \Pr[\boldsymbol{D} = \boldsymbol{d}'|\boldsymbol{Z}], \tag{4.1}$$

---

[7]A single-treatment version of the latter assumption appears in Vytlacil and Yildiz (2007) (Assumption A-4), which is weaker than assuming $\tilde{\theta}(t, \epsilon)$ is strictly increasing (decreasing) a.e. ϵ; see Vytlacil and Yildiz (2007) for related discussions.

where the first equality and $\Pr[\boldsymbol{D} = \boldsymbol{d}|\boldsymbol{Z}, X] = \Pr[\boldsymbol{D} = \boldsymbol{d}|\boldsymbol{Z}]$ in the second equality are by Assumption IN. In this expression, the counterfactual term $E[Y_{\boldsymbol{d}}|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z}, X]$ can be bounded as long as Y is bounded by a known interval (Manski (1990)) and instruments in $\boldsymbol{Z}$ that are excluded from the equation for Y can then be used to narrow the bounds. The goal of our analysis is to derive tighter bounds on the ATT's $E[Y_{\boldsymbol{d}}|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z}, X]$ by fully exploiting the structure of the model under the above assumptions, without necessarily requiring Y to be bounded by a known interval. These bounds then can be used to construct bounds on the ATE.

4.2 Analysis with Binary Y

As a leading case, we first consider model (2.1)–(2.2) with binary Y (consistent with Assumption M^*(i)) and no X to illustrate the main idea of our bound analysis. Moreover, with binary Y sharp bounds on the mean treatment parameters can be obtained in this model of a triangular structure. Consider

$$Y = 1[\mu(\boldsymbol{D}) \geq \epsilon_{\boldsymbol{D}}], \tag{4.2}$$

where, again, W is suppressed for succinctness.

We first define quantities that are identified directly from the data. For two realization of $\boldsymbol{z}, \boldsymbol{z}'$ of $\boldsymbol{Z}$, define

$$\begin{aligned} h(\boldsymbol{z}, \boldsymbol{z}') &\equiv E[Y|\boldsymbol{Z} = \boldsymbol{z}] - E[Y|\boldsymbol{Z} = \boldsymbol{z}'] \\ &= \Pr[Y = 1|\boldsymbol{Z} = \boldsymbol{z}] - \Pr[Y = 1|\boldsymbol{Z} = \boldsymbol{z}'], \end{aligned} \tag{4.3}$$

which record the change in the distributions of Y as $\boldsymbol{Z}$ changes. To see this change relative to the change in the distribution of $\boldsymbol{D}$, define a *joint propensity score* as $p_M(\boldsymbol{z}) \equiv \Pr[\boldsymbol{D} \in M|\boldsymbol{Z} = \boldsymbol{z}]$ for $M \subset \mathcal{D}$ and consider

$$p_{M^{\leq j}}(\boldsymbol{z}) < p_{M^{\leq j}}(\boldsymbol{z}') \ \forall j = 0, ..., S - 1, \tag{4.4}$$

where $M^{\leq j} \equiv \bigcup_{k=0}^{j} M_k$. Under Assumption EX, the existence of $\boldsymbol{z}, \boldsymbol{z}'$ that satisfy (4.4) is guaranteed by Theorem 3.1, since $p_{M^{\leq j}}(\boldsymbol{z}) - p_{M^{\leq j}}(\boldsymbol{z}') = \Pr[\boldsymbol{U} \in \boldsymbol{R}^{\leq j}(\boldsymbol{z})] - \Pr[\boldsymbol{U} \in \boldsymbol{R}^{\leq j}(\boldsymbol{z}')]$ by Assumption IN. Let a function $sgn\{h\}$ take values $-1, 0, 1$ when h is negative, zero and positive, respectively.

Lemma 4.1. *In model (4.2) and (2.2), suppose Assumptions SS, SY1, M1, IN, E, EX, M^* and SY hold. For $\boldsymbol{z}, \boldsymbol{z}'$ such that (4.4) holds, it satisfies that*

$$sgn\{h(\boldsymbol{z}, \boldsymbol{z}')\} = sgn\left\{\mu(\boldsymbol{d}^j) - \mu(\boldsymbol{d}^{j-1})\right\}$$

for $\boldsymbol{d}^j \in M_j$ and $\boldsymbol{d}^{j-1} \in M_{j-1}$ with $j = 1, ..., S$.

Given the result of this lemma, we recover the signs of $\mu(\boldsymbol{d}^j) - \mu(\boldsymbol{d}^{j-1})$, i.e., the direction of monotonicity in Assumption M^*(ii). This knowledge is useful to calculate bounds on the unknown conditional mean terms (the ATT's) in (4.1).

To illustrate the proof of this lemma, suppose $S = 2$; a formal proof can be found in Section 4.3 in a more general setting. By Proposition 3.1, $(1, 0)$ and $(0, 1)$ are the values of

$\boldsymbol{D}$ that can be realized as possible multiple equilibria. Given this knowledge, we define

$$
\begin{aligned}
h_M(\boldsymbol{z}, \boldsymbol{z}') \equiv{} & \Pr[Y = 1, \boldsymbol{D} \in \{(1,0), (0,1)\} | \boldsymbol{Z} = \boldsymbol{z}] \\
& - \Pr[Y = 1, \boldsymbol{D} \in \{(1,0), (0,1)\} | \boldsymbol{Z} = \boldsymbol{z}'],
\end{aligned}
$$

and

$$
\begin{aligned}
h_{11}(\boldsymbol{z}, \boldsymbol{z}') &\equiv \Pr[Y = 1, \boldsymbol{D} = (1,1) | \boldsymbol{Z} = \boldsymbol{z}] - \Pr[Y = 1, \boldsymbol{D} = (1,1) | \boldsymbol{Z} = \boldsymbol{z}'], \\
h_{00}(\boldsymbol{z}, \boldsymbol{z}') &\equiv \Pr[Y = 1, \boldsymbol{D} = (0,0) | \boldsymbol{Z} = \boldsymbol{z}] - \Pr[Y = 1, \boldsymbol{D} = (0,0) | \boldsymbol{Z} = \boldsymbol{z}'],
\end{aligned}
$$

so that $h(\boldsymbol{z}, \boldsymbol{z}') = h_{11}(\boldsymbol{z}, \boldsymbol{z}') + h_{00}(\boldsymbol{z}, \boldsymbol{z}') + h_M(\boldsymbol{z}, \boldsymbol{z}')$. Making use of the conditional symmetry assumption (SY), combining $\boldsymbol{D} = (1,0)$ and $\boldsymbol{D} = (0,1)$ will conveniently manage the multiple equilibria problem. Define

$$
\begin{aligned}
R_{11}(\boldsymbol{z}) &\equiv \left\{ \boldsymbol{U} : U_1 \le \nu_1^1(z_1), U_2 \le \nu_1^2(z_2) \right\}, \\
R_{00}(\boldsymbol{z}) &\equiv \left\{ \boldsymbol{U} : U_1 > \nu_0^1(z_1), U_2 > \nu_0^2(z_2) \right\}, \\
R_{10}(\boldsymbol{z}) &\equiv \left\{ \boldsymbol{U} : U_1 \le \nu_0^1(z_1), U_2 > \nu_1^2(z_2) \right\}, \\
R_{01}(\boldsymbol{z}) &\equiv \left\{ \boldsymbol{U} : U_1 > \nu_1^1(z_1), U_2 \le \nu_0^2(z_2) \right\}.
\end{aligned}
$$

Let $\mu_{\boldsymbol{d}} \equiv \mu(\boldsymbol{d})$ for brevity. Given Assumption M*(i), let ϵ be a r.v. such that $F_{\epsilon|U} = F_{\epsilon_{\boldsymbol{d}}|U}$ for any $\boldsymbol{d} \in \mathcal{D}$. By Assumption IN,

$$
\begin{aligned}
h_{11}(\boldsymbol{z}, \boldsymbol{z}') + h_{00}(\boldsymbol{z}, \boldsymbol{z}') ={} & \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in R_{11}(\boldsymbol{z})] - \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in R_{11}(\boldsymbol{z}')] \\
& + \Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in R_{00}(\boldsymbol{z})] - \Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in R_{00}(\boldsymbol{z}')], \qquad (4.5)
\end{aligned}
$$

where the equality uses R_{11} and R_{00} being disjoint and regions of unique equilibrium. By Assumption SY that $\mu_{10} = \mu_{01}$, we have

$$
h_M(\boldsymbol{z}, \boldsymbol{z}') = \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in R_{10}(\boldsymbol{z}) \cup R_{01}(\boldsymbol{z})] - \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in R_{10}(\boldsymbol{z}') \cup R_{01}(\boldsymbol{z}')]. \quad (4.6)
$$

The main insight to obtain the results of Lemma 4.1 is as follows. By (4.3), h captures how $\Pr[Y = 1 | \boldsymbol{Z} = \boldsymbol{z}]$ changes in $\boldsymbol{z}$. By $h = h_{11} + h_{00} + h_M$ and (4.5)–(4.6), such a change can be translated into shifts in the regions of equilibria while the thresholds of ϵ in each of h_{11}, h_{00} and h_M remaining unchanged by the exclusion restriction. Therefore by inspecting how $\Pr[Y = 1 | \boldsymbol{Z} = \boldsymbol{z}]$ changes in $\boldsymbol{z}$ (i.e., the sign of h) relative to the changes in the equilibrium regions R_{11} and R_{00} (i.e., the signs of h_{11}^D and h_{00}^D), we recover the signs of $\mu_{11} - \mu_{01}$ and $\mu_{10} - \mu_{00}$. In doing so, we use a crucial fact that the changes in the region $R_{10} \cup R_{01}$ are offset with the changes in R_{11} and R_{00}.

To be specific, since $(\boldsymbol{z}, \boldsymbol{z}')$ are chosen such that (4.4) holds, it satisfies that $R_{11}(\boldsymbol{z}) \supset$

(a) When $\boldsymbol{Z} = \boldsymbol{z}$ (b) When $\boldsymbol{Z} = \boldsymbol{z}'$ (c) Difference of (a) and (b)
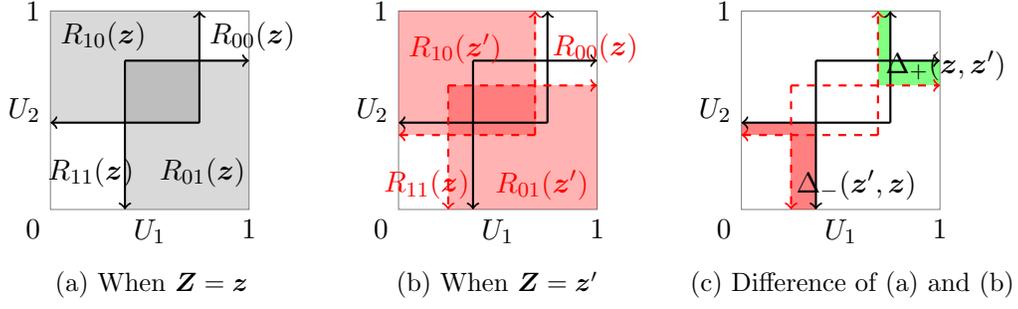
Figure 2: Inflow and outflow at change in $\boldsymbol{Z}$ in calculating h.

$R_{11}(\boldsymbol{z}')$ and $R_{00}(\boldsymbol{z}) \subset R_{00}(\boldsymbol{z}')$ by Theorem 3.1.[8] Then

$$\Delta_+(\boldsymbol{z}, \boldsymbol{z}') \equiv \{R_{10}(\boldsymbol{z}) \cup R_{01}(\boldsymbol{z})\} \setminus \{R_{10}(\boldsymbol{z}') \cup R_{01}(\boldsymbol{z}')\} = R_{00}(\boldsymbol{z}') \setminus R_{00}(\boldsymbol{z}), \qquad (4.7)$$

$$\Delta_-(\boldsymbol{z}, \boldsymbol{z}') \equiv \{R_{10}(\boldsymbol{z}') \cup R_{01}(\boldsymbol{z}')\} \setminus \{R_{10}(\boldsymbol{z}) \cup R_{01}(\boldsymbol{z})\} = R_{11}(\boldsymbol{z}) \setminus R_{11}(\boldsymbol{z}'), \qquad (4.8)$$

because, as $\boldsymbol{z}$ changes, an inflow of one region is an outflow of a region next to it. This set algebra is illustrated in Figure 2. Then (4.6) becomes

$$h_M(\boldsymbol{z}, \boldsymbol{z}') = \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in \Delta_+(\boldsymbol{z}, \boldsymbol{z}')] - \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in \Delta_-(\boldsymbol{z}, \boldsymbol{z}')], \qquad (4.9)$$

by the following general rule: for a uniform random vector $\tilde{\boldsymbol{U}}$ and two sets B and B' contained in $\tilde{\mathcal{U}}$ and for a r.v. ϵ and set $A \subset \mathcal{E}$,

$$\Pr[\epsilon \in A, \tilde{\boldsymbol{U}} \in B] - \Pr[\epsilon \in A, \tilde{\boldsymbol{U}} \in B'] = \Pr[\epsilon \in A, \tilde{\boldsymbol{U}} \in B \setminus B'] - \Pr[\epsilon \in A, \tilde{\boldsymbol{U}} \in B' \setminus B]. \qquad (4.10)$$

Therefore by combining (4.9) with (4.5) applying (4.10) once more, we have

$$h(\boldsymbol{z}, \boldsymbol{z}') = \Pr[\epsilon \le \mu_{11}, \boldsymbol{U} \in \Delta_-(\boldsymbol{z}, \boldsymbol{z}')] - \Pr[\epsilon \le \mu_{00}, \boldsymbol{U} \in \Delta_+(\boldsymbol{z}, \boldsymbol{z}')]$$
$$- \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in \Delta_-(\boldsymbol{z}, \boldsymbol{z}')] + \Pr[\epsilon \le \mu_{10}, \boldsymbol{U} \in \Delta_+(\boldsymbol{z}, \boldsymbol{z}')]. \qquad (4.11)$$

Now, given Assumption E, Assumption M*(ii) holds with $\mu(1, d_{-s}) > \mu(0, d_{-s})$ for any d_{-s} if and only if

$$h(\boldsymbol{z}, \boldsymbol{z}') = \Pr[\mu_{01} \le \epsilon \le \mu_{11}, \boldsymbol{U} \in \Delta_-(\boldsymbol{z}, \boldsymbol{z}')] + \Pr[\mu_{00} \le \epsilon \le \mu_{10}, \boldsymbol{U} \in \Delta_+(\boldsymbol{z}, \boldsymbol{z}')],$$

which is positive as is the sum of two probabilities. One can analogously show this for other signs and we have the result of Lemma 4.1.[9] Lastly, to gain efficiency in determining the sign of $h(\boldsymbol{z}, \boldsymbol{z}')$ for $\boldsymbol{z}, \boldsymbol{z}' \in \mathcal{Z}$, define the integrated version of h as

$$H \equiv E\left[h(\boldsymbol{Z}, \boldsymbol{Z}') \,\big|\, p_{M \le j}(\boldsymbol{Z}) < p_{M \le j}(\boldsymbol{Z}') \,\forall j = 0, ..., S - 1\right]. \qquad (4.12)$$

---

[8]We assume for simplicity that this choice of $\boldsymbol{z}$ and $\boldsymbol{z}'$ satisfies $A^* = \emptyset$, where A^* is defined in the proof of a more general case (Lemma 4.2).

[9]Note that in deriving the result of the lemma, a player-specific exclusion restriction is not crucial and one may be able to relax it.

Then $sgn\{H\} = sgn\{\mu_{11} - \mu_{01}\} = sgn\{\mu_{10} - \mu_{00}\}$ in this illustration.

Using 4.1, now consider calculating the upper bound on $\Pr[Y_{00} = 1]$. Suppose $H \geq 0$. Then by Lemma 4.1, $\mu_{00} \leq \mu_{10}$, $\mu_{00} \leq \mu_{01}$, and $\mu_{00} \leq \mu_{10} \leq \mu_{11}$. Then we can derive the upper bound on, e.g., $\Pr[Y_{00} = 1|\boldsymbol{D} = (1,0), Z]$ as

$$
\begin{aligned}
\Pr[Y_{00} = 1|\boldsymbol{D} = (1,0), Z = z] &= \Pr[\epsilon \leq \mu_{00}|\boldsymbol{D} = (1,0), \boldsymbol{Z} = \boldsymbol{z}] \\
&\leq \Pr[\epsilon \leq \mu_{10}|\boldsymbol{D} = (1,0), \boldsymbol{Z} = \boldsymbol{z}] \\
&= \Pr[Y = 1|\boldsymbol{D} = (1,0), \boldsymbol{Z} = \boldsymbol{z}],
\end{aligned}
\tag{4.13}
$$

which is smaller than one, the upper bound without the knowledge of the direction. Likewise, using $\mu_{00} \leq \mu_{01}$ and $\mu_{00} \leq \mu_{11}$, we can calculate upper bounds on the other unobserved terms $\Pr[Y_{00} = 1|\boldsymbol{D} = \boldsymbol{d}, \boldsymbol{Z}]$ for $\boldsymbol{d} \neq (0,0)$ analogous to the ones in (4.1). Consequently we have

$$
\Pr[Y_{00} = 1] \leq \Pr[Y = 1|\boldsymbol{Z} = \boldsymbol{z}].
$$

Likewise, we can derive the lower bounds on $\Pr[Y_{00} = 1]$ when $H \leq 0$.[10]

To be more general, we calculate the bounds on $E[Y_{\boldsymbol{d}^j}] = \Pr[Y_{\boldsymbol{d}^j} = 1]$ for given $\boldsymbol{d}^j \in M_j$ and $j = 0, ..., S$. We also show that the bounds are sharp. We consider the case $H > 0$; the case $H < 0$ is symmetric and the case $H = 0$ is straightforward. Recall $M^{\leq j} \equiv \bigcup_{k=0}^{j} M_k$ and let $M^{>j} \equiv \bigcup_{k=j+1}^{S} M_k = \mathcal{D} \backslash M^{\leq j}$, which are understood to be empty sets for unconforming values of j. Then one can show that $L_{\boldsymbol{d}^j} \leq \Pr[Y_{\boldsymbol{d}^j} = 1] \leq U_{\boldsymbol{d}^j}$ with

$$
\begin{aligned}
U_{\boldsymbol{d}^j} \equiv \inf_{\boldsymbol{z} \in \mathcal{Z}} \Bigg\{ &\Pr[Y = 1, \boldsymbol{D} \in M_j | \boldsymbol{Z} = \boldsymbol{z}] + \sum_{\boldsymbol{d}' \in M^{>j}} \Pr[Y = 1, \boldsymbol{D} = \boldsymbol{d}' | \boldsymbol{Z} = \boldsymbol{z}] \\
&+ \sum_{\boldsymbol{d}' \in M^{\leq j-1}} \Pr[\boldsymbol{D} = \boldsymbol{d}' | \boldsymbol{Z} = \boldsymbol{z}] \Bigg\},
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
L_{\boldsymbol{d}^j} \equiv \sup_{\boldsymbol{z} \in \mathcal{Z}} \Bigg\{ &\Pr[Y = 1, \boldsymbol{D} \in M_j | \boldsymbol{Z} = \boldsymbol{z}] + \sum_{\boldsymbol{d}' \in M^{\leq j-1}} \Pr[Y = 1, \boldsymbol{D} = \boldsymbol{d}' | \boldsymbol{Z} = \boldsymbol{z}] \\
&+ \sum_{\boldsymbol{d}' \in M^{>j}} \Pr[\boldsymbol{D} = \boldsymbol{d}' | \boldsymbol{Z} = \boldsymbol{z}] \Bigg\}.
\end{aligned}
\tag{4.15}
$$

We can simplify these bounds and show that they are sharp under the following assumption.

Assumption C. *(i) $\mu_{\boldsymbol{d}}(\cdot)$ and $\nu_{\boldsymbol{d}_{-s}}(\cdot)$ are continuous; (ii) $\mathcal{Z}$ is compact.*

For $j' = 0, ..., S - 1$, the joint propensity score with $M^{>j'}$ satisfies

$$
p_{M^{>j'}}(\boldsymbol{z}) = \Pr[\boldsymbol{U} \in \mathcal{U} \backslash \boldsymbol{R}^{\leq j'}(\boldsymbol{z})].
\tag{4.16}
$$

Under Assumption C and by Theorem 3.1, there exist vectors $\bar{\boldsymbol{z}} \equiv (\bar{z}_1, ..., \bar{z}_S)$ and $\underline{\boldsymbol{z}} \equiv$

---

[10] When $H \geq 0$, the lower bounds on $\Pr[Y_{00} = 1]$ is trivially zero.

$(\underline{z}_1, ..., \underline{z}_S)$ that satisfy

$$p_{M>j'}(\bar{z}) = \max_{z \in \mathcal{Z}} p_{M>j'}(z), \qquad p_{M>j'}(\underline{z}) = \min_{z \in \mathcal{Z}} p_{M>j'}(z), \tag{4.17}$$

$\forall j' = 0, ..., S-1$.

Theorem 4.1. *Given model* (4.2) *and* (2.2), *suppose the assumptions of Lemma 4.1 and Assumption C hold. Also suppose $H \geq 0$. Then the bounds U_{d^j} and L_{d^j} in* (4.14) *and* (4.15) *simplify as*

$$U_{d^j} = \Pr[Y = 1, \boldsymbol{D} \in M^{>j-1} | \boldsymbol{Z} = \bar{z}] + \Pr[\boldsymbol{D} \in M^{\leq j-1} | \boldsymbol{Z} = \bar{z}],$$
$$L_{d^j} = \Pr[Y = 1, \boldsymbol{D} \in M^{\leq j} | \boldsymbol{Z} = \underline{z}] + \Pr[\boldsymbol{D} \in M^{>j} | \boldsymbol{Z} = \underline{z}],$$

and these bounds and thus the bounds on the ATE are sharp.

In a single treatment model, Shaikh and Vytlacil (2011) use the propensity score as a scalar conditioning variable, which summarizes all the exogenous variation in the selection process and is convenient in simplifying the bounds and proving sharpness. In the context of the current paper, however, this approach is invalid since $\Pr[D_s = 1 | Z_s = z_s, \boldsymbol{D}_{-s} = \boldsymbol{d}_{-s}]$ cannot be written in terms of a propensity score of player s as $\boldsymbol{D}_{-s}$ is endogenous. We instead use vector $\boldsymbol{Z}$ as conditioning variables and establish partial ordering for the relevant conditional probabilities (that define the lower and upper bounds) w.r.t. the joint propensity score (4.16). In proving the sharpness of the bounds, Theorem 3.1 plays an important role. Even though $\boldsymbol{D}$ is a vector that is determined by simultaneous decisions, Theorem 3.1 combined with the partial ordering above establishes "monotonicity" of the event $\boldsymbol{U} \in \boldsymbol{R}^j(z)$ (and $\boldsymbol{U} \in \mathcal{U} \backslash \boldsymbol{R}^j(z)$) w.r.t. z.

Bounds when X is present in the model and its variation is additionally exploited will be narrower than the bounds in Theorem 4.1, but showing sharpness of these bounds requires a different approach of expressing bounds. This is discussed in the next section.

4.3 General Analysis

In this section we consider the full model (2.1)–(2.2), in which Y may no longer be binary and the number of players may exceeds two. We also exploit additional exogenous variation that is generated from X conditional on $\boldsymbol{Z}$. The existence of such variation is motivated by the examples of externalities we discussed. We first introduce a generalized version of the sign matching results (Lemma 4.1). For realizations x of X and z, z' of $\boldsymbol{Z}$, define

$$h(z, z', x) \equiv E[Y | \boldsymbol{Z} = z, X = x] - E[Y | \boldsymbol{Z} = z', X = x], \tag{4.18}$$
$$h_j(z, z', x) \equiv E[Y | \boldsymbol{D} \in M_j, \boldsymbol{Z} = z, X = x] \Pr[\boldsymbol{D} \in M_j | \boldsymbol{Z} = z]$$
$$\qquad - E[Y | \boldsymbol{D} \in M_j, \boldsymbol{Z} = z', X = x] \Pr[\boldsymbol{D} \in M_j | \boldsymbol{Z} = z']. \tag{4.19}$$

The introduction of the quantity (4.19) is motivated by Proposition 3.1.[11] Also, since M_j's are disjoint, $\sum_{j=0}^{S} \Pr[\boldsymbol{D} \in M_j | \boldsymbol{Z} = \cdot] = 1$ and thus $h(z, z', x) = \sum_{j=0}^{S} h_j(z, z', x)$. Let

---

[11]Even if $\Pr[\boldsymbol{D} = \boldsymbol{d}^j | \boldsymbol{Z} = z] \neq \Pr[\boldsymbol{U} \in R_{d^j}(z)]$ due to multiple equilibria, it satisfies that $\Pr[\boldsymbol{D} \in M_j | \boldsymbol{Z} = z] = \Pr[\boldsymbol{U} \in \boldsymbol{R}_j(z)]$.

$\boldsymbol{x} = (x_0, ..., x_S)$ be an array of (possibly different) realizations of X, i.e., each x_j for $j = 0, ..., S$ is a realization of X, and define

$$h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{x}) \equiv \sum_{j=0}^{S} h_j(\boldsymbol{z}, \boldsymbol{z}'; x_j).$$

Recall $\vartheta(\boldsymbol{d}, x; \boldsymbol{u}) \equiv E[\theta(\boldsymbol{d}, x, \epsilon)|\boldsymbol{U} = \boldsymbol{u}]$, and for succinctness let $\vartheta_j(x; \boldsymbol{u}) \equiv \vartheta(\boldsymbol{e}^j, x; \boldsymbol{u})$ as $\boldsymbol{e}^j$ is the only relevant set of treatments under Assumption SY. We state the main lemma of this section.

Lemma 4.2. *In model* (2.1)–(2.2), *suppose Assumptions SS, SY1, IN, E, EX, M and SY hold, and $h(\boldsymbol{z}, \boldsymbol{z}', x)$ and $h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{x})$ are well-defined. For $\boldsymbol{z}, \boldsymbol{z}'$ such that* (4.4) *holds, it satisfies that, for $j = 1, ..., S$,*
(i) $sgn\{h(\boldsymbol{z}, \boldsymbol{z}', x)\} = sgn\{\vartheta_j(x; \boldsymbol{u}) - \vartheta_{j-1}(x; \boldsymbol{u})\}$ a.e. $\boldsymbol{u}$;
(ii) for $\iota \in \{-1, 0, 1\}$, if $sgn\{h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{x})\} = sgn\{\vartheta_{k-1}(x_{k-1}; \boldsymbol{u}) - \vartheta_k(x_k; \boldsymbol{u})\} = \iota \ \forall k \neq j$, then $sgn\{\vartheta_j(x_j; \boldsymbol{u}) - \vartheta_{j-1}(x_{j-1}; \boldsymbol{u})\} = \iota$ a.e. $\boldsymbol{u}$.

Part (i) parallels Lemma 4.1. To show Lemma 4.2, we track the inflow and outflow in each $\boldsymbol{R}_j(\boldsymbol{Z})$ when the value of $\boldsymbol{Z}$ changes. Specifically, based on Theorem 3.1 we equate the inflow and outflow of $\boldsymbol{R}_j$ with those of $\boldsymbol{R}^{\leq j}$'s in calculating (4.19) (and thus $h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{x})$), which can be written as

$$h_j(\boldsymbol{z}, \boldsymbol{z}', x) = E[Y|\boldsymbol{U} \in \boldsymbol{R}_j(\boldsymbol{z}), \boldsymbol{Z} = \boldsymbol{z}, X = x]\Pr[\boldsymbol{U} \in \boldsymbol{R}_j(\boldsymbol{z})]$$
$$- E[Y|\boldsymbol{U} \in \boldsymbol{R}_j(\boldsymbol{z}'), \boldsymbol{Z} = \boldsymbol{z}', X = x]\Pr[\boldsymbol{U} \in \boldsymbol{R}_j(\boldsymbol{z}')], \qquad (4.20)$$

by Assumption IN. This approach is analogous to the simpler analysis shown in Section 4.2.

For part (i) of Lemma 4.2, suppose that $\vartheta_j(x; \boldsymbol{u}) - \vartheta_{j-1}(x; \boldsymbol{u}) > 0$ a.e. $\boldsymbol{u} \ \forall j = 1, ..., S$. Then by (D.7), $h > 0$. Conversely, if $h > 0$ then it should be that $\vartheta_j(x; \boldsymbol{u}) - \vartheta_{j-1}(x; \boldsymbol{u}) > 0$ a.e. $\boldsymbol{u} \ \forall j = 1, ..., S$. Suppose not and suppose $\vartheta_j(x; \boldsymbol{u}) - \vartheta_{j-1}(x; \boldsymbol{u}) \leq 0$ with positive measure for some j. Then by Assumption M, this implies that $\vartheta_j(x; \boldsymbol{u}) - \vartheta_{j-1}(x; \boldsymbol{u}) \leq 0 \ \forall j$ a.e. $\boldsymbol{u}$, and thus $h \leq 0$ which is contradiction. By applying similar arguments for other signs, we have the desired result. The proof for Lemma 4.2(ii) is in Appendix D.

Using Lemma 4.2, note first that the sign of the ATE is identified by Lemma 4.2(i) since $E[Y_{\boldsymbol{d}}|X = x] = E[\vartheta(\boldsymbol{d}, x; \boldsymbol{U})]$. Next, we calculate the bounds on $E[Y_{\boldsymbol{d}}|X = x]$ with $\boldsymbol{d} = \boldsymbol{d}^j$ for a given $\boldsymbol{d}^j \in M_j$ for some $j = 0, ..., S$. Consider

$$E[Y_{\boldsymbol{d}^j}|X = x] = E[Y|\boldsymbol{D} = \boldsymbol{d}^j, \boldsymbol{Z} = \boldsymbol{z}, X = x]\Pr[\boldsymbol{D} = \boldsymbol{d}^j|\boldsymbol{Z} = \boldsymbol{z}]$$
$$+ \sum_{\boldsymbol{d}' \neq \boldsymbol{d}^j} E[Y_{\boldsymbol{d}^j}|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z} = \boldsymbol{z}, X = x]\Pr[\boldsymbol{D} = \boldsymbol{d}'|\boldsymbol{Z} = \boldsymbol{z}]. \qquad (4.21)$$

Note that for $\boldsymbol{d}' \in M_j$,

$$E[Y_{\boldsymbol{d}^j}|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z} = \boldsymbol{z}, X = x] = E[Y|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z} = \boldsymbol{z}, X = x] \qquad (4.22)$$

by Assumption SY. In order to bound $E[Y_{\boldsymbol{d}^j}|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z} = \boldsymbol{z}, X = x]$ for $\boldsymbol{d}' \notin M_j$ in (4.21), we systematically use the results of Lemma 4.2. First, analogous to (4.12), define the integrated

version of $h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{x})$ as

$$H(\boldsymbol{x}) \equiv E\left[h(\boldsymbol{Z}, \boldsymbol{Z}'; \boldsymbol{x}) \big| p_{M \leq j}(\boldsymbol{Z}) < p_{M \leq j}(\boldsymbol{Z}') \ \forall j = 0, ..., S-1\right].$$

Then define the following sets of two consecutive elements of $\boldsymbol{x}$ that satisfy the conditions in Lemma 4.2: for $j = 1, ..., S$,

$$\mathcal{X}_{j,j-1}^0(\iota) \equiv \{(x_j, x_{j-1}) : sgn\{H(\boldsymbol{x})\} = \iota, x_0 = \cdots = x_S\},$$
$$\mathcal{X}_{j,j-1}^1(\iota) \equiv \{(x_j, x_{j-1}) : sgn\{H(\boldsymbol{x})\} = \iota, (x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}^0(-\iota) \ \forall k \neq j\} \cup \mathcal{X}_{j,j-1}^0(\iota),$$
$$\vdots$$
$$\mathcal{X}_{j,j-1}^t(\iota) \equiv \{(x_j, x_{j-1}) : sgn\{H(\boldsymbol{x})\} = \iota, (x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}^{t-1}(-\iota) \ \forall k \neq j\} \cup \mathcal{X}_{j,j-1}^{t-1}(\iota).$$

Note that $\mathcal{X}_{j,j-1}^t(\iota) \subset \mathcal{X}_{j,j-1}^{t+1}(\iota)$ for any t. Define $\mathcal{X}_{j,j-1}(\iota) \equiv \lim_{t \to \infty} \mathcal{X}_{j,j-1}^t(\iota)$.[12] Then by Lemma 4.2,

$$\text{if } (x_j, x_{j-1}) \in \mathcal{X}_{j,j-1}(\iota), \text{ then } sgn\{\vartheta_j(x_j; \boldsymbol{u}) - \vartheta_{j-1}(x_{j-1}; \boldsymbol{u})\} = \iota \text{ a.e. } \boldsymbol{u}. \tag{4.23}$$

Consider $j' < j$ for $E[Y_{\boldsymbol{d}^j}|\boldsymbol{D} = \boldsymbol{d}^{j'}, \boldsymbol{Z}, X]$ in (4.21). Then, for example, if $(x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}(-1) \cup \mathcal{X}_{k,k-1}(0)$ for $j' + 1 \leq k \leq j$, then $\vartheta_j(x; \boldsymbol{u}) \leq \vartheta_{j'}(x'; \boldsymbol{u})$ where $x = x_j$ and $x' = x_{j'}$ by transitively applying (4.23). Therefore

$$E[Y_{\boldsymbol{d}^j}|\boldsymbol{D} = \boldsymbol{d}^{j'}, \boldsymbol{Z} = \boldsymbol{z}, X = x] = E[\theta(\boldsymbol{d}^j, x, \epsilon)|\boldsymbol{U} \in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z}), \boldsymbol{Z} = \boldsymbol{z}, X = x]$$
$$= \frac{1}{\Pr[\boldsymbol{U} \in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})]} \int_{R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})} \vartheta_j(x; \boldsymbol{u})d\boldsymbol{u}$$
$$\leq \frac{1}{\Pr[\boldsymbol{U} \in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})]} \int_{R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})} \vartheta_{j'}(x'; \boldsymbol{u})d\boldsymbol{u}$$
$$= E[\theta(\boldsymbol{d}^{j'}, x', \epsilon)|\boldsymbol{U} \in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z}), \boldsymbol{Z} = \boldsymbol{z}, X = x']$$
$$= E[Y|\boldsymbol{D} = \boldsymbol{d}^{j'}, \boldsymbol{Z} = \boldsymbol{z}, X = x']. \tag{4.24}$$

Symmetrically, for $j' > j$, if $(x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}(1) \cup \mathcal{X}_{k,k-1}(0)$ for $j + 1 \leq k \leq j'$, then $\vartheta_j(x; \boldsymbol{u}) \leq \vartheta_{j'}(x'; \boldsymbol{u})$ where $x = x_j$ and $x' = x_{j'}$. Therefore the same bound as (4.24) is derived. Given these results, to collect all $x' \in \mathcal{X}$ that yield $\vartheta_j(x; \boldsymbol{u}) \leq \vartheta_{j'}(x'; \boldsymbol{u})$, we can construct a set

$$x' \in \left\{x_{j'} : (x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}(-1) \cup \mathcal{X}_{k,k-1}(0) \text{ for } j'+1 \leq k \leq j, x_j = x\right\}$$
$$\cup \left\{x_{j'} : (x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}(1) \cup \mathcal{X}_{k,k-1}(0) \text{ for } j+1 \leq k \leq j', x_j = x\right\}.$$

Then we can further shrink the bound in (4.24) by taking infimum over all x' in this set. The lower bound on $E[Y_{\boldsymbol{d}^j}|\boldsymbol{D} = \boldsymbol{d}^{j'}, \boldsymbol{Z} = \boldsymbol{z}, X = x]$ can be constructed by simply choosing the opposite signs in the preceding argument.

---

[12] In practice, the formula for $\mathcal{X}_{j,j-1}^t$ provides a natural algorithm to construct the set $\mathcal{X}_{j,j-1}$ for the computation of the bounds. The calculation of each $\mathcal{X}_{j,j-1}^t$ is straightforward as it is a search over a two-dimensional space for (x_j, x_{j-1}) once the set $\mathcal{X}_{j,j-1}^{t-1}$ from the previous step is obtained. Practitioners can employ truncation $t \leq T$ for some T and use $\mathcal{X}_{j,j-1}^T$ as an approximation for $\mathcal{X}_{j,j-1}$.

In conclusion, for bounds on the ATE $E[Y_{\boldsymbol{d}^j}|X=x]$, we can introduce the sets $\mathcal{X}^L_{\boldsymbol{d}^j}(x;\boldsymbol{d}')$ and $\mathcal{X}^U_{\boldsymbol{d}^j}(x;\boldsymbol{d}')$ for $\boldsymbol{d}' \neq \boldsymbol{d}^j$ as follows: for $\boldsymbol{d}' \in M_{j'}$ with $j' \neq j$,

$$
\begin{aligned}
\mathcal{X}^L_{\boldsymbol{d}^j}(x;\boldsymbol{d}') &\equiv \left\{ x_{j'} : (x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}(-1) \cup \mathcal{X}_{k,k-1}(0) \text{ for } j'+1 \leq k \leq j, x_j = x \right\} \\
&\quad \cup \left\{ x_{j'} : (x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}(1) \cup \mathcal{X}_{k,k-1}(0) \text{ for } j+1 \leq k \leq j', x_j = x \right\}, \quad (4.25) \\
\mathcal{X}^U_{\boldsymbol{d}^j}(x;\boldsymbol{d}') &\equiv \left\{ x_{j'} : (x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}(1) \cup \mathcal{X}_{k,k-1}(0) \text{ for } j'+1 \leq k \leq j, x_j = x \right\} \\
&\quad \cup \left\{ x_{j'} : (x_k, x_{k-1}) \in \mathcal{X}_{k,k-1}(-1) \cup \mathcal{X}_{k,k-1}(0) \text{ for } j+1 \leq k \leq j', x_j = x \right\}, \quad (4.26)
\end{aligned}
$$

and for $\boldsymbol{d}' \in M_j$,

$$
\mathcal{X}^L_{\boldsymbol{d}^j}(x;\boldsymbol{d}') = \mathcal{X}^U_{\boldsymbol{d}^j}(x;\boldsymbol{d}') \equiv \{x\}, \tag{4.27}
$$

where the last display is by (4.22). The following theorem summarize our results:

Theorem 4.2. *In model* (2.1)–(2.2), *suppose the assumptions of Lemma 4.2 hold. Then the sign of the ATE is identified, and the upper and lower bounds on the ASF and ATE with $\boldsymbol{d}, \tilde{\boldsymbol{d}} \in \mathcal{D}$ are*

$$
L_{\boldsymbol{d}}(x) \leq E[Y_{\boldsymbol{d}}|X=x] \leq U_{\boldsymbol{d}}(x)
$$

and

$$
L_{\boldsymbol{d}}(x) - U_{\tilde{\boldsymbol{d}}}(x) \leq E[Y_{\boldsymbol{d}} - Y_{\tilde{\boldsymbol{d}}}|X=x] \leq U_{\boldsymbol{d}}(x) - L_{\tilde{\boldsymbol{d}}}(x)
$$

where, for given $\boldsymbol{d}^\dagger \in \mathcal{D}$,

$$
\begin{aligned}
U_{\boldsymbol{d}^\dagger}(x) &\equiv \inf_{\boldsymbol{z} \in \mathcal{Z}} \left\{ E[Y|\boldsymbol{D}=\boldsymbol{d}^\dagger, \boldsymbol{Z}=\boldsymbol{z}, X=x] \Pr[\boldsymbol{D}=\boldsymbol{d}^\dagger|\boldsymbol{Z}=\boldsymbol{z}] \right. \\
&\quad \left. + \sum_{\boldsymbol{d}' \neq \boldsymbol{d}^\dagger} \inf_{x' \in \mathcal{X}^U_{\boldsymbol{d}^\dagger}(x;\boldsymbol{d}')} E[Y|\boldsymbol{D}=\boldsymbol{d}', \boldsymbol{Z}=\boldsymbol{z}, X=x'] \Pr[\boldsymbol{D}=\boldsymbol{d}'|\boldsymbol{Z}=\boldsymbol{z}] \right\}, \\
L_{\boldsymbol{d}^\dagger}(x) &\equiv \sup_{\boldsymbol{z} \in \mathcal{Z}} \left\{ E[Y|\boldsymbol{D}=\boldsymbol{d}^\dagger, \boldsymbol{Z}=\boldsymbol{z}, X=x] \Pr[\boldsymbol{D}=\boldsymbol{d}^\dagger|\boldsymbol{Z}=\boldsymbol{z}] \right. \\
&\quad \left. + \sum_{\boldsymbol{d}' \neq \boldsymbol{d}^\dagger} \sup_{x' \in \mathcal{X}^L_{\boldsymbol{d}^\dagger}(x;\boldsymbol{d}')} E[Y|\boldsymbol{D}=\boldsymbol{d}', \boldsymbol{Z}=\boldsymbol{z}, X=x'] \Pr[\boldsymbol{D}=\boldsymbol{d}'|\boldsymbol{Z}=\boldsymbol{z}] \right\}.
\end{aligned}
$$

When the variation of $\boldsymbol{Z}$ is only used in deriving the bounds, $\mathcal{X}_{k,k-1}(\iota)$ should simply reduce down to $\mathcal{X}^0_{k,k-1}(\iota)$ in the definition of $\mathcal{X}^L_{\boldsymbol{d}^j}(x;\boldsymbol{d}')$ and $\mathcal{X}^U_{\boldsymbol{d}^j}(x;\boldsymbol{d}')$. When Y is binary with no X, such bounds are equivalent to (4.14) and (4.15). The variation of X given $\boldsymbol{Z}$ yields substantially narrower bounds than the sharp bounds established in Theorem 4.1 under Assumption C. The resulting bounds, however, are not automatically implied to be sharp from Theorem 4.1, since they are based on a different DGP and the additional exclusion restriction.

Remark 4.1. *Maintaining that Y is binary, sharp bounds on the ATE with variation in X can be derived assuming that the signs of $\vartheta(\boldsymbol{d}, x; \boldsymbol{u}) - \vartheta(\boldsymbol{d}', x'; \boldsymbol{u})$ are identified for $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{D}$*

and $x, x' \in \mathcal{X}$ via Lemma 4.2. To see this, define

$$\tilde{\mathcal{X}}_{\boldsymbol{d}}^U(x; \boldsymbol{d}') \equiv \left\{ x' : \vartheta(\boldsymbol{d}, x; \boldsymbol{u}) - \vartheta(\boldsymbol{d}', x'; \boldsymbol{u}) \leq 0 \ a.e. \ \boldsymbol{u} \right\},$$
$$\tilde{\mathcal{X}}_{\boldsymbol{d}}^L(x; \boldsymbol{d}') \equiv \left\{ x' : \vartheta(\boldsymbol{d}, x; \boldsymbol{u}) - \vartheta(\boldsymbol{d}', x'; \boldsymbol{u}) \geq 0 \ a.e. \ \boldsymbol{u} \right\},$$

which are identified by assumption. Then by replacing $\mathcal{X}_{\boldsymbol{d}}^i(x; \boldsymbol{d}')$ with $\tilde{\mathcal{X}}_{\boldsymbol{d}}^i(x; \boldsymbol{d}')$ (for $i \in \{U, L\}$) in Theorem 4.2, we may be able to show that the resulting bounds are sharp. Since Lemma 4.2 implies that $\mathcal{X}_{\boldsymbol{d}^j}^i(x; \boldsymbol{d}') \subset \tilde{\mathcal{X}}_{\boldsymbol{d}^j}^i(x; \boldsymbol{d}')$ but not necessarily $\mathcal{X}_{\boldsymbol{d}^j}^i(x; \boldsymbol{d}') \supset \tilde{\mathcal{X}}_{\boldsymbol{d}^j}^i(x; \boldsymbol{d}')$, these modified bounds and the original bounds in Theorem 4.2 do not coincide. This contrasts the result of Shaikh and Vytlacil (2011) for a single-treatment model, and the complication lies in the fact that we deal with an incomplete model with a vector treatment. When there is no X, Lemma 4.2(i) establishes equivalence between the two signs, and thus $\mathcal{X}_{\boldsymbol{d}^j}^i(x; \boldsymbol{d}') = \tilde{\mathcal{X}}_{\boldsymbol{d}^j}^i(x; \boldsymbol{d}')$ for $i \in \{U, L\}$, which results in Theorem 4.1. Relatedly, we can also exploit variation from W, namely variables that are common to both X and $\boldsymbol{Z}$ (with or without exploiting excluded variation of X). This is related to the analysis of Chiburis (2010) and Mourifié (2015) in a single-treatment setting. One caveat of this approach is that, similar to these papers, we need an additional assumption that $W \perp (\epsilon, \boldsymbol{U})$.

Remark 4.2. When X does not have enough variation, an assumption that $Y \in [\underline{Y}, \overline{Y}]$ with known endpoints can be introduced to calculate the bounds. To see this, suppose we do not use the variation in X and suppose $H(x) \geq 0$. Then $\vartheta_k(x; \boldsymbol{u}) \geq \vartheta_{k-1}(x; \boldsymbol{u}) \ \forall k = 1, ..., S$ by Lemma 4.2(i) and by transitivity, $\vartheta_{j'} \geq \vartheta_j$ for any $j' > j$. Therefore, we have

$$
\begin{aligned}
E[Y_{\boldsymbol{d}^j}|X = x] \leq{} & \sum_{\boldsymbol{d} \in M_j} E[Y|\boldsymbol{D} = \boldsymbol{d}, \boldsymbol{Z}, X = x] \Pr[\boldsymbol{D} = \boldsymbol{d}|\boldsymbol{Z}] \\
& + \sum_{\boldsymbol{d}' \in M_{j'}: j' > j} E[Y|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z}, X = x] \Pr[\boldsymbol{D} = \boldsymbol{d}'|\boldsymbol{Z}] \\
& + \sum_{\boldsymbol{d}' \in M_{j'}: j' < j} E[Y_{\boldsymbol{d}^j}|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z}, X = x] \Pr[\boldsymbol{D} = \boldsymbol{d}'|\boldsymbol{Z}]. \quad (4.28)
\end{aligned}
$$

Without using variation in X, we can bound the last term in (4.28) by $Y \in [\underline{Y}, \overline{Y}]$. This is done in Section 4.2 with $\theta(\boldsymbol{d}, x, \epsilon) = 1[\mu_{\boldsymbol{d}} \geq \epsilon]$ and $\vartheta_j(x; \boldsymbol{u}) = F_{\epsilon|\boldsymbol{U}}(\mu_{\boldsymbol{e}^j}|\boldsymbol{u})$. Another example would be when $Y \in [0, 1]$ as in Example 2.

Remark 4.3. It may be possible to point identify the ATE by extending the result of Theorem 4.2 using X with larger support. For example, Lemma 4.2 enables us to find x' such that $\vartheta_j(x; \boldsymbol{u}) = \vartheta_{j'}(x'; \boldsymbol{u})$ $(j \neq j')$, from which we can point identify the ATT:

$$
\begin{aligned}
E[Y_{\boldsymbol{d}^j}|\boldsymbol{D} = \boldsymbol{d}^{j'}, \boldsymbol{Z} = \boldsymbol{z}, X = x] &= \frac{1}{\Pr[\boldsymbol{U} \in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})]} \int_{R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})} \vartheta_j(x; \boldsymbol{u}) d\boldsymbol{u} \\
&= \frac{1}{\Pr[\boldsymbol{U} \in R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})]} \int_{R_{\boldsymbol{d}^{j'}}(\boldsymbol{z})} \vartheta_{j'}(x'; \boldsymbol{u}) d\boldsymbol{u} \\
&= E[Y|\boldsymbol{D} = \boldsymbol{d}^{j'}, \boldsymbol{Z} = \boldsymbol{z}, X = x'].
\end{aligned}
$$

The existence of such x' requires sufficient variation of X conditional on $\boldsymbol{Z}$, which is rem-

Figure 3: Illustration of Assumptions ASY and ASY*.

iniscent of *Vytlacil and Yildiz (2007)*. *This approach is alternative to the use of the large variation of $\mathbf{Z}$ for point identification, which is discussed in Section 6.3 below.*

5 Relaxing Symmetry

We propose two different ways of relaxing the conditional symmetry assumption in the outcome function (Assumption SY) introduced in the preceding section.

5.1 Compensation of Strategic Substitutability

Assumption SY can be relaxed when there exists variation in $\mathbf{Z}$ that offsets the effect of strategic substitutability. With such variation, we show that regions of multiple equilibria are not involved in calculating $h(\mathbf{z}, \mathbf{z}'; x)$ and thus Assumption SY is no longer required in the bound analysis of the ATE.

Assumption ASY. *For $j = 1, ..., S-1$, there exist $\mathbf{z}, \mathbf{z}' \in \mathcal{Z}$ such that $\nu_{j-1}^s(z_s') \leq \nu_j^s(z_s)$ $\forall s$.*

Assumption ASY states that there exists variation in $\mathbf{Z}$ that offsets the effect of strategic substitutability (Assumption SS), which can be stated as $\nu_{j-1}^s(z_s) > \nu_j^s(z_s)$. For example in an entry game with Z_s being cost shifters, Assumption ASY may hold with $z_s' > z_s$ $\forall s$. In this example, all players may become less profitable with the increase in cost by government regulations, while one player becomes unprofitable to enter whose absence does not help overturn the decrease of other firms' profits. Assumption ASY is illustrated in Figure 3 with $\nu_0^s(z_s') < \nu_1^s(z_s)$ for $s = 1, 2$. Assumption ASY has a simple testable sufficient condition provided that the unobservables in the payoffs are independent to one another.

Assumption ASY*. *There exist $\mathbf{z}, \mathbf{z}' \in \mathcal{Z}$ such that*

$$\Pr[\mathbf{D} = (0, ..., 0)|\mathbf{Z} = \mathbf{z}] + \Pr[\mathbf{D} = (1, ..., 1)|\mathbf{Z} = \mathbf{z}'] > 1. \tag{5.1}$$

Lemma 5.1. *When $U_s \perp U_t$ for all $s \neq t$, Assumption ASY* implies Assumption ASY.*

The mutual independence of U_s's (conditional on W) is useful in inferring the relationship between players' interaction and instruments from the observed decisions of players. The intuition for the sufficiency of Assumption ASY2 is as follows. As long as there is no dependence in unobserved types, (5.1) dictates that the variation of $\mathbf{Z}$ is large enough to

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offset strategic substitutability, because otherwise the payoffs of players cannot move in the same direction, thus cannot resulting in same decisions.[13]

Under Assumption ASY, we can apply an analogous strategy as in the symmetric case in Section 4 to determine the direction of monotonicity and ultimately calculate the bounds on the ATE. For example, the following lemma replaces Lemma 4.2(i):

Lemma 5.2. *In model* (2.1)–(2.2), *suppose Assumptions SS, SY1, M1, IN, E, EX and M hold, and* $h(\boldsymbol{z}, \boldsymbol{z}', x)$ *is well-defined. For* $\boldsymbol{z}, \boldsymbol{z}'$ *such that Assumption ASY and* (4.4) *hold, it satisfies that*

$$sgn\{h(\boldsymbol{z}, \boldsymbol{z}', x)\} = sgn\left\{\vartheta(1, \boldsymbol{d}_{-s}, x; \boldsymbol{u}) - \vartheta(0, \boldsymbol{d}_{-s}, x; \boldsymbol{u})\right\}$$

a.e. $\boldsymbol{u}$ $\forall \boldsymbol{d}_{-s} \in \mathcal{D}_{-s}$ *and* $\forall s = 1, ..., S$.

Lemma 4.2(ii) can be similarly modified. When Assumption ASY hold, it can be shown that

$$R^*_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R^*_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = R^*_{\boldsymbol{d}^j}(\boldsymbol{z}') \cap R^*_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}) = \emptyset \tag{5.2}$$

for $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$, where $R^*_{\boldsymbol{d}}(\cdot)$ is the region that predicts $\boldsymbol{D} = \boldsymbol{d}$.[14] This is shown as part of the proof of above lemma. The result (5.2) liberates us from concerning about the regions of multiple equilibria and about a possible change in equilibrium selection at the change in $\boldsymbol{Z}$. Therefore we can separately consider each $\boldsymbol{d}^j$ when calculating $h(\boldsymbol{z}, \boldsymbol{z}', x)$.

Remark 5.1. *The condition* (5.2) *is related to stability in the equilibrium selection mechanism at the change in* $\boldsymbol{Z}$*: For* $j = 1, ..., S - 1$*, there exist* $\boldsymbol{z}, \boldsymbol{z}' \in \mathcal{Z}$ *such that the region that predicts* $\boldsymbol{D} = \boldsymbol{d}^j$ *is invariant for* $\boldsymbol{Z} \in \{\boldsymbol{z}, \boldsymbol{z}'\}$ *within* $\boldsymbol{R}_j(\boldsymbol{z}) \cap \boldsymbol{R}_j(\boldsymbol{z}')$ $\forall \boldsymbol{d}^j \in M_j$*. In fact, this condition is equivalent to* (5.2) *and trivially holds when* $\boldsymbol{Z}$ *varies sufficiently enough that the regions of multiple equilibria do not intersect to each other. This occurs when Assumption ASY holds.*

5.2 Partial Symmetry: Interactions Within Groups

In some cases, strategic interactions may occur within groups of players (i.e., treatments). In the airline example, it may be the case that larger airlines interact to one another as a group, so do smaller airlines as a different group, but there may be no interaction across the groups.[15] In general for K groups of players/treatments, we consider, with player index $s = 1, ..., S_g$ and group index $g = 1, ..., G$,

$$Y = \theta(\boldsymbol{D}^1, ..., \boldsymbol{D}^G, X, \epsilon_{\boldsymbol{D}}), \tag{5.3}$$

$$D^g_s = 1\left[\nu^{s,g}(\boldsymbol{D}^g_{-s}, Z^g_s) \geq U^g_s\right], \tag{5.4}$$

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[13]The requirement of $\boldsymbol{Z}$ variation in (5.1) is significantly weaker than the large support assumption invoked for an identification at infinity argument (Assumption EX* below).

[14]Unlike $R_{\boldsymbol{d}}(\boldsymbol{z})$ which is purely determined by the payoffs $\nu^s_{\boldsymbol{d}_{-s}}(z_s)$, $R^*_{\boldsymbol{d}}(\boldsymbol{z})$ is unknown to the econometrician even if all the players' payoffs had been known, since the equilibrium selection rule is unknown.

[15]We can also easily extend the model so that smaller airlines take larger airlines' entry decisions as given and play their own entry game, which may be more reasonable to assume.

where each $\boldsymbol{D}^g \equiv (D_1^g, ..., D_{S_k}^g)$ is the treatment vector of group g and $\boldsymbol{D} \equiv (\boldsymbol{D}^1, ..., \boldsymbol{D}^G)$. This model generalizes the model (2.1)–(2.2). It can also be seen as a special case of exogenously endowing an incomplete undirected network structure, where players interact to one another within each of complete sub-networks. In this model each group can differ in its number (S_g) and identity of players (under which the entry decision is denoted by D_s^g). Also, the unobservables $\boldsymbol{U}^g \equiv (U_1^g, ..., U_S^g)$ can be arbitrarily correlated across groups, in addition to the fact that U_s^g's can be correlated within group g and $\boldsymbol{U} \equiv (\boldsymbol{U}^1, ..., \boldsymbol{U}^G)$ can be correlated with $\epsilon_{\boldsymbol{D}}$. This partly relaxes the independence assumption across markets, which is frequently imposed in the entry game literature.

To calculate the bounds on the ATE $E[Y_{\boldsymbol{d}} - Y_{\boldsymbol{d}'}|X = x]$ we apply the results in Theorem 4.2, by adapting those assumptions to the current extension. In what follow is the modification of Assumption SY that (the conditional mean of) the outcome function is symmetric within each group but not across groups. This in turn can be seen as relaxation of Assumption SY. In terms of notation, let $\boldsymbol{D}^{-g} \equiv (\boldsymbol{D}^1, ..., \boldsymbol{D}^{g-1}, \boldsymbol{D}^{g+1}, ..., \boldsymbol{D}^G)$ and its realization be $\boldsymbol{d}^{-g}$. Then such an assumption would be stated as follows.

Assumption SY*. *For $g = 1, ..., G$ and every $x \in \mathcal{X}$, $\vartheta(\boldsymbol{d}^g, \boldsymbol{d}^{-g}, x; \boldsymbol{u}) = \vartheta(\tilde{\boldsymbol{d}}^g, \boldsymbol{d}^{-g}, x; \boldsymbol{u})$ a.e. $\boldsymbol{u}$ for any permutation $\tilde{\boldsymbol{d}}^g$ of $\boldsymbol{d}^g$.*

Under this *partial conditional symmetry* assumption, the bound on the ASF can be calculated by iteratively applying the previous results to each group.[16] Assumptions SS, SY1, EX and M can be modified so that they satisfy for within-group treatments and interaction. In particular, Assumption EX can be modified as follows: for each $\boldsymbol{d}_{-s}^g \in \mathcal{D}_{-s}^g$, $\nu^{s,g}(\boldsymbol{d}_{-s}^g, Z_s^g)|X, \boldsymbol{Z}^{-g}$ is nondegenerate, where $\boldsymbol{Z} \equiv (\boldsymbol{Z}^g, \boldsymbol{Z}^{-g})$. That is, there must be group-specific instruments that are excluded from other groups.[17]

We briefly show how to modify the previous bound analysis with binary Y and no X for simplicity. Analogous to the previous notation, let M_j^g be the set of equilibria with j entrants in group g and let $M^{g,\leq j} \equiv \bigcup_{k=0}^j M_k^g$. Suppose $G = 2$, and $\boldsymbol{d}^1 \in \{0,1\}^{S_1}$ and $\boldsymbol{d}^2 \in \{0,1\}^{S_2}$. Consider the ASF $E[Y_{\boldsymbol{d}}] = E[Y_{\boldsymbol{d}^1, \boldsymbol{d}^2}]$ with $\boldsymbol{d}^1 \in M_{j-1}^1$ and $\boldsymbol{d}^2 \in M_{k-1}^2$ for some $j = 1, ..., S_1$ and $k = 1, ..., S_2$. To calculate its bounds, we can bound $E[Y_{\boldsymbol{d}}|\boldsymbol{D} = \boldsymbol{d}', \boldsymbol{Z}]$ in (4.1) for $\tilde{\boldsymbol{d}} \neq \boldsymbol{d}$ by sequentially applying the analysis of Section 4 in each group. First consider $\tilde{\boldsymbol{d}} = (\tilde{\boldsymbol{d}}^1, \boldsymbol{d}^2)$ with $\tilde{\boldsymbol{d}}^1 \in M_j^1$. We apply Lemma 4.2 for the $\boldsymbol{D}^1$ portion after holding $\boldsymbol{D}^2 = \boldsymbol{d}^2$. Suppose

$$\Pr[Y = 1|\boldsymbol{D}^2 = \boldsymbol{d}^2, \boldsymbol{Z}^1 = \boldsymbol{z}^1, \boldsymbol{Z}^2 = \boldsymbol{z}^2] - \Pr[Y = 1|\boldsymbol{D}^2 = \boldsymbol{d}^2, \boldsymbol{Z}^1 = \boldsymbol{z}^{1\prime}, \boldsymbol{Z}^2 = \boldsymbol{z}^2] \geq 0,$$
$$\Pr[\boldsymbol{D}^1 \in M^{1,>j-1}|\boldsymbol{Z}^1 = \boldsymbol{z}^1] - \Pr[\boldsymbol{D}^1 \in M^{1,>j-1}|\boldsymbol{Z}^1 = \boldsymbol{z}^{1\prime}] > 0,$$

then we have $\mu_{\tilde{\boldsymbol{d}}^1, \boldsymbol{d}^2} \geq \mu_{\boldsymbol{d}^1, \boldsymbol{d}^2}$. The proof of Lemma 4.2 can be adapted by holding $\boldsymbol{D}^2 = \boldsymbol{d}^2$ in this case, because there is no strategic interaction across groups and therefore the multiple equilibria problem only occurs within each group. Note that this strategy still allows that there is dependence between $\boldsymbol{D}^1$ and $\boldsymbol{D}^2$ even after conditioning on $\boldsymbol{Z}$ due to dependence

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[16]This assumption can be further relaxed by adapting Assumption ASY in the framework of this section.

[17]We maintain Assumption R in the current setting since the assumption is equivalent to assuming a rank invariance within each group, i.e., $\epsilon_{\boldsymbol{d}^g, \boldsymbol{d}^{-g}} = \epsilon_{\tilde{\boldsymbol{d}}^g, \boldsymbol{d}^{-g}} \ \forall \boldsymbol{d}^g, \tilde{\boldsymbol{d}}^g \in \{0,1\}^{S_g}$ and $g = 1, ..., G$.

between $\boldsymbol{U}^1$ and $\boldsymbol{U}^2$. Then,

$$\Pr[Y_{\boldsymbol{d}^1,\boldsymbol{d}^2}=1|\boldsymbol{D}=(\tilde{\boldsymbol{d}}^1,\boldsymbol{d}^2),\boldsymbol{Z}=\boldsymbol{z}]=\Pr[\epsilon\leq\mu_{\boldsymbol{d}^1,\boldsymbol{d}^2}|\boldsymbol{D}=(\tilde{\boldsymbol{d}}^1,\boldsymbol{d}^2),\boldsymbol{Z}=\boldsymbol{z}]$$
$$\leq\Pr[\epsilon\leq\mu_{\tilde{\boldsymbol{d}}^1,\boldsymbol{d}^2}|\boldsymbol{D}=(\tilde{\boldsymbol{d}}^1,\boldsymbol{d}^2),\boldsymbol{Z}=\boldsymbol{z}]\qquad(5.5)$$
$$=\Pr[Y=1|\boldsymbol{D}=(\tilde{\boldsymbol{d}}^1,\boldsymbol{d}^2),\boldsymbol{Z}=\boldsymbol{z}].$$

Next, consider $\boldsymbol{d}=(\boldsymbol{d}^1,\boldsymbol{d}^2)$ and $\tilde{\boldsymbol{d}}=(\tilde{\boldsymbol{d}}^1,\tilde{\boldsymbol{d}}^2)$ with $\tilde{\boldsymbol{d}}^2\in M_k^2$ and the other elements as previously determined. Then by applying Lemma 4.2 this time for the $\boldsymbol{D}^2$ portion after holding $\boldsymbol{D}^1=\tilde{\boldsymbol{d}}^1$, we have $\mu_{\tilde{\boldsymbol{d}}^1,\tilde{\boldsymbol{d}}^2}\geq\mu_{\tilde{\boldsymbol{d}}^1,\boldsymbol{d}^2}$ by supposing

$$\Pr[Y=1|\boldsymbol{D}^1=\tilde{\boldsymbol{d}}^1,\boldsymbol{Z}^1=\boldsymbol{z}^1,\boldsymbol{Z}^2=\boldsymbol{z}^2]-\Pr[Y=1|\boldsymbol{D}^1=\tilde{\boldsymbol{d}}^1,\boldsymbol{Z}^1=\boldsymbol{z}^1,\boldsymbol{Z}^2=\boldsymbol{z}^{2\prime}]\geq0,$$
$$\Pr[\boldsymbol{D}^2\in M^{2,>j-1}|\boldsymbol{Z}^2=\boldsymbol{z}^2]-\Pr[\boldsymbol{D}^2\in M^{2,>j-1}|\boldsymbol{Z}^2=\boldsymbol{z}^{2\prime}]>0.$$

Then

$$\Pr[Y_{\boldsymbol{d}^1,\boldsymbol{d}^2}=1|\boldsymbol{D}=(\tilde{\boldsymbol{d}}^1,\tilde{\boldsymbol{d}}^2),\boldsymbol{Z}=\boldsymbol{z}]\leq\Pr[\epsilon\leq\mu_{\tilde{\boldsymbol{d}}^1,\boldsymbol{d}^2}|\boldsymbol{D}=(\tilde{\boldsymbol{d}}^1,\tilde{\boldsymbol{d}}^2),\boldsymbol{Z}=\boldsymbol{z}]$$
$$\leq\Pr[\epsilon\leq\mu_{\tilde{\boldsymbol{d}}^1,\tilde{\boldsymbol{d}}^2}|\boldsymbol{D}=(\tilde{\boldsymbol{d}}^1,\tilde{\boldsymbol{d}}^2),\boldsymbol{Z}=\boldsymbol{z}]\qquad(5.6)$$
$$=\Pr[Y=1|\boldsymbol{D}=(\tilde{\boldsymbol{d}}^1,\tilde{\boldsymbol{d}}^2),\boldsymbol{Z}=\boldsymbol{z}],$$

where the first inequality is by (5.5). Note that in deriving the upper bound in (5.6), it is important that at least the two groups share the same signs of within-group h's and h^D's. This is clearly a weaker requirement than imposing Assumption SY.

6 Discussions

6.1 Player-Specific Outcomes

Henceforth, we considered a scalar Y that may represent an outcome common to all players in a given market or a geographical region. The outcome, however, can also be an outcome that is specific to each player. In this regard, consider a vector of outcomes $\boldsymbol{Y}=(Y_1,...,Y_S)$ where each element Y_s is a player-specific outcome. An interesting example of this setting may be where $\boldsymbol{Y}$ is also an equilibrium outcome from strategic interaction not only through $\boldsymbol{D}$ but also through itself. In this case, it would become important to have a vector of unobservables even after assuming e.g., rank invariance, since we may want to include $\boldsymbol{\epsilon_D}=(\epsilon_{1,\boldsymbol{D}},...,\epsilon_{S,\boldsymbol{D}})$, where $\epsilon_{s,\boldsymbol{D}}$ is an unobservable directly affecting Y_s.[18] We may also want to include a vector of observables of all players $\boldsymbol{X}=(X_1,...,X_S)$, where X_s directly affects Y_s. Then interaction among Y_s can be modeled via a reduced-form representation:

$$Y_s=\theta_s(\boldsymbol{D},\boldsymbol{X},\boldsymbol{\epsilon_D}),\qquad s\in\{1,...,S\}.$$

In firms' entry, the first-stage scalar unobservable U_s may represent each firm's unobserved fixed cost (while Z_s captures observed fixed cost). The vector of unobservables in the player-specific outcome equation represents multiple shocks, such as the player's demand shock and

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[18]In this case, Assumption R should be imposed on $\epsilon_{s,\boldsymbol{D}}$ for each s.

variable cost shock, and other firms' variable cost shocks and demand shocks. Unlike in a linear model, it would be hard to argue that these errors are all aggregated in a scalar variable in this nonlinear outcome model, since it is not known in which fashion they enter the equation.

6.2 Relation to Manski's Work

Manski (2013) introduces a framework for social interaction where responses (i.e., outcomes) of agents are dependent on one another through their treatments. The framework relaxes the stable unit treatment value assumption (SUTVA) by allowing interaction across the units. Our framework is similar to Manski (2013) in that we also allow interaction among outcomes of players through their treatments, as we discuss in Section 6.1. The difference is that we consider interaction across treatment/player unit s, whereas he considers interaction across observational unit i. Furthermore, we explicitly model the selection process of how treatments are determined simultaneously through players' strategic interaction. His model, following his earlier work (Manski (1997) and Manski and Pepper (2000)), stays silent about the process. Despite the difference, the two settings share a similar spirit of departing from the SUTVA. The shape restrictions we impose are related to the assumptions of Manski (2013) for the treatment response, which we compare here. First of all, Assumption SY appears in Manski as an anonymity assumption. Also, we find that Assumptions SY and SY* are related to the constant TR (CTR) assumption in Manski, although he assumes anonymity separate from this assumption. The CTR assumption states that, with $\boldsymbol{d} = (d_i)_{i=1}^N$,

$$c(\boldsymbol{d}) = c(\boldsymbol{d}') \Longrightarrow Y_{\boldsymbol{d}} = Y_{\boldsymbol{d}'}.$$

As noted in Manski, $c(\boldsymbol{d})$ is an *effective treatment* in that, as long as $c(\boldsymbol{d})$ stays constant, the response does not change. SY and SY* can be restated using this concept with a particular choice of $c(\boldsymbol{d})$: with $\boldsymbol{d} = (d_s)_{s=1}^S$,

$$c(\boldsymbol{d}) = c(\boldsymbol{d}') \Longrightarrow E[Y_{\boldsymbol{d}}|X = x, \boldsymbol{U} = \boldsymbol{u}] = E[Y_{\boldsymbol{d}'}|X = x, \boldsymbol{U} = \boldsymbol{u}] \tag{6.1}$$

for given $x \in \mathcal{X}$ and a.e. $\boldsymbol{u}$, where $c(\boldsymbol{d})$ is chosen such that *the game for treatment decisions has a unique equilibrium in terms of $c(\boldsymbol{d})$*. The conditional symmetry assumption (Assumption SY) can be seen as one example of this, where the game has a unique equilibrium in terms of $c(\boldsymbol{d})$ that is invariant to permutation, such as the number of players who choose to take the action ($c(\boldsymbol{d}) = \sum_{s=1}^S d_s$). Likewise, SY* corresponds to $c(\boldsymbol{d}) = (c_1(\boldsymbol{d}), ..., c_G(\boldsymbol{d}))$ with $c_g(\boldsymbol{d}) = \sum_{s=1}^{S_g} d_s^g$. There can certainly be other choices of $c(\boldsymbol{d})$ that delivers a unique equilibrium in the game, although we do not explore this further.

6.3 Point Identification of the ATE

When there exist player-specific excluded instruments of large support, we point identify the ATEs. In this case, the shape restrictions (especially on the outcome function) are not needed. The following assumption holds for each $s \in \{1, ..., S\}$.

Assumption EX*. *For each $\boldsymbol{d}_{-s} \in \mathcal{D}_{-s}$, $\nu^s(\boldsymbol{d}_{-s}, Z_s)|(X, \boldsymbol{Z}_{-s})$ has an everywhere positive Lebesgue density.*

Assumption EX* is stronger than Assumption EX. It imposes not only the exclusion restriction of EX but also a player-specific exclusion restriction and large support.

Theorem 6.1. *In model* (2.1) *and* (2.2)*, suppose Assumptions IN, E and EX* hold. Then the ATE in* (2.3) *is identified.*

The identification strategy is to exploit the large variation of player specific instruments based on Assumption EX*, which simultaneously solves the multiple equilibria problem and the endogeneity problem. Suppose $S = 2$ and Z_s is scalar for illustration; the general case can be proved analogously. For example, to identify $E[Y_{11}|X]$, consider

$$
\begin{aligned}
E[Y|\boldsymbol{D} = (1,1), X = x, \boldsymbol{Z} = \boldsymbol{z}] &= E[Y_{11}|\boldsymbol{D} = (1,1), X = x, \boldsymbol{Z} = \boldsymbol{z}] \\
&= E[\theta(1,1,x,\epsilon_{11})|\nu^1(1,z_1) \geq U_1, \nu^2(1,z_2) \geq U_2] \\
&\to E[\theta(1,1,x,\epsilon_{11})] = E[Y_{11}|X = x],
\end{aligned}
$$

where the second equation is by Assumption IN, and the convergence is by Assumption EX* with $z_1 \to \infty$ and $z_2 \to \infty$. Likewise, $E[Y_{00}|X = x]$ can be identified. The identification of $E[Y_{10}|X = x]$ and $E[Y_{01}|X = x]$ can be achieved by similar reasoning. Note that $\boldsymbol{D} = (1,0)$ or $\boldsymbol{D} = (0,1)$ can be predicted as an outcome of multiple equilibria. When either $(z_1, z_2) \to (\infty, -\infty)$ or $(z_1, z_2) \to (-\infty, \infty)$ occurs, however, a unique equilibrium is guaranteed as a dominant strategy, i.e., $\boldsymbol{D} = (1,0)$ or $\boldsymbol{D} = (0,1)$, respectively. Based on these results, we can (point) identify all the ATE's.

7   The LATE

The result of Theorem 3.1 on the equilibrium regions can be used to establish a framework that defines the LATE parameter for multiple treatments that are generated by strategic interaction. In this section, given model (2.1)–(2.2), we only maintain the assumptions on the payoff functions in the equations for D_s, but not the assumptions on the outcome functions in the equation for Y. In particular, we no longer require Assumptions M and SY. In the case of a single binary treatment, there is well-known equivalence between the LATE monotonicity assumption and the specification of a selection equation (Vytlacil (2002)). This equivalence result is inapplicable to our setting due to the simultaneity in the first stage.[19] But Proposition 3.1 implies that, under Assumptions SS and SY1, there is in fact a monotonic pattern in the way the equilibrium regions lie in the space of $\boldsymbol{U}$ as written in (3.4). This generalized monotonicity, formalized in Theorem 3.1, allows us to establish equivalence between a version of the LATE monotonicity assumption and the simultaneous selection model (2.2).

We first introduce a relevant counterfactual outcome that can be used in defining the LATE parameter. For $M \subseteq \mathcal{D}$, introduce a selection variable $\boldsymbol{D}_M \in M$ that selects an equilibrium $\boldsymbol{D}_M = \boldsymbol{d}$ when facing a set of equilibria, M. This variable is useful in decomposing the event $\boldsymbol{D} = \boldsymbol{d}$ into two sequential events: $\boldsymbol{D} = \boldsymbol{d}$ is equivalent to an event that $\boldsymbol{D} \in M$ and

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[19]For instance in a two-player entry game, when cost shifters Z_1 and Z_2 increase, it may be the case that in one market only the first player enters given this increase as her monopolistic profit offsets the increased cost, while in another market only the second player enters by the same reason applied to this player. The direction of monotonicity is reversed in these two markets.

$\boldsymbol{D}_M = \boldsymbol{d}$. Trivially, we have $\boldsymbol{D}_{\mathcal{D}} = \boldsymbol{D}$. When $M \subsetneq \mathcal{D}$ is not a singleton, $\boldsymbol{D}_M$ is not observed precisely because the equilibrium selection mechanism is not observed in general.[20] Using $\boldsymbol{D}_M$, we define a *joint counterfactual outcome* Y_M as an outcome had $\boldsymbol{D}$ been an element in M:

$$Y_M = \sum_{\boldsymbol{d} \in M} 1[\boldsymbol{D}_M = \boldsymbol{d}] Y_{\boldsymbol{d}}. \tag{7.1}$$

Conditional on $\boldsymbol{D} \in M$, Y_M is assigned to be one of the usual counterfactual outcome $Y_{\boldsymbol{d}}$ based on the equilibrium being selected. When $M = \mathcal{D}$, we can write $Y = Y_{\mathcal{D}} = \sum_{\boldsymbol{d} \in \mathcal{D}} 1[\boldsymbol{D} = \boldsymbol{d}] Y_{\boldsymbol{d}}$, which yields the standard expression that relates the observed outcome with the potential outcomes. Moreover, for any partition $\{\tilde{M}_k\}_{k=1}^K$ such that $\bigcup_{k=1}^K \tilde{M}_k = \mathcal{D}$, we can express

$$\sum_{\boldsymbol{d} \in \mathcal{D}} 1[\boldsymbol{D} = \boldsymbol{d}] Y_{\boldsymbol{d}} = \sum_{k=1}^K \sum_{\boldsymbol{d} \in \tilde{M}_k} 1[\boldsymbol{D} \in \tilde{M}_k] 1[\boldsymbol{D}_{\tilde{M}_k} = \boldsymbol{d}] Y_{\boldsymbol{d}} = \sum_{k=1}^K 1[\boldsymbol{D} \in \tilde{M}_k] Y_{\tilde{M}_k},$$

where the first equality is by the equivalence of the events mentioned above and the second equality is by (7.1). Therefore, we can establish the following relationship:

$$Y = \sum_{k=1}^K 1[\boldsymbol{D} \in \tilde{M}_k] Y_{\tilde{M}_k}, \tag{7.2}$$

that is, $Y_{\tilde{M}_k}$ is observed when $D \in \tilde{M}_k$.

Now, consider a *treatment of dichotomous states* (e.g., dichotomous market structures): for $j = 0, ..., S-1$,

$$\boldsymbol{D} \in M^{>j} \text{ vs. } \boldsymbol{D} \in M^{\leq j},$$

where $M^{\leq j} \equiv \bigcup_{k=0}^j M_k$ and $M^{>j} \equiv \bigcup_{k=j+1}^S M_k$ are previously defined; e.g., for $S = 2$ and $j = 1$, $M^{\leq 1} = \{(1,0), (0,1), (0,0)\}$ and $M^{>1} = \{(1,1)\}$. Consider a corresponding treatment effect:

$$Y_{M^{>j}} - Y_{M^{\leq j}},$$

where $Y = 1[\boldsymbol{D} \in M^{>j}] Y_{M^{>j}} + 1[\boldsymbol{D} \in M^{\leq j}] Y_{M^{\leq j}}$ by (7.2). This quantity is the effect of being treated with an equilibrium of at least $j+1$ entrants relative to being treated with an equilibrium of at most j entrants. We now establish that a version of the LATE monotonicity assumption for this treatment $1[\boldsymbol{D} \in M^{>j}]$ of dichotomous states is implied by the model specification (2.2), using Theorem 3.1. Recall $\boldsymbol{D}(\boldsymbol{z}) \equiv (D_1(z_1), ..., D_S(z_S))$ where $D_s(z_s)$ is the potential treatment.

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[20]Alternatively, following the notation of Heckman et al. (2006), we can introduce a equilibrium selection indicator $D_{M,\boldsymbol{d}}$ that indicates that an equilibrium $\boldsymbol{d}$ is selected among equilibria in a set M:

$$D_{M,\boldsymbol{d}} = \begin{cases} 1 & \text{if } \boldsymbol{d} \in M \text{ is selected}, \\ 0 & \text{o.w.} \end{cases}$$

Then, $D_M = \boldsymbol{d}$ if and only if $D_{M,\boldsymbol{d}} = 1$.

Lemma 7.1. *Under Assumptions SS, SY1 and M1, the first-stage game* (2.2) *implies that, for any* $\boldsymbol{z}, \boldsymbol{z}' \in \mathcal{Z}$ *and* $j = 0, ..., S - 1$,

$$\Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{\leq j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{>j}] = 0$$
$$or \tag{7.3}$$
$$\Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{\leq j}] = 0.$$

The condition (7.3) is a generalized version of Imbens and Angrist (1994)'s monotonicity assumption.

Proof. For given $\boldsymbol{z}, \boldsymbol{z}' \in \mathcal{Z}$, suppose without loss of generality that in Assumption M1, $\nu_{\boldsymbol{d}_{-s}}^s(z_s) \geq \nu_{\boldsymbol{d}_{-s}}^s(z'_s) \; \forall \boldsymbol{d}_{-s}$ and $\forall s$. Then by Theorem 3.1, it follows that $\boldsymbol{R}^{>j}(\boldsymbol{z}) \supseteq \boldsymbol{R}^{>j}(\boldsymbol{z}')$. Then

$$\Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{\leq j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{>j}] = \Pr[\boldsymbol{U} \in \boldsymbol{R}^{\leq j}(\boldsymbol{z}) \cap \boldsymbol{R}^{>j}(\boldsymbol{z}')] = 0.$$

$\square$

Lemma 7.1 allows us to give the IV estimand a LATE interpretation in our model:

Theorem 7.1. *Given model* (2.1)–(2.2), *suppose Assumptions SS, SY1, M1, IN and EX hold. Then it satisfies that, for any* $j = 0, ..., S - 1$,

$$\frac{h(\boldsymbol{z}, \boldsymbol{z}')}{p_{M^{>j}}(\boldsymbol{z}) - p_{M^{>j}}(\boldsymbol{z}')} = \frac{E[Y|\boldsymbol{Z} = \boldsymbol{z}] - E[Y|\boldsymbol{Z} = \boldsymbol{z}']}{\Pr[\boldsymbol{D} \in M^{>j}|\boldsymbol{Z} = \boldsymbol{z}] - \Pr[\boldsymbol{D} \in M^{>j}|\boldsymbol{Z} = \boldsymbol{z}']}$$
$$= E[Y_{M^{>j}} - Y_{M^{\leq j}}|\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{\leq j}].$$

The LATE parameter $E[Y_{M^{>j}} - Y_{M^{\leq j}}|\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{\leq j}]$ is the average of treatment effect $Y_{M^{>j}} - Y_{M^{\leq j}}$ for a subgroup of "markets" that form more competitive markets (with at least $j + 1$ entrants) when players face $\boldsymbol{Z} = \boldsymbol{z}$, but form less competitive markets (with at most j entrants) when players face $\boldsymbol{Z} = \boldsymbol{z}'$. For concreteness, suppose $S = 2$, $j = 1$, Z_s is each airline company's cost shifters and Y is the pollution level in a market. The LATE

$$E[Y_{\{(1,1)\}} - Y_{\{(1,0),(0,1),(0,0)\}}|\boldsymbol{D}(\boldsymbol{z}) = (1,1), \boldsymbol{D}(\boldsymbol{z}') \in \{(1,0),(0,1),(0,0)\}]$$

is the effect of the existence of competition on pollution levels for markets consist of "compliers."[21] It is the average difference of potential pollution levels in a duopoly market (i.e., duopolistic competition) versus a monopoly or non-operating market (i.e., no competition) for the subgroups of markets that form a duopoly when companies are facing low cost ($\boldsymbol{Z} = \boldsymbol{z}$) but form a monopoly or do not operate when facing high cost ($\boldsymbol{Z} = \boldsymbol{z}'$). Figure 6 depicts this subgroup of markets. In this example, the LATE monotonicity assumption (implied by the entry game of strategic substitutes with symmetric payoffs) rules out those markets that respond to

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[21]In this multi-agent multi-treatment scenario, compliers are defined as those *players* whose behaviors are such that market structures are formed in conformance with the LATE monotonicity assumption (7.3). Unlike in the traditional setting (Imbens and Angrist (1994)) where compliers are defined in terms of the subset of population, the subpopulation in the present setting is the collection of the markets consist of the complying players.

cost shifters as "defiers." The LATE becomes the ATE when $1 = \Pr[\boldsymbol{D}(\boldsymbol{z}) = (1,1), \boldsymbol{D}(\boldsymbol{z}') \in \{(1,0),(0,1),(0,0)\}] = \Pr[\boldsymbol{D} = (1,1)|\boldsymbol{Z} = \boldsymbol{z}] - \Pr[\boldsymbol{D} \in \{(1,0),(0,1),(1,1)\}|\boldsymbol{Z} = \boldsymbol{z}']$, which is related to the large-support argument in Theorem 6.1.

In general, the LATE can be defined with $Y_M - Y_{M'}$ for any two partitioning sets M and M' of $\mathcal{D}$ (i.e., $\mathcal{D} = M \cup M'$ with $M \cap M' = \emptyset$) as long as $1[\boldsymbol{D}(\boldsymbol{z}) \in M] = 1 - 1[\boldsymbol{D}(\boldsymbol{z}) \in M']$ satisfies the LATE monotonicity assumption. Lemma 7.1 ensures that our simultaneous selection model imposes this monotonicity for a particular partition, $M = M^{>j}$ and $M' = M^{\leq j}$. Also the LATE using a more general function of the potential outcomes can be recovered analogous to Abadie (2003): $E[g(Y_{M^{>j}}, X) - g(Y_{M^{\leq j}}, X)|\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{\leq j}]$ for a measurable function $g(\cdot)$ such that $E|g(\cdot)| < \infty$.

Remark 7.1. *Similarly, it may be possible to recover the marginal treatment effect (MTE) of Heckman and Vytlacil (1999, 2005, 2007). Given our setting, it should be a transition-specific MTE for $Y_{M_j} - Y_{M_{j-1}}$. The identification of this MTE would require continuous variation of $\boldsymbol{Z}$. For discrete $\boldsymbol{Z}$, the approach by Brinch et al. (2017) can be applied by imposing structures on the MTE function.*

Remark 7.2. *The equilibrium selection mechanism may differ across different counterfactual worlds. In terms of our notation, $\boldsymbol{D}_M(\boldsymbol{z})$ may differ from $\boldsymbol{D}_M(\boldsymbol{z}')$, where $\boldsymbol{D}_M(\boldsymbol{z})$ is the counterfactual variable of $\boldsymbol{D}_M$. Note that not only the equilibrium being selected is different but also the selection mechanism can be different. This feature may be emphasized by writing $\boldsymbol{D}_M(\boldsymbol{z}) = \lambda_{\boldsymbol{z}}(\boldsymbol{z}, \boldsymbol{U})$ where the functional form of the equilibrium selection function may also change in $\boldsymbol{z}$. By considering Y_M instead of $Y_{\boldsymbol{d}}$, however, we can be agnostic about the selection mechanism, i.e., about the specification of $\lambda_{\boldsymbol{z}}(\cdot, \cdot)$. The definition (7.1) asserts that $Y_{\boldsymbol{d}}$ can be meaningfully analyzed within the current framework only when the equilibrium being selected is known.*

8 Numerical Studies

To illustrate the main results of this paper, we calculate the bounds on the ATE using the following data generating process:

$$Y_{\boldsymbol{d}} = 1[\tilde{\mu}_{\boldsymbol{d}} + \beta X \geq \epsilon],$$
$$D_1 = 1[\delta_2 D_2 + \gamma_1 Z_1 \geq V_1],$$
$$D_2 = 1[\delta_1 D_1 + \gamma_2 Z_2 \geq V_2],$$

where (ϵ, V_1, V_2) are drawn from a joint normal distribution with zero means and each correlation coefficient being 0.5, and drawn independent of $(X, \boldsymbol{Z})$. We draw Z_s ($s = 1, 2$) and X from multinomial, allowing Z_s to take two values, $\mathcal{Z}_s = \{-1, 1\}$ and X to take either three values, $\mathcal{X} = \{-1, 0, 1\}$, or fifteen values, $\mathcal{X} = \{-1, -\frac{6}{7}, -\frac{5}{7}, ..., \frac{5}{7}, \frac{6}{7}, 1\}$. Being consistent with Assumptions M and SY, we choose $\tilde{\mu}_{11} > \tilde{\mu}_{10} = \tilde{\mu}_{01} > \tilde{\mu}_{00}$, and with Assumption SS, we choose $\delta_1 < 0$ and $\delta_2 < 0$. Without loss of generality, we choose positives values for γ_1, γ_2, and β. Specifically, $\tilde{\mu}_{11} = 0.25$, $\tilde{\mu}_{10} = \tilde{\mu}_{01} = 0$ and $\tilde{\mu}_{00} = -0.25$. For default values, $\delta_1 = \delta_2 \equiv \delta = -0.1$, $\gamma_1 = \gamma_2 \equiv \gamma = 1$ and $\beta = 0.5$.

In this exercise, we focus on the ATE $E[Y_{11} - Y_{00}|X = 0]$ whose true value is 0.2 given the parameter values. For $h(\boldsymbol{z}, \boldsymbol{z}', x)$, we consider $\boldsymbol{z} = (1, 1)$ and $\boldsymbol{z}' = (-1, -1)$. Note that

$H(x) = h(\boldsymbol{z}, \boldsymbol{z}', x)$ and $H(x, x', x'') = h(\boldsymbol{z}, \boldsymbol{z}'; x, x', x'')$ since Z_s is binary. Then we can derive the sets $\mathcal{X}_{\boldsymbol{d}}^U(0; \boldsymbol{d}')$ and $\mathcal{X}_{\boldsymbol{d}}^L(0; \boldsymbol{d}')$ for each $\boldsymbol{d} \in \{(1,1), (0,0)\}$ and $\boldsymbol{d}' \neq \boldsymbol{d}$ in Theorem 4.2.

Based on our design, $H(0) > 0$ and thus the bounds when we use Z only are, with $x = 0$,

$$\max_{\boldsymbol{z} \in \mathcal{Z}} \Pr[Y = 1, \boldsymbol{D} = (0,0) | \boldsymbol{z}, x] \leq \Pr[Y_{00} = 1 | x] \leq \min_{\boldsymbol{z} \in \mathcal{Z}} \Pr[Y = 1 | \boldsymbol{z}, x],$$

and

$$\max_{\boldsymbol{z} \in \mathcal{Z}} \Pr[Y = 1 | \boldsymbol{z}, x] \leq \Pr[Y_{11} = 1 | x] \leq \min_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \Pr[Y = 1, \boldsymbol{D} = (1,1) | \boldsymbol{z}, x] + 1 - \Pr[\boldsymbol{D} = (1,1) | \boldsymbol{z}, x] \right\}.$$

Using both $\boldsymbol{Z}$ and X, we have narrower bounds. For example when $|\mathcal{X}| = 3$, with $H(0, -1, -1) < 0$, the lower bound on $\Pr[Y_{00} = 1 | X = 0]$ becomes

$$\max_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \Pr[Y = 1, \boldsymbol{D} = (0,0) | \boldsymbol{z}, 0] + \Pr[Y = 1, \boldsymbol{D} \in \{(1,0), (0,1)\} | \boldsymbol{z}, -1] \right\}.$$

With $H(1, 1, 0) < 0$, the upper bound on $\Pr[Y_{11} = 1 | X = 0]$ becomes

$$\min_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \Pr[Y = 1, \boldsymbol{D} = (1,1) | \boldsymbol{z}, 0] + \Pr[Y = 1, \boldsymbol{D} \in \{(1,0), (0,1)\} | \boldsymbol{z}, 1] + \Pr[\boldsymbol{D} = (0,0) | \boldsymbol{z}, 0] \right\}.$$

For comparison, we calculate the bounds by Manski (1990) using $\boldsymbol{Z}$. The Manski's bounds are

$$\max_{\boldsymbol{z} \in \mathcal{Z}} \Pr[Y = 1, \boldsymbol{D} = (0,0) | \boldsymbol{z}, x] \leq \Pr[Y_{00} = 1 | x]$$
$$\leq \min_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \Pr[Y = 1, \boldsymbol{D} = (0,0) | \boldsymbol{z}, x] + 1 - \Pr[\boldsymbol{D} = (0,0) | \boldsymbol{z}] \right\},$$

and

$$\max_{\boldsymbol{z} \in \mathcal{Z}} \Pr[Y = 1, \boldsymbol{D} = (1,1) | \boldsymbol{z}, x] \leq \Pr[Y_{11} = 1 | x]$$
$$\leq \min_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \Pr[Y = 1, \boldsymbol{D} = (1,1) | \boldsymbol{z}, x] + 1 - \Pr[\boldsymbol{D} = (1,1) | \boldsymbol{z}] \right\}.$$

We also compare the estimated ATE using the specification of a standard linear IV model where the nonlinearity of the true DGP are ignored:

$$Y = \pi_0 + \pi_1 D_1 + \pi_2 D_2 + \beta X + \epsilon,$$
$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} \gamma_{10} \\ \gamma_{20} \end{pmatrix} + \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

Here the first stage is the reduced-form representation of the linear simultaneous equations model for strategic interaction. Under this specification, the ATE becomes $E[Y_{11} - Y_{00} | X = 0] = \pi_1 + \pi_2$, which is estimated via two-stage least squares (TSLS).

The bounds calculated for the ATE are shown in Figures 7–10. Figure 7 shows how the bounds on the ATE change as the value of γ changes from 0 to 2.5. The larger γ is the stronger the instrument $\boldsymbol{Z}$ is. The first conspicuous result is that the TSLS estimate of the ATE is biased due the the problem of misspecification. Next, as expected, the Manski's bounds and our proposed bounds converge to the true value of the ATE as the instrument

becomes stronger. Overall, our bounds, with or without exploiting the variation of X, are much narrower than the Manski bounds.[22] Notice that the sign of the ATE is identified in the whole range of γ as predicted by the first part of Theorem 4.2, in contrast to the Manski's bounds. By using the additional variation from X with $|\mathcal{X}| = 3$, the width of the bounds is decreased, particularly with the smaller upper bounds on the ATE in this simulation design. Figure 8 depicts the bounds using X with $|\mathcal{X}| = 15$, which yields narrower bounds than using X with $|\mathcal{X}| = 3$ and substantially narrower than those using only $\boldsymbol{Z}$.

Figure 9 shows how the bounds change as the value of β changes from 0 to 1.5, where a larger β corresponds to a stronger exogenous variable X. The jumps in the upper bound are associated with the sudden changes in the signs of $H(-1, 0, -1)$ and $H(0, 1, 1)$. At least in this simulation design, the strength of X is not a crucial factor to obtain narrower bounds. In fact, based other simulation results (which are omitted in the paper), we conclude that the number of values X can take matters more than the dispersion of X (unless we pursue point identification of the ATE).

Figure 10 shows how the width of the bounds is related to the extent to which the opponents' actions D_{-s} affect one's payoff, captured in δ. We vary the value of δ from -2 to 0, and when $\delta = 0$, the players solve a single-agent optimization problem. Thus, heuristically, the bound at this point would be similar to the ones that can be obtained when Shaikh and Vytlacil (2011) is extended to a multiple-treatment setting with no simultaneity. In the figure, as the value of δ gets smaller, the bounds get narrower.

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[22] Although we do not make a rigorous comparison of the assumptions here, note that the bounds by Manski and Pepper (2000) under the semi-MTR is expected to be similar to ours. Their bounds, however, need to assume the direction of the monotonicity.

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A Partial ATE

Define a *partial counterfactual outcome* as follows: with a partition $\boldsymbol{D} = (\boldsymbol{D}^1, \boldsymbol{D}^2) \in \mathcal{D}^1 \times \mathcal{D}^2 = \mathcal{D}$ and its realization $\boldsymbol{d} = (\boldsymbol{d}^1, \boldsymbol{d}^2)$,

$$Y_{\boldsymbol{d}^1,\boldsymbol{D}^2} \equiv \sum_{\boldsymbol{d}^2 \in \mathcal{D}^2} 1[\boldsymbol{D}^2 = \boldsymbol{d}^2] Y_{\boldsymbol{d}^1,\boldsymbol{d}^2}. \tag{A.1}$$

This is a counterfactual outcome that is fully observed once $\boldsymbol{D}^1 = \boldsymbol{d}^1$ is realized. Then for each $\boldsymbol{d}^1 \in \mathcal{D}^1$, the partial ASF can be defined as

$$E[Y_{\boldsymbol{d}^1,\boldsymbol{D}^2}] = \sum_{\boldsymbol{d}^2 \in \mathcal{D}^2} E[Y_{\boldsymbol{d}^1,\boldsymbol{d}^2}|\boldsymbol{D}^2 = \boldsymbol{d}^2] \Pr[\boldsymbol{D}^2 = \boldsymbol{d}^2] \tag{A.2}$$

and the *partial ATE* between $\boldsymbol{d}$ and $\boldsymbol{d}'$ as

$$E[Y_{\boldsymbol{d}^1,\boldsymbol{D}^2} - Y_{\boldsymbol{d}^{1\prime},\boldsymbol{D}^2}]. \tag{A.3}$$

Using this concept, we can consider complementarity concentrated on, e.g., the first two treatments: $E\left[Y_{11,\boldsymbol{D}^2} - Y_{01,\boldsymbol{D}^2}\right] > E\left[Y_{10,\boldsymbol{D}^2} - Y_{00,\boldsymbol{D}^2}\right]$.

B More Examples

Example 3 (Incumbents' response to potential entrants)**.** *In this example, we are interested in how market i's incumbents respond to the threat of entry of potential competitors. Let Y_i be an incumbent firm's pricing or investment decision and $D_{s,i}$ be an entry decision by firm s in "nearby" markets, which can be formally defined in each context. For example, in airline entry, nearby markets are defined as city pairs that share the endpoints with the city pair of an incumbent (Goolsbee and Syverson (2008)). That is, potential entrants are airlines that operate in one (or both) of the endpoints of the incumbent's market i, but who have not connected these endpoints. Then the parameter $E[Y_{\boldsymbol{d},i} - Y_{\boldsymbol{d}',i}]$ captures the incumbent's response to the threat, specifically whether it responds by lowering the price or making an investment. As in Example 1, $Z_{s,i}$ are cost shifters and X_i are other factors affecting price of the incumbent, excluded from nearby markets, conditional of W_i. The characteristics of the incumbent's market can be a candidate of X_i, such as the distance between the endpoints of the incumbent's market in the airline example.*

Example 4 (Food desert)**.** *Let Y_i denote a health outcome, such as diabetes prevalence, in region i, and $D_{s,i}$ be the exit decision by large supermarket s in the region. Then $E[Y_{\boldsymbol{d},i} - Y_{\boldsymbol{d}',i}]$ measures the effects of absence of supermarkets on health of the residents. Conditional on*

other factors W_i, the instrument $Z_{s,i}$ can include changes in local government's zoning plans and X_i can include the region's health-related variables, such as the number of hospitals and the obesity rate. This problem is related to the literature on "food desert" (e.g., Walker et al. (2010)).

Example 5 (Ground water and agriculture). *In this example, we are interested in the impact of access to groundwater on economic outcomes in rural areas (Foster and Rosenzweig (2008)). In each Indian village i, symmetric wealthy farmers (of the same caste) make irrigation decisions $D_{s,i}$, i.e., whether or not to buy motor pumps, in the presence of peer effects and learning spillovers. Since ground water is a limited resource that is seasonally recharged and depleted, other farmers' entry may negatively affects one's payoff. The adoption of the technology affects Y_i, which can be the average of local wages of peasants or prices of agricultural products, or a village development or poverty level. In this example, continuous or binary instrument $Z_{s,i}$ can be the depth to groundwater, which is exogenously given (Sekhri (2014)), or provision of electricity for pumping in a randomized field experiment. X_i can be village-level characteristics that villagers do not know ex ante or do not concern about.*[23]

C   Model with Common Z

Consider model (2.1)–(2.2) but with instruments common to all players/treatments, i.e., $Z_1 = \cdots = Z_S$:

$$Y = \theta(\boldsymbol{D}, X, \epsilon_{\boldsymbol{D}}),$$
$$D_s = 1\left[\nu^s(\boldsymbol{D}_{-s}, Z_1) \geq U_s\right], \qquad s \in \{1, ..., S\}.$$

This setting can be motivated by such instruments as appeared in Example 2. Given this model, Assumptions SS, SY1, M1, IN, EX and C will be understood with $Z_1 = \cdots = Z_S$ imposed.[24] Then the bound analysis for the ATE including sharpness as well as the LATE result will naturally follow. The intuition of this straightforward extension is as follows. As a generalized version of monotonicity in the treatment selection process is restored (Theorem 3.1), model (2.1)–(2.2) can essentially be seen as a triangular model with an ordered-choice type of a first-stage. Therefore an instrument that "shift" the entire first-stage process is sufficient for the purpose of our analyses. Player-specific instruments do introduce an additional source of variation, as it is crucial for the point identification of the ATE that employs identification at infinity.

D   Proofs

D.1   Proof of Proposition 3.1

The following proposition is useful in proving Proposition 3.1:

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[23]Especially in this example, the number of players/treatments S_i is allowed to vary across villages. We assume in this case that players/treatments are symmetric (in a sense that becomes clear later) and $\nu^1(\cdot) = \cdots = \nu^{S_i}(\cdot) = \nu(\cdot)$.

[24]Assumption ASY may be slightly harder to justify with a common instrument.

Proposition D.1. *Let R and Q be sets defined by Cartesian products: $R = \prod_{s=1}^{S} r_s$ and $Q = \prod_{s=1}^{S} q_s$ where r_s and q_s are intervals in $\mathbb{R}$. Then the following holds:*
(i) If $r_s \cap q_s = \emptyset$ for some s, then $R \cap Q = \emptyset$;
(ii) If $r_s \sim q_s \ \forall s$, then $R \sim Q$;
(iii) If $r_s \nsim q_s$ for some s, then $R \nsim Q$;
(iv) $R \cap Q = \prod_{s=1}^{S} r_s \cap q_s$;
(v) $cl(R) = \prod_{s=1}^{S} cl(r_s)$ where $cl(\cdot)$ is the closure of its argument.

The proof of this proposition follows directly from the definition of R and Q. To utilize Proposition D.1, we show that Proposition 3.1(i)–(iii) are implied by similar statements that satisfy for all individual pairs between two regions: (i') $R_{\boldsymbol{d}^j} \cap R_{\boldsymbol{d}^{j'}} = \emptyset \ \forall d^j \in M_j$ and $\forall d^{j'} \in M_{j'}$ with $j \neq j'$; (ii') $R_{\boldsymbol{d}^j}$ and $R_{\boldsymbol{d}^{j-1}}$ are neighboring sets $\forall d^j \in M_j$ and $\forall d^{j-1} \in M_{j-1}$; (iii') $R_{\boldsymbol{d}^j}$ and $R_{\boldsymbol{d}^{j-t}}$ are not neighboring sets $\forall d^j \in M_j$ and $\forall d^{j-t} \in M_{j-t}$ with $t \geq 2$.

Before proving Proposition 3.1(i), we prove (i'). We first show a simple case as a reference: $R_{\boldsymbol{e}^j} \cap R_{\boldsymbol{e}^{j-1}} = \emptyset$ for $j = 1, ..., S$. Note that

$$R_{\boldsymbol{e}^j}(\boldsymbol{z}) = \left\{ \prod_{s=1}^{j} \left(0, \nu_{j-1}^s(z_s)\right] \right\} \times \left\{ \prod_{s=j+1}^{S} \left(\nu_j^s(z_s), 1\right] \right\}$$

$$R_{\boldsymbol{e}^{j-1}}(\boldsymbol{z}) = \left\{ \prod_{s=1}^{j-1} \left(0, \nu_{j-2}^s(z_s)\right] \right\} \times \left\{ \prod_{s=j}^{S} \left(\nu_{j-1}^s(z_s), 1\right] \right\}$$

and the j-th coordinates are $\left(0, \nu_{j-1}^j(z_j)\right]$ in $R_{\boldsymbol{e}^j}$ and $\left(\nu_{j-1}^j(z_j), 1\right]$ in $R_{\boldsymbol{e}^{j-1}}$. Since these two intervals are disjoint, by Proposition D.1(i), we can conclude that $R_{\boldsymbol{e}^j} \cap R_{\boldsymbol{e}^{j-1}} = \emptyset$. Now to prove (i'), we equivalently prove $R_{\boldsymbol{d}^j} \cap R_{\boldsymbol{d}^{j-t}} = \emptyset$ for $t \geq 1$ and $0 \leq j - t \leq S - t$, and draw insights from the simple case. Note that $\boldsymbol{d}^{j-t}$ contains $S - j + t$ zeros. Then there exists s^* such that $d_{s^*}^j = 1$ but $d_{s^*}^{j-t} = 0$, i.e., $U_{s^*} \in \left(0, \nu_{j-1}^{s^*}(z_{s^*})\right]$ in $R_{\boldsymbol{d}^j}$ but $U_{s^*} \in \left(\nu_{j-t}^{s^*}(z_{s^*}), 1\right]$ in $R_{\boldsymbol{d}^{j-t}}$. Suppose not. Then $\forall s$ such that $d_s^j = 1$, it must hold that $d_s^{j-t} = 1$. This implies that $\boldsymbol{d}^{j-t}$ has at least as many elements of unity as $\boldsymbol{d}^j$, which is contradiction as $t \geq 1$. Therefore since $\left(0, \nu_{j-1}^{s^*}(z_{s^*})\right]$ and $\left(\nu_{j-t}^{s^*}(z_{s^*}), 1\right]$ are disjoint, $R_{\boldsymbol{d}^j}$ and $R_{\boldsymbol{d}^{j-t}}$ are disjoint. When $t \geq 2$, by Assumption SS, $\nu_{j-t}^{s^*}(z_{s^*}) > \nu_{j-1}^{s^*}(z_{s^*})$ and therefore $\left(\nu_{j-t}^{s^*}(z_{s^*}), 1\right]$ and $\left(0, \nu_{j-1}^{s^*}(z_{s^*})\right]$ are disjoint and thus the same conclusion follows. Also when $t = 1$, $\left(\nu_{j-1}^{s^*}(z_{s^*}), 1\right]$ and $\left(0, \nu_{j-1}^{s^*}(z_{s^*})\right]$ are obviously disjoint. This proves (i').

For Proposition 3.1(i), one can conclude from (i') that $R_{\boldsymbol{d}^j}$ is disjoint to $R_{\boldsymbol{d}^{j'}}$ for any $\boldsymbol{d}^{j'} \in M_{j'}$ and hence is disjoint to $\bigcup_{\boldsymbol{d} \in M_{j'}} R_{\boldsymbol{d}}$. This is true $\forall d^j \in M_j$, and therefore $\bigcup_{\boldsymbol{d} \in M_j} R_{\boldsymbol{d}}$ is disjoint to $\bigcup_{\boldsymbol{d} \in M_{j'}} R_{\boldsymbol{d}}$.

To prove (ii'), by Proposition D.1(ii), one needs to show that each pair of intervals of the same coordinate are neighboring intervals. This is immediately true for $R_{\boldsymbol{e}^j}$ and $R_{\boldsymbol{e}^{j-1}}$ above, since (a) for coordinates $1 \leq s \leq j-1$, $\left(0, \nu_{j-1}^s(z_s)\right] \sim \left(0, \nu_{j-2}^s(z_s)\right]$ with a nonempty intersection since $\left(0, \nu_{j-1}^s(z_s)\right] \subset \left(0, \nu_{j-2}^s(z_s)\right]$; (b) for coordinate $s = j$, $\left(0, \nu_{j-1}^j(z_j)\right] \sim$

$\left(\nu_{j-1}^j(z_j), 1\right]$ and they are disjoint; and (c) for coordinates $j+1 \le s \le S$, $\left(\nu_j^s(z_s), 1\right] \sim \left(\nu_{j-1}^s(z_s), 1\right]$ with a nonempty intersection since $\left(\nu_j^s(z_s), 1\right] \supset \left(\nu_{j-1}^s(z_s), 1\right]$. Now consider $R_{\boldsymbol{d}^j}$ and $R_{\boldsymbol{d}^{j-1}}$. In $\boldsymbol{d}^j$ and $\boldsymbol{d}^{j-1}$, there exists s^* such that $d_{s^*}^j = 1$ but $d_{s^*}^{j-1} = 0$ by the same argument as above with $t=1$. The rest of the elements in $\boldsymbol{d}^j$ and $\boldsymbol{d}^{j-1}$ fall into one of the four types: for $s \ne s^*$, (a') $d_s^j = d_s^{j-1} = 1$; (b') $d_s^j = 1$ but $d_s^{j-1} = 0$; (c') $d_s^j = d_s^{j-1} = 0$; and (d') $d_s^j = 0$ but $d_s^{j-1} = 1$. See Table 1 in the main text for an example of this result. We aim to express the corresponding intervals of U_s that generate these values of d_s^j and d_s^{j-1}. By definition, the number of ones (and zeros) in $\boldsymbol{d}^j$ and $\boldsymbol{d}^{j-1}$ differs only by one, which happens in each vector's s^*-th element. Knowing this, for these pairs of d_s^j and d_s^{j-1} in (a')–(d'), we can determine the decision of the opponents of player s (i.e., the value of j in ν_j^s) which is useful to construct the payoff of s, and thus the corresponding interval of U_s. Specifically, we can determine that the corresponding interval pairs are: (a'') $\left(0, \nu_{j-1}^s(z_s)\right]$ and $\left(0, \nu_{j-2}^s(z_s)\right]$; (b'') $\left(0, \nu_{j-1}^s(z_s)\right]$ and $\left(\nu_{j-1}^s(z_s), 1\right]$; (c'') $\left(\nu_j^s(z_s), 1\right]$ and $\left(\nu_{j-1}^s(z_s), 1\right]$; (d'') $\left(\nu_j^s(z_s), 1\right]$ and $\left(0, \nu_{j-2}^s(z_s)\right]$. It is straightforward that the pairs in (a'')–(c'') are neighboring sets by the same arguments as for (a)–(c). The pair in (d'') are also neighboring sets because $\nu_j^s(z_s) < \nu_{j-2}^s(z_s)$ by Assumption SS. Lastly, for coordinate s^*, $\left(0, \nu_{j-1}^{s^*}(z_{s^*})\right] \sim \left(\nu_{j-1}^{s^*}(z_{s^*}), 1\right]$ as in (b''). Therefore, $R_{\boldsymbol{d}^j} \sim R_{\boldsymbol{d}^{j-1}}$.

For Proposition 3.1(ii), one can conclude from (ii') that $R_{\boldsymbol{d}^j}$ neighbors $R_{\boldsymbol{d}^{j-1}}$ for any $\boldsymbol{d}^{j-1} \in M_{j-1}$ and hence neighbors $\bigcup_{\boldsymbol{d} \in M_{j-1}} R_{\boldsymbol{d}}$. This is true $\forall \boldsymbol{d}^j \in M_j$, and therefore $\bigcup_{\boldsymbol{d} \in M_j} R_{\boldsymbol{d}} \sim \bigcup_{\boldsymbol{d} \in M_{j'}} R_{\boldsymbol{d}}$.

The result in Proposition 3.1(iii) follows from the proof of (i') above that there exists s^* such that $d_{s^*}^j = 1$ but $d_{s^*}^{j-t} = 0$, i.e., $U_{s^*} \in \left(0, \nu_{j-1}^{s^*}(z_{s^*})\right]$ in $R_{\boldsymbol{d}^j}$ but $U_{s^*} \in \left(\nu_{j-t}^{s^*}(z_{s^*}), 1\right]$ in $R_{\boldsymbol{d}^{j-t}}$. When $t \ge 2$, by Assumption SS, $\nu_{j-t}^{s^*}(z_{s^*}) > \nu_{j-1}^{s^*}(z_{s^*})$ and therefore $\left(0, \nu_{j-1}^{s^*}(z_{s^*})\right] \not\sim \left(\nu_{j-t}^{s^*}(z_{s^*}), 1\right]$ which implies that, by Proposition D.1(iii), their Cartesian products are not neighboring sets.

Lastly, we prove Proposition 3.1(iv). We consider a S-dimensional hyper-grid for $(0,1]^S$ that runs through all possible values of ν_j^s across $j = 0, ..., S$ for each $s = 1, ..., S$. Specifically, under Assumption SS and by conveniently letting $\nu_S^s = 0$ and $\nu_{-1}^s = 1$, the hyper-grid is a Cartesian product of 1-dimensional grids defined by $0 = \nu_S^s < \nu_{S-1}^s < \cdots < \nu_0^s < \nu_{-1}^s = 1$ for each coordinate s. Let each hyper-cube in this hyper-grid be represented as

$$r_1(j_1) \times r_2(j_2) \times \cdots \times r_S(j_S) \equiv \left(\nu_{j_1}^1, \nu_{j_1-1}^1\right] \times \left(\nu_{j_2}^2, \nu_{j_2-1}^2\right] \times \cdots \times \left(\nu_{j_S}^S, \nu_{j_S-1}^S\right],$$

where $r_s(\cdot)$ are intervals implicitly defined in the equation and labeled with $j_s = 0, ..., S$.

Using these notations, R_{e^j} for $j = 0, ..., S$ can be expressed as

$$R_{e^j} = \left\{ U : (U_1, ..., U_S) \in \left\{ \prod_{s=1}^{j} \bigcup_{k=j}^{S} r_s(k) \right\} \times \left\{ \prod_{s=j+1}^{S} \bigcup_{k=0}^{j} r_s(k) \right\} \right\}$$

$$= \left\{ U : (U_1, ..., U_S) \in \bigcup_{j_1=j}^{S} \cdots \bigcup_{j_j=j}^{S} \cdot \bigcup_{j_{j+1}=0}^{j} \cdots \bigcup_{j_S=0}^{j} r_1(j_1) \times \cdots \times r_S(j_S) \right\}, \qquad \text{(D.1)}$$

where the second equality is by iteratively applying the following: for sets A, B and C being Cartesian products (including intervals as a trivial case),

$$(A \cup B) \times C = (A \times C) \cup (B \times C).$$

More generally, $R_{d_\sigma^j}$ for some $\sigma(\cdot) \in \Sigma$ can be defined as

$$R_{d_\sigma^j} = \left\{ U : (U_{\sigma(1)}, ..., U_{\sigma(S)}) \in \bigcup_{j_1=j}^{S} \cdots \bigcup_{j_j=j}^{S} \cdot \bigcup_{j_{j+1}=0}^{j} \cdots \bigcup_{j_S=0}^{j} r_{\sigma(1)}(j_1) \times \cdots \times r_{\sigma(S)}(j_S) \right\}.$$

$$\text{(D.2)}$$

Below we show that any hyper-cube $r_1(j_1) \times r_2(j_2) \times \cdots \times r_S(j_S)$ is contained in one of $R_{d_\sigma^j}$'s for some j and $\sigma(\cdot)$. We first proceed by showing that there are hyper-cubes that are contained in R_{e^j}'s. We then show that any hyper-cube can be transformed using a permutation function into a hyper-cube contained in R_{e^j}, which means that the original hyper-cube is contained in some R_{d^j} which is a "permutated version" of R_{e^j}.

Claim 1: For $j_1 \geq j_2 \geq \cdots \geq j_S$, $r_1(j_1) \times r_2(j_2) \times \cdots \times r_S(j_S)$ is contained in R_{e^j} for some $j \leq j_1$.
Claim 2: For any $\{j_1, ..., j_S\}$, $r_1(j_1) \times r_2(j_2) \times \cdots \times r_S(j_S)$ is contained in R_{d^j} for $j \leq \max\{j_1, ..., j_S\}$.

Proof of Claim 1: Start with a hyper-cube at a corner:

$$r_1(0) \times r_2(0) \times \cdots \times r_S(0) \equiv \left(\nu_0^1, 1 \right] \times \left(\nu_0^2, 1 \right] \times \cdots \times \left(\nu_0^S, 1 \right].$$

This hyper-cube is contained in R_{e^0} as the two in fact coincide. Consider the next hyper-cube on the grid along the 1-st coordinate: $r_1(1) \times r_1(0) \cdots \times r_S(0)$. This hyper-cube is contained in R_{e^1} as

$$R_{e^1} = \bigcup_{j_1=1}^{S} \cdot \bigcup_{j_2=0}^{1} \cdots \bigcup_{j_S=0}^{1} r_1(j_1) \times \cdots \times r_S(j_S).$$

We move to the 2-nd coordinate holding the 1-st coordinate fixed. Then $r_1(1) \times r_2(1) \times r_3(0) \times \cdots \times r_S(0)$ is still contained in R_{e^1}. Likewise, from $r_1(1) \times r_2(1) \times r_3(1) \times r_4(0) \times \cdots \times r_S(0)$ all the way to $r_1(1) \times \cdots \times r_S(1)$, these hyper-cubes are contained in R_{e^1}.

Now consider the next hyper-cube along the 1-st coordinate, i.e., $r_1(2) \times r_2(0) \times \cdots \times r_S(0)$.

This is contained in $R_{\boldsymbol{e}^1}$. We move to the next coordinates holding the 1-st coordinate fixed. Then $r_1(2) \times r_2(1) \times r_3(0) \times \cdots \times r_S(0)$, $r_1(2) \times r_2(1) \times r_3(1) \times r_4(0) \times \cdots \times r_S(0)$ to $r_1(2) \times r_1(1) \times \cdots \times r_S(1)$ are still contained in $R_{\boldsymbol{e}^1}$. But the next $r_1(2) \times r_2(2) \times r_3(0) \times \cdots \times r_S(0)$ is no longer contained in $R_{\boldsymbol{e}^1}$ but is contained in

$$R_{\boldsymbol{e}^2} = \bigcup_{j_1=2}^{S} \cdot \bigcup_{j_2=2}^{S} \cdot \bigcup_{j_3=0}^{2} \cdots \bigcup_{j_S=0}^{2} r_1(j_1) \times r_2(j_2) \times \cdots \times r_S(j_S).$$

Likewise, following the same sequential rule, $r_1(2) \times r_2(2) \times r_3(1) \times r_4(0) \times \cdots \times r_S(0)$, $r_1(2) \times r_2(2) \times r_3(1) \times r_4(1) \times r_5(0) \times \cdots \times r_S(0)$ to $r_1(2) \times \cdots \times r_S(2)$ are all contained in $R_{\boldsymbol{e}^2}$. This argument can iteratively be applied to all other hyper-cubes $r_1(j_1) \times r_2(j_2) \times \cdots \times r_S(j_S)$ generated by the same sequential rule maintaining $j_1 \geq j_2 \geq \cdots \geq j_S$. This proves Claim 1.

Proof of Claim 2: In general, consider $r_1(j_1) \times \cdots \times r_S(j_S)$ for given $j_1, ..., j_S$. There exists permutation $\sigma(\cdot)$ and a sequence $\{k_s\}_{s=1}^{S}$ such that $j_s = k_{\sigma(s)}$ and $k_1 \geq k_2 \geq \cdots \geq k_S$. Then a hyper-cube

$$r_{\sigma(1)}(j_1) \times \cdots \times r_{\sigma(S)}(j_S) = r_{\sigma(1)}(k_{\sigma(1)}) \times \cdots \times r_{\sigma(S)}(k_{\sigma(S)})$$

in the space of $(U_{\sigma(1)}, ..., U_{\sigma(S)})$, or equivalently $r_1(k_1) \times \cdots \times r_S(k_S)$ in the space of $(U_1, ..., U_S)$, satisfies the condition in Claim 1 and thus is contained in $R_{\boldsymbol{e}^j}$ for some $j \leq k_S$ by Claim 1. Let $\sigma^{-1}(\cdot)$ be the inverse of $\sigma(\cdot)$. Note that $\sigma^{-1}(\cdot)$ itself is a permutation function. In general, for permutation $\tilde{\sigma}(\cdot)$, if $r_1(k_1) \times \cdots \times r_S(k_S)$ is contained in $R_{\boldsymbol{e}^j}$ for some j, then $r_{\tilde{\sigma}(1)}(k_1) \times \cdots \times r_{\tilde{\sigma}(S)}(k_S)$ is contained in $R_{\boldsymbol{d}_{\tilde{\sigma}}^j}$ by definition. Therefore, since $r_{\sigma^{-1}(\sigma(s))}(j_s) = r_s(j_s) \; \forall s$, we can conclude that $r_1(j_1) \times \cdots \times r_S(j_S)$ is contained in $R_{\boldsymbol{d}_{\sigma^{-1}}^j}$ for $j \leq k_S = j_{\sigma^{-1}(S)}$. This proves Claim 2.

D.2   Proof of Theorem 3.1

We prove the theorem by showing the following lemma:

Lemma D.1. *Under Assumptions SS, SY1 and M1 and for $j = 0, ..., S - 1$, $\boldsymbol{R}^{\leq j}(\boldsymbol{z})$ is expressed as a union across $\sigma(\cdot) \in \Sigma$ of Cartesian products, each of which is a product of intervals that are either $(0, 1]$ or $\left(\nu_j^{\sigma(s)}(z_{\sigma(s)}), 1 \right]$ for some $s = 1, ...S$.*

This lemma asserts that the region which predicts all equilibria with at most j entrants is solely determined by the payoffs of players who stay out facing j entering opponents. Given this lemma, (3.5) holds by Assumption M1.

To prove Lemma D.1, first, consider a pair of $R_{\boldsymbol{d}^{j+1}}(\boldsymbol{z})$ and $R_{\boldsymbol{d}^j}(\boldsymbol{z})$ (for $\boldsymbol{d}^{j+1} \in M_{j+1}$ and $\boldsymbol{d}^j \in M_j$) in $\boldsymbol{R}_{j+1}(\boldsymbol{z})$ and $\boldsymbol{R}_j(\boldsymbol{z})$, respectively. From the proof of Proposition 3.1(ii), we know that the elements in $\boldsymbol{d}^{j+1}$ and $\boldsymbol{d}^j$ fall into one of the four types (a′)–(d′) (including s^*), and thus the corresponding pairs of intervals fall into one of the four types: (a†) $\left(0, \nu_j^s(z_s) \right]$ and $\left(0, \nu_{j-1}^s(z_s) \right]$; (b†) $\left(0, \nu_j^s(z_s) \right]$ and $\left(\nu_j^s(z_s), 1 \right]$; (c†) $\left(\nu_{j+1}^s(z_s), 1 \right]$ and $\left(\nu_j^s(z_s), 1 \right]$; (d†) $\left(\nu_{j+1}^s(z_s), 1 \right]$ and $\left(0, \nu_{j-1}^s(z_s) \right]$.

Definition D.1. *For two Cartesian products R and Q such that $R \sim Q$ and $R \cap Q = \emptyset$, their border $R \parallel Q$ is a set that satisfies $R \parallel Q \equiv cl(R) \cap cl(Q)$. Also, the border $R \parallel Q$ is a hyper-surface that is common to $cl(R)$ and $cl(Q)$.*

By Proposition 3.1, $R_{\boldsymbol{d}^{j+1}}(\boldsymbol{z}) \sim R_{\boldsymbol{d}^j}(\boldsymbol{z})$ and $R_{\boldsymbol{d}^{j+1}}(\boldsymbol{z}) \cap R_{\boldsymbol{d}^j}(\boldsymbol{z}) = \emptyset$, and thus their border can be properly defined. Given $(\text{a}^\dagger)$–$(\text{d}^\dagger)$, we show that $R_{\boldsymbol{d}^{j+1}}(\boldsymbol{z}) \parallel R_{\boldsymbol{d}^j}(\boldsymbol{z})$ is a Cartesian product of $\left(0, \nu_j^s(z_s)\right] \parallel \left(\nu_j^s(z_s), 1\right] = \{\nu_j^s(z_s)\}$ (for some s) and other intervals. Specifically, by applying Proposition D.1(iv) and (v) with $R = cl(R_{\boldsymbol{d}^{j+1}}(\boldsymbol{z}))$, $Q = cl(R_{\boldsymbol{d}^j}(\boldsymbol{z}))$, and r_s and q_s being the closures of the intervals in $(\text{a}^\dagger)$–$(\text{d}^\dagger)$, we have

$$R_{\boldsymbol{d}^{j+1}}(\boldsymbol{z}) \parallel R_{\boldsymbol{d}^j}(\boldsymbol{z}) = \{\nu_j^s(z_s)\} \times \prod_{k \neq s} r_k \cap q_k, \tag{D.3}$$

for some s, where each $r_k \cap q_k$ is one of $\left[0, \nu_j^k(z_k)\right]$, $\{\nu_j^k(z_k)\}$, $\left[\nu_j^k(z_k), 1\right]$, and $\left[\nu_{j+1}^k(z_k), \nu_{j-1}^k(z_k)\right]$. Observe that $R_{\boldsymbol{d}^{j+1}}(\boldsymbol{z}) \parallel R_{\boldsymbol{d}^j}(\boldsymbol{z})$ is therefore a lower-dimensional Cartesian product (with dimension less than S), which is consistent with the notion of a border or a hyper-surface. Also, observe that this hyperspace is located at $\nu_j^s(z_s)$ in the s-coordinate. Likewise, (D.3) holds for any $R_{\boldsymbol{d}^{j+1}}(\boldsymbol{z})$ and $R_{\boldsymbol{d}^j}(\boldsymbol{z})$ pair with a different value of s and different choice for each $r_k \cap q_k$. But, since $cl(A \cup B) = cl(A) \cup cl(B)$ for any sets A and B, $cl(\boldsymbol{R}_{j+1}(\boldsymbol{z})) = \bigcup_{\boldsymbol{d} \in M_{j+1}} cl(R_{\boldsymbol{d}}(\boldsymbol{z}))$ and $cl(\boldsymbol{R}_j(\boldsymbol{z})) = \bigcup_{\boldsymbol{d} \in M_j} cl(R_{\boldsymbol{d}}(\boldsymbol{z}))$, and thus

$$\boldsymbol{R}_{j+1}(\boldsymbol{z}) \parallel \boldsymbol{R}_j(\boldsymbol{z}) = \bigcup_{\boldsymbol{d}^{j+1} \in M_{j+1}} \bigcup_{\boldsymbol{d}^j \in M_j} \left(R_{\boldsymbol{d}^{j+1}}(\boldsymbol{z}) \parallel R_{\boldsymbol{d}^j}(\boldsymbol{z})\right). \tag{D.4}$$

Now, let $\boldsymbol{R}^{>j}(\boldsymbol{z}) \equiv \bigcup_{\boldsymbol{d} \in M^{>j}} R_{\boldsymbol{d}}(\boldsymbol{z}) = \mathcal{U} \backslash \boldsymbol{R}^{\leq j}(\boldsymbol{z})$ where $M^{>j} \equiv \bigcup_{k=j+1}^S M_k$. Note that $\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \sim \boldsymbol{R}^{>j}(\boldsymbol{z})$ and $\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \cap \boldsymbol{R}^{>j}(\boldsymbol{z}) = \emptyset$ by Proposition 3.1. Then $\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \parallel \boldsymbol{R}^{>j}(\boldsymbol{z}) = \boldsymbol{R}_{j+1}(\boldsymbol{z}) \parallel \boldsymbol{R}_j(\boldsymbol{z})$ by the discussions around (3.4). Since $\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \cup \boldsymbol{R}^{>j}(\boldsymbol{z}) = \mathcal{U}$ by definition, $\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \parallel \boldsymbol{R}^{>j}(\boldsymbol{z})$ is the only nontrivial hyper-surface of $cl(\boldsymbol{R}^{\leq j}(\boldsymbol{z}))$ (and of $cl(\boldsymbol{R}^{>j}(\boldsymbol{z}))$), i.e., a surface that is not part of the surface of $cl(\mathcal{U})$. Therefore by (D.3) and (D.4), we can conclude that $cl(\boldsymbol{R}^{\leq j}(\boldsymbol{z}))$ and hence $\boldsymbol{R}^{\leq j}(\boldsymbol{z})$ is a function of $\boldsymbol{z}$ only through $\nu_j^s(z_s)$ $\forall s$. Moreover, in the expression of $R_{\boldsymbol{d}^k}(\boldsymbol{z})$ in (3.2) with $k \leq j-1$ (and hence in the expression of $\boldsymbol{R}^{\leq j-1}(\boldsymbol{z})$), there is no interval with $\nu_j^s(z_s)$ in its endpoint by definition.[25] Also, the interval in the expression of $R_{\boldsymbol{d}^j}(\boldsymbol{z})$ in (3.2) (and hence in the expression of $\boldsymbol{R}_j(\boldsymbol{z})$) that has $\nu_j^s(z_s)$ in its endpoints is $\left(\nu_j^s(z_s), 1\right]$ $\forall s$. Consequently, $\boldsymbol{R}^{\leq j}(\boldsymbol{z}) = \boldsymbol{R}^{\leq j-1}(\boldsymbol{z}) \cup \boldsymbol{R}_j(\boldsymbol{z})$ is only expressed with $\left(\nu_j^s(z_s), 1\right]$ $\forall s$ and $(0, 1]$. If $\boldsymbol{R}^{\leq j}(\boldsymbol{z})$ is expressed using other intervals whose endpoints are functions of z_s, then it contradicts the fact that $\boldsymbol{R}^{\leq j}(\boldsymbol{z})$ is a function of $\boldsymbol{z}$ only through $\nu_j^s(z_s)$. This completes the proof.

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[25]That is, the payoff $\nu_j^s(z_s)$ is not relevant in defining markets with fewer than j entrants.

D.3 Proof of Theorem 4.1

Recall $M^{\leq j} \equiv M^j$ and $M^{>j} \equiv \bigcup_{k=j+1}^{S} M_k$. Then the bounds (4.14) and (4.15) can be rewritten as

$$U_{\boldsymbol{d}^j} = \inf_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \tilde{p}_{M>j-1}(\boldsymbol{z}) + p_{M \leq j-1}(\boldsymbol{z}) \right\}, \qquad L_{\boldsymbol{d}^j} = \sup_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \tilde{p}_{M \leq j}(\boldsymbol{z}) + p_{M>j}(\boldsymbol{z}) \right\},$$

where for a set $M \subset \mathcal{D}$, $\tilde{p}_M(\boldsymbol{z}) \equiv \Pr[Y = 1, \boldsymbol{D} \in M | \boldsymbol{Z} = \boldsymbol{z}]$ and $p_M(\boldsymbol{z}) \equiv \Pr[\boldsymbol{D} \in M | \boldsymbol{Z} = \boldsymbol{z}]$. Since $\mathcal{D} = M^{\leq \tilde{j}} \cup M^{>\tilde{j}}$ for some $\tilde{j}$, note that $\tilde{p}_{M>\tilde{j}}(\boldsymbol{z}) = \Pr[Y = 1 | \boldsymbol{Z} = \boldsymbol{z}] - \tilde{p}_{M \leq \tilde{j}}(\boldsymbol{z})$. Using this result, for $\boldsymbol{z}, \boldsymbol{z}'$ such that $\sum_{k=j'+1}^{S} h_k^D(\boldsymbol{z}, \boldsymbol{z}') = p_{M>j'}(\boldsymbol{z}) - p_{M>j'}(\boldsymbol{z}') > 0$ ($j' = 0, ..., S-1$), observe that each term in $U_{\boldsymbol{d}^j}$ satisfies

$$\tilde{p}_{M>j-1}(\boldsymbol{z}) - \tilde{p}_{M>j-1}(\boldsymbol{z}') = -\tilde{p}_{M \leq j-1}(\boldsymbol{z}) + \tilde{p}_{M \leq j-1}(\boldsymbol{z}') = \Pr[\epsilon \leq \mu_{\boldsymbol{D}}, \boldsymbol{U} \in \Delta^j(\boldsymbol{z}', \boldsymbol{z})]$$

$$p_{M \leq j-1}(\boldsymbol{z}) - p_{M \leq j-1}(\boldsymbol{z}') = -\Pr[\boldsymbol{U} \in \Delta^j(\boldsymbol{z}', \boldsymbol{z})]$$

by (D.8) and (D.11), and thus

$$\tilde{p}_{M>j-1}(\boldsymbol{z}) + p_{M \leq j-1}(\boldsymbol{z}) - \left\{ \tilde{p}_{M>j-1}(\boldsymbol{z}') + p_{M \leq j-1}(\boldsymbol{z}') \right\} = -\Pr[\epsilon > \mu_{\boldsymbol{D}}, \boldsymbol{U} \in \Delta^j(\boldsymbol{z}', \boldsymbol{z})] < 0.$$

Then this relationship creates a *partial ordering* of $\tilde{p}_{M>j-1}(\boldsymbol{z}) + p_{M \leq j-1}(\boldsymbol{z})$ as a function of $\boldsymbol{z}$ in terms of $p_{M>j'}(\boldsymbol{z})$ (for any j'). According to this ordering, $\tilde{p}_{M>j-1}(\boldsymbol{z}) + p_{M \leq j-1}(\boldsymbol{z})$ takes its smallest value as $p_{M>j'}(\boldsymbol{z})$ takes its largest value. Therefore, by (4.17),

$$U_{\boldsymbol{d}^j} = \inf_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \tilde{p}_{M>j-1}(\boldsymbol{z}) + p_{M \leq j-1}(\boldsymbol{z}) \right\} = \tilde{p}_{M>j-1}(\bar{\boldsymbol{z}}) + p_{M \leq j-1}(\bar{\boldsymbol{z}}).$$

By a symmetric argument, $L_{\boldsymbol{d}^j} = \sup_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \tilde{p}_{M \leq j}(\boldsymbol{z}) + p_{M>j}(\boldsymbol{z}) \right\} = \tilde{p}_{M \leq j}(\underline{\boldsymbol{z}}) + p_{M>j}(\underline{\boldsymbol{z}})$.

To prove that these bounds on $E[Y_{\boldsymbol{d}^j}]$ are sharp, it suffices to show that for $s_j \in [L_{\boldsymbol{d}^j}, U_{\boldsymbol{d}^j}]$, there exists a density function $f^*_{\epsilon, \boldsymbol{U}}$ such that the following claims hold:
(A) $f^*_{\epsilon | \boldsymbol{U}}$ is strictly positive on $\mathbb{R}$.
(B) The proposed model is consistent with the data: $\forall j = 0, ..., S$

$$\Pr[\boldsymbol{D} \in M^{\leq j} | \boldsymbol{Z} = \boldsymbol{z}] = \Pr[\boldsymbol{U}^* \in \boldsymbol{R}^{\leq j}(\boldsymbol{z})],$$

$$\Pr[Y = 1 | \boldsymbol{D} \in M^{\leq j}, \boldsymbol{Z} = \boldsymbol{z}] = \Pr[\epsilon^* \leq \mu_{\boldsymbol{D}} | \boldsymbol{U}^* \in \boldsymbol{R}^{\leq j}(\boldsymbol{z})],$$

$$\Pr[Y = 1 | \boldsymbol{D} \in M^{>j}, \boldsymbol{Z} = \boldsymbol{z}] = \Pr[\epsilon^* \leq \mu_{\boldsymbol{D}} | \boldsymbol{U}^* \in \boldsymbol{R}^{>j}(\boldsymbol{z})].$$

(C) The proposed model is consistent with the specified values of $E[Y_{\boldsymbol{d}^j}]$: $\Pr[\epsilon^* \leq \mu_{\boldsymbol{d}^j}] = s_j$.

Theorem 3.1 and the partial ordering above establishes monotonicity of the event $\boldsymbol{U} \in \boldsymbol{R}^{\leq j}(\boldsymbol{z})$ (and $\boldsymbol{U} \in \boldsymbol{R}^{>j}(\boldsymbol{z})$) w.r.t. $\boldsymbol{z}$. For example, for $\boldsymbol{z}, \boldsymbol{z}'$ such that $p_{M>j}(\boldsymbol{z}) > p_{M>j}(\boldsymbol{z}')$, Theorem 3.1 implies that $\boldsymbol{R}^{\leq j}(\boldsymbol{z}) \subset \boldsymbol{R}^{\leq j}(\boldsymbol{z}')$ and hence

$$1[\boldsymbol{U} \in \boldsymbol{R}^{\leq j}(\boldsymbol{z}')] - 1[\boldsymbol{U} \in \boldsymbol{R}^{\leq j}(\boldsymbol{z})] = 1[\boldsymbol{U} \in \boldsymbol{R}^{\leq j}(\boldsymbol{z}') \backslash \boldsymbol{R}^{\leq j}(\boldsymbol{z})]. \tag{D.5}$$

Given $1[\boldsymbol{D} \in M^{\leq j}] = 1[\boldsymbol{U} \in \boldsymbol{R}^{\leq j}(\boldsymbol{Z})]$, (D.5) is analogous to a scalar treatment decision $\tilde{D} = 1[\tilde{D} = 1] = 1[\tilde{U} \leq \tilde{P}]$ with a scalar instrument $\tilde{P}$, where $1[\tilde{U} \leq p'] - 1[\tilde{U} \leq p] = 1[p \leq \tilde{U} \leq p']$ for $p' > p$. Based on this result and the results for the first part of Theorem 4.1, we can modify the proof of Theorem 2.1(iii) in Shaikh and Vytlacil (2011) to show (A)–(C).

D.4   Proof of Lemma 4.2

We introduce a lemma that establishes the connection between Theorem 3.1 and Lemma 4.2.

Lemma D.2. *Based on the results in Proposition 3.1, $h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{x}) \equiv \sum_{j=0}^{S} h_j(\boldsymbol{z}, \boldsymbol{z}', x_j)$ satisfies*

$$h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{x}) = \sum_{j=1}^{S} \int_{\Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z})} \{\vartheta_j(x_j; \boldsymbol{u}) - \vartheta_{j-1}(x_{j-1}; \boldsymbol{u})\} \, d\boldsymbol{u}, \tag{D.6}$$

where $\Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z}) \equiv \boldsymbol{R}^{\leq j-1}(\boldsymbol{z}') \backslash \boldsymbol{R}^{\leq j-1}(\boldsymbol{z})$.

As a special case of this lemma, $h(\boldsymbol{z}', \boldsymbol{z}; x, ..., x) = h(\boldsymbol{z}', \boldsymbol{z}, x) = \sum_{j=0}^{S} h_j(\boldsymbol{z}', \boldsymbol{z}, x)$ can be expressed as

$$h(\boldsymbol{z}', \boldsymbol{z}, x) = \sum_{j=1}^{S} \int_{\Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z})} \{\vartheta_j(x; \boldsymbol{u}) - \vartheta_{j-1}(x; \boldsymbol{u})\} \, d\boldsymbol{u}. \tag{D.7}$$

We prove Lemma D.2 by drawing on the results of Proposition 3.1. By Theorem 3.1, for $\boldsymbol{z}$ and $\boldsymbol{z}'$ such that (4.4) holds, we have

$$\boldsymbol{R}^j(\boldsymbol{z}) \subseteq \boldsymbol{R}^j(\boldsymbol{z}') \tag{D.8}$$

for $j = 0, ..., S$, including $\boldsymbol{R}^S(\boldsymbol{z}) = \boldsymbol{R}^S(\boldsymbol{z}') = \mathcal{U}$ as a trivial case. For those $\boldsymbol{z}$ and $\boldsymbol{z}'$, introduce notations[26]

$$\Delta_{j,+}(\boldsymbol{z}, \boldsymbol{z}') \equiv \boldsymbol{R}_j(\boldsymbol{z}) \backslash \boldsymbol{R}_j(\boldsymbol{z}'), \tag{D.9}$$
$$\Delta_{j,-}(\boldsymbol{z}, \boldsymbol{z}') \equiv \boldsymbol{R}_j(\boldsymbol{z}') \backslash \boldsymbol{R}_j(\boldsymbol{z}), \tag{D.10}$$

and

$$\Delta^j(\boldsymbol{z}', \boldsymbol{z}) \equiv \boldsymbol{R}^j(\boldsymbol{z}') \backslash \boldsymbol{R}^j(\boldsymbol{z}). \tag{D.11}$$

Note that, for $j = 1, ..., S$,

$$\boldsymbol{R}_j(\cdot) = \boldsymbol{R}^j(\cdot) \backslash \boldsymbol{R}^{j-1}(\cdot), \tag{D.12}$$

since $\boldsymbol{R}^j(\boldsymbol{z}) \equiv \bigcup_{k=0}^{j} \boldsymbol{R}_k(\boldsymbol{z})$. Fix $j = 1, ..., S$. Consider

$$
\begin{aligned}
\Delta_{j,+}(\boldsymbol{z}, \boldsymbol{z}') &= \left(\boldsymbol{R}^j(\boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c\right) \cap \left(\boldsymbol{R}^j(\boldsymbol{z}') \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}')^c\right)^c \\
&= \left(\boldsymbol{R}^j(\boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c\right) \cap \left(\boldsymbol{R}^j(\boldsymbol{z}')^c \cup \boldsymbol{R}^{j-1}(\boldsymbol{z}')\right) \\
&= \left(\boldsymbol{R}^j(\boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c \cap \boldsymbol{R}^j(\boldsymbol{z}')^c\right) \cup \left(\boldsymbol{R}^j(\boldsymbol{z}) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c \cap \boldsymbol{R}^{j-1}(\boldsymbol{z}')\right) \\
&= \left\{\left(\boldsymbol{R}^j(\boldsymbol{z}) \backslash \boldsymbol{R}^j(\boldsymbol{z}')\right) \cap \boldsymbol{R}^{j-1}(\boldsymbol{z})^c\right\} \cup \left\{\left(\boldsymbol{R}^{j-1}(\boldsymbol{z}') \backslash \boldsymbol{R}^{j-1}(\boldsymbol{z})\right) \cap \boldsymbol{R}^j(\boldsymbol{z})\right\} \\
&= \Delta^{j-1}(\boldsymbol{z}', \boldsymbol{z}) \cap \boldsymbol{R}^j(\boldsymbol{z}),
\end{aligned}
$$

---

[26]Note that $\Delta_+(\boldsymbol{z}, \boldsymbol{z}')$ and $\Delta_-(\boldsymbol{z}, \boldsymbol{z}')$ defined in Section 4.2 for the $S = 2$ are simplified versions of these notations: $\Delta_+(\boldsymbol{z}, \boldsymbol{z}') = \Delta_{1,+}(\boldsymbol{z}, \boldsymbol{z}')$ and $\Delta_-(\boldsymbol{z}, \boldsymbol{z}') = \Delta_{1,-}(\boldsymbol{z}, \boldsymbol{z}')$.

where the first equality is by plugging in (D.12) into (D.9), the third equality is by the distributive law, and the last equality is by (D.8) and hence $\left(\boldsymbol{R}^j(\boldsymbol{z})\backslash\boldsymbol{R}^j(\boldsymbol{z}')\right)\cap\boldsymbol{R}^{j-1}(\boldsymbol{z})^c=\emptyset$. But

$$\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})\cap\boldsymbol{R}^j(\boldsymbol{z})=\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})\backslash\left(\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})\backslash\boldsymbol{R}^j(\boldsymbol{z})\right).$$

Symmetrically, by changing the role of $\boldsymbol{z}$ and $\boldsymbol{z}'$, consider

$$\begin{aligned}\Delta_{j,-}(\boldsymbol{z}',\boldsymbol{z})&=\left(\boldsymbol{R}^j(\boldsymbol{z}')\cap\boldsymbol{R}^{j-1}(\boldsymbol{z}')^c\right)\cap\left(\boldsymbol{R}^j(\boldsymbol{z})\cap\boldsymbol{R}^{j-1}(\boldsymbol{z})^c\right)^c\\&=\left\{\left(\boldsymbol{R}^j(\boldsymbol{z}')\backslash\boldsymbol{R}^j(\boldsymbol{z})\right)\cap\boldsymbol{R}^{j-1}(\boldsymbol{z}')^c\right\}\cup\left\{\left(\boldsymbol{R}^{j-1}(\boldsymbol{z})\backslash\boldsymbol{R}^{j-1}(\boldsymbol{z}')\right)\cap\boldsymbol{R}^j(\boldsymbol{z}')\right\}\\&=\Delta^j(\boldsymbol{z}',\boldsymbol{z})\cap\boldsymbol{R}^{j-1}(\boldsymbol{z}')^c,\end{aligned}$$

where the last equality is by (D.8) that $\boldsymbol{R}^{j-1}(\boldsymbol{z})\subset\boldsymbol{R}^{j-1}(\boldsymbol{z}')$. But

$$\Delta^j(\boldsymbol{z}',\boldsymbol{z})\cap\boldsymbol{R}^{j-1}(\boldsymbol{z}')^c=\Delta^j(\boldsymbol{z}',\boldsymbol{z})\backslash\left(\Delta^j(\boldsymbol{z}',\boldsymbol{z})\cap\boldsymbol{R}^{j-1}(\boldsymbol{z}')\right).$$

Note that

$$\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})\backslash\boldsymbol{R}^j(\boldsymbol{z})=\Delta^j(\boldsymbol{z}',\boldsymbol{z})\backslash\boldsymbol{R}^{j-1}(\boldsymbol{z}')\equiv A^*,\tag{D.13}$$

because

$$\begin{aligned}\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})\backslash\boldsymbol{R}^j(\boldsymbol{z})&=\boldsymbol{R}^{j-1}(\boldsymbol{z}')\cap\boldsymbol{R}^{j-1}(\boldsymbol{z})^c\cap\boldsymbol{R}^j(\boldsymbol{z})^c=\boldsymbol{R}^{j-1}(\boldsymbol{z}')\cap\boldsymbol{R}^j(\boldsymbol{z})^c\\&=\boldsymbol{R}^j(\boldsymbol{z}')\cap\boldsymbol{R}^j(\boldsymbol{z})^c\cap\boldsymbol{R}^{j-1}(\boldsymbol{z}')=\Delta^j(\boldsymbol{z}',\boldsymbol{z})\cap\boldsymbol{R}^{j-1}(\boldsymbol{z}'),\end{aligned}$$

where the second equality is by $\boldsymbol{R}^{j-1}(\boldsymbol{z})\subset\boldsymbol{R}^j(\boldsymbol{z})$ and the third equality is by $\boldsymbol{R}^{j-1}(\boldsymbol{z}')\subset\boldsymbol{R}^j(\boldsymbol{z}')$. In sum,

$$\Delta_{j,+}(\boldsymbol{z},\boldsymbol{z}')=\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})\backslash A^*,\qquad\Delta_{j,-}(\boldsymbol{z},\boldsymbol{z}')=\Delta^j(\boldsymbol{z}',\boldsymbol{z})\backslash A^*.\tag{D.14}$$

(D.14) shows how the outflow $(\Delta_{j,+}(\boldsymbol{z},\boldsymbol{z}'))$ and inflow $(\Delta_{j,-}(\boldsymbol{z},\boldsymbol{z}'))$ of $\boldsymbol{R}_j$ can be written in terms of the inflows of $\boldsymbol{R}^{j-1}$ and $\boldsymbol{R}^j$, respectively. And figuratively, A^* adjusts for the "leakage" when the change from $\boldsymbol{z}$ to $\boldsymbol{z}'$ is relatively large. Now, with $\vartheta_j(\boldsymbol{u})\equiv\vartheta_j(x;\boldsymbol{u})\equiv\vartheta(\boldsymbol{e}^j,x;\boldsymbol{u})$, (4.20) can be expressed as

$$\begin{aligned}&\int_{\boldsymbol{R}_j(\boldsymbol{z})}\vartheta_j(\boldsymbol{u})d\boldsymbol{u}-\int_{\boldsymbol{R}_j(\boldsymbol{z}')}\vartheta_j(\boldsymbol{u})d\boldsymbol{u}\\&=\int_{\Delta_{j,+}(\boldsymbol{z},\boldsymbol{z}')}\vartheta_j(\boldsymbol{u})d\boldsymbol{u}+\int_{\boldsymbol{R}_j(\boldsymbol{z})\cap\boldsymbol{R}_j(\boldsymbol{z}')}\vartheta_j(\boldsymbol{u})d\boldsymbol{u}-\int_{\Delta_{j,-}(\boldsymbol{z},\boldsymbol{z}')}\vartheta_j(\boldsymbol{u})d\boldsymbol{u}-\int_{\boldsymbol{R}_j(z)\cap\boldsymbol{R}_j(z')}\vartheta_j(\boldsymbol{u})d\boldsymbol{u}\\&=\int_{\Delta_{j,+}(\boldsymbol{z},\boldsymbol{z}')}\vartheta_j(\boldsymbol{u})d\boldsymbol{u}-\int_{\Delta_{j,-}(\boldsymbol{z},\boldsymbol{z}')}\vartheta_j(\boldsymbol{u})d\boldsymbol{u},\end{aligned}\tag{D.15}$$

where the last equality is derived by IN and SY. First, for $j = 1, ..., S$, by (D.14),

$$\int_{\Delta_{j,+}(\boldsymbol{z},\boldsymbol{z}')} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u} - \int_{\Delta_{j,-}(\boldsymbol{z},\boldsymbol{z}')} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u} = \int_{\Delta^j(\boldsymbol{z}',\boldsymbol{z})\backslash A^*} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u} - \int_{\Delta^j(\boldsymbol{z}',\boldsymbol{z})\backslash A^*} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u}$$

$$= \int_{\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})\backslash A^*} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u} + \int_{A^*} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u} - \int_{A^*} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u}$$

$$- \left\{ \int_{\Delta^j(\boldsymbol{z}',\boldsymbol{z})\backslash A^*} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u} + \int_{A^*} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u} - \int_{A^*} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u} \right\}$$

$$= \int_{\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u} - \int_{\Delta^j(\boldsymbol{z}',\boldsymbol{z})} \vartheta_j(\boldsymbol{u}) d\boldsymbol{u}, \tag{D.16}$$

where the last equality is because $\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z}) \supset A^*$ and $\Delta^j(\boldsymbol{z}',\boldsymbol{z}) \supset A^*$ by the definition of A^*.

For $j = 0$,

$$\int_{\Delta_{0,+}(\boldsymbol{z},\boldsymbol{z}')} \vartheta_0(\boldsymbol{u}) d\boldsymbol{u} - \int_{\Delta_{0,-}(\boldsymbol{z},\boldsymbol{z}')} \vartheta_0(\boldsymbol{u}) d\boldsymbol{u} = - \int_{\Delta^0(\boldsymbol{z}',\boldsymbol{z})} \vartheta_0(\boldsymbol{u}) d\boldsymbol{u}, \tag{D.17}$$

since $\Delta_{0,+}(\boldsymbol{z},\boldsymbol{z}') = \emptyset$ by the choice of $(\boldsymbol{z},\boldsymbol{z}')$ and $\Delta_{0,-}(\boldsymbol{z},\boldsymbol{z}') = \Delta^0(\boldsymbol{z}',\boldsymbol{z})$. For $j = S$,

$$\int_{\Delta_{S,+}(\boldsymbol{z},\boldsymbol{z}')} \vartheta_S(\boldsymbol{u}) d\boldsymbol{u} - \int_{\Delta_{S,-}(\boldsymbol{z},\boldsymbol{z}')} \vartheta_S(\boldsymbol{u}) d\boldsymbol{u} = \int_{\Delta^{S-1}(\boldsymbol{z}',\boldsymbol{z})} \vartheta_S(\boldsymbol{u}) d\boldsymbol{u}, \tag{D.18}$$

since $\Delta_{S,-}(\boldsymbol{z},\boldsymbol{z}') = \emptyset$ by the choice of $(\boldsymbol{z},\boldsymbol{z}')$ and $\Delta_{S,+}(\boldsymbol{z},\boldsymbol{z}') = \Delta^{S-1}(\boldsymbol{z}',\boldsymbol{z})$. Then combining (4.20) and (D.15)–(D.18) evaluated at $x = x_j$,

$$h(\boldsymbol{z},\boldsymbol{z}';\boldsymbol{x}) \equiv \sum_{j=0}^{S} h_j(\boldsymbol{z},\boldsymbol{z}',x_j) = \sum_{j=1}^{S} \int_{\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})} \{ \vartheta_j(x_j;\boldsymbol{u}) - \vartheta_{j-1}(x_{j-1};\boldsymbol{u}) \} d\boldsymbol{u}.$$

This completes the proof of Lemma D.2.

Now we prove 4.2. Part (i) is already shown in the text, so we prove part (ii) here. By Lemma D.2, $h(\boldsymbol{z},\boldsymbol{z}';\boldsymbol{x}) = \sum_{j=1}^{S} \int_{\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})} \{ \vartheta_j(x_j;\boldsymbol{u}) - \vartheta_{j-1}(x_{j-1};\boldsymbol{u}) \} d\boldsymbol{u}$ with $\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z}) \equiv \bar{R}^{j-1}(\boldsymbol{z}')\backslash\bar{R}^{j-1}(\boldsymbol{z})$, which can be rewritten as

$$h(\boldsymbol{z},\boldsymbol{z}';\boldsymbol{x}) - \sum_{k \neq j} \int_{\Delta^{k-1}(\boldsymbol{z}',\boldsymbol{z})} \{ \vartheta_k(x_k;u) - \vartheta_{k-1}(x_{k-1};\boldsymbol{u}) \} d\boldsymbol{u}$$

$$= \int_{\Delta^{j-1}(\boldsymbol{z}',\boldsymbol{z})} \{ \vartheta_j(x_j;\boldsymbol{u}) - \vartheta_{j-1}(x_{j-1};\boldsymbol{u}) \} d\boldsymbol{u}. \tag{D.19}$$

We prove the case $\iota = 1$; the proof for the other cases follows symmetrically. For $k \neq j$, when $\vartheta_{k-1}(x_{k-1};\boldsymbol{u}) - \vartheta_k(x_k;\boldsymbol{u}) > 0$ a.e. $\boldsymbol{u}$, it satisfies $- \int_{\Delta^{k-1}(\boldsymbol{z}',\boldsymbol{z})} \{ \vartheta_k(x_k;\boldsymbol{u}) - \vartheta_{k-1}(x_{k-1};\boldsymbol{u}) \} d\boldsymbol{u} > 0$. Combining with $h(\boldsymbol{z},\boldsymbol{z}';\boldsymbol{x}) > 0$ implies that the l.h.s. of (D.19) is positive. This implies that $\vartheta_j(x;\boldsymbol{u}) - \vartheta_{j-1}(x;\boldsymbol{u}) > 0$ a.e. $\boldsymbol{u}$. Suppose not and suppose $\vartheta_j(x_j;\boldsymbol{u}) - \vartheta_{j-1}(x_{j-1};\boldsymbol{u}) \leq 0$ with positive probability. Then by Assumption Y, $\vartheta_j(x;\boldsymbol{u}) - \vartheta_{j-1}(x;\boldsymbol{u}) \leq 0$ a.e. $\boldsymbol{u}$, which is contradiction.

D.5 Proof of Lemma 5.1

The claim is that when (5.2) holds, it satisfies $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = \emptyset$ for $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$. But the latter is equivalent to Assumption ASY by the first part of the proof of Lemma 5.2 below.

We first prove the claim for $S = 2$ and then generalize it. The probabilities in (5.2) equal

$$\Pr[\boldsymbol{D} = (0,0)|\boldsymbol{Z} = \boldsymbol{z}] = \Pr[\boldsymbol{U} \in R_{00}(\boldsymbol{z})],$$
$$\Pr[\boldsymbol{D} = (1,1)|\boldsymbol{Z} = \boldsymbol{z}'] = \Pr[\boldsymbol{U} \in R_{11}(\boldsymbol{z}')].$$

Under independent unobserved types, these probabilities are equivalent to the volume of $R_{00}(\boldsymbol{z})$ and $R_{11}(\boldsymbol{z}')$, respectively. We consider two isoquant curves: a curve that delivers the same volume as $R_{00}(\boldsymbol{z})$ with origin $(1,1)$ and a curve for $R_{11}(\boldsymbol{z}')$ with origin $(0,0)$ in $\mathcal{U}$. Consider an extreme scenario along these isoquant curves. Namely, consider the situation that player 1 is unprofitable to enter irrespective of player 2's decisions when $\boldsymbol{Z} = \boldsymbol{z}$. Then $\mathcal{U} = \tilde{R}_{00}(\boldsymbol{z}) \cup \tilde{R}_{01}(\boldsymbol{z})$ where $\Pr[\boldsymbol{U} \in \tilde{R}_{00}(\boldsymbol{z})] = \Pr[\boldsymbol{U} \in R_{00}(\boldsymbol{z})]$. Also, consider a situation that player 1 is profitable to enter irrespective of player 2's decisions when $\boldsymbol{Z} = \boldsymbol{z}'$. Then $\mathcal{U} = \tilde{R}_{10}(\boldsymbol{z}') \cup \tilde{R}_{11}(\boldsymbol{z}')$ where $\Pr[\boldsymbol{U} \in \tilde{R}_{11}(\boldsymbol{z}')] = \Pr[\boldsymbol{U} \in R_{11}(\boldsymbol{z}')]$. In order for $\tilde{R}_{01}(\boldsymbol{z}) \cap \tilde{R}_{10}(\boldsymbol{z}') = \emptyset$, it must be that

$$1 - \Pr[\boldsymbol{U} \in \tilde{R}_{00}(\boldsymbol{z})] = \Pr[\boldsymbol{U} \in \tilde{R}_{01}(\boldsymbol{z})] < 1 - \Pr[\boldsymbol{U} \in \tilde{R}_{10}(\boldsymbol{z}')] = \Pr[\boldsymbol{U} \in \tilde{R}_{11}(\boldsymbol{z}')]$$

or equivalently, $1 - \Pr[\boldsymbol{U} \in R_{00}(\boldsymbol{z})] < \Pr[\boldsymbol{U} \in R_{11}(\boldsymbol{z}')]$ should hold. But note that if $\tilde{R}_{01}(\boldsymbol{z}) \cap \tilde{R}_{10}(\boldsymbol{z}') = \emptyset$, then $R_{01}(\boldsymbol{z}) \cap R_{10}(\boldsymbol{z}') = \emptyset$ for any $R_{01}(\boldsymbol{z})$ and $R_{10}(\boldsymbol{z}')$ along the isoquant curves, since $R_{01}(\boldsymbol{z}) \subset \tilde{R}_{01}(\boldsymbol{z})$ and $R_{10}(\boldsymbol{z}') \subset \tilde{R}_{10}(\boldsymbol{z}')$. Symmetrically one can show $R_{10}(\boldsymbol{z}) \cap R_{01}(\boldsymbol{z}') = \emptyset$.

To prove the general case for $S > 2$, we iteratively apply the argument for the two players case. Consider two isoquant hyper-surfaces, one with origin $(1, ..., 1)$ for $R_{\boldsymbol{d}^0}(\boldsymbol{z})$ and the other with origin $(0,...,0)$ for $R_{\boldsymbol{d}^S}(\boldsymbol{z})$. Consider a scenario where the first $S - 1$ players are unprofitable to enter irrespective of the remaining player's decision when $\boldsymbol{Z} = \boldsymbol{z}$. Then $\mathcal{U} = \tilde{R}_{\boldsymbol{d}^0}(\boldsymbol{z}) \cup \tilde{R}_{0,...,0,1}(\boldsymbol{z})$ where $\Pr[\boldsymbol{U} \in \tilde{R}_{\boldsymbol{d}^0}(\boldsymbol{z})] = \Pr[\boldsymbol{U} \in R_{\boldsymbol{d}^0}(\boldsymbol{z})]$. Also, consider a situation where the first $S - 1$ players are profitable irrespective of the remaining player's decision when $\boldsymbol{Z} = \boldsymbol{z}'$. Then $\mathcal{U} = \tilde{R}_{1,...,1,0}(\boldsymbol{z}') \cup \tilde{R}_{\boldsymbol{d}^S}(\boldsymbol{z}')$ where $\Pr[\boldsymbol{U} \in \tilde{R}_{\boldsymbol{d}^S}(\boldsymbol{z}')] = \Pr[\boldsymbol{U} \in R_{\boldsymbol{d}^S}(\boldsymbol{z}')]$. Then when (5.1) holds, $\tilde{R}_{0,...,0,1}(\boldsymbol{z}) \cap \tilde{R}_{1,...,1,0}(\boldsymbol{z}') = \emptyset$. Note that $\tilde{R}_{0,...,0,1}(\boldsymbol{z}) \supset R_{\boldsymbol{d}_{-s},1}(\boldsymbol{z})$ for any $R_{\boldsymbol{d}_{-s},1}(\boldsymbol{z})$ with $\boldsymbol{d}_{-s} \neq (1, ..., 1)$ and $\tilde{R}_{1,...,1,0}(\boldsymbol{z}') \supset R_{\boldsymbol{d}_{-s},0}(\boldsymbol{z}')$ for any $R_{\boldsymbol{d}_{-s},0}(\boldsymbol{z}')$ with $\boldsymbol{d}_{-s} \neq (0, ..., 0)$ by Proposition 3.1. Therefore $R_{\boldsymbol{d}_{-s},1}(\boldsymbol{z}) \cap R_{\boldsymbol{d}_{-s},0}(\boldsymbol{z}') \neq 0$ for $\boldsymbol{d}^j$ and $\tilde{\boldsymbol{d}}^j$ such that $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$, $\boldsymbol{d}^j = (\boldsymbol{d}_{-s}, 1)$ and $\tilde{\boldsymbol{d}}^j = (\boldsymbol{d}_{-s}, 0)$ for $j = 1, ..., S - 1$. Since the same argument applies irrelevant of which $S - 1$ players we choose from the outset, $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = \emptyset$ for $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$ as it is desired.

D.6 Proof of Lemma 5.2

The first part proves the claim in Remark 5.1. For any $\boldsymbol{d}^j$ and $\tilde{\boldsymbol{d}}^j$ ($\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$), the expression of $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}')$ can be inferred as follows. First, there exists s^* such that $d_{s^*}^j = 1$ and $\tilde{d}_{s^*}^j = 0$, otherwise it contradicts $\boldsymbol{d}^j \neq \tilde{\boldsymbol{d}}^j$. That is, $U_{s^*} \in \left(0, \nu_{j-1}^{s^*}(z_{s^*})\right]$ in $R_{\boldsymbol{d}^j}(\boldsymbol{z})$ and $U_{s^*} \in \left(\nu_j^{s^*}(z'_{s^*}), 1\right]$ in $R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}')$. For other $s \neq s^*$, the pair is realized to be one of the four

types: (i) $d_s^j = 1$ and $\tilde{d}_s^j = 0$; (ii) $d_s^j = 0$ and $\tilde{d}_s^j = 1$; (iii) $d_s^j = 1$ and $\tilde{d}_s^j = 1$; (iv) $d_s^j = 0$ and $\tilde{d}_s^j = 0$. Then the corresponding pair of intervals for $R_{\boldsymbol{d}^j}(\boldsymbol{z})$ and $R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}')$, respectively, falls into one of the four types: (i) $\left(0, \nu_{j-1}^s(z_s)\right]$ and $\left(\nu_j^s(z_s'), 1\right]$; (ii) $\left(\nu_j^s(z_s), 1\right]$ and $\left(0, \nu_{j-1}^s(z_s')\right]$; (iii) $\left(0, \nu_{j-1}^s(z_s)\right]$ and $\left(0, \nu_{j-1}^s(z_s')\right]$; (iv) $\left(\nu_j^s(z_s), 1\right]$ and $\left(\nu_j^s(z_s'), 1\right]$. Then by Proposition D.1(iv), $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z})$ is a product of the intersections of the interval pairs. But the intersections resulting from (i) and (ii) are empty and hence $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = \emptyset$ if and only if $\nu_{j-1}^s(z_s) \leq \nu_j^s(z_s') \ \forall s$. Finally, note that $R_{\boldsymbol{d}^j}(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}(\boldsymbol{z}') = \emptyset$ implies $R_{\boldsymbol{d}^j}^*(\boldsymbol{z}) \cap R_{\tilde{\boldsymbol{d}}^j}^*(\boldsymbol{z}') = \emptyset$.

Now, we prove Lemma 5.2 with binary Y, no X and $S = 2$ for simplicity; the general case can be easily shown by analogously modifying the proof of Lemma 4.2. In place of $h_M(\boldsymbol{z}, \boldsymbol{z}')$ that is used to prove Lemma 4.1, introduce

$$h_{10}(\boldsymbol{z}, \boldsymbol{z}') \equiv \Pr[Y = 1, \boldsymbol{D} = (1, 0)|\boldsymbol{Z} = \boldsymbol{z}] - \Pr[Y = 1, \boldsymbol{D} = (1, 0)|\boldsymbol{Z} = \boldsymbol{z}'],$$
$$h_{01}(\boldsymbol{z}, \boldsymbol{z}') \equiv \Pr[Y = 1, \boldsymbol{D} = (0, 1)|\boldsymbol{Z} = \boldsymbol{z}] - \Pr[Y = 1, \boldsymbol{D} = (0, 1)|\boldsymbol{Z} = \boldsymbol{z}'].$$

Then h defined in (4.3) satisfies $h = h_{11} + h_{00} + h_{10} + h_{01}$; in fact, $h_M = h_{10} + h_{01}$. Let R_{10}^* and R_{01}^* be the regions that predict $\boldsymbol{D} = (1, 0)$ and $\boldsymbol{D} = (0, 1)$, respectively. For $(\boldsymbol{z}, \boldsymbol{z}')$ such that (4.4) holds, we have $R_{11}(\boldsymbol{z}) \supset R_{11}(\boldsymbol{z}')$ and $R_{00}(\boldsymbol{z}) \subset R_{00}(\boldsymbol{z}')$, respectively, by Theorem 3.1. Since $R_{10}^* \cup R_{01}^* = R_{10} \cup R_{01} = \boldsymbol{R}_1$, (4.7) and (4.8) can alternatively be expressed as

$$\Delta_+(\boldsymbol{z}, \boldsymbol{z}') \equiv \{R_{10}^*(\boldsymbol{z}) \cup R_{01}^*(\boldsymbol{z})\} \backslash \boldsymbol{R}_1(\boldsymbol{z}'), \tag{D.20}$$
$$\Delta_-(\boldsymbol{z}, \boldsymbol{z}') \equiv \{R_{10}^*(\boldsymbol{z}') \cup R_{01}^*(\boldsymbol{z}')\} \backslash \boldsymbol{R}_1(\boldsymbol{z}). \tag{D.21}$$

Consider partitions $\Delta_+(\boldsymbol{z}, \boldsymbol{z}') = \Delta_+^1(\boldsymbol{z}, \boldsymbol{z}') \cup \Delta_+^2(\boldsymbol{z}, \boldsymbol{z}')$ and $\Delta_-(\boldsymbol{z}, \boldsymbol{z}') = \Delta_-^1(\boldsymbol{z}, \boldsymbol{z}') \cup \Delta_-^2(\boldsymbol{z}, \boldsymbol{z}')$ such that

$$\Delta_+^1(\boldsymbol{z}, \boldsymbol{z}') \equiv R_{10}^*(\boldsymbol{z}) \backslash \boldsymbol{R}_1(\boldsymbol{z}'), \quad \Delta_+^2(\boldsymbol{z}, \boldsymbol{z}') \equiv R_{01}^*(\boldsymbol{z}) \backslash \boldsymbol{R}_1(\boldsymbol{z}'),$$
$$\Delta_-^1(\boldsymbol{z}, \boldsymbol{z}') \equiv R_{10}^*(\boldsymbol{z}') \backslash \boldsymbol{R}_1(\boldsymbol{z}), \quad \Delta_-^2(\boldsymbol{z}, \boldsymbol{z}') \equiv R_{01}^*(\boldsymbol{z}') \backslash \boldsymbol{R}_1(\boldsymbol{z}).$$

That is, $\Delta_+^1(\boldsymbol{z}, \boldsymbol{z}')$ and $\Delta_-^1(\boldsymbol{z}, \boldsymbol{z}')$ are regions of R_{10}^* exchanged with the regions for $\boldsymbol{D} = (0, 0)$ and $\boldsymbol{D} = (1, 1)$, respectively, and $\Delta_+^2(\boldsymbol{z}, \boldsymbol{z}')$ and $\Delta_-^2(\boldsymbol{z}, \boldsymbol{z}')$ are for R_{01}^*.

By Assumption IN,

$$
\begin{aligned}
h_{10}(\boldsymbol{z}, \boldsymbol{z}') &= \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in R_{10}^*(\boldsymbol{z})] - \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in R_{10}^*(\boldsymbol{z}')] \\
&= \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in R_{10}^*(\boldsymbol{z}) \backslash R_{10}^*(\boldsymbol{z}')] - \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in R_{10}^*(\boldsymbol{z}') \backslash R_{10}^*(\boldsymbol{z})] \\
&= \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in \Delta_+^1(\boldsymbol{z}, \boldsymbol{z}')] - \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in \Delta_-^1(\boldsymbol{z}, \boldsymbol{z}')] \\
&= \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in \Delta_+^1(\boldsymbol{z}, \boldsymbol{z}') \cup A^*] - \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in \Delta_-^1(\boldsymbol{z}, \boldsymbol{z}') \cup A^*],
\end{aligned}
$$

where A^* is defined in (D.13), the second equality is by (4.10) and the third equality is by the following derivation:

$$
\begin{aligned}
R_{10}^*(\boldsymbol{z}) \backslash R_{10}^*(\boldsymbol{z}') &= \left[\{R_{10}^*(\boldsymbol{z}) \cap \boldsymbol{R}_1(\boldsymbol{z}')^c\} \backslash R_{10}^*(\boldsymbol{z}')\right] \cup \left[\{R_{10}^*(\boldsymbol{z}) \cap \boldsymbol{R}_1(\boldsymbol{z}')\} \backslash R_{10}^*(\boldsymbol{z}')\right] \\
&= \left[\{R_{10}^*(\boldsymbol{z}) \cap \boldsymbol{R}_1(\boldsymbol{z}')^c\}\right] \cup \left[\{R_{10}^*(\boldsymbol{z}') \cap \boldsymbol{R}_1(\boldsymbol{z})\} \backslash R_{10}^*(\boldsymbol{z}')\right] \\
&= \Delta_+^1(\boldsymbol{z}, \boldsymbol{z}'),
\end{aligned}
$$

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where the first equality is by the distributive law and $\mathcal{U} = \boldsymbol{R}_1(\boldsymbol{z}')^c \cup \boldsymbol{R}_1(\boldsymbol{z}')$, the second equality is by $\boldsymbol{R}_1(\boldsymbol{z}')^c = R_{10}^*(\boldsymbol{z}')^c \cap R_{01}^*(\boldsymbol{z}')^c$ (the first term) and by Assumption ASY (the second term), and the last equality is by the definition of $\Delta_+^1(\boldsymbol{z}, \boldsymbol{z}')$ and $\{R_{10}^*(\boldsymbol{z}') \cap \boldsymbol{R}_1(\boldsymbol{z})\} \backslash R_{10}^*(\boldsymbol{z}')$ being empty. Analogously, one can show that $R_{10}^*(\boldsymbol{z}') \backslash R_{10}^*(\boldsymbol{z}) = \Delta_-^1(\boldsymbol{z}, \boldsymbol{z}')$ using Assumption ASY and the definition of $\Delta_-^1(\boldsymbol{z}, \boldsymbol{z}')$. Likewise,

$$
\begin{aligned}
h_{01}(\boldsymbol{z}, \boldsymbol{z}') &= \Pr[\epsilon \leq \mu_{01}, \boldsymbol{U} \in R_{01}^*(\boldsymbol{z})] - \Pr[\epsilon \leq \mu_{01}, \boldsymbol{U} \in R_{01}^*(\boldsymbol{z}')] \\
&= \Pr[\epsilon \leq \mu_{01}, \boldsymbol{U} \in R_{01}^*(\boldsymbol{z}) \backslash R_{01}^*(\boldsymbol{z}')] - \Pr[\epsilon \leq \mu_{01}, \boldsymbol{U} \in R_{01}^*(\boldsymbol{z}') \backslash R_{01}^*(\boldsymbol{z})] \\
&= \Pr[\epsilon \leq \mu_{01}, \boldsymbol{U} \in \Delta_+^2(\boldsymbol{z}, \boldsymbol{z}')] - \Pr[\epsilon \leq \mu_{01}, \boldsymbol{U} \in \Delta_-^2(\boldsymbol{z}, \boldsymbol{z}')].
\end{aligned}
$$

Also, by the definitions of the partitions,

$$
\begin{aligned}
h_{11}(\boldsymbol{z}, \boldsymbol{z}') &= \Pr[\epsilon \leq \mu_{11}, \boldsymbol{U} \in \Delta_-(\boldsymbol{z}, \boldsymbol{z}') \cup A^*] \\
&= \Pr[\epsilon \leq \mu_{11}, \boldsymbol{U} \in \Delta_-^1(\boldsymbol{z}, \boldsymbol{z}') \cup A^*] + \Pr[\epsilon \leq \mu_{11}, \boldsymbol{U} \in \Delta_-^2(\boldsymbol{z}, \boldsymbol{z}')]
\end{aligned}
$$

and

$$
\begin{aligned}
h_{00}(\boldsymbol{z}, \boldsymbol{z}') &= -\Pr[\epsilon \leq \mu_{00}, \boldsymbol{U} \in \Delta_+(\boldsymbol{z}, \boldsymbol{z}') \cup A^*] \\
&= -\Pr[\epsilon \leq \mu_{00}, \boldsymbol{U} \in \Delta_+^1(\boldsymbol{z}, \boldsymbol{z}') \cup A^*] - \Pr[\epsilon \leq \mu_{00}, \boldsymbol{U} \in \Delta_+^2(\boldsymbol{z}, \boldsymbol{z}')].
\end{aligned}
$$

Now combining all the terms yields

$$
\begin{aligned}
h(\boldsymbol{z}, \boldsymbol{z}') &= \Pr[\epsilon \leq \mu_{11}, \boldsymbol{U} \in \Delta_-^1(\boldsymbol{z}, \boldsymbol{z}') \cup A^*] - \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in \Delta_-^1(\boldsymbol{z}, \boldsymbol{z}') \cup A^*] \\
&\quad + \Pr[\epsilon \leq \mu_{11}, \boldsymbol{U} \in \Delta_-^2(\boldsymbol{z}, \boldsymbol{z}')] - \Pr[\epsilon \leq \mu_{01}, \boldsymbol{U} \in \Delta_-^2(\boldsymbol{z}, \boldsymbol{z}')] \\
&\quad + \Pr[\epsilon \leq \mu_{10}, \boldsymbol{U} \in \Delta_+^1(\boldsymbol{z}, \boldsymbol{z}') \cup A^*] - \Pr[\epsilon \leq \mu_{00}, \boldsymbol{U} \in \Delta_+^1(\boldsymbol{z}, \boldsymbol{z}') \cup A^*] \\
&\quad + \Pr[\epsilon \leq \mu_{01}, \boldsymbol{U} \in \Delta_+^2(\boldsymbol{z}, \boldsymbol{z}')] - \Pr[\epsilon \leq \mu_{00}, \boldsymbol{U} \in \Delta_+^2(\boldsymbol{z}, \boldsymbol{z}')].
\end{aligned}
$$

Then by Assumption M, $\mu_{1,\boldsymbol{d}_{-s}} - \mu_{0,\boldsymbol{d}_{-s}}$ share the same signs for all s and $\forall \boldsymbol{d}_{-s} \in \{0,1\}$ and therefore $sgn\{h(\boldsymbol{z}, \boldsymbol{z}')\} = sgn\left\{\mu_{1,\boldsymbol{d}_{-s}} - \mu_{0,\boldsymbol{d}_{-s}}\right\}$.

D.7 Proof of Theorem 7.1

For given $j = 0, ..., S-1$, consider

$$
\begin{aligned}
&E[Y|\boldsymbol{Z} = \boldsymbol{z}] - E[Y|\boldsymbol{Z} = \boldsymbol{z}'] \\
&= E\left[Y_{M \leq j} + 1[\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}]\{Y_{M > j} - Y_{M \leq j}\}\right] - E\left[Y_{M \leq j} + 1[\boldsymbol{D}(\boldsymbol{z}') \in M^{>j}]\{Y_{M > j} - Y_{M \leq j}\}\right] \\
&= E\left[\{1[\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}] - 1[\boldsymbol{D}(\boldsymbol{z}') \in M^{>j}]\}\{Y_{M > j} - Y_{M \leq j}\}\right] \\
&= E[Y_{M > j} - Y_{M \leq j}|\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{\leq j}]\Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{\leq j}] \\
&\quad - E[Y_{M > j} - Y_{M \leq j}|\boldsymbol{D}(\boldsymbol{z}) \in M^{\leq j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{>j}]\Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{\leq j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{>j}] \\
&= E[Y_{M > j} - Y_{M \leq j}|\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{\leq j}]\Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{\leq j}], \qquad \text{(D.22)}
\end{aligned}
$$

where the first equality plugs in $Y = 1[\boldsymbol{D} \in M^{>j}]Y_{M > j} + \{1 - 1[\boldsymbol{D} \in M^{>j}]\}Y_{M \leq j}$ and applies Assumption IN, and the last equality is by supposing that the result of Lemma 7.1 is satisfied

with

$$\Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{\leq j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{>j}] = 0. \tag{D.23}$$

But note that

$$\Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{\leq j}] = \Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}] - \Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{>j}],$$

where $\Pr[\boldsymbol{D}(\boldsymbol{z}) \in M^{>j}, \boldsymbol{D}(\boldsymbol{z}') \in M^{>j}] = \Pr[\boldsymbol{D}(\boldsymbol{z}') \in M^{>j}]$ by (D.23). Combining this result with (D.22) yields the desired result.

(a) R_{000} (b) R_{100} (c) R_{010} (d) R_{001}

(e) R_{101} (f) R_{011} (g) R_{110} (h) R_{111}
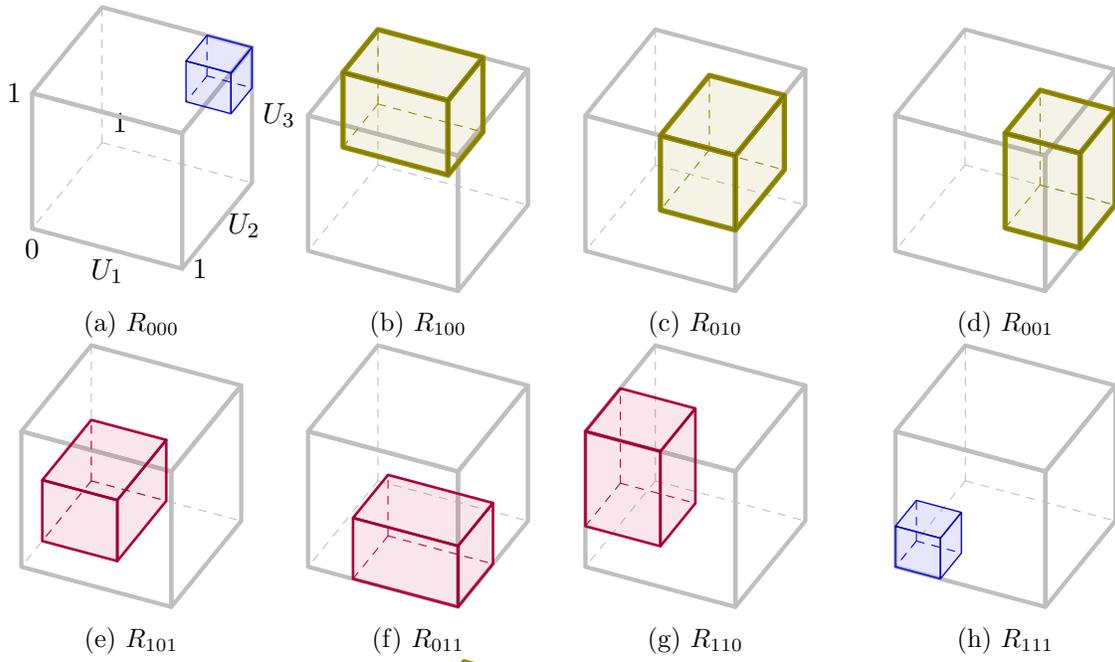
(i) $\bigcup_{0 \leq j \leq 3} \left\{ \bigcup_{\boldsymbol{d} \in M_j} R_{\boldsymbol{d}} \right\} = \mathcal{U} \equiv (0,1]^3$

Figure 4: Illustration of equilibrium regions in treatment selection process (Proposition 3.1) for three players ($S = 3$).
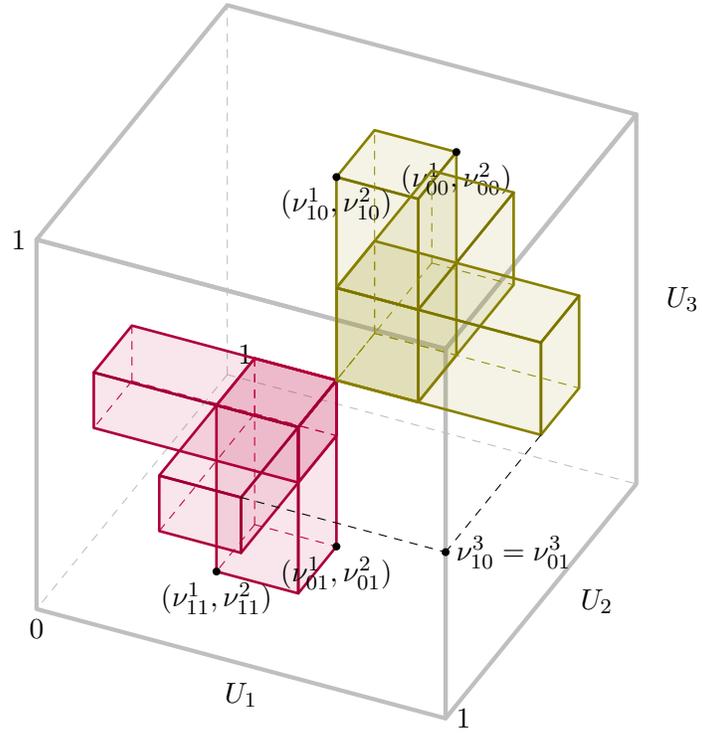
Figure 5: Depicting the regions of multiple equilibria for three players ($S = 3$).
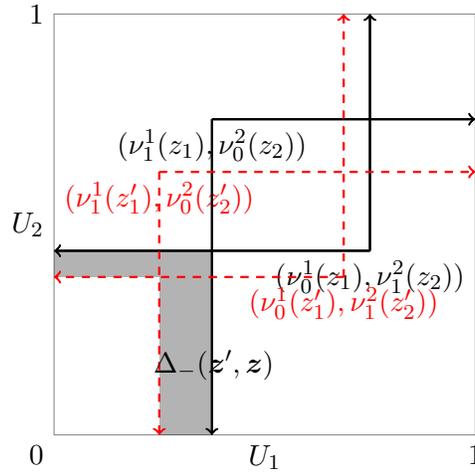
Figure 6: The region of LATE subgroup for two players ($S = 2$).
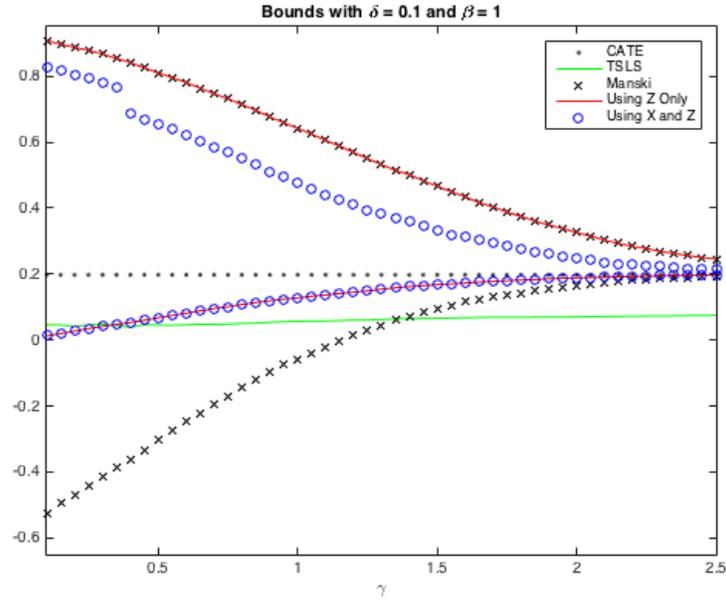
Figure 7: Bounds on the ATE with different strength of vector $\boldsymbol{Z} = (Z_1, Z_2)$ of binary instruments when X takes three different values ($|\mathcal{X}| = 3$).

This figure (and the next) depicts the simulated bounds for $E[Y_{11} - Y_{00}|X = 0] = 0.2$ (the straight dotted line). The horizontal axis is the value of the coefficients on the instruments ($\gamma_1 = \gamma_2 = \gamma$). The stronger the instruments, the narrower the bounds are. The cross lines are Manski (1990)'s bounds. The red solid lines are our bounds using only the variation of $\boldsymbol{Z}$, which identify the sign of the ATE. The blue circle lines are bounds where the variation of X, the exogenous variable excluded from the treatment selection process, is also used. Lastly, the green solid line is the simulated TSLS estimand assuming a linear simultaneous equations model.
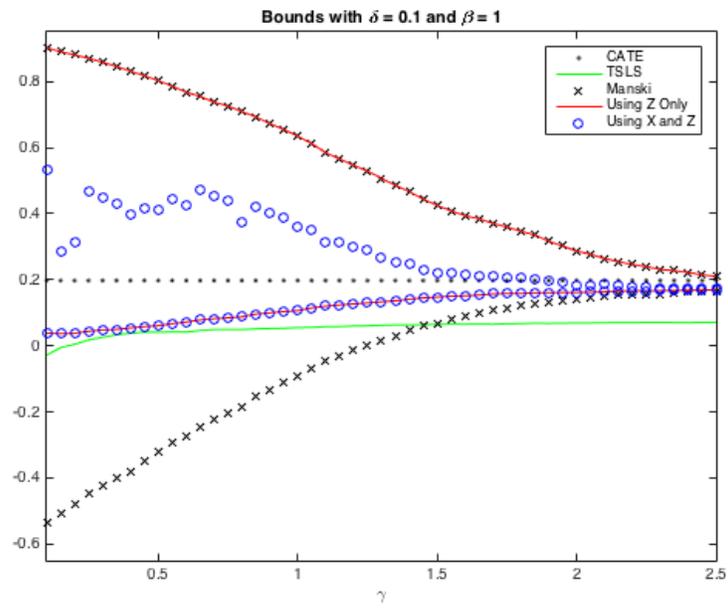
Figure 8: Bounds with different strength of vector $\boldsymbol{Z} = (Z_1, Z_2)$ of binary instrument when X takes fifteen different values ($|\mathcal{X}| = 15$).
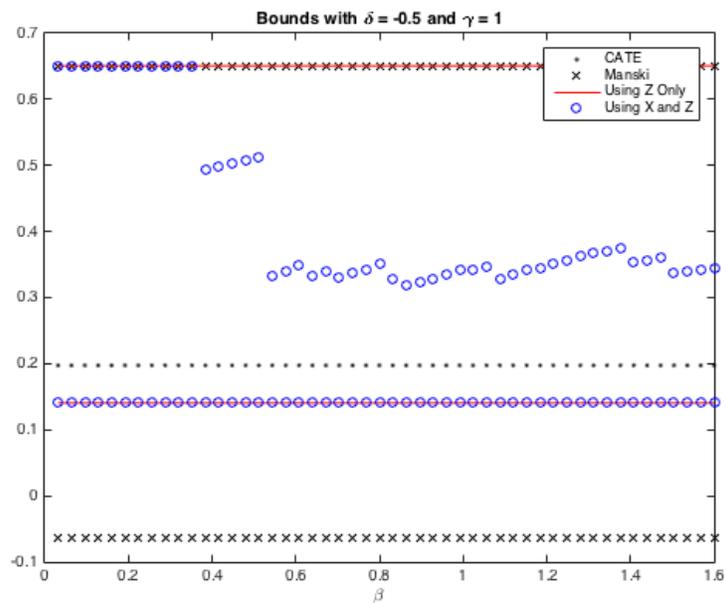
Figure 9: Bounds under Different Strength of X with $|\mathcal{X}| = 15$.

The horizontal axis is the value of the coefficient on the exogenous variable X excluded from the treatment selection process. The jumps in the bounds when both the variations of $\boldsymbol{Z}$ and X are used (the blue circle lines) are because different inequalities are involved for different values of the coefficient; see the text for details.
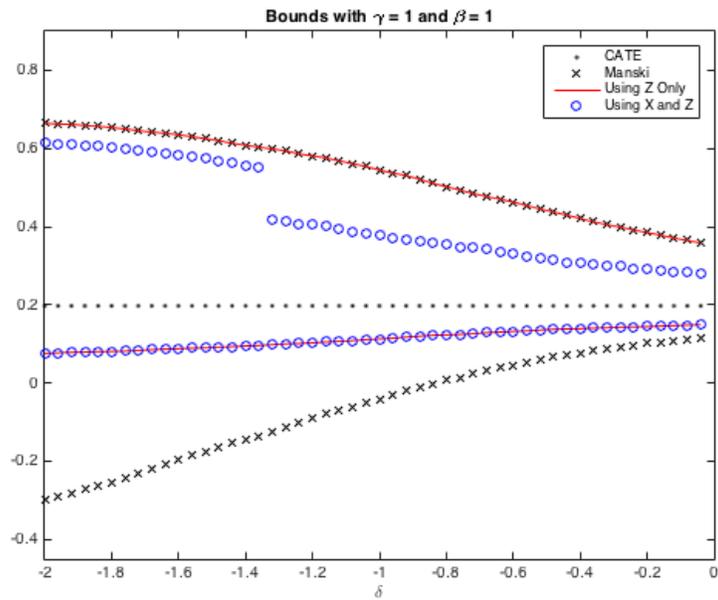
Figure 10: Bounds under Different Strength of Interaction with $|\mathcal{X}| = 3$.

The horizontal axis is the value of the coefficients on the opponents' decisions ($\delta_1 = \delta_2 = \delta$). The smaller the interaction effects, the narrower the bounds are. Again, the jumps in the bounds when both the variations of $\mathbf{Z}$ and X are used (the blue circle lines) are because different inequalities are involved for different values of the coefficient.