Asset Pricing under Rational Learning
about Rare Disasters

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Abstract

Why is investment in stocks so persistently weak after a rare disaster? If investors erroneously become pessimistic about stock returns, how long does it take until negative market outcomes due to self-fulfilling expectations revert? Rational-expectations models with variable disaster risk often fail to tightly connect disaster episodes with post-disaster expectations. We demonstrate that introducing limited information and learning about rare disaster risk, while retaining full rationality, can generate stock-investment behavior that seems similar to persistent investor fear after a rare disaster. We build a workable asset-pricing model of variable disaster risk under rational expectations (RE) and we introduce two limited-information variations to it: (a) rational learning for state verification (RLS), in which investors know the data-generating process of disaster riskiness but cannot observe whether the economy is in a riskier state or not, and (b) rational learning about the data-generating process (RLP) of disaster risk, an environment in which investors neither know the data-generating process of disaster riskiness nor they can observe the state of disaster riskiness. We provide analytical results for all setups (RE, RLS, and RLP), and examine both the transitional dynamics of asset prices after a disaster and their long-run behavior.

Keywords: beliefs, Bayesian learning, controlled diffusions and jump processes, learning about jumps, adaptive learning, rational learning

JEL classification: D83, G11, C11, D91, E21, D81, C61

1. Introduction

In the past twenty-five years, stock prices in the United States and other markets around the world experienced one boom and two busts. The boom took place in the second half of the 1990s, the first bust in the year 2000 and the second one at the end of 2008. **Figure 1** shows that, in the US market, fundamentals have played a role in both bust episodes. In both cases, dividends and earnings exhibited a drop of 20 to 30 percent within a short period of time. Strikingly, price-dividend (P-D) and price-earnings (P-E) ratios started to decline massively shortly after the drop in dividends. The fall in prices over and above the reduction in dividends or earnings was particularly pronounced and rapid after the 2008 episode. Clearly, achieving a better understanding of the factors that drive the movement of asset prices following a rare stock market crash is of great importance not only for researchers and investors, but also for policymakers keen to assess the extent of negative impact on overall economic activity. For example, they may wonder whether these asset price movements reflect information about the duration and frequency of such crashes, or whether they are driven by irrational fear and panic among investors.

This paper presents a new approach for modeling investor fear under uncertainty about the likelihood of rare disasters. It relates two different literatures on asset-pricing that have proceeded mostly separately from one another. First, there is the large literature on learning under parameter uncertainty. It recognizes that market participants lack knowledge of many key parameters +-characterizing financial markets. In a recent survey, Pastor and Veronesi (2009a), point out that *"many facts (in financial markets) that appear baffling at first sight seem less puzzling once we recognize that parameters are uncertain and subject to learning"*.[1] The other literature aims to explain asset pricing puzzles as a consequence of disaster risk

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[1] Recent studies include Pastor and Veronesi (2009b), who investigate the emergence of bubbles when average productivity of a new technology is uncertain and subject to learning, and Weitzman (2007), who argues

while maintaining the assumption of rational expectations. First proposed by Rietz (1988), this idea received renewed interest once Longstaff and Piazzesi (2004) and Barro (2006, 2009) showed that empirically plausible disaster probabilities provide a powerful explanation of historical equity premia.[2] Weitzman (2007), however, questions the disaster risk literature by pointing to the "*inherent implausibility of being able to meaningfully calibrate rational-expectations-equilibria objective frequency distributions of rare disasters because the rarer the event the more uncertain is our estimate of its probability.*"

Our paper addresses Weitzman's criticism head-on and incorporates parameter uncertainty and Bayesian learning in a disaster-risk asset-pricing model. Similarly to Longstaff and Piazzesi (2004), Barro (2006, 2009) and Weitzman (2007), we use the Lucas (1978) exchange-economy asset-pricing model as a vehicle for conducting our analysis. As in Longstaff and Piazzesi (2004) and Barro (2006, 2009) we assume that dividends follow a jump-diffusion process in continuous time. This process includes a standard Brownian motion with drift that is interrupted by rare downward jumps. In the disaster risk literature, the probability that such a crash occurs within a given period of time may be fixed or time-varying but its stochastic properties are always assumed known to investors. We refer to this measure of the frequency of disasters as the hazard rate and consider a time-variable setup with the hazard rate switching between a high and low value at a given probability.[3] Under

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that learning about the parameter that controls the spread of the distribution of future consumption growth helps explain equity premia and excess volatility relative to the rational expectations benchmark. Early on, Timmermann (2001) studied the problem of learning investors that have imperfect information about the rate of dividend growth following a structural break.

[2] Longstaff and Piazzesi (2004) calibrated a Rietz-type model with large downward dividend jumps using data from the Great Depression. More recently, Barro and Ursua (2008) put together a large international dataset on consumption and disasters and Barro et al. (2010) provided new estimates of the variability and persistence of disaster risk. Gabaix (2008, 2010), Wachter (2013) and Gourio (2008a,b) show that variable disaster risk serves to explain excessive volatility of price-dividend ratios. See also, LeRoy (2008) for a survey on asset-pricing excess volatility and existing approaches for diagnosing it in the data.

[3] The inverse of the hazard rate is the number of periods it takes for a jump to occur on average. So, if a hazard rate is high, then rare disasters are more frequent on average, meaning that any underlying sources of rare disasters create a more hazardous environment. Technically, nature's true disaster-shock process is a

rational expectations—the starting point of our analysis—this probability is known, and consequently also the average hazard rate, that is the average frequency of disasters. Following Weitzman's critique, we proceed by treating this probability as unknown and model investors' beliefs and learning about the average hazard rate explicitly.[4] In other words, we consider Bayesian learning about the key parameter governing the frequency of disasters.[5]

Bayes' rule implies that investors' beliefs abruptly drop to a more pessimistic level following a stock market crash. Investors suddenly fear that such disasters will occur much more frequently in the future than they had thought in the past. Here, pessimism or fear is not meant to suggest investor irrationality. Rather, increased pessimism simply means that investors' perceived value of the probability assigned to the high hazard rate case has risen. The exact definition of investor rationality under Bayesian learning will be laid out further below. Over time, investors revise their beliefs by repeatedly applying Bayes' rule. Bayesian learning makes efficient use of historical information and new data. In the absence of another crash, beliefs slowly turn more optimistic and learning implies a smooth reduction in the perceived probability of the high-hazard-rate case. Thus, investors' pessimistic beliefs exhibit a certain degree of persistence after a disaster has occurred.

In our model, asymptotic beliefs are unbiased. However, even infinitely-lived investors would never reach full confidence about the average frequency of disasters, as would be the case under rational expectations. This result is due to the slow arrival of information about the frequency of rare disasters. Despite using Bayes' rule for updating priors, posteriors

random mixture of two Poisson processes, one with a high (constant) hazard rate and one with a low one. The probability that the disaster is drawn from the high or low hazard rate process is also constant.

[4] An alternative approach to modeling the impact of uncertainty about rare events on asset prices is offered by robust control in the presence of Knightian model uncertainty (see Liu, Pan and Wang (2005) for an implementation).

[5] The recent contribution by Pettenuzo and Timmermann (2011) provides empirical support that infrequent breaks are a major source of investment risk and that it is important to model investors belief formation regarding future breaks.

never catch up with nature's parameters with infinite precision, even if an infinitely long history of actual data has been processed. These belief dynamics lend additional support to Weitzman's claim that analysis of rare-disaster-risk requires the modeling of subjective expectations and investor learning.

Asset prices depend on investors' beliefs about the unknown parameter governing the frequency of rare disasters. We solve analytically for the relationship between asset prices and beliefs regarding disaster probability. Using this asset pricing formula, we find that the abrupt increase in pessimism following a crash causes a sudden drop in price-dividend ratios. The extent of the decline in the asset price over and above the fall in dividends is entirely due to the shift in beliefs. Under rational expectations, jumps in prices and dividends would be proportional and the ratio would remain the same.[6] The subsequent persistence in pessimistic beliefs implies that the price-dividend ratio remains depressed for some time. It recovers slowly as long as no other crash occurs and investors' beliefs assign successively lower probabilities to the high-frequency-disaster case.

The link between asset prices and beliefs is derived by solving the dynamic optimization problem of the representative investor/household in our asset pricing model. In this context, we distinguish between a fully rational investor and one who learns in an adaptive fashion. Both, the rational and the adaptive learner base their decision on current beliefs regarding disaster probabilities that were obtained by applying Bayes rule to available data. The distinction between rational and adaptive Bayesian learning depends on whether or not the decision maker takes into account the dynamic transition equations of beliefs in her

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[6] Here our setup differs from other studies that introduce time-variable disaster risk in rational-expectations asset-pricing models (cf. Gabaix (2008, 2010), Gourio (2008b) and Wachter (2013)). While their specifications of time-variable disaster risk are useful for explaining excess volatility under rational expectations, we aim to show that such variations in P-D ratios could even be exclusively due to changes in investors' subjective perceptions of disaster risk. Our setup implies a constant P-D ratio under rational expectations, because the average hazard rate is known and independent of past developments.

optimization problem, in addition to the other recursions governing laws of motion of state variables such as the dividend process. In other words, a rational learner knows that her beliefs will change in the future as new information arrives. In particular, as long as no crash occurs the perceived hazard rate will smoothly decline. The adaptive learner acts as if her beliefs will never change. Only as time advances, she re-calculates her estimate of the average hazard rate according to Bayes rule.[7] In this manner, adaptive Bayesian learning represents a well-defined deviation from fully rational behavior. Any difference between asset prices under adaptive versus rational Bayesian learning could then be characterized as being due to overly pessimistic or optimistic views.[8]

Under certain plausible conditions, we find that asset prices under rational learning are always higher than prices that follow from the behavior of adaptively learning investors for any given prior belief. This finding can be attributed to the fact that rational learners take into account that their estimates of disaster probabilities will change in the future. Specifically, in the absence of another crash they anticipate the gradual emergence of a more optimistic outlook. Thus, they demand more of the risky asset.

A recent paper that also investigates Bayesian learning about rare jumps is Benzoni, Colline-Dufresne and Goldstein (2011). These authors aim to explain the dramatic and lasting steepening of the implied volatility curve for equity index options after the 1987 stock

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[7] For other work distinguishing adaptive and rational Bayesian learning see Guidolin and Timmermann (2007), Cogley and Sargent (2008) and Koulovatianos, Mirman, and Santugini (2009). Adaptive learning reflects the anticipated utility concept studied, e.g., by Kreps (1998), Cogley and Sargent (2008), and Koulovatianos and Wieland (2011). Rational Bayesian learning may involve active experimentation, for example in the presence of multiplicative parameter uncertainty (see Mirman, Urbano and Samuelson (1993), Wieland (2000a,b) and Beck and Wieland (2002)). Recent work on asset pricing explores the role of Bayesian learning in booms and busts (see Benhabib and Dave (2011) for adaptive learning and Adam and Marcet (2010) for rational beliefs). Bansal and Shaliastovich (2011) present a model in which income and dividends are smooth but asset prices exhibit large moves. These jumps arise from rational learning by investors about an unobserved state.

[8] Our model with subjective beliefs about disaster risk and learning may also offer a more useful reference point for comparison with behavioral finance research on the consequences of investor sentiment and overreaction (cf. Barberis et al. (1998)) than the standard rational expectations benchmark.

market crash despite minimal changes in aggregate consumption. Similar to our approach, they consider learning about high versus low disaster intensity, but in their model jumps are in perceived dividends while the actual process is smooth. Benzoni et al. (2011) provide a numerical approximation to the solution for a given parameterization of their model, while we obtain an analytical solution to our model and derive price-dividend ratios explicitly as a function of investor beliefs. Furthermore, we distinguish between rational and adaptive learning and investigate the pricing implications.

Furthermore, we discovered an older yet unpublished study by Comon (2001) which also introduces rational Bayesian learning with extreme events.[9] Comon introduces parameter uncertainty regarding the hazard rate of rare dividend jumps in a variant of the Cox, Ingersoll and Ross (1985) exchange economy. Contrary to our approach, he assumes that prior subjective hazard rates of investors are Gamma distributed. One consequence of his framework is that learning only matters in influencing price-dividend ratios during the transition to rational expectations, which complicates empirical identification. On this we contribute two insights. First, we communicate that typical time-series models that investors may consider as data-generating processes of disasters have the property of exchangeability. We then refer to De Finetti's (1931, 1937, 1964) theorem on exchangeable 0-1 processes, which demonstrates that beliefs about the frequency of rare disasters should be beta-distributed with well-specified hyperparameters, as long as one assumes beta-distributed non-informative beliefs (i.e., beliefs before any observation is available). Björk and Johansson (1993) have demonstrated that De Finetti's (1931, 1937, 1964) theorem holds in continuous time, with beliefs about the frequency of rare disasters being gamma-distributed with well-specified hyperparameters. Second, we provide an exact solution for asset-pricing which demonstrates that

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[9] We are grateful to Pietro Veronesi for mentioning it in commenting on the first version of our paper and to Comon's adviser at Harvard, John Campbell, for scanning and sending us chapter 1 of his dissertation.

6

the asset-pricing mechanics. While the belief and asset-pricing mechanics of parameter learning are similar to these of state-verification learning, these mechanics disappear in the long run. For this reason, we think that a combination of both learning setups, the state-verification and the parameter-learning setup may be ideal in asset-pricing research that focuses on understanding weak investment after a rare disaster.

Finally, we calibrate our model and conduct dynamic simulations that illustrate the model's potential to capture key elements of the dynamic path of price-dividend ratios following the two crashes in the U.S. stock market in 2000 and 2008 shown in **Figure 1**. The calibration requires setting an initial prior belief on the average hazard rate of disasters. It turns out that it is possible to generate sudden drops followed by slowly improving P-D ratios under rational and adaptive learning. However, the adaptive learning simulation requires a prior belief that is roughly twice as optimistic as under rational learning. Since beliefs are the main driver of P-D dynamics in our model, it is of great interest to compare the behavior of model beliefs with survey data on investors' perception of the threat of a crash. Fortunately, such data is available in the form of Robert Shiller's Crash Confidence Index. The questionnaire underlying this data is explained in Shiller, Kon-Ya and Tsutsui (1996). Interestingly, our simulations of the boom and busts in the U.S. stock market are broadly consistent with dynamics of the beliefs indicated by the survey.

The remainder of the paper proceeds as follows. Section 2 presents the asset pricing model under rational expectations and demonstrates our solution approach. In Section 3, we then analyze and compare the decision making of households/investors that learn in an adaptive or rational Bayesian fashion. In the fourth section, closed-form solutions for asset prices are derived. To illustrate the power of the model to fit the behavior of price-dividend ratios and survey measures of beliefs following the last two crashes in the U.S. stock market,

we present dynamic simulations conditional on given priors and the timing of these two busts. An extension to Epstein-Zin-Weil preferences, which do not restrict the intertemporal elasticity of substitution to be equal to the inverse of the coefficient of relative risk aversion, is discussed in Section 5. It is meant to address concerns regarding the special nature of the standard asset-pricing model with constant relative risk aversion (CRRA) preferences (see, for example, Barro (2009) and Wachter (2013)). Section 6 concludes.

2. The Model

We use a simple representative-agent Lucas (1978) tree economy with disaster shocks hitting the dividend process in order to achieve two goals. First, we provide analytical results throughout the paper, which facilitate a clearer understanding of the model's mechanics in each setting of learning. Second, following the spirit of Barro (2006), we show that this simplest possible version of asset-pricing models can put the theoretical analysis in the correct ballpark of matching empirical observations. Such quantitative properties of simple models corroborate that the potential for key future extensions is open.

2.1 Stochastic structure

The dividend process is given by,

$$\frac{dD\left(t\right)}{D\left(t\right)} = \mu dt + \sigma dz\left(t\right) - \zeta\left(t\right)dN\left(t\right) \ , \tag{1}$$

in which $dz\left(t\right)$ is a standard Brownian motion, i.e., $dz\left(t\right) = \varepsilon\left(t\right)\sqrt{dt}$, with $\varepsilon\left(t\right) \sim N\left(0,1\right)$, for all $t \geq 0$. Moreover, $N\left(t\right)$ is a counting process driving random downward jumps in dividends of size $\zeta\left(t\right) \cdot D\left(t\right)$, where $\zeta\left(t\right) \in \left(0,1\right)$ is a random variable with given time-invariant distribution having compact support, $\mathcal{Z} \subset \left(0,1\right)$. The counting process $N\left(t\right)$ is

characterized by,

$$dN(t) = \begin{cases} 1 & \text{with Probability } \lambda(t)\,dt \\ 0 & \text{with Probability } 1 - \lambda(t)\,dt \end{cases}. \tag{2}$$

Following and extending Gabaix (2012) and Wachter (2013), a central feature of our analysis is that variable disaster risk is allowed. This disaster-risk variability means that $\lambda(t)$ in equation (2), is also random. In order to enable exact solutions for our model, we approximate any stochastic process driving $\lambda(t)$ by a two-point Markov chain in continuous time. Specifically, we assume that $\lambda(t)$ is allowed to take two values only, a high value, λ_h, and a low value, λ_l, with $\lambda_h > \lambda_l > 0$.[10] The Markov-chain transition probabilities are given by,

$$\Pr\left(\lambda(t+dt) = \lambda_h \mid \lambda(t) = \lambda_l\right) = \omega_{lh}dt\,, \text{ and } \Pr\left(\lambda(t+dt) = \lambda_l \mid \lambda(t) = \lambda_h\right) = \omega_{hl}dt\,, \tag{3}$$

in which $0 \le \omega_{lh}, \omega_{hl} \le 1$. We assume that this two-point Markov chain is a time-invariant data-generating process of $\lambda(t)$ governed by the continuous-time stochastic matrix

$$\Omega = \begin{bmatrix} -\omega_{hl} & \omega_{lh} \\ \omega_{hl} & -\omega_{lh} \end{bmatrix}, \tag{4}$$

that asymptotically converges to the (unconditional) binomial distribution,

$$\lambda(t) = \begin{cases} \lambda_h & \text{with Probability } \pi^* \\ \lambda_l & \text{with Probability } 1 - \pi^* \end{cases}, \tag{5}$$

for all $t \ge 0$, and with $\pi^* \in (0,1)$.[11] A particularly interesting case is this of a data-generating process for $\lambda(t)$ without autocorrelation, referring to a special matrix,

$$\bar{\Omega} \equiv \begin{bmatrix} -(1-\pi^*) & \pi^* \\ 1-\pi^* & -\pi^* \end{bmatrix}, \tag{6}$$

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[10]So our analysis nests specific stochastic processes that imply a persistent $\lambda(t)$, as, for example, in Wachter (2013, eq. 2, p. 990). Our Markov chain is more general at the negligible expense of the two-point approximation.

[11]Equation (5) means that $(\pi^*, 1 - \pi^*)$ is the normalized eigenvector corresponding to the 0-valued eigenvalue of matrix Ω.

in which with $\omega_{lh} = \pi^*$ and $\omega_{hl} = 1 - \pi^*$, i.e., the current state does not matter for shaping the probability of moving to another state.[12] Finally, *we assume that variables $z(t)$, $N(t)$, $\lambda(t)$, and $\zeta(t)$ are all independent from each other at all times.*

2.2   Utility and budget constraint

The Lucas-tree-fruit economy is an exchange economy of a large number of identical infinitely-lived agents of total mass equal to one. Agents trade only one risky asset, the market portfolio, which has returns given by equation (1). The representative agent maximizes expected lifetime utility given by,

$$E_0 \left[\int_0^\infty e^{-\rho t} \frac{c(t)^{1-\gamma} - 1}{1 - \gamma} dt \right] \tag{7}$$

with $c(t)$ denoting an individual's consumption, with $\rho > 0$ being the rate of time preference, and with $\gamma \geq 0$ being the coefficient of relative risk aversion (the special case of $\gamma = 0$ corresponds to a risk-neutral investor, and is possible to be studied in the context of a representative-agent Lucas-tree economy). With time-separable utility given by (7), coefficient γ is also equal to the inverse of the elasticity of intertemporal substitution. As in Gabaix (2012), for the main body of our analysis we use time-separable utility in order to obtain analytical results, but for the simpler case of adaptive learning we use Epstein-Zin-Weil (EZW) preferences based on the work of Epstein and Zin (1989), Weil (1990), and Epstein-Duffie (1992a,b).

At any time $t \geq 0$, an individual holds $s(t) \geq 0$ shares of the risky asset. At time $t = 0$, the aggregate supply of the asset is $S(0) > 0$, and there is no new issuing of shares, so $S(t) = S(0)$ for all $t \geq 0$. Moreover, at time $t = 0$, the endowment of a representative

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[12]Notice that the normalized eigenvector corresponding to the 0-valued eigenvalue of matrix $\bar{\Omega}$ is $(\pi^*, 1 - \pi^*)$. A previous version of this paper (see Koulovatianos and Wieland, 2011) has focused exclusively on this special case of a process governing $\lambda(t)$ without autocorrelation, assuming also that learners try to verify the state of $\lambda(t) \in \{\lambda_h, \lambda_l\}$ assuming that the data-generating process of $\lambda(t)$ is non-autocorrelated, driven by some (possibly unknown) matrix $\bar{\Omega}$.

individual is $s(0) = S(0)$. The budget constraint in continuous time is,[13]

$$ds(t) = \frac{1}{P(t)} \left[s(t) D(t) - c(t) \right] dt .$$

(8)

Our solution approach relies on Hamilton-Jacobi-Bellman (HJB) equations. HJB equations introduce a common solution technique through undetermined coefficients and a common recursive language that efficiently describes conceptual distinctions among asset-pricing problems in four environments: (a) rational expectations (RE), (b) rational learning on state verification (RLS), (c) rational learning on the data-generating process (RLP), and (d) adaptive learning on state verification (ALS).

3. Rational Expectations (RE)

There are two aspects of knowledge about the stochastic structure that are known under rational learning: (a) the data-generating processes of all risks, including the data-generating process of variable parameter $\lambda(t)$, are all known, and (b) the current state of disaster risk, $\lambda(t) \in \{\lambda_h, \lambda_l\}$, is also observed and known.

In order to solve the asset-pricing problem an individual agent must determine her demand for the risky asset at any time $t \geq 0$. So, given any possible path $(P(t))_{t \geq 0}$ generated by a price function with $P(t) > 0$ for all $t \geq 0$, the agent must pick the paths $(s(t), c(t))_{t \geq 0}$ that maximize her utility given by (7), subject to (8) and (1). Yet, the determination of these demand functions is a stationary discounted dynamic programming problem that can

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[13]For the derivation of equation (8) from its discrete-time counterpart, which is

$$P_t s_t = s_{t-1} (P_t + D_t) - c_t ,$$

notice that the above equation can be re-written as

$$\Delta s_t \equiv s_t - s_{t-1} = \left(s_{t-1} \frac{D_t}{P_t} - \frac{c_t}{P_t} \right) \Delta t ,$$

where $\Delta t = 1$ under the convention that the discrete-time period length is unity. In continuous time, taking the limit $\Delta t \to 0$ results in (8).

be solved through a recursive time-invariant functional-choice problem with a pricing rule,

$$P = \Psi^{RE}(D, \lambda) \ .$$

We denote a value function that is subject to the pricing rule $\Psi^{RE}(D, \lambda)$ by $J^{RE}\left(s, D, \lambda \mid \Psi^{RE}\right)$. The two possible states of λ are two, λ_h and λ_l. So, asset prices conditional on state λ_i with $i \in \{h, l\}$ are jointly determined by the pair of two HJB equations given by,

$$\rho J^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) = \max_{c \geq 0} \left\{ \frac{c^{1-\gamma} - 1}{1 - \gamma} + J_s^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) \cdot \left[\frac{1}{\Psi^{RE}(D, \lambda_i)}(sD - c) \right] + \right.$$

$$+ J_D^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) \cdot \mu D + J_{DD}^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) \frac{(\sigma D)^2}{2} +$$

$$+ \lambda_i \left\{ (1 - \omega_{ij}) E_\zeta \left[J^{RE}\left(s, (1 - \zeta) D, \lambda_i \mid \Psi^{RE}\right) \right] + \right.$$

$$\left. \left. + \omega_{ij} E_\zeta \left[J^{RE}\left(s, (1 - \zeta) D, \lambda_j \mid \Psi^{RE}\right) \right] - J^{RE}\left(s, D \mid \Psi^{RE}\right) \right\} \right\} \ , \quad i, j \in \{h, l\}, \ i \neq j, \quad (9)$$

with E_ζ denoting the expectations operator with respect to the random variable ζ only. In the Appendix we show that the price-dividend (P-D) ratio conditional on state λ_i with $i \in \{h, l\}$ is given by,

$$\frac{P}{D} = \frac{\Psi^{RE}(D, \lambda_i)}{D} =$$

$$= \frac{Q + \lambda_j \xi + (\lambda_j \omega_{ji} + \lambda_i \omega_{ij})(1 - \xi)}{(Q + \lambda_i \xi)(Q + \lambda_j \xi) + [(Q + \lambda_i \xi) \lambda_j \omega_{ji} + (Q + \lambda_j \xi) \lambda_i \omega_{ij}](1 - \xi)} \ , \quad (10)$$

in which

$$Q \equiv \rho - (1 - \gamma)\left(\mu - \gamma \frac{\sigma^2}{2}\right) \ , \quad \text{and} \quad \xi \equiv 1 - E_\zeta\left[(1 - \zeta)^{1-\gamma}\right] \ .$$

In the special case of $\gamma = 1$, $Q = \rho$ and $\xi = 0$, so the pricing function given by (10) implies $\Psi^{RE}(D, \lambda_i)/D = 1/\rho$, $i \in \{h, l\}$. This means that if $\gamma = 1$ the presence of risk does not affect pricing, no matter if this risk stems from the diffusion or from the jump process. For

$\gamma \neq 1$, notice that all terms in (10) are symmetric across states $i \in \{h,l\}$, except the term "$\lambda_j \xi$" in the numerator.

In general,

$$\xi \overset{\leq}{\underset{>}{\gtrless}} 0 \Leftrightarrow \gamma \overset{\geq}{\underset{<}{\lessgtr}} 1 . \tag{11}$$

The parametric relationship given by (11) implies that in the state of increased disaster risk, λ_h, the P-D ratio decreases prices only if $\gamma < 1$. If $\gamma > 1$, which can be loosely interpreted as having higher risk aversion, increased disaster risk implies higher P-D ratios. This paradoxical result has been discussed by Bansal and Yaron (2004, p. 1487), and also by Barro (2009, p. 249). Both of these studies attribute the paradox to the fact that, with power utility, the coefficient of relative risk aversion and the elasticity of intertemporal substitution cannot be disentangled, as the one equals the reciprocal of the other. As a resolution to this rigid feature of constant-relative-risk-aversion (CRRA) preferences, the studies by Bansal and Yaron (2004) and Barro (2009) suggest the use of Epstein-Zin (1989) and Weil (1990) utility functions. In our learning application, expected hazard rates will be moving over time together with beliefs, so a parameter-value choice $\gamma < 1$ vs. $\gamma > 1$ becomes important for some additional mechanics related to learning. To tackle such calibration concerns we discuss possible extensions to Duffie-Epstein (1992a,b) preferences in a later section. For the time being, we note that the model exhibits plausible mechanics if $\gamma < 1$.

4. Rational Learning for State Verification (RLS)

4.1 Characterization of Beliefs

Here we take only a small but still very influential step away from rational expectations. Our investors cannot observe which hazard rate (λ_h vs. λ_l) is triggered by nature at any point in time. An investor observes stock market crashes but does not have data on histories of

disaster-risk realizations $\lambda(t)$ (dates and number of instances in which λ_h vs. λ_l have been triggered in the past). So state verification takes place through the history of past disaster events. This is a standard filtering problem.[14]

In a finite-state-verification problem a rational learner assigns subjective probabilities to each state, expressing beliefs on the likelihood of being at a particular state. In our two-state verification problem, such subjective beliefs are given by the time-specific Bernoulli distribution

$$\tilde{\lambda}(t) = \begin{cases} \lambda_h & \text{with Probability } \pi(t) \\ \\ \lambda_l & \text{with Probability } 1 - \pi(t) \end{cases}, \tag{12}$$

in which the tilde denotes random variables governed by distributions that depend on subjective beliefs. Based on (12), the ex-ant subjective perception of process $N(t)$ by the investor at time $t \geq 0$ is,

$$d\tilde{N}(t) = \begin{cases} 1 & \text{with Probability } \Lambda(\pi(t))\, dt \\ \\ 0 & \text{with Probability } 1 - \Lambda(\pi(t))\, dt \end{cases}, \tag{13}$$

in which

$$\Lambda(\pi(t)) \equiv \lambda_h \pi(t) + \lambda_l [1 - \pi(t)] \ , \tag{14}$$

i.e., the perceived disaster risk equals the expected hazard rate according to priors $\pi(t)$, $E_\pi\left(\tilde{\lambda}(t)\right) = \Lambda(\pi(t))$. Proposition 1 characterizes the dynamics of these subjective beliefs.

Proposition 1 *If learners are aware of the transition probabilities $0 \leq \omega_{lh}, \omega_{hl} \leq 1$ given by (3), then Bayesian updating implies that the dynamics of ex-ante perceived beliefs π about the current state of disaster risk $\lambda(t) \in \{\lambda_h, \lambda_l\}$ (denoted by $d\tilde{\pi}$) are governed by,*

$$d\tilde{\pi}(t) = \{-\delta\pi(t)[1 - \pi(t)] + \omega_{lh}[1 - \pi(t)] - \omega_{hl}\pi(t)\}\, dt +$$

---

[14]See, for example, Liptser and Shiryaev (2001).

$$+ \left[\frac{\lambda_h \pi(t)}{\Lambda(\pi(t))} - \pi(t) \right] d\tilde{N}(t) \ , \qquad (15)$$

in which $\delta \equiv (\lambda_h - \lambda_l)$. If learners assume that the data generating process of $\lambda(t)$ is non-autocorrelated and is based on a (not necessarily known) Markov transition matrix with the same structure as $\bar{\Omega}$ in equation (6), then the dynamics of ex-ante perceived beliefs π about the current state of disaster risk $\lambda(t)$ are governed by,

$$d\tilde{\pi}(t) = -\delta \pi(t) [1 - \pi(t)] dt + \left[\frac{\lambda_h \pi(t)}{\Lambda(\pi(t))} - \pi(t) \right] d\tilde{N}(t) \ . \qquad (16)$$

The jump process $d\tilde{N}(t)$ in both (15) and (16) is given by (13).

Proof This is a direct application of Liptser and Shiryaev (2001, Theorem 19.6, p. 332), after following the arguments outlined in the proofs of Liptser and Shiryaev (2001, Examples 1-3, pp. 333-5). □

Equation (16) corresponds to equation (19.86) in Liptser and Shiryaev (2001, p. 333).[15] Equation (15) has also been used by Benzoni et al. (2011, eq. 8, p. 556).

4.1.1 Belief dynamics

Both equations (15) and (16) imply that whenever a rare disaster occurs subjective beliefs, $\pi(t)$, jump upward, towards pessimism. This is evident by the fact that the term $[\lambda_h / \Lambda(\pi) - 1] \pi$ that multiplies the jump process of dividends, $d\tilde{N}$, is strictly positive. During times that there are no disasters the negatively-valued drift terms in both (15) and (16) imply a smooth transition to optimism. These belief dynamics play a central role in explaining why investment is so persistently weak after a rare disaster.

---

[15]A similar characterization of beliefs has been used by Keller and Rady (2010). They study a game-theoretical application of learning with Poisson differential equations (Poisson bandits) analytically.

4.2 Asset Pricing by Rational Learners Verifying States (RLS)

We are interested in analyzing both state-verification-learning setups indicated by Proposition 1: one in which learners assume that the data generating process of $\lambda(t)$ is governed by some matrix Ω as in (4), exhibiting autocorrelation, and the other case in which learners assume a data-generating process without autocorrelation. A mechanistic way of joining equations (15) and (16) together can be achieved through an indicator function I^A with,

$$
I^A = \begin{cases} 1 & \text{if the data-generating process of } \lambda(t) \text{ has autocorrelation} \\ 0 & \text{else} \end{cases} .
$$

Combining I^A with equations (15) and (16) we obtain

$$
d\tilde{\pi} = \left[-\delta\pi(1-\pi) + I^A\omega_{lh}(1-\pi) - I^A\omega_{hl}\pi \right] dt + \left[\frac{\lambda_h\pi}{\Lambda(\pi)} - \pi \right] d\tilde{N} . \tag{17}
$$

Introducing equation (17) into the HJB of a rational learner who verifies states (denoted by RLS), we can cover the two cases of data-generating processes indicated by I^A. Specifically, the HJB equation of RLS is,

$$
\rho J^{RLS}\left(s, D, \pi \mid \Psi^{RLS}\right) = \max_{c \geq 0} \left\{ \frac{c^{1-\gamma}-1}{1-\gamma} + J_s^{RLS}\left(s, D, \pi \mid \Psi^{RLS}\right) \cdot \left[\frac{1}{\Psi^{RLS}(D,\pi)}(sD-c) \right] + \right.
$$

$$
+ J_D^{RLS}\left(s, D, \pi \mid \Psi^{RL}\right) \cdot \mu D + J_{DD}^{RLS}\left(s, D, \pi \mid \Psi^{RLS}\right) \frac{(\sigma D)^2}{2} +
$$

$$
+ J_\pi^{RLS}\left(s, D, \pi \mid \Psi^{RLS}\right) \cdot \left[-\delta\pi(1-\pi) + I^A\omega_{lh}(1-\pi) - I^A\omega_{hl}\pi \right] +
$$

$$
\left. + \Lambda(\pi) \left\{ E_\zeta \left[J^{RLS}\left(s, (1-\zeta)D, \frac{\lambda_h\pi}{\Lambda(\pi)} \mid \Psi^{RLS}\right) \right] - J^{RLS}\left(s, D, \pi \mid \Psi^{RLS}\right) \right\} \right\} ,
$$

$$
\tag{18}
$$

given a pricing rule $P = \Psi^{RLS}(D, \pi)$, and while the dynamics of π are driven by equation (17).

In the Appendix we prove that the P-D ratio implied by the RLS problem is.

$$
\frac{P}{D} = \frac{\Psi^{RLS}(D,\pi)}{D} = \frac{\pi(Q+\lambda_l\xi) + (1-\pi)(Q+\lambda_h\xi) + I^A \cdot (\omega_{hl}+\omega_{lh})}{(Q+\lambda_h\xi)(Q+\lambda_l\xi) + I^A \cdot [\omega_{hl}(Q+\lambda_l\xi) + \omega_{lh}(Q+\lambda_h\xi)]} . \tag{19}
$$

The case of a non-autocorrelated data-generated process for $\lambda(t)$ gives a remarkably simple formula. After setting $I^A = 0$ in (19) we obtain,

$$\frac{P}{D} = \frac{\Psi^{RLS}(D, \pi)}{D} = \left[\pi \frac{1}{Q + \lambda_h \xi} + (1 - \pi) \frac{1}{Q + \lambda_l \xi}\right] . \tag{20}$$

It is also notable that in the case $\gamma = 1$, the P-D ratio is equal to $1/\rho$, as was the case under rational expectations. Logarithmic utility seems to balance out any conflicting risk considerations intertemporally, so the effects of incomplete information about risk also seem to be balanced out in asset pricing. As in the rational-expectations case, we focus on examining the special case of $\gamma < 1$.

4.3 Mechanics of RLS

The key message from equations (19) and (20) is that as π jumps upward with every disaster event and then slowly drops during periods without disasters, with $\gamma < 1$, the P-D ratio also jumps downward and abruptly after a disaster, while it slowly rebounds during periods without disasters. Unlike the rational-expectations environment in which often weak investment can be attributed to changing expectations without observing any disasters, in the rational-learning case weak investment follows a disaster. This synchronization between disaster events and pessimistic spells under rational learning can explain why stock prices can remain low for long periods after a disaster. In our toy calibration exercise below we examine whether formulas as simple as (20) have the potential of capturing observed stock-market dynamics. Before that exercise we examine rational learning about the data-generating process of $\lambda(t)$.

5. Rational Learning About the Data-Generating Process (RLP)

In the setting of rational learning with state verification (RLS) we have assumed that an investor has at least partial knowledge about the structure of the data-generating process, but cannot observe the states $\lambda(t) \in \{\lambda_h, \lambda_l\}$. Here, apart from assuming again that the observation of the state $\lambda(t)$ is again impossible, we also relax the assumption that investors know the structure of the data-generating process. *We only make a loose assumption, that all learners assume that the data-generating process of $\lambda(t)$ is exchangeable.*[16]

5.1 Theoretical Underpinnings Regarding Exchangeability: De Finetti's Theorem

We have assumed that nature's data-generating process is some Markov chain with transition probability matrices given either by (4) or (6). Our additional assumption that all learners believe that the data-generating process is loosely related to nature's stochastic structure since weakly convergent Markov chains are exchangeable processes.[17] Nevertheless, a very broad class of time-series processes generates exchangeable realizations, so investors may have truly different data-generating processes in mind.

The weak assumption of exchangeability has quite precise implications about how RLP agents form beliefs. De Finetti's (1931, 1937, 1964) theorem states that any infinite sequence of exchangeable "0-1" random variables is a unique mixture of independent Bernoulli

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[16]Based on Heath and Sudderth (1976, p. 188),

> **Definition of exchangeability in discrete time:** *The 0-1 random variables $q_1, ..., q_n$ are exchangeable if the $n!$ permutations $\left(q_{\pi(1)}, ..., q_{\pi(n)}\right)$ have the same n-dimensional probability distribution. The variables of an infinite sequence, $\{q_n\}_{n=0}^{\infty}$, are exchangeable if $q_1, ..., q_m$ are exchangeable for each m.*

[17]The proof of the exchangeability of Markov chains is intuitive, considering that future outcomes generated by Markov chains are conditionally dependent on the current state. For a general treatment that immediately implies this result see Chow and Teicher (1988, p. 222). See also Diaconis and Fredman (1980) for more results regarding exchangeability and Markov chains.

sequences.[18] This means that if we want to learn the probability that parameter λ takes specific values based on prior disaster observations in a discrete-time setting, our priors can be quite specific: every investor with the same beta-distributed non-informative priors would have the same beta beliefs with the same hyperparameters (common priors). In fact, even if non-informative priors are heterogeneous, after some disasters (about three disaster episodes), the non-informative priors become dominated by the beta-distributed informative priors.

Importantly for our analysis, Björk and Johansson (1993) show that De Finetti's (1931, 1937, 1964) theorem also holds in continuous time. Specifically, if sampling is infinite and the data-generating '0-1' process produces sequences of exchangeable events, then beliefs about λ based on information $\mathcal{I}_t$ by time t are given by,

$$\Pr\left(\lambda \mid \mathcal{I}_t, p_0\left(\cdot\right)\right) \propto \underbrace{\hat{T}\left(t\right) e^{-\hat{T}(t)\lambda} \left[\hat{T}\left(t\right)\lambda\right]^{\hat{N}(t)-1}}_{\textbf{data-driven beliefs}} \cdot \underbrace{p_0\left(\lambda\right)}_{\textbf{noninformative priors}}$$

in which $\hat{N}\left(t\right)$ is the cumulative sum of disasters since sampling started and $\hat{T}\left(t\right) = T\left(0\right) + t$ is the elapsed sampling time since sampling started at time $T\left(0\right)$. For example, with non-informative priors distributed as $Gamma\left(\rho_0, 0\right)$, which are "agnostic" since the second hyperparameter is set equal to zero, beliefs are given in the section below. As we have

---

[18]Based on Heath and Sudderth (1976, p. 189), a statement of de Finetti's theorem in discrete time is,

Theorem (de Finetti, taken from Heath and Sudderth, 1976, p. 189) *To any infinite sequence of exchangeable random variables, $\{q_n\}_{n=0}^{\infty}$, having values in $\{0,1\}$, there corresponds a probability density $p_0\left(\cdot\right)$ concentrated on the interval $[0,1]$, such that,*

$$\Pr\left(q_1 = 1, ..., q_{\hat{N}_t} = 1, q_{\hat{N}_t+1} = 0, ..., q_{\hat{T}_t} = 0\right)$$

$$= \int_0^1 \underbrace{\theta^{\hat{N}_t-1}\left(1-\theta\right)^{\hat{T}_t-\hat{N}_t-1}}_{\text{informative priors}} \cdot \underbrace{p_0\left(\theta\right)}_{\text{noninformative priors}} \, d\theta \, , \qquad (21)$$

for all $\hat{T}_t$, and all $0 \leq \hat{N}_t \leq \hat{T}_t$, with $\hat{N}_t$ being the cumulative count of past jumps up to period t, $\hat{N}_t = \hat{N}_0 + \sum_{i=0}^{t} q_i$, and with $\hat{T}_t = \hat{T}_0 + t$ being the elapsed sampling time up to period t.

assumed common priors for all investors in the case of RLS, we also assume common non-informative priors here.[19]

5.2   Posterior beliefs, posterior-belief moments, and dynamics

From this section and on we view the elapsed sampling time as a state variable, for convenience in dynamic-programming applications. We denote the length of the elapsed sampling time by $T(t)$ for all $t \geq 0$. In addition, we assume that $T(0) > 0$ and $N(0) \geq 1$. Theorem 1 implies that the posterior distribution of the virtual econometrician's beliefs at any time $t \geq 0$, is given by,

$$
\Pr(\lambda \mid \mathcal{F}_t) = f_{\left(N(t) \,,\, [T(0)+t]^{-1}\right)}(\lambda) = \begin{cases} [T(0)+t]\, e^{-[T(0)+t]\lambda}\, \dfrac{\{[T(0)+t]\lambda\}^{N(t)-1}}{\Gamma(N(t))} & , \text{ if } \lambda \geq 0 \\ 0 & , \text{ if } \lambda < 0 \end{cases},
$$
(22)

in which $\Gamma(a) \equiv \int_0^\infty e^{-v} v^{a-1} dv$ is the Gamma function, and $\mathcal{F}_t$ is the filtration at time $t \geq 0$. Equation (22) demonstrates the well-known result that the posterior distribution of a Gamma prior is also Gamma.[20] Based on standard results about the moments of Gamma-distributed variables, the mean and variance of the posterior distribution for all $t \geq 0$ are,[21]

$$
E\left[\tilde{\lambda}(t) \mid N(t) = N(0) + n\right] = \frac{N(0)+n}{T(0)+t},
$$
(23)

and

$$
Var\left[\tilde{\lambda}(t) \mid N(t) = N(0) + n\right] = \frac{N(0)+n}{[T(0)+t]^2},
$$
(24)

for all $n \in \{0, 1, ...\}$.

---

[19]Comon (2001) has assumed gamma-distributed priors without pointing out the theoretical background and learning intuition that stems from de Finetti's theorem.

[20]See, for example, Gelman et al. (2004, p. 53).

[21]See, for example, Papoulis and Pillai (2002, p. 154) for the moment-generating function of the Gamma distribution.

In the context of optimization through HJB equations, the key result to use is the mean jump-frequency belief. Specifically, learners' perceived counting process $\tilde{N}(t)$ is given by,

$$d\tilde{N}(t) = \begin{cases} 1 & \text{with Probability } \frac{N(t)}{T}dt \\ 0 & \text{with Probability } 1 - \frac{N(t)}{T}dt \end{cases}. \tag{25}$$

In the related formula given by (23), the denominator is a continuously and linearly-growing variable, while the numerator is a discrete point process. The point process in the numerator means that once a jump occurs, average jump frequency beliefs jump upwards to a pessimistic level. After a period without further busts, average jump frequency beliefs decay, implying that the learning agents become more optimistic. In brief, the trajectory of average jump-frequency beliefs will exhibit spikes which coincide with the occurrence of busts. Qualitatively, the belief dynamics of $E\left(\tilde{\lambda}(t)\right)$ under RLP, are similar to the belief dynamics of $\pi(t)$ under RLS.

5.3 Asset Pricing by Rational Learners of Data-Generating Processes (RLP)

Given a history of disaster observations characterized by the pair (N, T), for some state-space represented pricing rule, $P = \Psi^{RLP}(D, N, T)$, the Hamilton-Jacobi-Bellman (HJB) equation is,

$$\rho J^{RLP}(s, D, N, T) = \max_{c \geq 0} \left\{ \frac{c^{1-\gamma} - 1}{1 - \gamma} + J_s^{RLP}(s, D, N, T) \frac{1}{\Psi^{RLP}(D, N, T)}(sD - c) \right.$$

$$+ J_D^{RLP}(s, D, N, T) \mu D + J_{DD}^{RLP}(s, D, N, T) \frac{(\sigma D)^2}{2} + J_T^{RLP}(s, D, N, T)$$

$$\left. + \frac{N}{T} E_\zeta \left[J^{RLP}(s, D(1 - \zeta), N + 1, T) \right] - J^{RLP}(s, D, N, T) \right\}. \tag{26}$$

In the Appendix we prove that the P-D ratio based on $\Psi^{RLP}(D, N, T)$ is given by,

$$\frac{P}{D} = \frac{\Psi^{RLP}(D, N, T)}{D} = \frac{1}{Q}\left(\frac{Q}{\xi}T\right)^N e^{\frac{Q}{\xi}T}\Gamma\left(1 - N, \frac{Q}{\xi}T\right) , \qquad (27)$$

in which $\Gamma(a, b) \equiv \int_b^\infty e^{-v}v^{a-1}dv$ is the incomplete gamma function.

5.4   Mechanics of RLP

While equation (27) is an exact solution of the P-D ratio, it is not as tractable as the other closed-form solutions we have achieved for P-D ratios so far. Demonstrating that the mechanics of the P-D ratio in the RLP case are similar to those of the RLS case using analytical methods is not straightforward. The P-D ratio given by equation (27) needs computation, so, in a later section where we numerically compare alternative learning setups we provide numerical results for equation (27). The general finding is that, qualitatively, the P-D ratio in the RLP case works in exactly the same way as the P-D ratio in the RLS case with $\gamma < 1$: after a rare disaster the P-D ratio jumps downward, and then slowly recovers after a long period without disaster episodes.

6.   Long-Run Dynamics: Comparing RLS and RLP

Muth (1961) has suggested the concept of asymptotic convergence to rational expectations. We prove that neither under RLS nor under RLP convergence to rational expectations takes place.

6.1   Long-run Dynamics in the case of RLS

Proposition 2 provides a characterization of long-run dynamics of subjective beliefs, as these are captured by π. The central message of Proposition 2 is that the limiting distribution of beliefs under RLS has non-zero variance. We focus on the case of non-autocorrelated

riskiness regimes, i.e., the special case in which the dynamics of subjective beliefs, π, are governed by (16).

Proposition 2 *Let beliefs about the occurrence of a jump event be given by equation (16). Then for all $\pi(0) = \pi_0 \in (0,1)$, $\pi(t) \in (0,1)$ for all $t \geq 0$, and*

$$\lim_{t \to \infty} E\left[\pi(t)\right] = \pi^*, \tag{28}$$

while

$$\lim_{t \to \infty} Var\left[\pi(t)\right] = \frac{\left\{\lambda^* - 4\lambda_l + \left[(\lambda^*)^2 + 8\lambda^*\lambda_h\right]^{\frac{1}{2}}\right\}^2}{16\delta^2} - (\pi^*)^2 > 0 \ . \tag{29}$$

Proof See the Appendix. □

Proposition 2 states that, even without persistence in transitory belief shocks ($\omega_{lh} = \omega_{hl} = 0$), after collecting infinite rare-disaster data drawn from nature's realizations, the beliefs of learning investors about π are asymptotically unbiased, but learners do not reach infinite precision about this limiting average parameter. The variance of belief parameter $\pi(t)$ is bounded away from 0, as indicated by equation (29). A direct implication from equation (20) is that the P-D ratios should also exhibit a non-zero asymptotic variance. We are confident that this qualitative result for non-autocorrelated riskiness regimes carries through to the case of serial autocorrelation in the riskiness regimes (governed by equation (15)), too.[22]

Empirically capturing priors on belief parameter $\pi(t)$ alone, is sufficient to describe the implied dynamics of learning in our model. In a section appearing below we use survey data collected through a questionnaire described in Shiller et al. (1996) that approximate belief parameter $\pi(t)$ in order to calibrate our model.

---

[22]The proof of Proposition 2 is involved and we do not think it is necessary to extend it to the case of dynamics governed by equation (15), since this is also verifiable numerically, e.g., through performing a Monte-Carlo simulation.

6.2 Long-run Dynamics in the case of RLP

In the Appendix we prove two interesting results. First, that the asymptotic distribution of beliefs about λ is degenerate, converging to $\lambda^* = \pi^* \lambda_h + (1 - \pi^*)$. Second, that the P-D ratio given by equation (27) converges to,

$$\lim_{t \to \infty} \frac{\Psi^{RLP}(D, N, T)}{D} = \frac{1}{Q + \lambda^* \xi} \ . \tag{30}$$

6.3 Summary of comparison between RLS and RLP

Under both RLS and RLP, the asymptotic mean of the distributions is learned, but state verification cannot be achieved in neither of the two learning setups in an environment of variable disaster risk. Interestingly RLP converges to degenerate beliefs, and to a constant P-D ratio, a limit that leads to large deviations from understanding some real-world process with latent variable-disaster risk. On the contrary, RLS leads to an asymptotic behavior according to which the occurrence of disasters still shifts investor priors. This limiting behavior of investors under RLS potentially leads to fewer perception mistakes, especially if disaster risk has high positive autocorrelation (persistent regimes of riskiness). This potentially higher performance of RLS in comparison with RLP is intuitive: RLP is an environment with less information compared to that of RLS.

7.   Transitional Dynamics of Learning

We perform simple simulations of the RLS model using survey data on disaster expectations. In addition we compute P-D ratios under the RLP setting. In order to better understand the RLS framework we compare our results with asset prices by adaptive learning under state verification (denoted by "ALS").The ALS framework corresponds to "anticipated utility" in

Cogley and Sargent (2008).[23] Given the attention the literature gives to the ALS framework we contrast it in the simulations.

In the following we use the continuous-time formulation and parameterization of recursive "Epstein-Zin-Weil" preferences, suggested by Duffie and Epstein (1992a,b) to specify the AL investor's utility, namely,

$$J^{ALS}\left(s\left(t\right),D\left(t\right),\pi\left(t\right)\mid\tilde{\Psi}^{ALS}\right)=\int_{t}^{\infty}f\left(c\left(\tau\right)\ ,\ J\left(s\left(\tau\right),D\left(\tau\right),\pi\left(\tau\right)\mid\tilde{\Psi}^{ALS}\right)\right)d\tau\ ,\tag{31}$$

with $f\left(c,J\right)$ being a normalized aggregator of continuation utility, J, and current consumption, c, with

$$f\left(c,J\right)\equiv\rho\left(1-\gamma\right)\cdot J\cdot\frac{\left\{\frac{c}{\left[\left(1-\gamma\right)J\right]^{\frac{1}{1-\gamma}}}\right\}^{1-\frac{1}{\eta}}-1}{1-\frac{1}{\eta}}\ ,\tag{32}$$

where $\eta>0$ denotes the investor's elasticity of intertemporal substitution, while $\gamma>0$ is the coefficient of relative risk aversion. Moreover, $\tilde{\Psi}^{AL}\left(D,\pi\right)$ denotes the pricing rule under Epstein-Zin-Weil preferences. Using a HJB solution approach, we show in the Appendix that,

$$\frac{\tilde{\Psi}^{ALS}\left(D,\pi\right)}{D}=\frac{1}{\rho-\left(1-\frac{1}{\eta}\right)\left(\mu-\gamma\frac{\sigma^{2}}{2}\right)+\Lambda\left(\pi\right)\frac{1-\frac{1}{\eta}}{1-\gamma}\left\{1-E_{\zeta}\left(1-\zeta\right)^{1-\gamma}\right\}}=$$

$$=\frac{1}{\rho-\frac{1-\frac{1}{\eta}}{1-\gamma}\left[\chi-\Lambda\left(\pi\right)\xi\right]}\ ,\tag{33}$$

in which $\chi=\left(1-\gamma\right)\left(\mu-\gamma\sigma^{2}/2\right)$. The special case of $\gamma=1/\eta$, corresponds to

$$\frac{\tilde{\Psi}^{ALS}\left(D,\pi\right)}{D}=\frac{1}{Q+\Lambda\left(\pi\right)\xi}\ .\tag{34}$$

---

[23]The conceptual distinction between an adaptive learner (AL) and a rational learner (RL) is based on how the decision maker accounts for her own ignorance regarding π^{*}. AL is aware of her ignorance at the current time instant. However, she simply assumes that her beliefs will not change in the future. By contrast, RL, apart from being aware of her ignorance at the current time, is also aware of the future evolution of her beliefs according to Bayes' rule. So, RL approaches her lack of information in a fully rational manner, while AL is boundedly rational, because she neglects the knowledge that beliefs will be revised in the future with new data.

Pricing implied by (34) above, since this is the case in which Epstein-Zin-Weil preferences collapse to standard time-separable preferences with constant relative-risk aversion.

7.1 Comparing dynamic model simulations to data on P-D ratios and survey expectations

This section serves two purposes. First, it uses dynamic simulations of a calibrated version of our asset pricing model to provide a simple visual illustration of the interaction of disaster risk, subjective beliefs of investors and price-dividend ratios. This illustration helps improve our understanding of the analytical results discussed in the preceding sections. Secondly, this section examines similarities between actual data on price-dividend ratios and survey expectations after a stock-market crash and such model simulations. In particular, we investigate whether the dynamics of subjective beliefs regarding disaster probabilities can cause P-D ratio drops and persistence similar to the U.S. stock market experience in the last two decades. We also check whether these belief dynamics remain roughly within the range of belief variations apparent in surveys of the perceived threat of such a crash. Such a comparison may help to motivate a thorough empirical investigation of the role of subjective beliefs relative to fundamentals in future research.

The U.S. data on P-D-ratios from **Figure 1**, is plotted again in the two top panels of **Figure 2** (dashed lines). The dashed lines in the two bottom panels of **Figure 2** represent survey-based beliefs about the likelihood of an imminent stock-market crash in the United States, produced using a survey method described in Shiller et al. (1996). Shiller's Crash Confidence Index (CCI) refers to the percentage of the respondents who stated that the probability of a stock-market crash occurring within the following semester is less than 10%.[24] So, the higher the CCI, the higher the optimism (more accurately, the higher the

---

[24]Data are taken from the website
   http://icf.som.yale.edu/stock-market-confidence-indices-united-states

fraction of non-pessimistic survey respondents). At the time when the two incidents of the massive drops in dividends occurred, the Crash Confidence Index (CCI) was at its lowest level. Most interestingly, after the November 2008 crash, the CCI continued dropping for almost a year. Stock-market prices broadly seemed to follow the change in beliefs and declined substantially.[25]

The spirit of our exercise is to initiate a model simulation in the second semester of the year 1989 setting a level of initial beliefs for agent RLS, π_{1989}^{RL}, which is close to values from CCI data. Then we impose two unforeseen jumps, one in the summer of year 2000, and one in the fall of 2008. To be able to simulate the model, we also need to calibrate a number of other parameters. In doing so, we choose values close to those used by Barro (2006) for explaining the equity premium puzzle. Investors' preference parameters are set to $\gamma = 0.2$ and $\rho = 2.5\%$. The parameters of the diffusion process with drift are set to $\mu = 2.62\%$ and $\sigma = 2\%$.[26] Regarding the magnitude of the impact of a disaster on dividends, ζ, we use a generic distribution, in which $\zeta = 20\%$ with probability 1, if a disaster occurs.

The hazard rates determining disaster risk are set to $\lambda_h = 1/5$, and $\lambda_l = 1/40$. $\lambda_h = 1/5$ implies an upper bound for pessimism, namely that sudden drops in the dividend process of magnitude 20% arrive once every five years on average. The most optimistic view, determined by $\lambda_l = 1/40$, is that such jumps arrive once every forty years on average. These values for λ_h and λ_l are not far from hazard rates motivated by rare-disaster data presented in Barro (2006, 2009). Moreover, the choice of upper bound provides a natural link to the Crash

---

[25]An alternative approach to measuring confidence using the cross-section of quarterly real GDP forecasts from the survey of professional forecasters is presented in Bansal and Shaliastovich (2010). They provide evidence that confidence and returns are negatively correlated and develop a model with jump-like confidence shocks and recency-biased learning.

[26]Barro (2006) uses similar values in order to match historical data on consumption growth and volatility. An alternative calibration that would match growth and volatility data of dividends during the examined period would be: $\mu = 9\%$, $\sigma = 10\%$ and, $\rho = 7.4\%$. It would imply the same dynamics, because the magnitude of the expression $\rho - \chi$ is the same.

Confidence Index. Since the hazard rate reflects the average rate of disasters per year, the value $\lambda_h = 1/5$ implies the perceived probability that a disaster may occur with probability 10% each semester. Since $\pi(t)$ is the probability placed on such an event, the value $1 - \pi(t)$ can serve as a proxy for the CCI index, viewed as the percentage of respondents who think that there is a lower than 10% probability of disaster (i.e., $1 - \pi(t)$ is interpreted as percentage of respondents who think that the hazard rate is lower than $\lambda_h = 1/5$)).

The two bottom panels of **Figure 2** plot the dynamics of $1 - \pi(t)$ (solid lines), under rational (bottom-left panel) and adaptive learning (bottom-right panel) that follow from equation (16) relative to the CCI index (dashed lines). We have assumed no autocorrelation in the underlying riskiness regimes, which makes the exercise potentially more demanding. The P-D ratio formulas that we use are (20) for RLS and (34) for ALS. The prior belief regarding the average hazard rate is to 85 percent for the RLS and ALS investors (all with time-separable preferences), that is $\pi_{1989}^{RL} = \pi_{1989}^{AL} = 79\%$. Thus, $1 - \pi_{1989} = 21\%$, which could be compared with a CCI index value indicating a 21% share of optimistic respondents. Since the dynamics of beliefs, $\pi(t)$, are driven by the same equation, and the disaster data is the same (namely, two crashes in 2000 and 2008 respectively), the evolution of beliefs is the same for both types of investors. These beliefs exhibit variations in the same range as the CCI index in the last two decades, namely between 20 to 50 percent. There is a gradual increase in optimism prior to the crashes in 2000 and 2008. A crash causes a drop to pessimistic levels that persists and is followed by a slow improvement. There are some differences and some similarities with the movements of the CCI index. This index did not rise so much before 2000. However, there is a local minimum around 2000, which is then followed by a slow improvement to optimistic heights prior to the global financial crisis. Then it rapidly declines reaching a minimum around the Lehman collapse, followed by another improvement.

The resulting dynamics of the price-dividend ratio are shown in the top panels of **Figure 2**. Though they share the same beliefs, asset demand by rational and adaptive learners is different and therefore also the evolution of price dividend ratios. As apparent in the top-left panel, RLS investors anticipate more optimistic perceptions in the absence of another crisis and value the risky asset more highly. The simulation under rational learning thereby exhibits a continuing increase in the price-dividend ratio throughout the 1990s that almost reaches the observed U.S. stock market peak prior to the crash in 2000. The simulated P-D ratio then slowly rises from this depressed level to a lower peak followed by the rash in 2008. The comparison with the actual U.S. P-D data serves to illustrate that variations in subjective beliefs may well be capable of causing such dramatic movements.

Under adaptive learning (top right-hand panel) the movements in the P-D ratio are much smaller. This observation is fully consistent with Corollary 2 in Section 4.4, given that the preference parameter γ is set to a value below unity. ALS investors act as if their beliefs will remain unchanged in the future. On balance they value the risky asset less than the RLS investors. Thus, the P-D ratio remains substantially smaller under adaptive learning and its dynamics less pronounced. However, this simulation is not meant to propose that the assumption of adaptive learning is necessarily inconsistent with observed behavior the P-D ratio in the U.S. stock market. It is possible to change the calibration so as to achieve more pronounced movements in the P-D ratio under adaptive learning. For example, a more optimistic prior would result in higher valuations of the risky asset from the ALS investors' perspective. As shown in the dynamic simulation reported in the right-hand-side panels of **Figure 3** (solid lines), an initial prior of $\pi_{1989}^{AL} = 47\%$, is sufficient to generate more dramatic rises and falls in the P-D ratio over time. This prior implies a level of optimism, $1 - \pi_{1989}^{AL} = 53\%$, that is more than double the value used in **Figure 2** and above the CCI

data of that period.

Clearly, these simulations indicate that subjective belief dynamics can play an important role in understanding P-D ratios after stock market crashes. A thorough empirical investigation should be the subject of a future study. Before closing, however, we want to address a possibly important concern with regard to the theoretical specification of preferences we have used. Barro (2009) and others have suggested that asset-pricing models with CRRA preferences have difficulty matching certain empirical regularities because they restrict the intertemporal elasticity of substitution to be equal to the inverse of the coefficient of relative risk aversion. Instead, Epstein-Zin (1989) and Weil (1990) preferences allow to differentiate between risk aversion and the elasticity of substitution.

Finally, **Figure 4** plots the computed P-D ratio based on equation (27). In the Online Appendix we show how this P-D ratio can be computed, based on both the formula of equation (27) and on HJB recursions which can numerically tackle Epstein-Zin-Weil preferences. **Figure 4** uses the benchmark calibrating parameters employed in Figures 1 and 2. Clearly, **Figure 4** demonstrates the same qualitative mechanics as in the RLS and ALS cases: the P-D ratio is decreasing in N, which implies that P-D ratios jump down every time that a disaster occurs, and also the P-D ratio is increasing in T, which means that P-D ratios slowly rebound after a long period of no disasters. Nevertheless, the P-D ratios in **Figure 4** seem too high. **Figure 5** shows P-D ratios in the case of setting the rate of time preference ρ to 3%. This increase in ρ brings the P-D ratio to the ballpark indicated by the data.

8. Conclusion

Recent research on rational-expectations asset-pricing models focuses on proposing variability in disaster risk as an explanation for several asset pricing puzzles and, in particular,

for excessively volatile price-dividend (P-D) ratios (see, for example, Gabaix (2008, 2011), Wachter (2013), and Gourio (2008) and Barro et al. (2010)). Another line of research focuses on subjective beliefs and learning by investors and questions the assumption of knowledge of objective frequency distributions of disasters (for example, Weitzman (2007)). We have developed an asset pricing model with time variable disaster risk and Bayesian learning by imperfectly informed investors. We have also shown that this model helps understand episodes in which P-D ratios drop both rapidly and massively, at times intimately connected with jumps in the dividend process (e.g., see Figure 1). Such observations have also motivated research on bounded rationality and investor sentiment (see, for example, Barberis et al. (1998)). Instead of following such a research approach, here, we have proposed a theory that does not require relaxing rationality. Our analysis has only assumed limited information, i.e. we have relaxed that investors know everything about the structure of disaster-risk variability and we have introduced rational Bayesian learning. In addition, we have defined and analyzed a particular deviation from fully rational behavior in the form of adaptive Bayesian learning.

A key reason motivating our limited-information approach has been the particular nature of rare disasters. Given the slow rate at which rare disasters arrive, it is rather difficult to argue that investors confidently reach rational expectations about the average frequency of arrival of disasters (hazard rate).

In our model, rational investors may be perfectly aware of their ignorance, and fully forward-looking, anticipating new information to arrive and future learning to take place. We show that in such an environment, Bayes' rule implies that beliefs jump to pessimistic levels after a rare disaster occurs. These jumps towards pessimism create massive jumps in demands for assets, and therefore imply massive downward jumps in P-D ratios. When

31

disasters take long to occur, optimism gradually takes over, and it can lead to high P-D ratios. These dynamics imitate behavior that is often attributed to investor psychology, such as sudden investment freezing due to fear after a sudden event with dramatic short-run consequences occurs, and slow restoration of confidence after a long period of no stock-market crashes. So, our findings suggest that, under the assumption of not knowing the stochastic structure of rare events with dramatic short-run consequences, emotion and logic may meet each other, in the sense that what is perceived as emotion can be fully rationalized. An evolutionary psychology perspective might suggest that our results formalize an argument that the instinct of fear is an endowment by nature that complements rationality.

Asset-pricing dynamics in our illustrative simulations are qualitatively similar between adaptive and rational learners. However, there are substantial quantitative differences. The study of such quantitative differences between adaptive and rational learning could be an interesting topic for future research in asset-pricing models. Other important extensions would concern belief-heterogeneity among investors and second-order learning about rare disasters and "black-swan" incidents as in Orlik and Veldkamp (2014). Our setup and analysis could also be generalized in order to include learning about the possibility of upward jumps. For example, the emergence of a new general-purpose technology, may motivate optimistic expectations for a "new economy" with sudden bursts of investor enthusiasm, triggered by rare upward jumps in the dividend process (sometimes triggered by the sudden massive entry of new firms).

9. Appendix

Proof of the asset pricing equation under rational expectations (RE – equation (10))

For the derivation of equation (10), we first show that J^{RE} is given by,

$$J^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) = \begin{cases} \frac{\Psi^{RE}(D,\lambda_i)}{D} \frac{(sD)^{1-\gamma}}{1-\gamma} - \frac{1}{\rho(1-\gamma)} & , \quad \text{if } \gamma \neq 1 \\ \frac{\Psi^{RE}(D,\lambda_i)}{D} \ln(sD) + \kappa_i^{RE}\left(\Psi^{RE}\right) & , \quad \text{if } \gamma = 1 \end{cases} \tag{35}$$

in which $\kappa_i^{RE}\left(\Psi^{RE}\right)$ is some constant that does not affect optimization. The first-order conditions of (9) are,

$$c^{-\gamma} = \frac{1}{\Psi^{RE}\left(D, \lambda_i\right)} \cdot J_s^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) . \tag{36}$$

In order to solve the differential equation given by (9) subject to (36), we take a guess on the general functional form of $J^{RE}\left(s, D \mid \Psi^{RE}\right)$ with undetermined coefficients. First, we examine the case $\gamma \neq 1$, taking the guess,

$$J^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) = a + b_i \frac{(sD)^{1-\gamma}}{1-\gamma} , \quad i \in \{h, l\} , \tag{37}$$

in which the undetermined coefficients may depend on Ψ^{RE}, and thus be functionals of the form $a\left(\Psi^{RE}\right)$ and $b_i\left(\Psi^{RE}\right)$. We drop the dependence of α and b on Ψ^{RE} for notational simplicity. Equation (37) implies,

$$J_s^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) = b_i s^{-\gamma} D^{1-\gamma} , \tag{38}$$

$$J_D^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) = b_i s^{1-\gamma} D^{-\gamma} ,$$

and,

$$J_{DD}^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) = -\gamma b_i s^{1-\gamma} D^{-\gamma-1} . \tag{39}$$

Combining equation (36) with (38) gives,

$$c = \left[b_i \frac{D}{\Psi^{RE}(D, \lambda_i)} \right]^{-\frac{1}{\gamma}} sD \ . \tag{40}$$

Since all agents are identical, in equilibrium there is no trade among individuals, and the representative agent's demand for assets is $s(t) = S(t) = S(0)$ for all $t \geq 0$. This means that $ds(t) = 0$ for all $t \geq 0$. So, the budget constraint, equation (8), implies that, for all $t \geq 0$, each household consumes her dividend, i.e.

$$c = sD \ . \tag{41}$$

From (41) and (40) we obtain,

$$b_i = \frac{\Psi^{RE}(D)}{D} \ , \tag{42}$$

which reconfirms the claim in (35), that b_i equals the P-D ratio. Plugging equations (37) through (42) into the HJB equation given by (9), we arrive at,

$$\rho \left[a + \frac{1}{\rho(1-\gamma)} \right] + \left[1 - \left(\phi_{ii} b_i + \phi_{ij} b_j \right) \right] \frac{(sD)^{1-\gamma}}{1-\gamma} = 0 \ , \tag{43}$$

in which b_j corresponds to $J^{RE}\left(s, D, \lambda_j \mid \Psi^{RE}\right)$, and with

$$\phi_{ii} = Q + \lambda_i \left[\omega_{ij} (1 - \xi) + \xi \right] \ , \quad \text{and} \quad \phi_{ij} = -\lambda_i \omega_{ij} (1 - \xi) \ . \tag{44}$$

Following the same procedure for (9) in the case of $\lambda = \lambda_j$, we obtain

$$\rho \left[a + \frac{1}{\rho(1-\gamma)} \right] + \left[1 - \left(\phi_{ji} b_i + \phi_{jj} b_j \right) \right] \frac{(sD)^{1-\gamma}}{1-\gamma} = 0 \ , \tag{45}$$

in which

$$\phi_{ji} = -\lambda_j \omega_{ji} (1 - \xi) \ , \quad \text{and} \quad \phi_{jj} = Q + \lambda_j \left[\omega_{ji} (1 - \xi) + \xi \right] \ . \tag{46}$$

Setting both the constants and the factors of $(sD)^{1-\gamma}$ in equations (43) and (45) equal to zero, we have to solve the linear system of equations

$$\begin{bmatrix} \phi_{ii} & \phi_{ij} \\ \phi_{ji} & \phi_{jj} \end{bmatrix} \begin{bmatrix} b_i \\ b_j \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ ,$$

which leads to the pricing function given by (10) and reconfirms the functional form of $J^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right)$ given by the branch of (35) corresponding to the case of $\gamma \neq 1$.

For the case in which $\gamma = 1$, the guess for $J^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right)$ is,

$$J^{RE}\left(s, D, \lambda_i \mid \Psi^{RE}\right) = a_1 + b_{1,i} \ln\left(sD\right) ,$$

and the same procedure as above leads to the expression given by the branch of (35) corresponding to the case of $\gamma \neq 1$. $\quad\square$

Proof of the asset pricing equation under rational learning for state verification $\left(RLS - \textbf{equation (19)}\right)$

We prove that J^{RLS} is given by,

$$J^{RLS}\left(s, D, \pi \mid \Psi^{RLS}\right) = \begin{cases} \frac{\Psi^{RLS}(D,\pi)}{D} \frac{(sD)^{1-\gamma}}{1-\gamma} - \frac{1}{\rho(1-\gamma)} & , \quad \text{if } \gamma \neq 1 \\ \frac{\Psi^{RLS}(D,\pi)}{D} \ln\left(sD\right) + \kappa^{RL}\left(\pi \mid \Psi^{RLS}\right) & , \quad \text{if } \gamma = 1 \end{cases} \tag{47}$$

in which $\kappa^{RL}\left(\pi \mid \Psi^{RLS}\right)$ is a constant that does not affect optimization. The first-order conditions of (18) are given by,

$$c^{-\gamma} = \frac{1}{\Psi^{RLS}\left(D, \pi\right)} \cdot J_s^{RLS}\left(s, D, \pi \mid \Psi^{RLS}\right) , \tag{48}$$

and the guess we take for the undetermined-coefficients functional form of J^{RL} in the case $\gamma \neq 1$ is

$$J^{RLS}\left(s, D, \pi \mid \Psi^{RLS}\right) = \kappa + (a + b\pi) \frac{(sD)^{1-\gamma}}{1 - \gamma} \tag{49}$$

in which the undetermined coefficients, κ, a, and b, may depend on Ψ^{RL}, but we do not denote this dependence for notational simplicity. Moreover, undetermined coefficients a and

b are different from these defined in other appendices. Equation (49) implies

$$J_s^{RLS}\left(s, D, \ \pi \mid \ \Psi^{RLS}\right) = (a + b\pi)\, s^{-\gamma} D^{1-\gamma} \ ,$$

and after this is combined with (48), the implied formula for consumption is,

$$c = \left[\frac{D}{\Psi^{RLS}\left(D, \ \pi\right)} \cdot (a + b\pi)\right]^{-\gamma} sD \ . \tag{50}$$

So, provided that the guess given by (49) proves to be correct, the market-clearing condition $c = sD$ combined with (50) implies that the P-D ratio is,

$$\frac{\Psi^{RLS}\left(D, \ \pi\right)}{D} = a + b\pi \ . \tag{51}$$

Focusing on the case $\gamma \neq 1$, and using the guess given by (49) in order to calculate J_s^{RLS}, J_D^{RLS}, J_{DD}^{RLS}, and J_π^{RLS}, substitution of the resulting functions into the HJB equation (18), together with the market-clearing condition $c = sD$, after some algebra, results in the following expression,

$$\nu_1 + (\nu_2 + \nu_3 \cdot \pi) \frac{(sD)^{1-\gamma}}{1 - \gamma} = 0 \ , \tag{52}$$

in which,

$$\nu_1 = \rho \left[\kappa + \frac{1}{\rho\,(1 - \gamma)}\right] \ ,$$

$$\nu_2 = (Q + \lambda_l \xi) \left(a - \frac{1 + b\omega_{lh}}{Q + \lambda_l \xi}\right) \ ,$$

and

$$\nu_3 = b\,(Q + \lambda_h \xi + \omega_{lh} + \omega_{hl}) + a\delta\xi \ .$$

Ideally, it should be possible to make equation (52) hold for any levels of the model's variables, π, s, and D. The functional form on the left-hand side of equation (52) reveals that the only way to have equation (52) hold for any arbitrary levels of π, s, and D is to set $\nu_1 = \nu_2 = \nu_3 = 0$. Indeed, there exist unique values for the undetermined coefficients κ, a,

and b, that make conditions $\nu_1 = \nu_2 = \nu_3 = 0$ to hold. These values are, $\kappa = -1/\left[\rho\left(1 - \gamma\right)\right]$, $a = \left(1 + b\omega_{lh}\right)/\left(Q + \lambda_l \xi\right)$, and $b = -a\delta\xi/\left(Q + \lambda_h \xi + \omega_{lh} + \omega_{hl}\right)$, and after these values are combined with (51), and (49), equation (19) and the part of equation (47) that refers to the case in which $\gamma \neq 1$ are both validated.

For the case $\gamma = 1$, we take a guess of the form

$$ J^{RLS}\left(s, D, \ \pi \mid \Psi^{RLS}\right) = a_1 \cdot \ln\left(sD\right) + b_1\pi + \kappa_1 \ , $$

and following the same procedure as above we arrive at the part of equation (47) that refers to the case in which $\gamma = 1$. $\square$

Proof of the asset pricing equation under rational learning for data-generating process (RLP – equation (27))

Unlike all other cases above in which we took a guess on the functional form of the value function, here we use the insight that in the standard representative-agent Lucas-tree model investors consume the dividend in equilibrium. So, we compute the utility function through stochastic integration.

Our first goal is to calculate the expectation $E_0\left[D\left(t\right)^{1-\gamma}\right]$ conditional on information available at time 0, which is given by $\left(N_0, T_0\right) = \left(N\left(0\right), T\left(0\right)\right)$. At any future point in time, $\left(N, T\right) = \left(N\left(t\right), T_0 + t\right)$. Denote a perceived value of $\lambda\left(t\right)$ by $\tilde{\lambda}\left(t\right)$. Remember that the belief density of $\tilde{\lambda}\left(t\right)$ is Gamma-distributed conditional on hyperparameters $\left(N\left(t\right), T\right)$, given by,

$$ f\left(\tilde{\lambda} \mid N, T\right) = \frac{T^N}{\Gamma\left(N\right)}e^{-\tilde{\lambda}T}\tilde{\lambda}^{N-1} = \frac{\left(T_0 + t\right)^{N(t)}}{\Gamma\left(N\left(t\right)\right)}e^{-\tilde{\lambda}\left(T_0+t\right)}\tilde{\lambda}^{N(t)-1} = f\left(\tilde{\lambda} \mid N\left(t\right), T_0 + t\right) \ . $$

$$ \tag{53} $$

A key insight from (53) is that by fixing a certain value for $\tilde{\lambda}$ over time, the resulting conditional jump process for $D(t)$ in equation (1) becomes Poisson with fixed jump frequency $\tilde{\lambda}$. This observation is useful for calculating $E_0\left[D(t)^{1-\gamma}\right]$ for any $t \geq 0$, using the law of total expectation,

$$E_0\left[D(t)^{1-\gamma}\right] = \int_0^\infty \left\{E_{N(t)}\left\{E\left[D(t)^{1-\gamma} \mid N(t), \tilde{\lambda}\right] f\left(\tilde{\lambda} \mid N(t), T_0 + t\right)\right\}\right\} d\tilde{\lambda}, \quad (54)$$

in which $E_{N(t)}(\cdot)$ refers to the expectation with respect to possible cumulative sums of disasters $N(t)$ within the time interval $[0,t]$. Applying Itô's lemma on (1) for a fixed $\tilde{\lambda}$ we obtain,

$$d\ln\left(D^{1-\gamma}\right) = (1-\gamma)\left(\mu - \frac{\sigma^2}{2}\right) dt + (1-\gamma)\sigma dz + (1-\gamma)\ln(1-\zeta) dN. \quad (55)$$

Because $z(t)$, $N(t)$, and $\zeta(t)$ are all independent among each other, we can condition equation (55) on fixed values of ζ. So, we stochastically integrate both sides of equation (55) with respect to time, we exponentiate both sides of the resulting equation, and we take the total expectation conditioning on the fixed value of $\tilde{\lambda}$ and on $N(t)$ arriving at,

$$E\left[D(t)^{1-\gamma} \mid N(t), \tilde{\lambda}\right] = e^{(Q-\rho)t} D(0)^{1-\gamma} \cdot (1-\xi)^{N(t)}. \quad (56)$$

Given that $\tilde{\lambda}$ is fixed, after applying Posch and Wälde (2006, Lemma 3, p. 23) on equation (56) we obtain,

$$E_{N(t)}\left\{E\left[D(t)^{1-\gamma} \mid N(t), \tilde{\lambda}\right] f\left(\tilde{\lambda} \mid N(t), T_0 + t\right)\right\} = e^{(Q-\rho-\tilde{\lambda}\xi)t} D(0)^{1-\gamma} \frac{T_0^{N_0}}{\Gamma(N_0)} e^{-\tilde{\lambda}T_0} \tilde{\lambda}^{N_0-1}. \quad (57)$$

Substituting (57) into (54) gives,

$$E_0\left[D(t)^{1-\gamma}\right] = e^{-(Q-\rho)t} \frac{T_0^{N_0}}{\Gamma(N_0)} D(0)^{1-\gamma} \underbrace{\int_0^\infty e^{-(T_0+\xi t)\tilde{\lambda}} \tilde{\lambda}^{N_0-1} d\tilde{\lambda}}_{\overset{\shortparallel}{A}},$$

which can be re-written as,

$$E_0 \left[D(t)^{1-\gamma} \right] = e^{-(Q-\rho)t} \frac{T^N}{\Gamma(N)} D(0)^{1-\gamma} \cdot A ,\qquad(58)$$

with

$$A \equiv \int_0^\infty e^{-(T+\xi t)\tilde{\lambda}} \tilde{\lambda}^{N-1} d\tilde{\lambda} ,$$

for notational simplicity. In order to calculate A, use the transformation $z = (T + \xi t)\tilde{\lambda}$, which implies, $\tilde{\lambda} = (T + \xi t)^{-1} z$ and $d\tilde{\lambda} = (T + \xi t)^{-1} dz$, leading to,

$$A = (T + \xi t)^{-N} \int_0^\infty e^{-z} z^{N-1} dz = (T + \xi t)^{-N} \Gamma(n) .\qquad(59)$$

Given that,

$$J^{RLP}(s, D, N, T) = E_0 \left\{ \int_0^\infty e^{-\rho t} \frac{[sD(t)]^{1-\gamma} - 1}{1 - \gamma} dt \;\middle|\; D(0) = D \right\} ,\qquad(60)$$

after combining (59) with (58) and substituting it into (60) we obtain,

$$J^{RLP}(s, D, N, T) = b(N, T) \frac{(sD)^{1-\gamma}}{1 - \gamma} - \frac{1}{\rho(1 - \gamma)} ,\qquad(61)$$

in which

$$b(N, T) = T^N \int_0^\infty e^{-Qt} (T + \xi t)^{-N} dt .\qquad(62)$$

Following the steps of the proofs for derive the P-D ratio in the RE case, notice that $b(N, T)$ is the P-D ratio. In order to simplify the expression for $b(N, T)$, use the transformation $y = (Q/\xi)(T + \xi t)$, which implies $t = y/Q - T/\xi$, $dt = dy/Q$, and also that $y_0 = QT/\xi$ is the value of y corresponding to $t = 0$, substitute all these values into (62) to obtain,

$$b(N, T) = \frac{e^{\frac{QT}{\xi}}}{Q} \left(\frac{QT}{\xi} \right)^N \underbrace{\int_{\frac{QT}{\xi}}^\infty e^{-y} y^{-N} dy}_{\overset{\shortparallel}{\Gamma\left(1-N, \frac{QT}{\xi}\right)}} ,$$

39

which proves equation (27). The proof is essentially complete here. In order to cross verify that equation (27) is also consistent with the HJB equation, use (61) above as the guess to substitute into (26) to obtain,

$$Qb\left(N,T\right) = 1 + b_T\left(N,T\right) + \frac{N}{T}\left[\left(1-\xi\right)b\left(N+1,T\right) - b\left(N,T\right)\right] \ . \tag{63}$$

Using the formula given by (62) for $b\left(N,T\right)$, we can show at a first stage that (63) simplifies to,

$$Qb\left(N,T\right) + \xi\frac{N}{T}b\left(N+1,T\right) = 1 \ . \tag{64}$$

Using either equation (27) or equation (62), after some integration by parts we can prove that (64) holds, completing the proof. $\square$

Proof of Proposition 2   Equation (16) has two parts, a deterministic part and a stochastic part. The deterministic part is the first term of the RHS of (16), which is equal to $-\delta\pi\left(1-\pi\right)$, and it defines a deterministic first-order differential equation,

$$\dot{\pi} = -\delta\pi\left(1-\pi\right) \ , \tag{65}$$

which can be re-written as,

$$\dot{\pi} = \delta\pi^2 - \delta\pi \ . \tag{66}$$

Equation (66) is a Bernoulli differential equation. So, we can use the Bernoulli transformation

$$z_\pi\left(t\right) \equiv \pi\left(t\right)^{-1} \quad \text{for all } t \geq 0 \ . \tag{67}$$

From (67) it is,

$$\dot{z}_\pi\left(t\right) = -\pi\left(t\right)^{-2}\cdot\dot{\pi}\left(t\right) \quad \text{for all } t \geq 0 \ . \tag{68}$$

So, after multiplying both sides of equation (66) by $-\pi^{-2}$ and also substituting (67) and (68) it is,

$$\dot{z}_\pi = \delta z_\pi - \delta \ . \tag{69}$$

The solution to equation (69) is,

$$z_\pi(t) - 1 = e^{\delta t}\left[z_\pi(0) - 1\right] \ ,$$

and substituting (67) gives

$$\pi(t) = \frac{1}{1 + e^{\delta t}\frac{1-\pi_0}{\pi_0}} \ , \quad \text{for all } t \geq 0 \ . \tag{70}$$

Equation (70) shows that, no matter how much time has passed without any jumps occurring, probability π always stays within the open interval $(0, 1)$.

The second part of equation (16) is stochastic and given by a jump process, such that the probability jumps from its original level π to the level given by $\lambda_h \cdot \pi / \Lambda(\pi)$. The statement

$$\pi \in (0, 1) \Rightarrow \frac{\lambda_h \pi}{\Lambda(\pi)} \in (0, 1) \tag{71}$$

holds, because $\lambda_h \cdot \pi / \Lambda(\pi) > 0$ for all $\pi \in (0, 1)$, and $\lambda_h \cdot \pi / \Lambda(\pi) < 1 \Leftrightarrow \pi < 1$, which is also true for all $\pi \in (0, 1)$. Combining (71) with (70) proves the part of the proposition which states that for all $\pi(0) = \pi_0 \in (0, 1)$, $\pi(t) \in (0, 1)$ for all $t \geq 0$.

Applying analytical techniques that pertain to Poisson differential equations on (16), we obtain,[27]

$$\frac{E(d\pi)}{dt} = -\delta\pi(1 - \pi) + \lambda^*\left[\frac{\lambda_h \pi}{\Lambda(\pi)} - \pi\right] \ . \tag{72}$$

Using the fact that $\Lambda(\pi^*) = \lambda^*$ implies $\pi^* = (\lambda^* - \lambda_l)/\delta$, after some algebra, equation (72) gives,

$$\dot{\pi}^e = \delta^2\pi(1 - \pi)\frac{\pi^* - \pi}{\Lambda(\pi)} \ . \tag{73}$$

[27]See, for example, Merton (1971, pp. 395-401) and Kushner (1967, pp. 16-22).

For calculating the limit $\lim_{t \to \infty} E\left[\pi(t)\right]$, notice that, according to Dynkin's formula (we denote the Dynkin operator by $\mathcal{D}$),

$$E\left[\pi(t)\right] = \pi(0) + E\left[\int_0^t \mathcal{D}\pi(\tau)\,d\tau\right], \tag{74}$$

in which,

$$\mathcal{D}\pi(\tau) = \delta^2 \pi(1-\pi)\frac{\pi^* - \pi}{\Lambda(\pi)}, \tag{75}$$

which is a formula based on equation (73). The expectations operator in the expression $E\left[\int_0^t \mathcal{D}\pi(\tau)\,d\tau\right]$ of the RHS of (74) transforms the dynamics of (73) into

$$\dot{\pi}^e = \delta^2 \pi^e (1-\pi^e)\frac{\pi^* - \pi^e}{\Lambda(\pi^e)}, \tag{76}$$

where $\pi^e \equiv E(\pi)$. To see that the expectations operator in the expression $E\left[\int_0^t \mathcal{D}\pi(\tau)\,d\tau\right]$ of the RHS of (74) leads to (76), fix any $t \geq 0$ and any $\Delta t > 0$, and consider equation (74) expressed as,

$$\pi^e(t+\Delta t) = E\left[\pi(t+\Delta t)\right] = \pi(t) + E\left[\int_t^{t+\Delta t} \mathcal{D}\pi(\tau)\,d\tau\right]. \tag{77}$$

If we set $\Delta t > 0$ arbitrarily small, then (77) and (75) give rise to an approximate recursion with respect to π^e, given by the difference equation,

$$\pi^e(t+\Delta t) = \pi^e(t) + \delta^2 \pi^e(t)\left[1 - \pi^e(t)\right]\frac{\pi^* - \pi^e(t)}{\Lambda(\pi^e(t))} \cdot \Delta t, \tag{78}$$

for all discrete periods with interval length $[t, \ t+\Delta t]$ and any $t \geq 0$. Equation (78) is a deterministic equation, since its initial conditions are non-stochastic ($\pi^e(0) = \pi(0) = \pi_0$). Equation (78) is a construction by approximation that leads to differential equation (76) by subtracting $\pi^e(t)$ from both sides of (78), dividing by Δt, and taking the limit $\Delta t \to 0$. Equation (76) implies dynamics given by,

$$\text{for all } \pi^e \in (0,1), \quad \dot{\pi}^e \gtreqless 0 \Leftrightarrow \pi^e \lesseqgtr \pi^*. \tag{79}$$

With the help of a one-dimensional phase diagram it can be verified that $\lim_{t\to\infty} E\left[\pi\left(t\right)\right] = \lim_{t\to\infty} \pi^e\left(t\right) = \pi^*$, which proves equation (28) of the proposition.

In order to prove equation (29), after applying analytical techniques that pertain to Poisson differential equations on (16), we obtain,

$$\frac{E\left(d\pi^2\right)}{dt} = -2\delta\pi^2\left(1-\pi\right) + \lambda^*\left\{\left[\frac{\lambda_h\pi}{\Lambda\left(\pi\right)}\right]^2 - \pi^2\right\} \, ,$$

which simplifies to,

$$\frac{E\left(d\pi^2\right)}{dt} = \delta\pi^2\left(1-\pi\right)\left[\frac{\lambda^*\left(\lambda_h + \lambda_l + \delta\pi\right)}{\Lambda\left(\pi\right)^2} - 2\right] \, . \tag{80}$$

Using the same argument as above, we can show that (80) yields

$$\frac{E\left(d\pi^2\right)}{dt} = \delta E\left(\pi^2\right)\left\{1 - \left[E\left(\pi^2\right)\right]^{\frac{1}{2}}\right\}\left[\frac{\lambda^*\left\{\lambda_h + \lambda_l + \delta\left[E\left(\pi^2\right)\right]^{\frac{1}{2}}\right\}}{\Lambda\left(\left[E\left(\pi^2\right)\right]^{\frac{1}{2}}\right)^2} - 2\right] \, . \tag{81}$$

For notational simplicity, we can use the transformation $z \equiv E\left(\pi^2\right)$, which makes (81) be expressed as,

$$\dot{z} = \delta z\left(1 - z^{\frac{1}{2}}\right)\left[\frac{\lambda^*\left(\lambda_h + \lambda_l + \delta z^{\frac{1}{2}}\right)}{\Lambda\left(z^{\frac{1}{2}}\right)^2} - 2\right] \, , \tag{82}$$

since $\pi \in (0,1)$. Because the term $\delta z\left(1 - z^{\frac{1}{2}}\right)$ in (82) is always positive for all $\pi \in (0,1)$, we can focus on the sign of the expression in the bracket of the right-hand side of equation (82), which is determined by the sign of the expression

$$f\left(z^{\frac{1}{2}}\right) \equiv z - \frac{\delta\pi^* - 3\lambda_l}{2\delta}z^{\frac{1}{2}} - \frac{\lambda^*\lambda_h - \lambda_l + \left(\lambda^* - 1\right)\lambda_l}{2\delta^2} \, , \tag{83}$$

and

$$f\left(z^{\frac{1}{2}}\right) \lesseqqgtr 0 \Leftrightarrow \dot{z} \gtreqqless 0 \, . \tag{84}$$

There exist two real roots for the quadratic form given by (83), namely,

$$z^{\frac{1}{2}}_{1,2} = \frac{\lambda^* - 4\lambda_l \pm \left(\lambda^*\right)^{\frac{1}{2}}\left(\lambda^* + 8\lambda_h\right)^{\frac{1}{2}}}{4\delta} \, ,$$

43

and it is easy to verify that one root is negative, while the other is positive. Since $\pi \in (0, 1)$ for all $t \geq 0$, we discard the negative root and we keep the positive root which is,

$$\hat{z}^{\frac{1}{2}} = \left[\widehat{E\left(\pi^2\right)} \right]^{\frac{1}{2}} = \frac{\lambda^* - 4\lambda_l + (\lambda^*)^{\frac{1}{2}} (\lambda^* + 8\lambda_h)^{\frac{1}{2}}}{4\delta} \ ,$$

implying that,

$$\hat{z} = \widehat{E\left(\pi^2\right)} = \left[\frac{\lambda^* - 4\lambda_l + (\lambda^*)^{\frac{1}{2}} (\lambda^* + 8\lambda_h)^{\frac{1}{2}}}{4\delta} \right]^2 . \tag{85}$$

Most importantly, $f\left(z^{\frac{1}{2}}\right) \lesseqqgtr 0 \Leftrightarrow z \lesseqqgtr \hat{z}$ for all $z \in (0, 1)$, so through the aid of a one-dimensional phase diagram, the relationship given by (84) confirms that $\hat{z}$ is globally stable, for all $z \in (0, 1)$. This means that as $t \to \infty$, $E\left(\pi^2\right) \to \widehat{E\left(\pi^2\right)}$. Using the fact that, asymptotically, $Var\left(\pi\right) = \widehat{E\left(\pi^2\right)} - (\pi^*)^2$, proves (29). Given that $\pi^* = (\lambda^* - \lambda_l)/\delta$, after some algebra, it can be shown that $\widehat{E\left(\pi^2\right)} - (\pi^*)^2 > 0 \Leftrightarrow \lambda_h > \lambda^*$. The right-hand side of this equivalence is a true statement, completing the proof of the proposition. $\square$

Proof of that under RLP beliefs converge to a degenerate distribution in the long run

Since the Markov transition matrix governing $\lambda(t)$ asymptotically leads to $(\pi^*, 1 - \pi^*)$, from a modeler's perspective, the expected value of $N(t)$ is $\lambda^* \cdot t$ ($\lambda^* = \pi^* \lambda_h + (1 - \pi^*) \lambda_l$. Denote the expected realization from a modeler's perspective by $E_m[N(t)] = \lambda^* \cdot t$). After applying the law of iterated expectations on equations (23) and (24) it is,

$$E_m\left[\tilde{\lambda}(t)\right] = \frac{N(0) + \lambda^* \cdot t}{T(0) + t} \ , \tag{86}$$

and

$$Var_m\left[\tilde{\lambda}(t)\right] = \frac{N(0) + \lambda^* \cdot t}{[T(0) + t]^2} \ . \tag{87}$$

The asymptotic distribution is directly characterized from (86) and (87) which imply,

$$\lim_{t\to\infty} E_m\left[\tilde{\lambda}(t)\right] = \lambda^* , \qquad (88)$$

and

$$\lim_{t\to\infty} Var_m\left[\tilde{\lambda}(t)\right] = 0 . \qquad (89)$$

Equation (89) implies infinite confidence asymptotically, and together with the unbiasedness implied by (86) the result is proved. $\square$

Proof of equation (30)

For notational simplicity, denote $\Psi^{RLP}(D, N, T)$ by $\Psi(D, N, T)$.

Using formulas, we show that,

$$\lim_{t\to\infty} \frac{\Psi\left(D(t), E_m[N(t)], T+t\right)}{D(t)} = \frac{1}{Q + \lambda^*\xi} , \qquad (90)$$

in which $E_m[N(t)] = N(0) + \lambda^* \cdot t$ denotes the expected realization from a modeler's perspective. Using $E_m[N(t)]$ in rational learner's expression for the P-D ratio given by (27) is equivalent to a Monte-Carlo simulation expression.[28] So, based on the theorem of the limit of composition of functions,

$$\lim_{t\to\infty} \frac{\Psi\left(D(t), E_m[N(t)], T+t\right)}{D(t)} = \lim_{t\to\infty} \frac{\Psi\left(D(t), N_\infty^e, T+t\right)}{D(t)} , \qquad (91)$$

in which,

$$N_\infty^e \equiv \lim_{t\to\infty} E_m[N(t)] ,$$

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[28]This equivalence can be proved using Dynkin's formula. The procedure of such a proof is explained in the proof of Proposition 2 above.

and notice that N_∞^e need not be finite. Based on (27),

$$\lim_{t\to\infty} \frac{\Psi\left(D\left(t\right), N_\infty^e, T+t\right)}{D\left(t\right)} = \lim_{t\to\infty} \frac{1}{Q} \left[\frac{Q}{\xi}\left(T+t\right)\right]^{N_\infty^e} e^{\beta l(t)^{-1}} \Gamma\left(1 - N_\infty^e, \frac{Q}{\xi}\left(T+t\right)\right) . \quad (92)$$

Let

$$x \equiv \frac{Q}{\xi}\left(T+t\right)$$

and notice that $t\to\infty$ implies $x\to\infty$. So, (92) implies,

$$\lim_{t\to\infty} \frac{\Psi\left(D\left(t\right), N_\infty^e, T+t\right)}{D\left(t\right)} = \frac{1}{Q} \lim_{x\to\infty} \frac{\Gamma\left(1 - N_\infty^e, x\right)}{e^{-x} \cdot x^{-N_\infty^e}} . \quad (93)$$

The incomplete-Gamma function corresponds to

$$\Gamma\left(1 - N_\infty^e, x\right) = \int_x^\infty e^{-z} z^{(1-N_\infty^e)-1} dz = \int_x^\infty e^{-z} z^{-N_\infty^e} dz ,$$

and $\lim_{x\to\infty} \Gamma\left(1 - N_\infty^e, x\right) = 0$, no matter if $N_\infty^e = \infty$ or if $N_\infty^e < \infty$. Similarly, $\lim_{x\to\infty} e^{-x} \cdot x^{-N_\infty^e} = 0$, no matter if $N_\infty^e = \infty$ or if $N_\infty^e < \infty$, too. So, we apply L'Hôpital's rule on the limit of expression (93) to obtain,

$$\lim_{t\to\infty} \frac{\Psi\left(D\left(t\right), N_\infty^e, T+t\right)}{D\left(t\right)} = \frac{1}{Q} \lim_{x\to\infty} \frac{\frac{\partial \Gamma(1-N_\infty^e, x)}{\partial x}}{\frac{\partial\left(e^{-x} \cdot x^{-N_\infty^e}\right)}{\partial x}}$$

which implies,

$$\lim_{t\to\infty} \frac{\Psi\left(D\left(t\right), N_\infty^e, T+t\right)}{D\left(t\right)} = \frac{1}{Q} \lim_{x\to\infty} \frac{-e^{-x} \cdot x^{-N_\infty^e}}{e^{-x} \cdot x^{-N_\infty^e}\left(-1 - \frac{N_\infty^e}{x}\right)} ,$$

or,

$$\lim_{t\to\infty} \frac{\Psi\left(D\left(t\right), N_\infty^e, T+t\right)}{D\left(t\right)} = \frac{1}{Q} \lim_{x\to\infty} \frac{1}{1 + \frac{N_\infty^e}{x}} . \quad (94)$$

Now we can restore that $x = \frac{Q}{\xi}\left(T+t\right)$ on the right-hand side of (94), keeping in mind that $x\to\infty \Leftrightarrow t\to\infty$, so,

$$\lim_{t\to\infty} \frac{\Psi\left(D\left(t\right), N_\infty^e, T+t\right)}{D\left(t\right)} = \frac{1}{Q} \lim_{t\to\infty} \frac{1}{1 + \frac{N_\infty^e}{\frac{Q}{\xi}\left(T+t\right)}} ,$$

which implies (after taking into account that $N_\infty^e = \lim_{t\to\infty} E_m[N(t)] = \lim_{t\to\infty} E_m[N(0) + \lambda^* \cdot t]$),

$$\lim_{t\to\infty} \frac{\Psi(D(t), N_\infty^e, T+t)}{D(t)} = \frac{1}{Q} \frac{1}{1 + \lim_{t\to\infty} \frac{N(0)+\lambda^* \cdot t}{\frac{Q}{\xi}(T+t)}} \ ,$$

i.e.,

$$\lim_{t\to\infty} \frac{\Psi(D(t), N_\infty^e, T+t)}{D(t)} = \frac{1}{Q} \frac{1}{1 + \frac{\lambda^*\xi}{Q}} \ , \tag{95}$$

which proves the statement.   □

Proof of the asset pricing equation under adaptive expectations and Epstein-Zin-Weil preferences (Equation (33))

For the derivation of equation (33), notice that the form of the HJB equation with Epstein-Zin-Weil preferences is,

$$0 = \max_{c \geq 0} \left\{ f\left(c, J\left(s, D, \pi \mid \tilde{\Psi}^{AL}\right)\right) + J_s\left(s, D, \pi \mid \tilde{\Psi}^{AL}\right) \cdot \left[\frac{1}{\tilde{\Psi}^{AL}(D, \pi)}(sD - c) \right] + \right.$$

$$+ J_D\left(s, D, \pi \mid \tilde{\Psi}^{AL}\right) \cdot \mu D + J_{DD}\left(s, D, \pi \mid \tilde{\Psi}^{AL}\right) \frac{(sD)^2}{2} +$$

$$\left. + \Lambda(\pi) \left\{ E_\zeta\left[J\left(s, (1-\zeta)D, \pi \mid \tilde{\Psi}^{AL}\right) \right] - J\left(s, D, \pi \mid \tilde{\Psi}^{AL}\right) \right\} \right\} \ . \tag{96}$$

First-order conditions are,

$$f_c\left(c, J\left(s, D, \pi \mid \tilde{\Psi}^{AL}\right)\right) = \frac{J_s\left(s, D, \pi \mid \tilde{\Psi}^{AL}\right)}{\tilde{\Psi}^{AL}(D, \pi)} \ . \tag{97}$$

Our guess for the functional form of J is,

$$J\left(s, D, \pi \mid \tilde{\Psi}^{AL}\right) = b(\pi) \frac{(sD)^{1-\gamma}}{1-\gamma} \ , \tag{98}$$

and we denote $b(\pi)$ by b for notational simplicity. Since $c = SD$ in equilibrium, (97) implies,

$$\frac{D}{\tilde{\Psi}^{AL}(D, \pi)} = \rho b^{\frac{\frac{1}{\eta}-1}{1-\gamma}} \ . \tag{99}$$

Substituting (98) and its implied derivatives into (96) gives, after some algebra,

$$\rho b^{\frac{\frac{1}{\eta}-1}{1-\gamma}} = \rho - \left(1 - \frac{1}{\eta}\right)\left(\mu - \gamma\frac{\sigma^2}{2}\right) + \frac{1 - \frac{1}{\eta}}{1 - \gamma}\Lambda\left(\pi\right)\left\{E_\zeta\left(1 - \zeta\right)^{1-\gamma} - 1\right\} \ . \qquad (100)$$

Combining (99) with (100) leads to (33).

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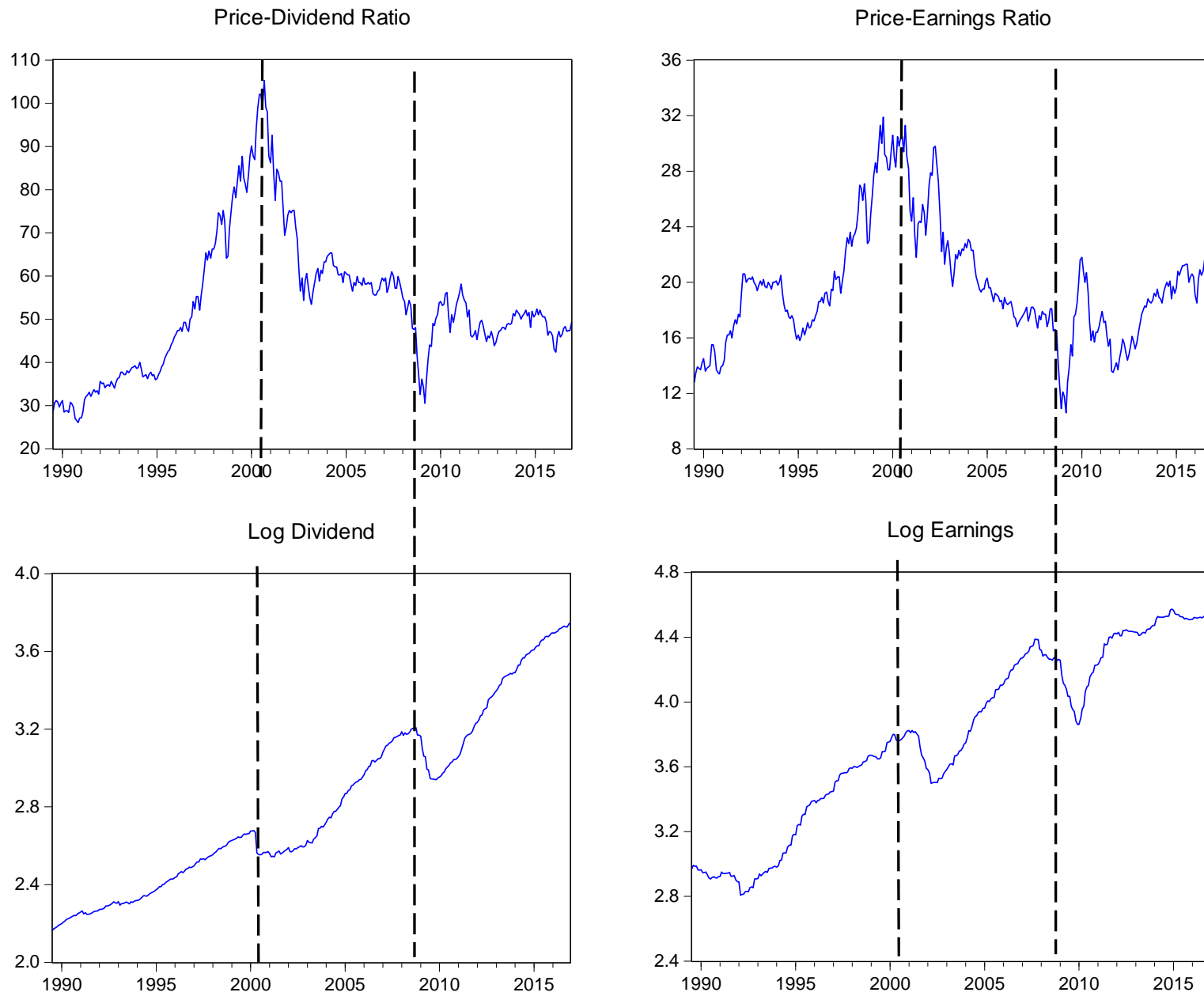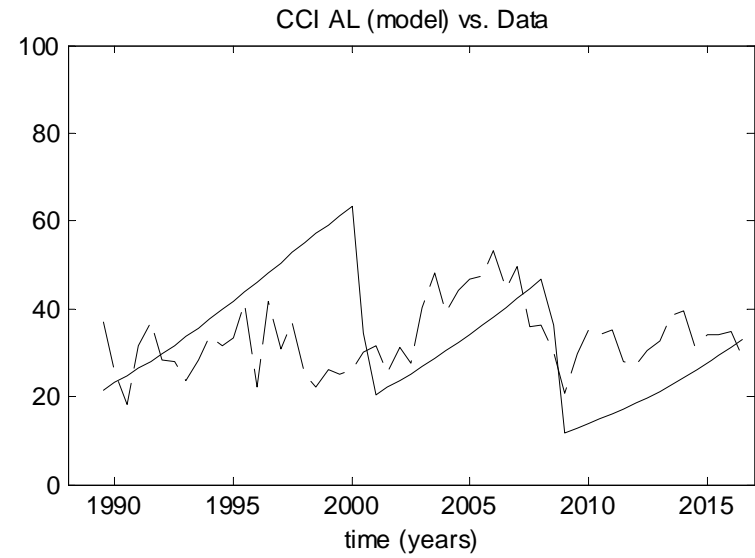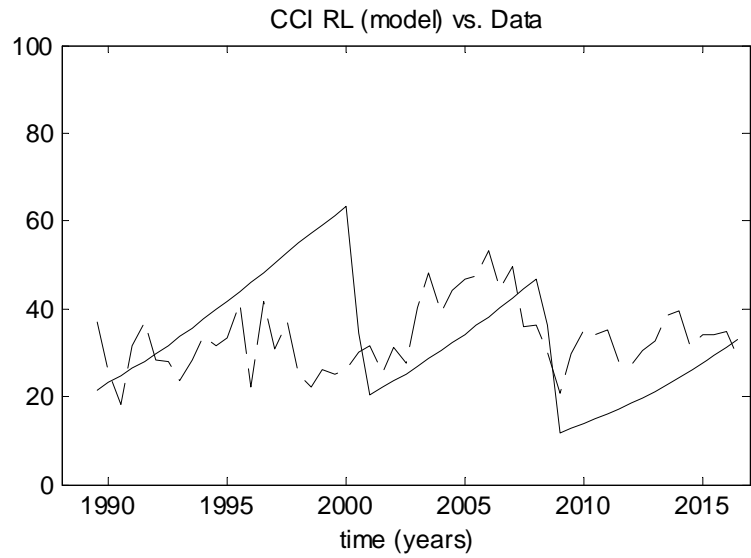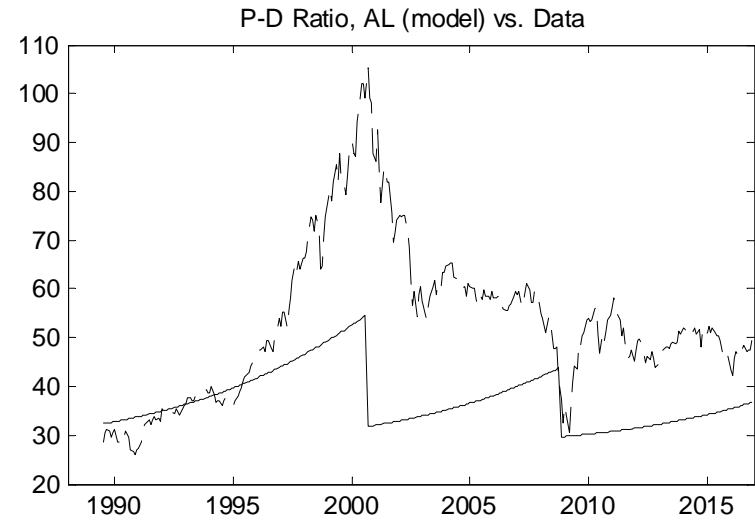**Figure 1**   Monthly US stock-market data 1989-2016. Source: Datastream (TOTMKUS).

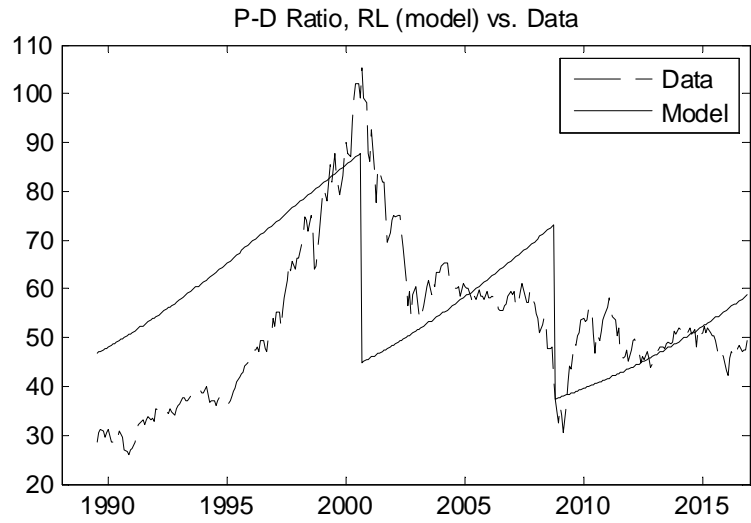**Figure 2**
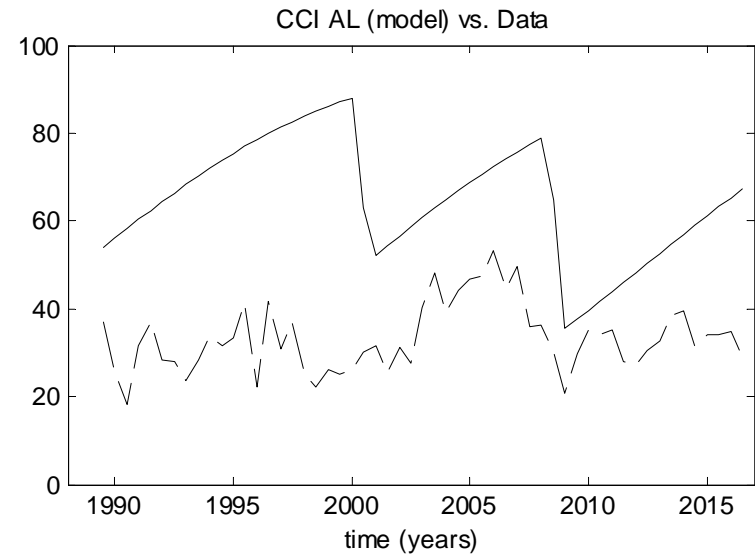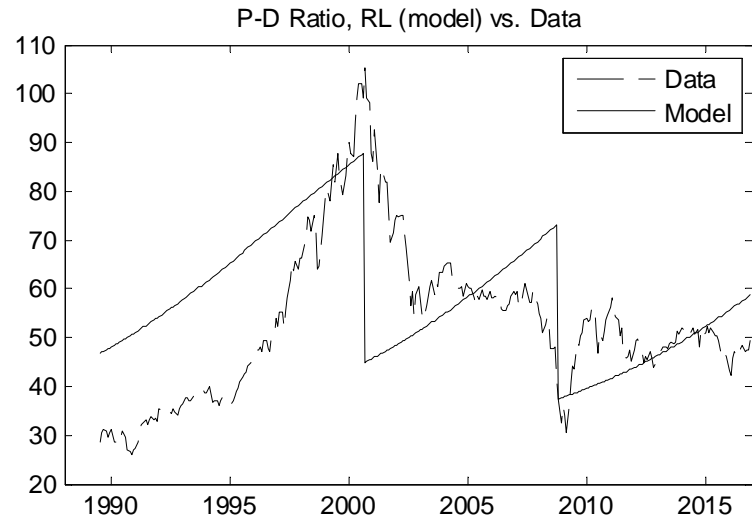
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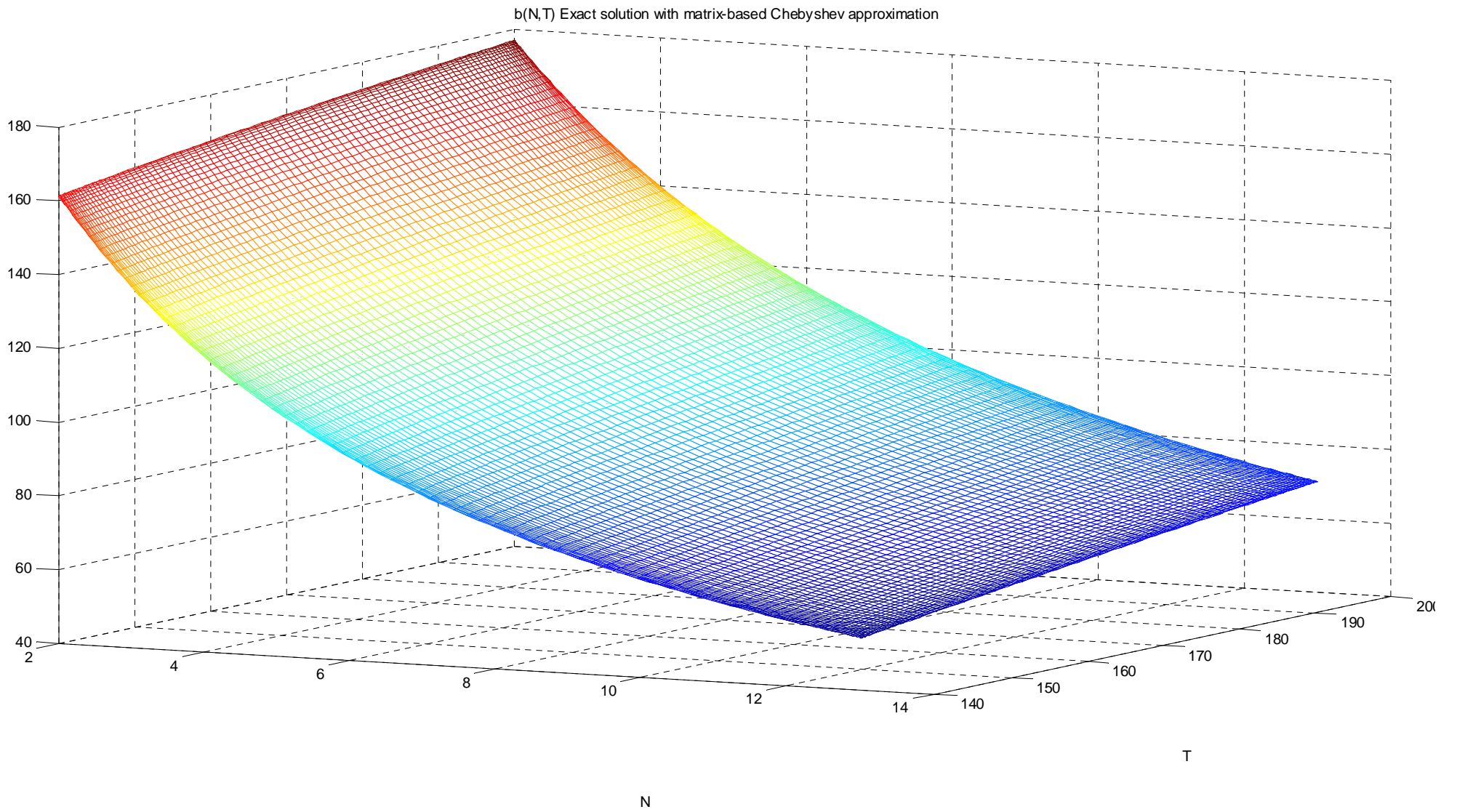
Figure 4 P-D ratio in the RLP case, using the benchmark calibrating parameters
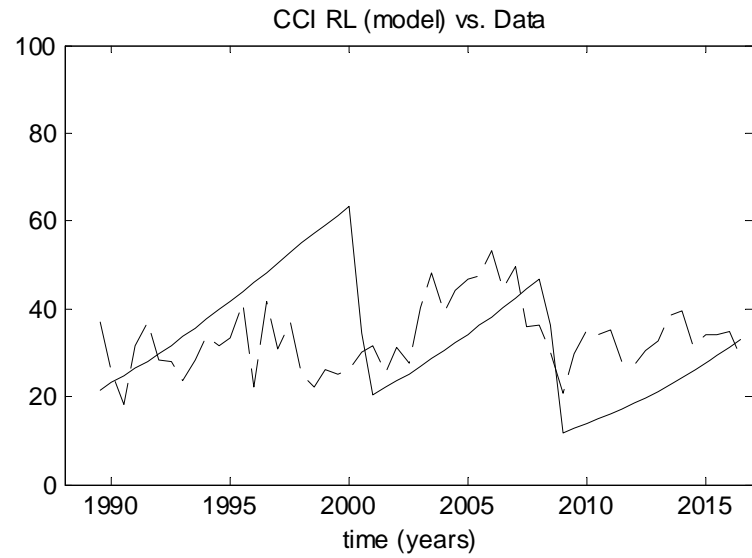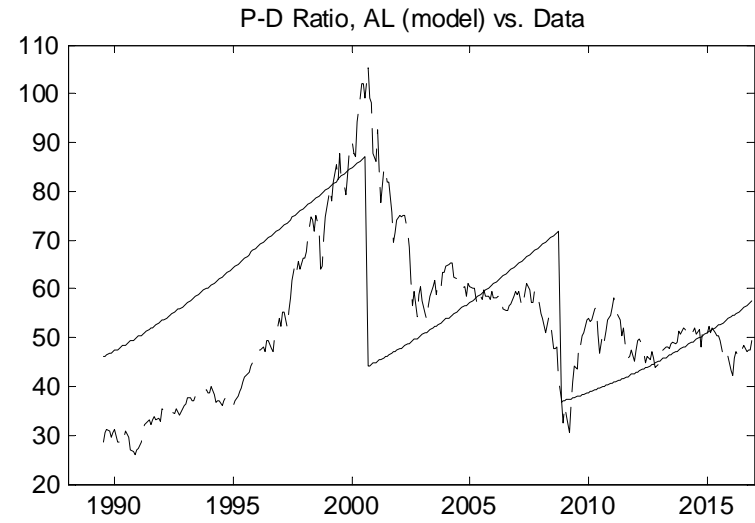
b(N,T) Exact solution with matrix-based Chebyshev approximation
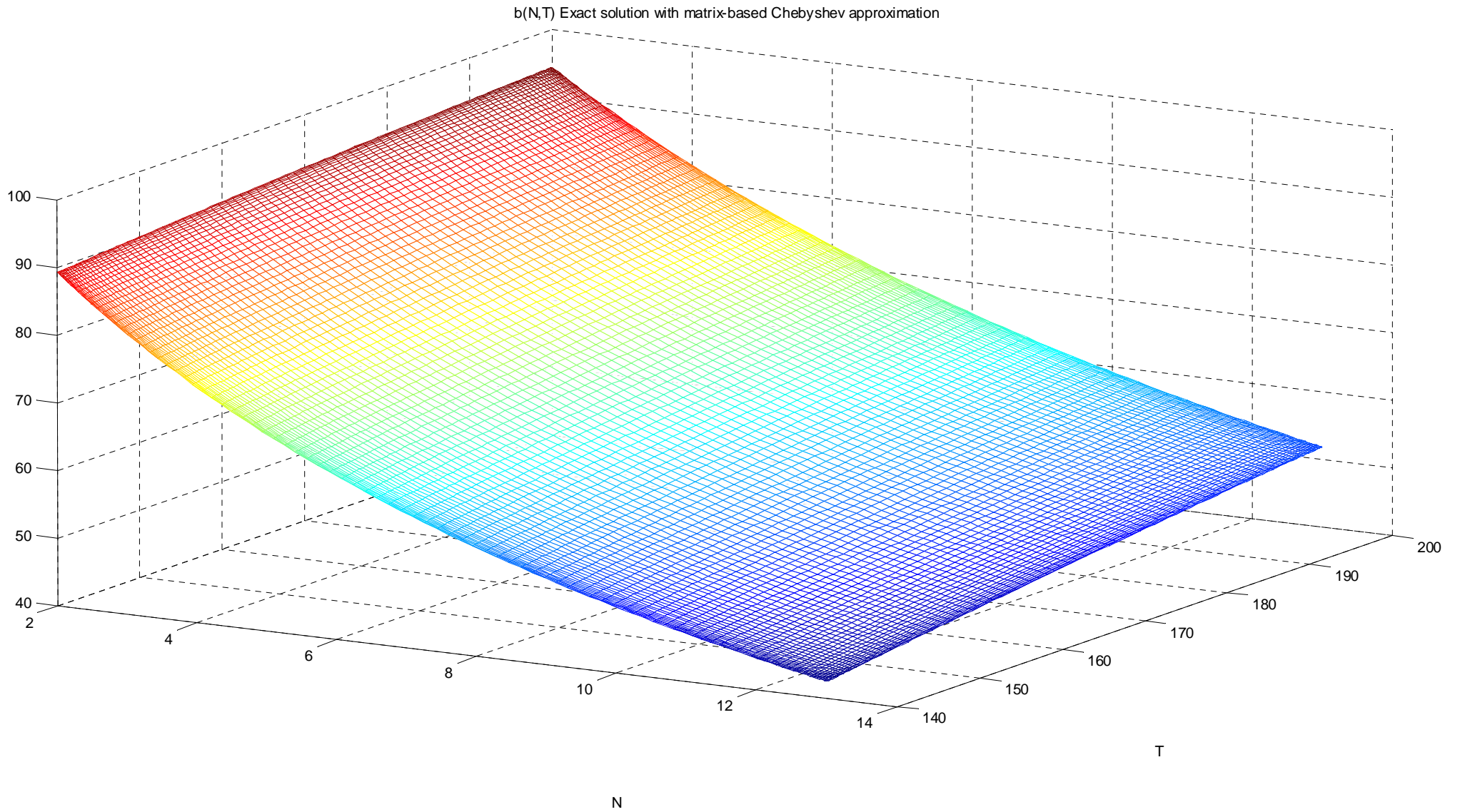
Figure 5 P-D ratio in the RLP case, setting ρ=3%

Online Appendix

for

Asset Pricing under Rational Learning about Rare Disasters

Christos Koulovatianos and Volker Wieland

Numerically Solving the Asset-Pricing Problem in the RLP case with Epstein-Zin-Weil Preferences

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1. The Hamilton-Jacobi-Bellman Equation

Let the continuous-time version of the Epstein-Zin-Weil expected utility function,

$$J(t) = E_t \left[\int_t^\infty f(c(\tau), J(\tau)) d\tau \right], \tag{1}$$

in which $f(c, J)$ is a normalized aggregator of continuation utility, J, and current consumption, c, with

$$f(c, J) \equiv \rho(1-\gamma) \cdot J \cdot \frac{\left\{ \frac{c}{[(1-\gamma)J]^{\frac{1}{1-\gamma}}} \right\}^{1-\frac{1}{\eta}} - 1}{1 - \frac{1}{\eta}}, \tag{2}$$

in which $\rho, \eta, \gamma > 0$. For notational simplicity we set,

$$\theta \equiv \frac{1 - \frac{1}{\eta}}{1 - \gamma}. \tag{3}$$

Given that in the Lucas-tree model $c = D$ at all times, the Hamilton-Jacobi-Bellman (HJB) equation is,

$$0 = f(D, J) + \mu D J_D + \frac{(\sigma D)^2}{2} J_{DD} + J_T + \frac{N}{T} \left\{ E_\zeta \left[J(N+1, T, (1-\zeta)D) \right] - J \right\}, \tag{4}$$

in which J denotes $J(N, T, D)$ for notational simplicity. Based on notation dictated by (3), the preference aggregator $f(D, J)$ can be written as,

$$f(D, J) = \frac{\rho}{\theta(1-\gamma)} \left\{ D^{\theta(1-\gamma)} \left[(1-\gamma)J \right]^{1-\theta} - (1-\gamma)J \right\}. \tag{5}$$

The solution of (4) is of the multiplicatively-separable form,

$$J(N, T, D) = b(N, T) \frac{D^{1-\gamma}}{1-\gamma}, \tag{6}$$

2

and throughout we will be using b for $b(N, T)$ unless it becomes necessary to distinguish that the inputs of function b are arguments taking values other than (N, T). Substituting (6) into (5) gives,

$$f(D, J) = \frac{\rho}{\theta} \left(b^{1-\theta} - b \right) \frac{D^{1-\gamma}}{1-\gamma} . \tag{7}$$

In turn, substituting (7) into (4) gives,

$$\frac{Q}{\theta} b = \frac{\rho}{\theta} b^{1-\theta} + b_T + \frac{N}{T} \left[(1-\xi) b(N+1, T) - b \right] , \tag{8}$$

in which

$$\xi \equiv 1 - E_\zeta \left[(1-\zeta)^{1-\gamma} \right] ,$$

and

$$Q \equiv \rho - \theta(1-\gamma) \left(\mu - \gamma \frac{\sigma^2}{2} \right) .$$

In light of equation (8), the goal of the recursive algorithm is to compute function $b(N, T)$.

2.   Numerical Computation of the Exact Solution in the case of $\theta = 1$

In order to test the effectiveness of the recursive algorithm it is useful to numerically compute the available exact solution when $\theta = 1$, as a yardstick for controlling approximation errors of the polynomial approximation in the recursive procedure outlined below. The closed-form solution for $b(N, T)$ in the case of $\theta = 1$ is,

$$b(N, T) = \frac{1}{Q} \left(\frac{Q}{\xi} T \right)^N e^{\frac{Q}{\xi} T} \Gamma \left(1 - N, \frac{Q}{\xi} T \right) , \tag{9}$$

in which $\Gamma(a, x)$ is the incomplete gamma function $\Gamma(a, x) = \int_x^\infty e^{-z} z^{a-1} dz$, i.e.,

$$\Gamma \left(1 - N, \frac{Q}{\xi} T \right) = \int_{\frac{Q}{\xi} T}^\infty e^{-z} z^{-N} dz .$$

3

Matlab does not have a built-in routine calculating incomplete gamma functions, and since (9) involves an integral, one idea is to use numerical integration. But Matlab has a built-in routine that calculates the exponential integral, which is closely related to the incomplete gamma function. In particular, according to Paris (2010, eq. 8.19.1, p. 185),

$$\mathbf{E}_N(x) = x^{N-1}\Gamma(1-N,x) \ , \tag{10}$$

in which $\mathbf{E}_N(x)$ denotes the generalized exponential integral of degree N (boldface is used in order to avoid confusion with expectations operators), given by,

$$\mathbf{E}_N(x) = \int_1^\infty e^{-xz} z^{-N} dz \ .$$

Combining (9) and (10), the computation of $b(N,T)$ boils down to computing the formula,

$$b(N,T) = \frac{1}{Q}\left(\frac{Q}{\xi}T\right) e^{\frac{Q}{\xi}T} \mathbf{E}_N\left(\frac{Q}{\xi}T\right) \ . \tag{11}$$

Matlab has a built-in routine under the command "`expint(x)`", that corresponds to the special case of the exponential integral of degree $N = 1$, namely,

$$\texttt{expint(x)} = \mathbf{E}_1(x) = \int_1^\infty e^{-xz} z^{-1} dz = \int_x^\infty e^{-y} y^{-1} dy \ ,$$

in which the last expression, resulting after a change-of-variables trasformation $(y = xz)$, coincides with the opening equation presented in Temme (2010, eq. 6.2.1, p. 150). Fast and efficient computation of generalized exponential integrals, $\mathbf{E}_N(x)$, is achieved through a recursion appearing in Paris (2010, eq. 8.19.12, p. 186),

$$\mathbf{E}_{N+1}(x) = \frac{1}{N}\left[e^{-x} - x\mathbf{E}_N(x)\right] \ . \tag{12}$$

3.   Chebyshev Algorithm

First, we need to pick zero-nodes for the Chebyshev algorithm. First, notice that the domain of $T_j(x)$ is $[-1, 1]$.[1] Thanks to linearity properties of vector spaces it is straightforward to implement the Chebyshev projection method to values $Z \in [\underline{Z}, \bar{Z}]$ through the linear transformation,

$$X(Z) = \frac{2}{\bar{Z} - \underline{Z}} \cdot Z - \frac{\bar{Z} + \underline{Z}}{\bar{Z} - \underline{Z}} \ . \tag{13}$$

Important is also the inverse transformation of (13), according to which,

$$Z(x) = \frac{(x+1)(\bar{Z} - \underline{Z})}{2} + \underline{Z} \ . \tag{14}$$

3.1   Zeros of the n-th degree Chebyshev polynomial

The idea here is that an n-th degree Chebyshev polynomial, $T_n(x) = \cos(n \cdot \arccos(x))$, is a periodic fuction with codomain $[-1, 1]$. We can minimize rounding errors even further if our gridpoints are endogenously chosen. Specifically rounding errors minimized if gridpoints correspond to values x_k, such that $T_n(x_k) = 0$, $k \in \{1, ..., n\}$.[2]

The usual transformation is $\theta_k = \arccos(x_k)$, so $T_n(x)$ can be expressed as $\cos(n \cdot \theta_k)$. We know where the $\cos(n \cdot \theta_k)$ cuts the abscissa, it is the values $\pi/2$, $3\pi/2$, $5\pi/2$, and so on, namely all odd positive integers divided by 2, which do not exceed n. With some algebra, we can find that x_k is given by,

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k \in \{1, ..., n\} \ . \tag{15}$$

The Matlab program "`Chebyshev_zeros.m`" implements the formula given by equation (15), using the straightforward commands,

---

[1] The Chebyshev polynomials algorithm follows closely Heer and Maußner (2005, Ch. 8).
[2] This error-minimizing property of gridpoints $\{x_k\}_{k=1}^{n}$ with $T_n(x_k) = 0$ can be proved formally. See, for example, Judd (1992) and further references therein.

```
function x_k = Chebyshev_zeros(n)

 k = 1:n;

 x_k = cos((2*k-1)./(2*n)*pi);

end
```

Notice a great advantage of this endogenous gridpoint choice procedure. In practice we deal with interpolating known functions and unknown functions. The choice of gridpoints $\{x_k\}_{k=1}^n$ with $T_n(x_k) = 0$ is independent of whether we know the function to be approximated or not. The choice of gridpoints only depends on the choice of the polynomial degree, n, of $T_n(x)$, because it is desirable to have more gridpoints than the highest degree n of a sequence of Chebyshev polynomials $\{T_j(x)\}_{j=0}^n$. These technical issues become clearer while implementing of the Chebyshev regression algorithm.

3.2 The two-dimensional grid in matrix form

The Chebyshev-approximated function will have the form,

$$f(z_1, z_2) \simeq \sum_{j_1=0}^{\nu_1-1} \sum_{j_1=0}^{\nu_1-1} \theta_{j_1,j_2} T_{j_1}(X(z_1)) T_{j_2}(X(z_2)) \ . \tag{16}$$

In order to take advantage of the discrete-orthogonality conditions we will approximate the Chebyshev form given by (16) through Chebyshev-zero grids. In Heer and Maußner (2005, Ch. 8, pp. 440-1) we can see that if we have a two-dimensional function $f(z_1, z_2)$, then the Chebyshev-zero grids, $(\bar{x}_{1,k_1}, \bar{x}_{2,k_2})$, will produce an $m_1 \times m_2$ matrix, $\bar{\mathbf{Y}}$, with generic element $\bar{y}_{k_1,k_2} = f(Z_1(\bar{x}_{1,k_1}), Z_2(\bar{x}_{2,k_2}))$, in which,

$$Z_i(x) = \frac{(x+1)(\bar{Z}_i - \underline{Z}_i)}{2} + \underline{Z}_i \ , \quad i = 1, 2 \ .$$

Heer and Maußner (2005, Ch. 8, p. 441) prove that the optimal Chebyshev-approximation estimator $\hat{\theta}_{j_1,j_2}$ is given by,

$$\hat{\theta}_{0,0} = \frac{1}{m_1 m_2} \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \bar{y}_{k_1,k_2} \tag{17}$$

$$\hat{\theta}_{j_1,0} = \frac{2}{m_1 m_2} \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \bar{y}_{k_1,k_2} T_{j_1} \left(\bar{x}_{1,k_1} \right) \tag{18}$$

$$\hat{\theta}_{0,j_2} = \frac{2}{m_1 m_2} \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \bar{y}_{k_1,k_2} T_{j_2} \left(\bar{x}_{2,k_2} \right) \tag{19}$$

$$\hat{\theta}_{j_1,j_2} = \frac{4}{m_1 m_2} \sum_{k_1=1}^{m_1} \sum_{k_2=1}^{m_2} \bar{y}_{k_1,k_2} T_{j_1} \left(\bar{x}_{1,k_1} \right) T_{j_2} \left(\bar{x}_{2,k_2} \right) \ , \tag{20}$$

for $j_1 \in \{1, ..., \nu_1 - 1\}$ and $j_2 \in \{1, ..., \nu_2 - 1\}$.

Consider the matrices,

$$\mathbf{T}_1 \left(X \left(\bar{\mathbf{Z}}_1 \right) \right) = \mathbf{T}_1 \left(\bar{\mathbf{x}}_1 \right) = \begin{bmatrix} T_0 \left(\bar{x}_{1,1} \right) & T_1 \left(\bar{x}_{1,1} \right) & \cdots & T_{\nu_1-1} \left(\bar{x}_{1,1} \right) \\ T_0 \left(\bar{x}_{1,2} \right) & T_1 \left(\bar{x}_{1,2} \right) & \cdots & T_{\nu_1-1} \left(\bar{x}_{1,2} \right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0 \left(\bar{x}_{1,m_1} \right) & T_1 \left(\bar{x}_{1,m_1} \right) & \cdots & T_{\nu_1-1} \left(\bar{x}_{1,m_1} \right) \end{bmatrix} ,$$

and

$$\mathbf{T}_2 \left(X \left(\bar{\mathbf{Z}}_2 \right) \right) = \mathbf{T}_2 \left(\bar{\mathbf{x}}_2 \right) = \begin{bmatrix} T_0 \left(\bar{x}_{2,1} \right) & T_1 \left(\bar{x}_{2,1} \right) & \cdots & T_{\nu_2-1} \left(\bar{x}_{2,1} \right) \\ T_0 \left(\bar{x}_{2,2} \right) & T_1 \left(\bar{x}_{2,2} \right) & \cdots & T_{\nu_2-1} \left(\bar{x}_{2,2} \right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0 \left(\bar{x}_{2,m_2} \right) & T_1 \left(\bar{x}_{2,m_2} \right) & \cdots & T_{\nu_2-1} \left(\bar{x}_{2,m_2} \right) \end{bmatrix} .$$

Notice that $\mathbf{T}_1 \left(\bar{\mathbf{x}}_1 \right)$ is of size $m_1 \times \nu_1$, while $\mathbf{T}_2 \left(\bar{\mathbf{x}}_2 \right)$ is an $m_2 \times \nu_2$ matrix. Consider also the two matrices,

$$\mathbf{I}_{m_1} = \begin{bmatrix} \frac{1}{m_1} & 0 & \cdots & 0 \\ 0 & \frac{2}{m_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{m_1} \end{bmatrix} ,$$

7

and

$$
\mathbf{I}_{m_2} =
\begin{bmatrix}
\frac{1}{m_2} & 0 & \cdots & 0 \\
0 & \frac{2}{m_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{2}{m_2}
\end{bmatrix} ,
$$

with $\mathbf{I}_{m_1}$ being of size $m_1 \times m_1$, and with $\mathbf{I}_{m_2}$ being of size $m_2 \times m_2$.

In the special case in which $\nu_1 = m_1$ and $\nu_2 = m_2$, we can verify that the $\nu_1 \times \nu_2$ $(= m_1 \times m_2)$ matrix $\hat{\boldsymbol{\Theta}}$ that contains all Chebyshev coefficients $\hat{\theta}_{j_1,j_2}$ for $j_1 \in \{0, ..., \nu_1 - 1\}$ and $j_2 \in \{0, ..., \nu_2 - 1\}$, as these are given by

$$
\hat{\boldsymbol{\Theta}} = \mathbf{I}_{m_1} \cdot \mathbf{T}_1 \left(\bar{\mathbf{x}}_1\right)^{\mathsf{T}} \cdot \bar{\mathbf{Y}} \cdot \mathbf{T}_2 \left(\bar{\mathbf{x}}_2\right) \cdot \mathbf{I}_{m_2} . \tag{21}
$$

Expression (21) is helpful for introduction to Matlab.

Finally, notice that

$$
\bar{\mathbf{Y}} = \mathbf{T}_1 \left(\bar{\mathbf{x}}_1\right) \cdot \hat{\boldsymbol{\Theta}} \cdot \mathbf{T}_2 \left(\bar{\mathbf{x}}_2\right)^{\mathsf{T}} , \tag{22}
$$

which is easy to verify from the expression given by (21) and the Chebyshev discrete-orthogonality conditions, which imply,

$$
\mathbf{T}_1 \left(\bar{\mathbf{x}}_1\right) \cdot \mathbf{I}_{m_1} \cdot \mathbf{T}_1 \left(\bar{\mathbf{x}}_1\right)^{\mathsf{T}} = \mathbf{I}_{(m_1 \times m_1)} \quad \text{and} \quad \mathbf{T}_2 \left(\bar{\mathbf{x}}_2\right) \cdot \mathbf{I}_{m_2} \cdot \mathbf{T}_2 \left(\bar{\mathbf{x}}_2\right)^{\mathsf{T}} = \mathbf{I}_{(m_2 \times m_2)} ,
$$

and in which $\mathbf{I}_{(m_1 \times m_1)}$ and $\mathbf{I}_{(m_2 \times m_2)}$ are identiy matrices of size $m_1 \times m_1$ and $m_2 \times m_2$.

3.3 The Chebyshev-approximated partial derivative

The Chebyshev-approximated partial derivative $f_{z_1} \left(z_1, z_2\right)$ for the special case in which $\nu_1 = m_1$, uses the formula,

$$
f_{z_1} \left(z_1, z_2\right) \simeq \underbrace{\left(\frac{2}{\left(\bar{Z}_1 - \underline{Z}_1\right)}\right)}_{\overset{\shortparallel}{X'(z_1)}} \cdot \mathbf{T}_1 \left(\bar{\mathbf{x}}_1\right) \cdot \left(\mathbf{I} - \mathbf{A}\right)^{-1} \mathbf{B} \cdot \hat{\boldsymbol{\Theta}} \cdot \mathbf{T}_2 \left(\bar{\mathbf{x}}_2\right)^{\mathsf{T}} , \tag{23}
$$

in which,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} , \tag{24}$$

$$\mathbf{B} = 2 \cdot \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \nu - 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} , \tag{25}$$

are $m_1 \times m_1$ matrices. The formulas given by (23), (24), and (25), implement the rearrangement of coefficients of matrix $\hat{\boldsymbol{\Theta}}$ suggested by Press et al. (2003, Ch. 5, §5.9, p. 240). The error-minimization properties of the suggestion in Press et al. (2003, Ch. 5, §5.9, p. 240) for univariate functions are formally studied by Bruno and Hoch (2012). Equation (23) implements the same matrix-rearrangement idea for partial derivatives.

Alternatively, for the calculation of Chebyshev-approximated partial derivatives one can use the formula,

$$T_j'(x) = \frac{\partial \cos\left(j \cdot \arccos\left(x\right)\right)}{\partial x} = j \frac{\sin\left(j \arccos\left(x\right)\right)}{\sqrt{1 - x^2}} , \tag{26}$$

9

in order to generate the matrix,

$$\mathbf{T}_1'\left(\bar{\mathbf{x}}_1\right) = \begin{bmatrix} T_0'\left(\bar{x}_{1,1}\right) & T_1'\left(\bar{x}_{1,1}\right) & \cdots & T_{\nu_1-1}'\left(\bar{x}_{1,1}\right) \\ T_0'\left(\bar{x}_{1,2}\right) & T_1'\left(\bar{x}_{1,2}\right) & \cdots & T_{\nu_1-1}'\left(\bar{x}_{1,2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_0'\left(\bar{x}_{1,m_1}\right) & T_1'\left(\bar{x}_{1,m_1}\right) & \cdots & T_{\nu_1-1}'\left(\bar{x}_{1,m_1}\right) \end{bmatrix} ,$$

in order to use the approximation,

$$\tilde{f}_{z_1}\left(z_1,z_2\right) \simeq \underbrace{\left(\frac{2}{\left(\bar{Z}_1-\underline{Z}_1\right)}\right)}_{X'(z_1)} \cdot \mathbf{T}_1'\left(\bar{\mathbf{x}}_1\right) \cdot \hat{\boldsymbol{\Theta}} \cdot \mathbf{T}_2\left(\bar{\mathbf{x}}_2\right)^\top . \tag{27}$$

3.4   Choice of grids

As an example, we use the Barro-Ursua (2008) values $\mu = 2.5\%$, $\sigma = 2\%$, $\zeta = 22\%$, $\rho = 4\%$, $\lambda^* = 1/28$, and initial conditions for the start date, $T(0) = \underline{T} = 140$ (the difference between year 2010 and 1870), and we consider 60 years ahead, i.e., the highest value of elapsed time in the grid for T is $\bar{T} = 200$. For implementing the additive-separability constraint by setting $\gamma = 1/\eta$, we pick $\gamma = 1/2$ and $\eta = 2$. Regarding the range of the N grid, $[\underline{N}, \bar{N}]$, it is driven by the 95% confidence interval around $N^*(T) \equiv \lambda^* T$, based on the Gamma density function,

$$\phi\left(\lambda \mid N, 1/T\right) = T \cdot e^{-\lambda t} \frac{\left(\lambda T\right)^{N-1}}{(N-1)!} ,$$

which has cumulative density,

$$\Pr\left(x \le \lambda\right) = \Phi\left(\lambda \mid N, 1/T\right) = \int_0^\lambda T \cdot e^{-xt} \frac{\left(xT\right)^{N-1}}{(N-1)!} \mathrm{d}x . \tag{28}$$

So,

$$\underline{N} = \min\left\{ \left\lfloor \Phi^{-1}\left(\Pr\left(x \le \lambda\right) = \frac{5\%}{2} \mid \lambda^* \bar{T}, \frac{1}{\bar{T}}\right) \cdot \bar{T} \right\rfloor , \left\lfloor \Phi^{-1}\left(\Pr\left(x \le \lambda\right) = \frac{5\%}{2} \mid \lambda^* \underline{T}, \frac{1}{\underline{T}}\right) \cdot \underline{T} \right\rfloor \right\}$$

and

$$\bar{N} = \max\left\{ \left\lceil \Phi^{-1}\left(1 - \frac{5\%}{2} \mid \lambda^*\bar{T}, \frac{1}{\bar{\bar{T}}}\right) \cdot \bar{T} \right\rceil , \left\lceil \Phi^{-1}\left(1 - \frac{5\%}{2} \mid \lambda^*\underline{T}, \frac{1}{\underline{T}}\right) \cdot \underline{T} \right\rceil \right\} ,$$

in which Φ^{-1} is the inverse of the function given by (28). For the above calibration, using the Matlab command "$\mathtt{gaminv}$", the results are, $\underline{N} = 2$ and $\bar{N} = 13$. Finally, well-behaved Chebyshev grid sizes and polynomial values are, $m_1 = m_2 = \nu_1 = \nu_2 = 100$. We start from a 61×12-point grid that uses (11) in order to compute $b\,(N, T)$. Comparing the exact solution from the paper with the results from the HJB equation we are able to examine both the correctness and the performance of the code that uses the HJB equation.

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