An Active-Learning Approach to Visualizing Multivariate Functions using Balloons

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Introduction

- This hands-on multi-class exercise helps students learn marginal analysis by doing marginal analysis on their own balloon
  - Use in both introductory and intermediate classes
- The marginal analysis at the core of microeconomics requires that students think in terms of marginal tradeoffs and understand how to balance those tradeoffs to achieve optimal outcomes
  - For example, one of the first things that economics students learn is that although individuals gross purchase behavior varies from person to person; all face the same marginal purchase behavior
- The *PowerPoint* protocol provides detailed student instructions to build the balloon (see 3 red-bordered slides below) as well as how to use the balloon to analyze economic concepts (see **Exercise Goals** slide)
  - Slides include additional instructor notes
- This geometric analysis may be used in a calculus-free Microeconomics or Managerial Economics course
- Algebraic counterparts are provided for those teaching a calculus enriched course (see bottom row slides)

Exercise Goals (color-coded to analytical tasks)

- The overarching goal is to understand how economists represent 3-D, (x, y, z), surfaces using 2-D, (x, y), graphs
  - z represents utility, profit, or output produced while x and y are outputs or inputs depending on economic situation under analysis

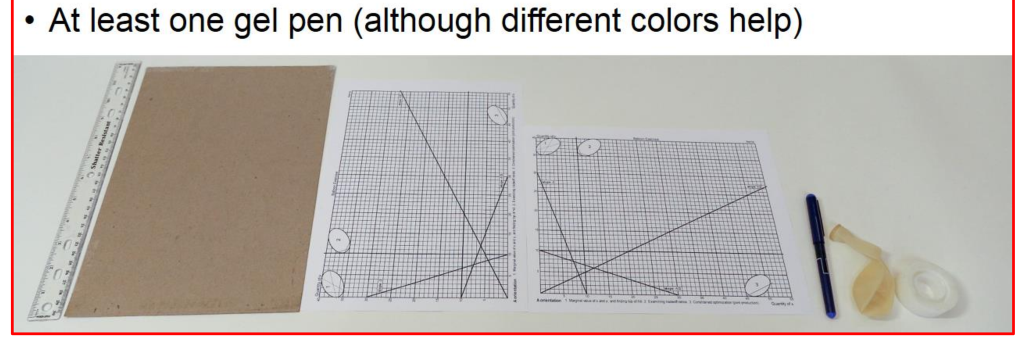
1
- To visualize marginal value of x and marginal value of y as slope in the x and y directions
  - This is the basis of the economist's notion of *ceteris paribus*
  - Marginal value is marginal utility, marginal profit, or marginal product depending on economic situation under analysis
- To find the top of a hill using $MV_x = 0$ and $MV_y = 0$ lines

2
- To visualize level sets on the balloon and on the graph
  - These are indifference curves, isoprofit contours, and isoquants
- To compare points where x is more valuable than y with ones where y is more valuable than x using tradeoff ratios
  - Do this on the balloon and on the 2-D graph paper

3
- To visualize constrained optimization on the balloon and on the graph as tangency between constraint and level set
  - Relate this point to level sets and marginal values

Each Student needs the following Materials

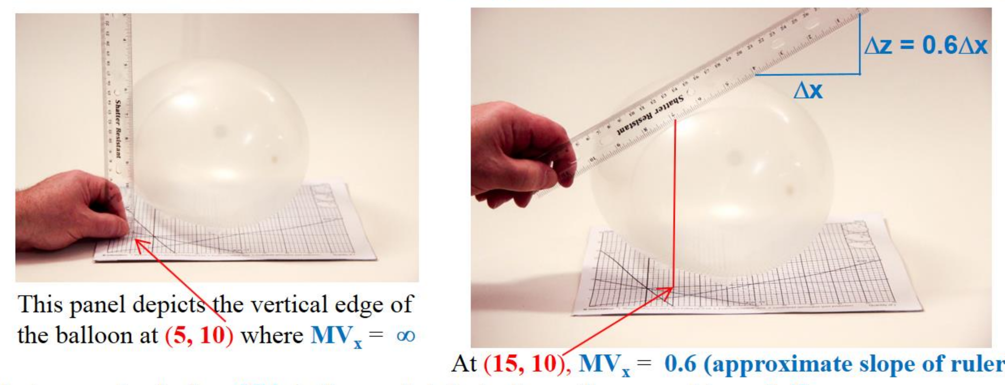
- One clear 11 or 12 inch balloon (don't blow it up yet)
- Two sheets of graph paper with the same orientation (A or B)
  - One provides a base, the other is for 2-D graphing CLICK for HANDOUT
  - The sheets have lines and "orientation guides" to use during the exercise
- One piece of backing board
- Tape for attaching graph paper to backing board and balloon to the base
  - 12 two-inch pieces of tape (8 attach paper to base, 4 attach balloon to base)
- Ruler (clear is best)
- At least one gel pen (although different colors help)



1A. Visualizing z Slope in the x Direction

- Marginal value of x, MV_x, is $\Delta z/\Delta x$, the z slope in the x direction, *holding y fixed* (Imagine walking W to E)*
  - This is done at y = 10 in the panels below (a B-style balloon shown)
    - The marginal value of x is infinite (i.e., the ruler is vertical) at x = 5
    - From there, marginal value of x declines as x increases
    - At x = 26, given y = 10, the marginal value of x is zero
    - For x > 26, given y = 10, the marginal value of x is negative

Note: This will occur at different x values for each person



This panel depicts the vertical edge of the balloon at (5, 10) where $MV_x = \infty$

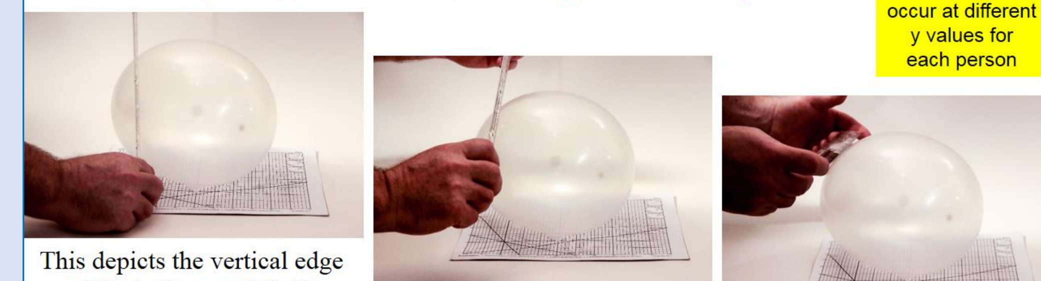
At (15, 10), $MV_x = 0.6$ (approximate slope of ruler)

*In terms of calculus, MV_x is the partial derivative with respect to x, $\partial z/\partial x$

1A. Visualizing z Slope in the y Direction

- Marginal value of y, MV_y, is $\Delta z/\Delta y$, the z slope in the y direction, holding x fixed (Imagine walking S to N)*
  - This is done at x = 10 in the figures below
    - The marginal value of y is infinite (i.e., the ruler is vertical) at y = 7
    - From there, marginal value of y declines as y increases
    - At y = 26, given x = 10, the marginal value of y is zero

Note: This will occur at different y values for each person



This depicts the vertical edge of the balloon at (10, 7) where $MV_y = \infty$

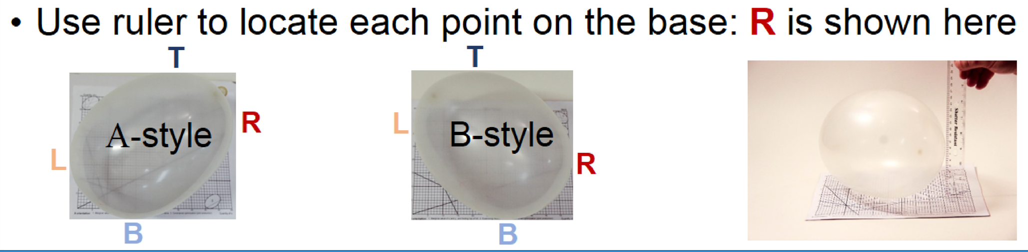
MV_y is positive for 7 < y < 26, given x = 10

$MV_y = 0$ at y = 26, given x = 10

*Similarly, MV_y is the partial derivative with respect to y, $\partial z/\partial y$, is written in shorthand as z_y

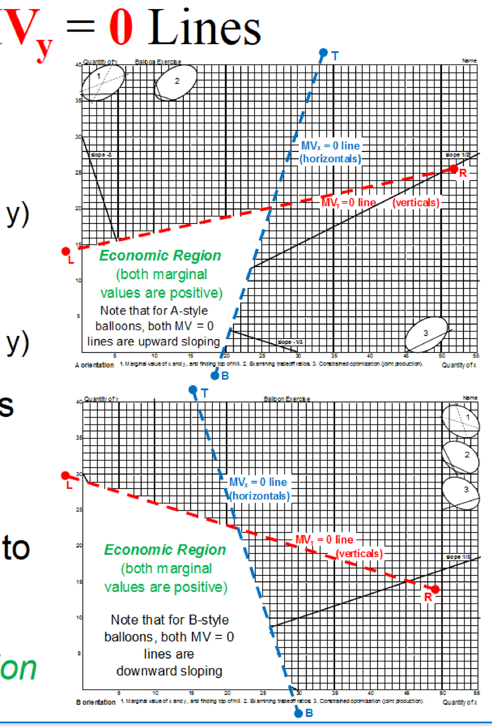
1B. *Finding* the Top of a Hill using $MV_x = 0$ and $MV_y = 0$ Lines

- To find MV = 0 lines, view the balloon from above (stand up)
- Find the point where $MV_x = 0$ at the bottom and top
  - At the Bottom, this is where the outline of the balloon goes from negative to positive slope as x increases (See B)
  - At the Top, this is where the outline of the balloon goes from positive to negative slope as x increases (See T)
- Find the point where $MV_y = 0$ at the left and right
  - At the Left, this is where the outline of the balloon goes from negative to positive slope as y increases (See L)
  - At the Right, this is where the outline of the balloon goes from positive to negative slope as y increases (See R)
- Use ruler to locate each point on the base: R is shown here



1B. *Graphing* the Top of a Hill using $MV_x = 0$ and $MV_y = 0$ Lines

- Transfer this information to graph paper
- Use your ruler to draw a dashed line between B and T
  - This line approximates the set of (x, y) bundles where $MV_x = 0$
- Use your ruler to draw a dashed line between L and R
  - This line approximates the set of (x, y) bundles where $MV_y = 0$
- The intersection of these two lines is the approximate top of the hill
- This will be more accurate, the closer is your balloon to an oval
- The area below red $MV_y = 0$ and to the left of the blue $MV_x = 0$ lines has both $MV_x > 0$ and $MV_y > 0$
  - This is called the *Economic Region* (of production or consumption)



2. Estimating Δz in any Direction

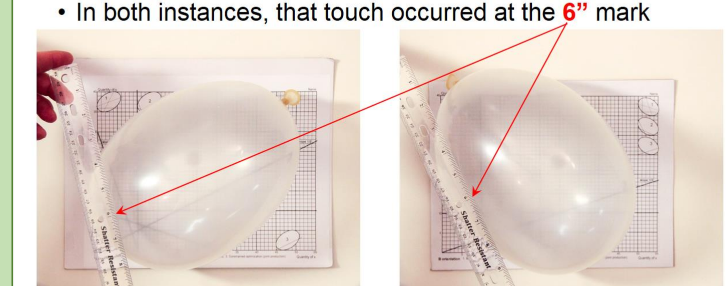
- The goal in 1A was to understand how one describes slope in two specific directions: x, MV_x; and y, MV_y
- We can use these slopes to describe the change in height on the balloon that would occur if we move in **any** direction
- One way to describe the move uses points of a compass; another uses change coordinates (Δx, Δy)
  - For example, a move to the North East (along a 45° line) involves moving (Δx, Δy) where Δx = Δy > 0
- What would happen to z if we move Δx in the x direction and Δy in the y direction on the surface of the balloon?
  - For small changes in x and y we (approximately) would have:

$$\Delta z = \Delta x \cdot MV_x + \Delta y \cdot MV_y$$

  - This change would be exact if the surface was a plane
- The first component is the change in z due to the change in x and the second is the change in z due to the change in y

2A. *Finding* a point where x is more valuable than y (high tradeoff ratio)

- Looking at the balloon from overhead, hold the ruler, parallel to the base, in line with the steep line starting at (0, 30) and lower the ruler until it just touches the balloon
  - In both instances, that touch occurred at the 6" mark



For A-style balloons the tradeoff ratio is 3 — For B-style balloons the tradeoff ratio is 2

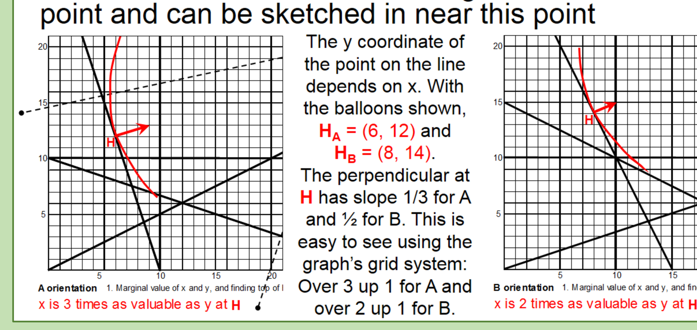
2A. *Marking* a point where x is more valuable than y (high tradeoff ratio)

- Use a gel pen to mark a bit on each side of point found
  - This line should be essentially flat when viewed from the side
- Mark a perpendicular from the middle of this point
  - This represents the steepest rate of increase from this point
  - The resulting mark should look something like this: ⊥

The x coordinate of this point is at x = 6 — The x coordinate of this point is at x = 8

2A. *Graphing* a point where x is more valuable than y (High tradeoff ratio)

- The x coordinate at the center of the ⊥ noted above is on the steeply sloped line starting at (0, 30)
- Mark the perpendicular from this point using the grid
- The level set of the balloon is tangent to the line at this point and can be sketched in near this point



The y coordinate of the point on the line depends on x. With the balloons shown, $H_A = (6, 12)$ and $H_B = (8, 14)$. The perpendicular at H has slope 1/3 for A and ½ for B. This is easy to see using the graph's grid system: Over 3 up 1 for A and over 2 up 1 for B.

x is 3 times as valuable as y at H — x is 2 times as valuable as y at H

2. Level Sets and Tradeoff Ratios

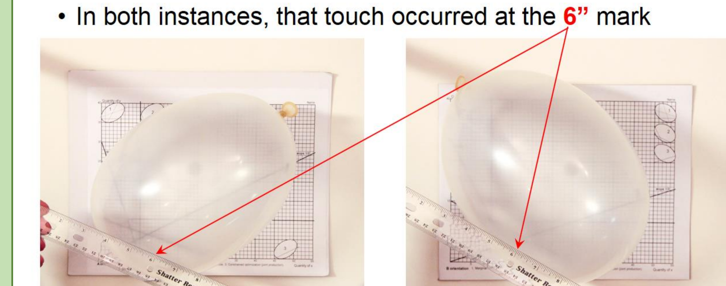
- If the change in z due to changes of Δx and Δy cancel one another, the net change in z is zero
  - In this case, (x_0, y_0) and $(x_0+\Delta x, y_0+\Delta y)$ are on the same *level set*
- Economists have a variety of terms for these level sets
  - *Indifference curves, isoquants and isoprofit contours* are the terms given for equal utility, equal production, and equal profit curves
- Consider an initial point in the Economic Region (where both marginal values are positive): If x increases from there then y must decrease if Δz = 0
  - Put another way, there must be a **tradeoff** between y and x
- Minus the slope of the level set describes the *tradeoff ratio*
  - Set Δz = 0 and solve for -Δy/Δx:

$$0 = \Delta x \cdot MV_x + \Delta y \cdot MV_y \quad \text{implies} \quad -\Delta y/\Delta x = MV_x / MV_y$$

  - The minus is added so we can talk about this tradeoff in positive terms (two units of y for one unit of x, for example)
- The ***tradeoff ratio*** tells us the rate at which y must be traded for x in order to maintain utility, or production, or profit at a given level
  - Economists often call this the *Marginal Rate of Substitution, MRS* (although other letters are often added to distinguish scenarios)

2B. *Finding* a point where y is more valuable than x (low tradeoff ratio)

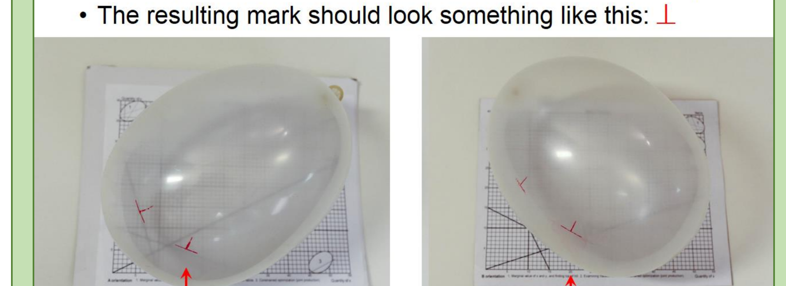
- Looking at the balloon from overhead, hold the ruler, parallel to the base, in line with the shallow line starting at (30, 0) and lower the ruler until it just touches the balloon
  - In both instances, that touch occurred at the 6" mark



For A-style balloons the tradeoff ratio is 1/3 — For B-style balloons the tradeoff ratio is 1/2

2B. *Marking* a point where y is more valuable than x (low tradeoff ratio)

- Use a gel pen to mark a bit on each side of point found
  - This line should be essentially flat when viewed from the side
- Mark a perpendicular from the middle of this point
  - This represents the steepest rate of increase from this point
  - The resulting mark should look something like this: ⊥

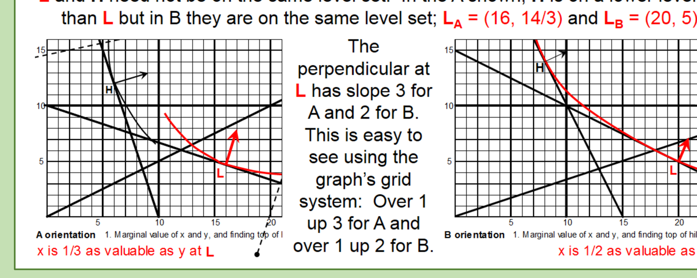


The x coordinate of this point is x = 16 — The x coordinate of this point is x = 20

2B. *Graphing* a point where y is more valuable than x (Low tradeoff ratio)

- The x coordinate at the center of the ⊥ noted above is on the shallowly sloped line starting at (30, 0)
- Mark the perpendicular from this point using the grid
- The level set of the balloon is tangent to the line at this point and can be sketched in near this point

L and H need not be on the same level set. In the A shown, H is on a lower level set than L but in B they are on the same level set; $L_A = (16, 14/3)$ and $L_B = (20, 5)$.



The perpendicular at L has slope 3 for A and 2 for B. This is easy to see using the graph's grid system: Over 1 up 3 for A and over 1 up 2 for B.

x is 1/3 as valuable as y at L — x is 1/2 as valuable as y at L

2C. *Decomposing* the Direction of Maximal Change

- The ***gradient*** (perpendicular to the level set at the point), **∇z(x, y)**, is the vector of marginal values

∇z(x, y) = (marginal value of x, marginal value of y)

- The components represent slopes in the x and y direction
  - The magnitude represents steepness of slope at that point
  - You can visualize these slopes on the balloon in each direction

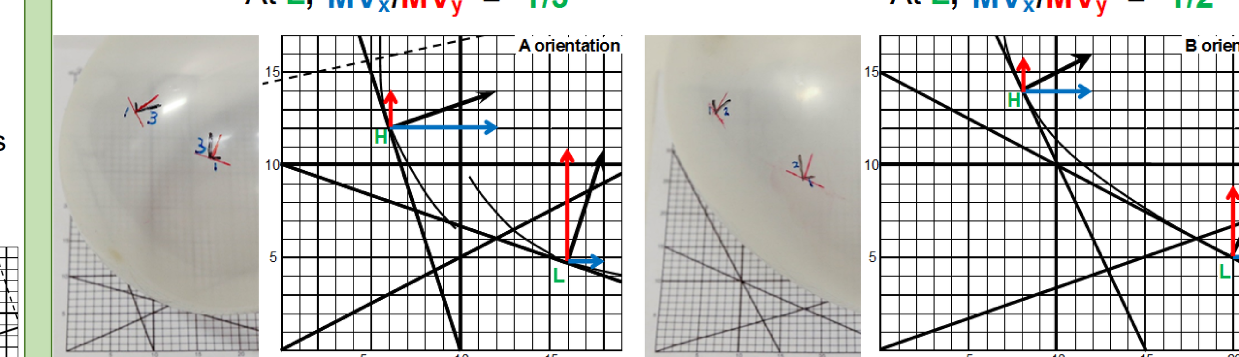
CLICK for a deeper geometric discussion of these slopes

CLICK for a deeper algebraic discussion

- The ratio of these slopes is the tradeoff ratio

A-style: At H, $MV_x/MV_y = 3$; At L, $MV_x/MV_y = 1/3$

B-style: At H, $MV_x/MV_y = 2$; At L, $MV_x/MV_y = 1/2$

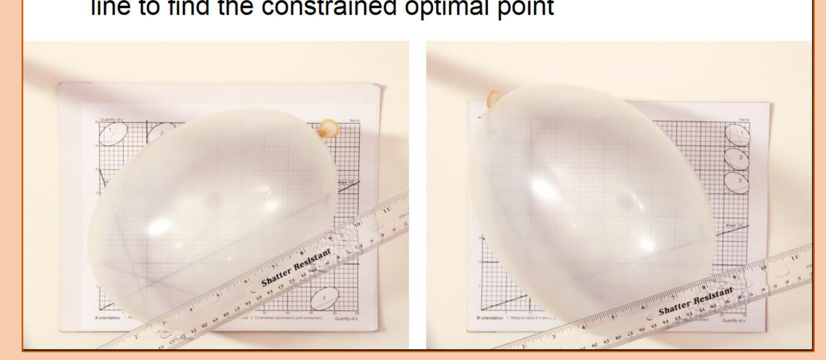


3A. Constrained Optimization

- Economic situations often involve constraints
  - A consumer maximizes utility subject to a budget constraint
  - A producer maximizes output subject to a cost constraint
  - A firm maximizes profit subject to a production constraint
- We can model each of these situations using the balloon but our initial focus will be on the third situation
- Assume a firm makes profits from two goods, x and y
  - The firm cannot attain the top of the profit hill because it can only produce x and y in fixed proportions
  - This is a condition called *joint production*
    - Common examples of joint production are beef and hides, bark and boards, gasoline and fuel oil
    - Such production can occur under fixed or variable proportions but the easiest to model is fixed proportion
- Imagine c units of x is produced for each unit of y produced
  - This constraint can be represented by the line: y = x/c

3B. *Visualizing* a Constrained Optimum

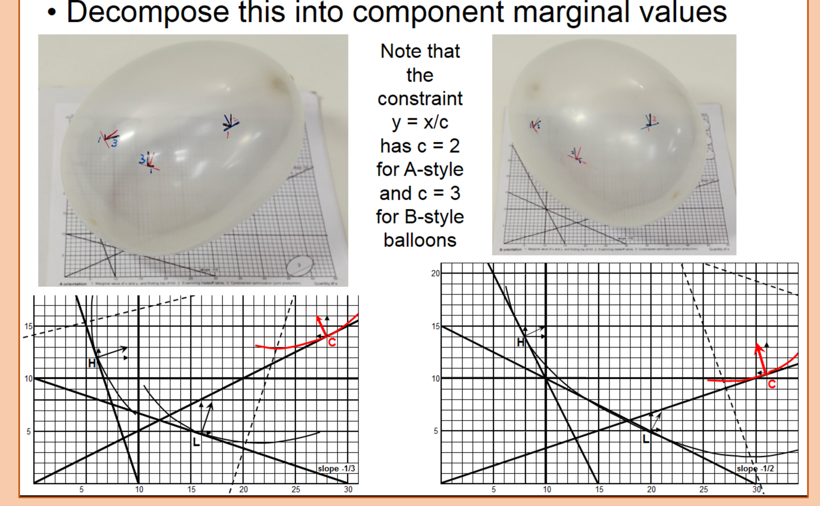
- We use the same strategy used in Part 2 to find the constrained optimal point on the balloon
- Find the highest point on the balloon subject to being on the constraint line
  - Use the ruler parallel to the surface and above the y = x/c line to find the constrained optimal point



3B. *Graphing* a Constrained Optimum

- Mark point C on the balloon and on the graph paper
- Decompose this into component marginal values

Note that the constraint y = x/c has c = 2 for A-style and c = 3 for B-style balloons



3B. *Analyzing* this Constrained Optimum

- The constrained optimum is no longer in the Economic Region
  - The constrained optimum is in the 4th quadrant relative to the top of the hill
  - In this region, $MV_x < 0$ and $MV_y > 0$ and the gradient points to the NW rather than NE as it did in Part 2
  - In this context, increasing x decreases profit but increasing y increases profit
- At the constrained optimum, the gain in profit from increased y should balance off the loss in profit from increased x
- Since each unit of y produces c units of x, **Δx = c·Δy**
- In terms of the equation we initial examined,

$$0 = \Delta x \cdot MV_x + \Delta y \cdot MV_y = c \cdot \Delta y \cdot MV_x + \Delta y \cdot MV_y$$

0 = Loss due to x + Gain due to y

- Reorganizing and dividing by Δy we obtain: $MV_y = -c \cdot MV_x$
- In words, one unit of y must be c times as valuable as the loss from one unit of x because each unit of y comes with c units of x
- Geometrically, the slope in the y direction is c times as steep as in the x direction

3C. Additional Constrained Optimization Problems

- You have already found solutions to six other constrained optimization problems in 2A-2C as tradeoff ratio tangencies between ruler & balloon

H is the solution to these three constrained optimization problems: At H, the tangency is $MV_x/MV_y = P_x/P_y$

Maximize*	Subject to:[1]	*In this instance, P_x depends on balloon style — Style: A B		A	B
z = Utility(x, y)	Income = \$90	P_x = \$9 \$6	P_y = \$3	3	2
z = Output(labor, capital)	Cost = \$60	P_{labor} = \$6 \$4	$P_{capital}$ = \$2	3	2
z = Profit(x, y)	$Time_{total\ hours}$ = 30h	Time/x = 3h 2h	Time/y = 1h	3	2

[1]Each constraint can be written as y = 30 - 3x for A-style & y = 30 - 2x for B-style prices and income.

L is the solution to these three constrained optimization problems: At L, the tangency is $MV_x/MV_y = P_x/P_y$

Maximize**	Subject to:[2]	**In this instance, P_y depends on balloon style — Style: A B		A	B
z = Utility(x, y)	Income = \$90	P_y = \$9 \$6	P_x = \$3	1/3	1/2
z = Output(labor, capital)	Cost = \$60	$P_{capital}$ = \$6 \$4	P_{labor} = \$2	1/3	1/2
z = Profit(x, y)	$Time_{total\ hours}$ = 30h	Time/y = 3h 2h	Time/x = 1h	1/3	1/2

[2]Each constraint can be written as y = 10 - x/3 for A-style & y = 15 - x/2 for B-style prices and income.

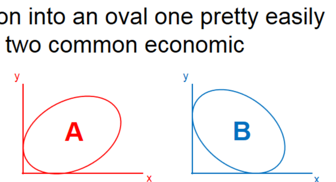
In each instance, the tangency condition can be recast as *equal bang for the buck*: $MV_x/P_x = MV_y/P_y$.

- You can readily create additional problems and find their solutions on the balloon by changing prices or income in the above constrained optimization problems

These 3 Slides Describe Making the Model

How to Create an Oval Balloon from a Round Balloon

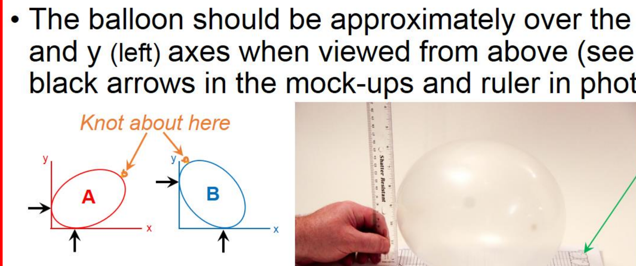
- Round balloons are readily available but not as interesting as ovals for representing economic models
  - Luckily, you can turn a round balloon into an oval one pretty easily
  - These ovals can be used to model two common economic concepts:
    - A) Imperfect complements — like peanut butter and jelly
    - B) Imperfect substitutes — like green peppers and onions



  - The height of the balloon above the base, z, can be used to represent utility or profits, or production (if x and y are inputs)
- Blow up the balloon while restricting the balloon in the middle with your fingers — Watch Here
  - Think of two balloon ends and restricted middle like a barbell
- You should end up with a balloon that is 9 to 10 inches long, and 6 to 7 inches wide

Orienting your Balloon on the Base

- Tape the corners and edges of a piece of graph paper to the backing board to create a base for the balloon
- Create 4 small *tape loops* and use them to fix the balloon to the base using the *orientation guides with knot in NE or NW corner as shown below*
- The balloon should be approximately over the x (bottom) and y (left) axes when viewed from above (see the black arrows in the mock-ups and ruler in photograph)



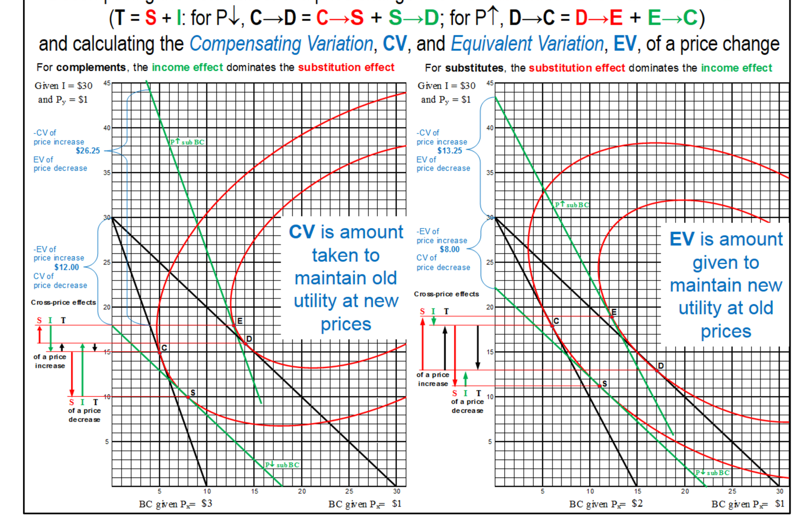
Two of the *tape loops* can be seen here

The three *orientation guides* highlight lines used in the three main analytical tasks in the exercise

Why A-style v. B-style are Complements v. Substitutes using Demand Decomposition

Intermediate Microeconomics Extensions

Decomposing the *Total Effect* of a price change into the *Substitution Effect* and *Income Effect* (T = S + I: for P↓, C→D = C→S + S→D; for P↑, D→C = D→E + E→C) and calculating the *Compensating Variation*, **CV**, and *Equivalent Variation*, **EV**, of a price change



Algebraic Counterparts to this Geometric Analysis (using Paraboloids)

This is a quiz question from the 3rd week of class

- This provides an algebraic counterpart to the slides shown with blue and green background (slides labelled 1 and 2)
- Partial answers are shown to the right
- Click here to see full answers for this question

1;30 total) FOC Enterprises estimates its profit, π, as a function of the two goods it produces, x and y, is:

$$\pi(x, y) = 400x - 20x^2 + 180y - 10y^2 + 10xy - 1{,}240.$$

A;8) How many units of x and y should FOC produce in order to maximize profits? What profits result?

B;4) FOC wishes to verify that it is, indeed, producing at the profit maximizing output level. To do this, FOC wishes to consider what profits will be if it produces Δx = 6 units more or less x than the amount determined in part A. FOC chooses the level of y that maximizes profits given these alternative x values, y(x). What profits result in each instance? Fill in the table to answer this question and provide the equation for y(x):

6 Δx	Part B	x	y(x)	π(x,y)
x is Δx more than x_{max}				
π maximizing x = x_{max}				
x is Δx less than x_{max}				

y(x) =

C;3) FOC decides to consider what profits will be if it produces Δy = 4 units more or less y than the amount determined in part A. FOC would then choose the level of x that maximizes profits given these alternative y values, x(y). What profits result in each instance? Fill in the table to answer this question and provide the equation for x(y):

4 Δy	Part C	x(y)	y	π(x,y)
y is Δy more than y_{max}				
π maximizing y = y_{max}				
y is Δy less than y_{max}				

x(y) =

D;13) Graph the profit hill using the graph paper provided. Make sure to include the iso-profit contours associated with your answers to B and C. Provide two lines on this graph: $\pi_x = 0$; $\pi_y = 0$.

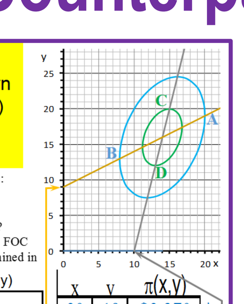
E;2) Is this more like an A-style or a B-style balloon? Briefly explain (no credit for simply asserting an answer).

x	y	π(x,y)	
20	19	\$2,370	A
14	16	\$ 3,000	
8	13	\$2,370	B

0.5 * x + 9 = y(x)

x	y	π(x,y)	
15	20	\$2,860	C
14	16	\$ 3,000	
13	12	\$2,860	D

0.25 * y + 10 = x(y)



1;50 total) FOC Enterprises estimates its profit, π, as a function of the two goods it produces, x and y is:

$$\pi(x, y) = 540x - 10x^2 + 600y - 10y^2 - 10xy - 4920.$$

A;9) What should FOC do to maximize profits? How many units of x, y, and π(x,y) result? (Show math below.) When are the isoprofit contours horizontal & vertical (write as x(y) or y(x) as appropriate)? Place equations in this box.

HORIZONTAL: VERTICAL:

2 FACTS: If $x^* = x_{max} \pm 4$ then $\pi(x^*, y(x^*)) = \$5{,}880$. If $y^* = y_{max} \pm 6$ then $\pi(x(y^*), y^*) = \$5{,}730$.

B;9) Suppose FOC has a capacity constraint and can produce no more than 24 total xs and ys. How much x and how much y should FOC produce given this constraint? Label this solution B = (x_b, y_b). What π results?

C;9) Suppose FOC has a production process that produces 0.5 units of y for every unit of x produced (i.e., if x = 10, y = 5 or if y = 10, x = 20). How much x and how much y should Cal produce given this constraint? What profits result? Label this solution C = (x_c, y_c).

D;6) Estimate the value of both partials at B and C. Explain what each estimate means and use these estimates to show that your answer really does maximize profits given the restrictions in B or C.

NOTE: All x, y, π, and partial values are whole numbers in parts A - D. Answer B-D in the boxes.

E;17) Graph the profit hill using the graph paper provided. Make sure to include isoprofit contours associated with points B and C (label on graph). It will help to provide 4 lines on this graph: $\pi_x = 0$; $\pi_y = 0$; and the constraint functions in parts B & C. (*Hint*: It may help to recreate the "± x and y boxes" from last week.)

Partial Answers Below and Bottom Left

x	y	π(x,y)
20	20	\$ 5,880
16	22	\$ 6,000
12	24	\$ 5,880

-0.5 * x + 30 = y(x)

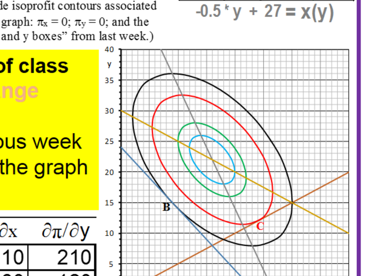
x	y	π(x,y)
13	28	\$ 5,730
16	22	\$ 6,000
19	16	\$ 5,730

-0.5 * y + 27 = x(y)

The question above is from the 4th week of class

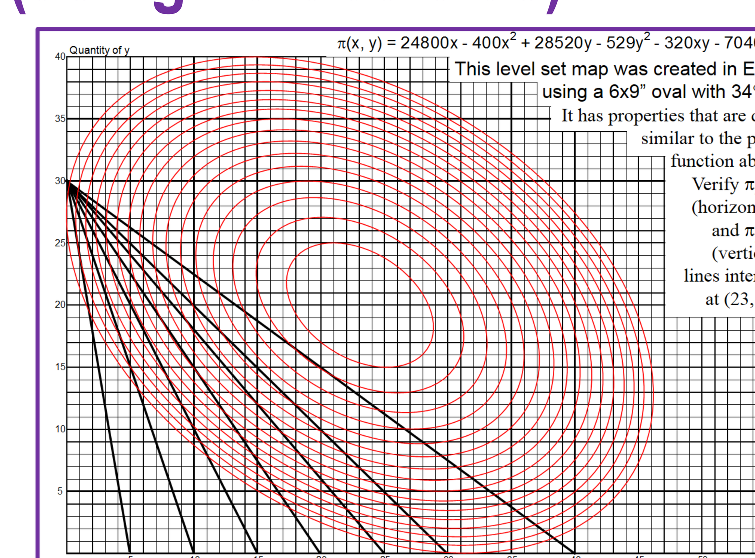
- This an algebraic counterpart to the orange background slides (labelled 3)
- The "± x and y boxes" (taught the previous week and shown to the right) help in drawing the graph
- Click to see full answer to this question

	x	y	π(x,y)	∂π/∂x	∂π/∂y
Value at (x_b, y_b)	9	15	4,530	210	210
Value at (x_c, y_c)	24	12	5,160	-60	120



$$\pi(x, y) = 24800x - 400x^2 + 28520y - 529y^2 - 320xy - 70400$$

This level set map was created in Excel using a 6x9" oval with 34° tilt. It has properties that are quite similar to the profit function above. Verify $\pi_x = 0$ (horizontals) and $\pi_y = 0$ (verticals) lines intersect at (23, 20).



Consider instead: $\pi(x, y) = 12000x - 400x^2 + 13800y - 529y^2 + 320xy - 76000$. What changes?

Presented at the January 2018 ASSA, AEA Committee on Economic Education Annual Conference, Philadelphia, PA.