Balanced Growth Despite Uzawa*

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Abstract

The evidence for the United States points to balanced growth despite falling investment-good prices and a less-than-unitary elasticity of substitution between capital and labor. This is inconsistent with the Uzawa Growth Theorem. We extend Uzawa's theorem to show that the introduction of human capital accumulation in the standard way does not resolve the puzzle. However, balanced growth is possible if education is endogenous and capital is more complementary with schooling than with raw labor. We present a class of aggregate production functions for which a neoclassical growth model with capital-augmenting technological progress and endogenous schooling converges to a balanced growth path.

Keywords: neoclassical growth, balanced growth, technological progress, capital-skill complementarity

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1 Introduction

Some key facts about economic growth have become common lore. Among those famously cited by Kaldor (1961) are the observation that output per worker and capital per worker have grown steadily, while the capital-output ratio, the real return on capital, and the shares of capital and labor in national income have remained fairly constant. Jones (2015) updates these facts using the latest available data. He reports that real per capita GDP in the United States has grown "at a remarkably steady average rate of around two percent per year" for a period of nearly 150 years, while the ratio of physical capital to output has remained nearly constant. The shares of capital and labor in total factor payments were very stable from 1945 through about 2000.[1]

These facts suggest to many the relevance of a "balanced growth path" and thus the need for models that predict sustained growth of output, consumption, and capital at constant rates. Indeed, neoclassical growth theory was developed largely with this goal in mind. Apparently, it succeeded. As Jones and Romer (2010, p.225) conclude: "There is no longer any interesting debate about the features that a model must contain to explain [the Kaldor facts]. These features are embedded in one of the great successes of growth theory in the 1950s and 1960s, the neoclassical growth model."

Alas, "all is not well," as Hamlet might say. Jones (2015) highlights yet another fact that was noted earlier by Gordon (1990), Greenwood et al. (1997), and others: the relative price of capital equipment, adjusted for quality, has been falling steadily and dramatically since at least 1960. Figure 1 reproduces two series from the FRED database.[2] In the period from 1947 to 2013, the relative price of investment goods declined at a compounded average rate of 2.0 percent per annum. The relative price of equipment declined at an even faster annual rate of 3.8 percent.

The observation of falling capital prices rests uncomfortably with features of the economy thought to be needed for balanced growth. As Uzawa (1961) pointed out, and Schlicht (2006) and Jones and Scrimgeour (2008) later clarified, a balanced growth path in the two-factor neoclassical growth model with a constant and exogenous rate of population growth and a constant rate of labor-augmenting technological progress requires either an aggregate production function with a unitary elasticity of substitution between capital and labor or else an absence of capital-augmenting technological progress. The size of the elasticity

[1] As is well known from Piketty (2014) and others before him and since, the capital share in national income has been rising, and that of labor falling, since around 2000; see, for example, Elsby et al. (2013), Karabarbounis and Neiman (2014), and Lawrence (2015). It is not clear yet whether this is a temporary fluctuation around the longstanding division, part of a transition to a new steady-state division, or perhaps (as Piketty asserts) a permanent departure from stable factor shares.

[2] The Federal Reserve Economic Data (FRED) are maintained by the Federal Reserve Bank of St. Louis. Their investment and equipment prices are based on updates of Gordon's (1990) series by Cummins and Violante (2002) and DiCecio (2009).

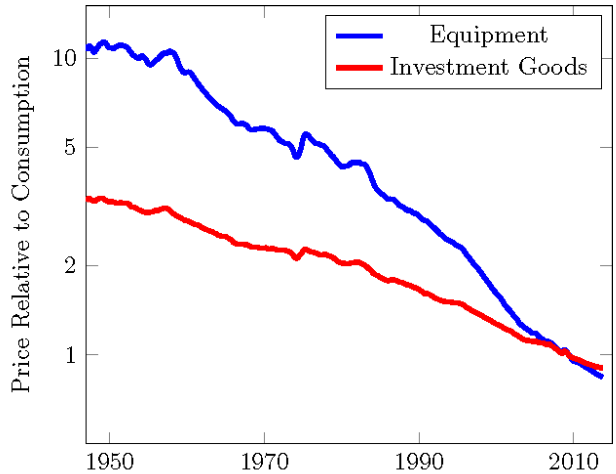

Figure 1: U.S. Relative Price of Equipment, 1947-2013
Source: Federal Reserve Bank Economic Data (FRED), Series PIRIC and PERIC.

of substitution is much debated and still controversial; yet, a preponderance of the evidence suggests an elasticity well below one.[3] And the decline in quality-adjusted prices of investment goods (and especially equipment) relative to final output suggests that capital-augmenting technological progress—embodied, for example, in each new generation of equipment—has been occurring.[4]

The Uzawa Growth Theorem rests on the impossibility of getting an endogenous rate of capital accumulation to line up with an exogenous growth rate of effective labor in the presence of capital-augmenting technological progress, unless the aggregate production function takes a Cobb-Douglas form. The "problem," it would seem, stems from the model's assumption of an inelastic supply of effective labor that does not respond to capital deepening, even over time. If human capital could be accumulated via, for example, investments in schooling, then perhaps effective labor growth would fall into line with growth in effective capital, and a balanced growth path would be possible in a broader set of circumstances. Seen in this light, another fact about the U.S. growth experience is encouraging. We reproduce—as did Jones (2015)—a figure from Goldin and Katz (2007). Figure 2 shows the average years of schooling measured at age thirty for all cohorts of native American workers born between 1876 and 1982.[5] Clearly, educational attainment has been rising steadily for more than a century. Put differently, there has been

[3] Chirinko (2008, p.671), for example, who surveyed and evaluated a large number of studies that attempted to measure this elasticity, concluded that "the weight of the evidence suggests a value of [the elasticity of substitution] in the range of 0.4 to 0.6." In research conducted since that survey, Karabarounis and Nieman (2014) estimate an elasticity of substitution greater than one, but Chirinko et al. (2011), Oberfield and Raval (2014), Chirinko and Mallick (2014), Herrendorf, et al. (2015), and Lawrence (2015) all estimate elasticities below one.

[4] Motivated by Uzawa's Growth Theorem, Acemoğlu (2003) and Jones (2005) develop theories of directed technical change in order to provide an explanation for the absence of capital-augmenting technical change. To be consistent with balanced growth, both look for restrictions that would lead endogenous technical change to be entirely labor-augmenting. Neither attempts to reconcile capital-augmenting technical change with balanced growth.

[5] We are grateful to Larry Katz for providing the unpublished data that allowed us to extend his earlier figure.

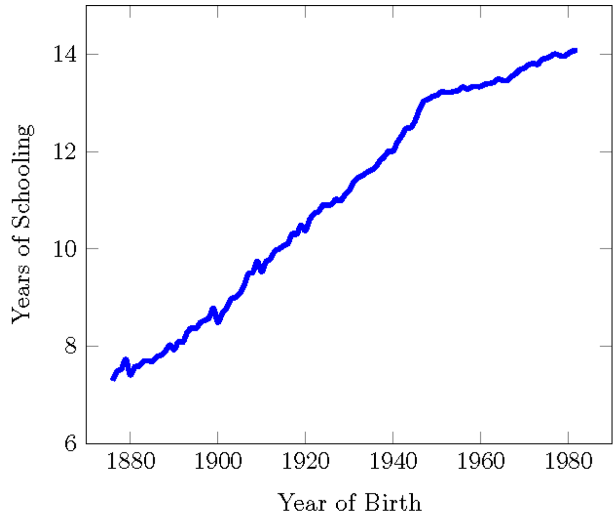

Figure 2: U.S. Education by Birth Cohort, 1876-1982
Source: Goldin and Katz (2007) and additional data from Lawrence Katz.

ongoing investment in "human capital." Indeed, Uzawa (1965), Lucas (1988), and others have established existence of balanced growth paths in neoclassical growth models that incorporate standard treatments of human capital accumulation, albeit in settings that lack capital-augmenting technological progress.[6]

Unfortunately, the usual formulation of human capital does not do the trick. In the next section, we prove an extended version of the Uzawa Growth Theorem that allows for education. We specify an aggregate production function that has effective capital (the product of physical capital and a productivity-augmenting technology term) and "human capital" as arguments. Human capital can be any increasing function of technology-augmented "raw labor" and a variable that measures cumulative investments in schooling. In this setting, we show that balanced growth again requires either a unitary elasticity of substitution between physical capital and human capital or an absence of capital-augmenting technological progress. The intuition is similar to that provided by Jones and Scrimgeour for the original Uzawa theorem. Along a balanced growth path, the value of physical capital that is produced from final goods inherits the trend in output.[7] But the growth rate of final output is a weighted average of the growth rates of effective capital and effective labor, with factor shares as weights. If these shares are to remain constant along a balanced growth path with an aggregate production function that is not Cobb-Douglas, then effective capital and effective labor must grow at common rates. So, the growth rate of output also

[6] Uzawa (1965) studies a model with endogenous accumulation of human capital in which education augments "effective labor supply" so as to generate convergence to a steady state. Lucas (1988) incorporates an externality in his measure of human capital, a possibility that we do not consider here. Acemoğlu (2009, pp. 371-374) characterizes a balanced growth path in a setting with overlapping generations.

[7] If the price of investment goods relative to consumption can change—something Jones and Scrimgeour did not consider—the analogous requirement is that the *value* of the capital stock inherits the growth rate of output.

mirrors the growth rate of effective capital. If the growth rate of output must be equal to both the growth rate of physical capital and that of effective capital, then there is no room for capital productivity to improve or for the cost of investment goods to fall.

But our findings in Section 2 also suggest a resolution to the puzzle. Ongoing increases in education potentially can reconcile the existence of a balanced growth path with a sustained rise in capital or investment productivity and an elasticity of substitution between capital and labor less than unity, provided that schooling enters the aggregate production function in a particular way. If the production technology is such that investments in schooling offset the change in the capital share that results from capital deepening, balanced growth can emerge. To be more precise, suppose that $F(K, L, s; t)$ is the output that can be produced with the technology available at time t by L units of "raw labor" and K units of physical capital, when the economy has an education level summarized by the scalar measure s. The measure might reflect, for example, the average years of schooling in the workforce or the relative supplies of skilled to unskilled hours. Suppose that $F(\cdot)$ has constant returns to scale in K and L and that $\sigma_{KL} < 1$, where $\sigma_{KL} \equiv F_L F_K / F F_{LK}$ is the elasticity of substitution between capital and labor, holding schooling constant. We will show that a balanced growth path with constant factor shares, positive capital-augmenting technological progress, and a rising index of educational attainment can emerge if and only if the ratio of the marginal product of schooling to the marginal product of labor rises as the capital stock grows; i.e., $\partial(F_s/F_L)/\partial K > 0$. Clearly, this precludes a production function of the form $F(K, H; t)$, where $H = G(L, s)$ is a standard measure of human capital, because then F_s/F_L is independent of K. A necessary condition for balanced growth in the presence of capital-augmenting technological progress and a non-unitary elasticity of substitution is a sufficient degree of complementarity between capital and education. Of course, many researchers have noted the empirical relevance of "capital-skill complementarity" (see, most prominently, Krusell, et al., 2000 and Autor, et al., 1998), albeit with varying interpretations of the word "skill" and of the word "complementarity." Our analysis makes clear that the appropriate sense of complementarity is a relative one: growth in the capital stock must raise the marginal product of schooling relatively more than it does the marginal product of raw labor.

The fact that schooling gains *can* offset the effects of capital-augmenting technological progress on the capital share does not of course mean that they *will* do so in a reasonable model of education decisions. We proceed in Section 3 to introduce optimizing behavior. We first solve a social planner's problem that incorporates a reduced-form specification of the trade-off between an index of an economy's education level and its labor supply. A simplifying assumption is that an economy's schooling can be represented by a scalar measure that can jump from one moment to the next. Under this assumption, when the aggregate production function belongs to a specified class, the optimal growth trajectory converges to a balanced-growth path with constant rates of growth of output, consumption and capital, and constant factor shares. Following the presentation of the planner's problem, we describe two distinct models in which the market equilibrium shares the dynamic properties of the efficient solution. In both models, the economy is populated by a continuum of similar dynasties, each comprising a sequence of family members who survive for only infinitesimal lifespans. In the "time-in-school" model of Section 3.2, each individual decides what fraction of her brief existence to devote to schooling, thereby determining her productivity in her remaining time as a worker. Firms allocate capital to their various employees as a function of their productivity levels and therefore their schooling. In the "manager-worker" model of Section 3.3, individuals instead make a discrete educational choice. Those who spend a fixed fraction of their life in school are trained to work as managers with their remaining time. Those who do not opt for management training have their full lives to serve as production workers. In this case, our measure of the economy's education level is the ratio of manager hours to worker hours. We take the productivity of a production unit (workers combined with equipment) as increasing in this ratio due to improved monitoring. In both models the economy converges to a balanced-growth path for a specified class of production functions, all of whose members are characterized by stronger complementarity between capital and schooling than between capital and raw labor.

The class of production functions that we describe in our Assumption 1 is not only sufficient for the emergence of balanced growth, but (essentially) necessary as well.[8] The endogenous gains in education must not only counteract the decline in capital share that would otherwise result from capital-augmenting technological progress with $\sigma_{KL} < 1$, but they must do so exactly. The requirements for balanced growth remain strong, but they are not obviously at odds with the empirical evidence. Moreover, the restrictions on technology are no stronger than those relating to preferences that are known to be needed for balanced growth. Importantly, our simplifying assumptions about demographics and education are not essential to the argument; we show in a companion paper (Grossman et al., 2016) that balanced growth can emerge in an overlapping-generations model with finite lives, wherein the economy's educational state is characterized by a distribution of schooling levels. The key assumption there is analogous to Assumption 1 and relates to how capital affects the productivity of education relative to that of raw labor.

[8]More precisely, we show in the online appendix that balanced growth in the presense of ongoing capital-augmenting technological progress requires that the technology has a representation with the form indicated in Assumption 1.

In the concluding section, we discuss how our findings relate to the large and still-growing literature on the long-run implications of investment-specific technological change.

2 The Extended Uzawa Growth Theorem and a Possible Way Out

In this section, we state and prove a version of the Uzawa Growth Theorem, using methods adapted from Schlicht (2006) and Jones and Scrimgeour (2008). We extend the theorem to allow for falling investment-good prices and the possible accumulation of human capital. We also show how investments in schooling can loosen the straitjacket of the theorem, but only if capital accumulation boosts the marginal product of education proportionally more than it does the marginal product of raw labor.

Let $Y_t = F(A_t K_t, B_t L_t, s_t)$ be a standard neoclassical production function with constant returns to scale in its first two arguments, where, as usual, Y_t is output, K_t is capital, L_t is labor, and where A_t and B_t characterize the state of (disembodied) technology at time t, augmenting respectively physical capital and raw labor.[9] We take s_t to be some scalar measure of the prevailing education level in the economy that is independent of the economy's size. For example, s_t might be the average years of schooling among workers, or the fraction of the labor force with a college degree, or the ratio of trained managers to production-line workers. The labor force L_t grows at some constant rate, g_L, that can be positive, negative, or zero.

At time t, the economy can convert one unit of output into q_t units of capital. Growth in q_t represents what Greenwood et al. (1997) have called "investment-specific technological change." This is a form of *embodied* technical change—familiar from the earlier work of Johansen (1959), Solow (1960) and others—inasmuch as new capital goods require less foregone consumption than did prior vintages of capital. The economy's resource constraint can be written as

$$Y_t = C_t + I_t/q_t \ ,$$

where C_t is consumption and I_t is the number of newly-installed units of capital. Investment in new capital augments the capital stock after replacing depreciation, which occurs at a fixed rate δ; i.e.,

$$\dot{K}_t = I_t - \delta K_t \ .$$

[9] For ease of exposition and for comparability with the literature, we treat technology as a combination of components that augment physical capital and raw labor. However, as we show in the online appendix, our Proposition 1 can readily be extended to any constant-returns to scale production function with the form $F(K_t, L_t, s_t; t)$. Indeed, Uzawa (1961) originally proved his theorem (without the education variable s_t) in this more general form.

We begin with a lemma that extends slightly the one proved by Jones and Scrimgeour (2008) so as to allow for investment-specific technological progress. Define a balanced-growth path (BGP) as a trajectory along which the economy experiences constant proportional rates of growth of Y_t, C_t, and K_t. Let $g_X = \dot{X}/X$ denote the growth rate of the variable X along a BGP. We have

Lemma 1 *Suppose g_q is constant. Then, along any BGP with $0 < C_t < Y_t$, $g_Y = g_C = g_K - g_q$.*

The proof, which closely follows Jones and Scrimgeour, is relegated to the online appendix. The lemma states that the growth rates of consumption and capital mirror that of total output. However, with the possibility of investment-specific technological progress, it is the *value* of the capital stock measured in units of the final good (and the resources used in investment) that grows at the same rate as output.[10]

Now define $\gamma_K \equiv g_A + g_q$. This can be viewed as the *total rate of capital-augmenting technological change*, combining the rate of disembodied progress (g_A) and the rate of embodied progress (g_q). Also, define, as we did before, $\sigma_{KL} \equiv (F_L F_K) / (F_{LK} F)$ to be the elasticity of substitution between capital and labor holding fixed the education index. In the online appendix we prove

Proposition 1 *Suppose q grows at constant rate g_q. If there exists a BGP along which factor shares are constant and strictly positive when the factors are paid their marginal products, then*

$$(1 - \sigma_{KL}) \gamma_K = \sigma_{KL} \frac{F_L}{F_K} \frac{\partial (F_s/F_L)}{\partial K} \dot{s} \ . \tag{1}$$

The proposition stipulates a relationship between the combined rate of capital-augmenting technological progress and the change in the education index that is needed to keep factor shares constant as the value of the capital stock and output grow at common rates.

We can now revisit the two cases that are familiar from the literature. First, suppose that there are no opportunities for investment in schooling, so that s remains constant. This is the setting considered by Uzawa (1961). Setting $\dot{s} = 0$ in (1) yields

Corollary 1 (Uzawa) *Suppose that s is constant. Then a BGP with constant and strictly positive factor shares can exist only if $\sigma_{KL} = 1$ or $\gamma_K = 0$.*

As is well known, balanced growth in a neoclassical economy with exogenous population growth and no investments in human capital requires either a Cobb-Douglas production function or an absence of

[10] When capital goods are valued, their price p_t in terms of final goods must equal the cost of new investment, i.e., $p_t = 1/q_t$.

capital-augmenting technological progress.[11]

Second, suppose that (effective) labor and schooling can be aggregated into an index of human capital, $H(BL, s)$, such that net output can be written as a function of effective physical capital and human capital, as in Uzawa (1965), Lucas (1988), or Acemoğlu (2009). Denote this production function by $\tilde{F}[AK, H(BL, s)] \equiv F(AK, BL, s)$. Then $F_s/F_L = H_s/H_L$, which is independent of K. Setting $\partial(F_s/F_L)/\partial K = 0$ in (1) yields

Corollary 2 (Human Capital) *Suppose that there exists a measure of human capital, $H(BL, s)$, such that $F(AK, BL, s) \equiv \tilde{F}[AK, H(BL, s)]$. Then a BGP with constant and strictly positive factor shares can exist only if $\sigma_{KL} = 1$ or $\gamma_K = 0$.*

In this case, ongoing accumulation of human capital cannot perpetually neutralize the effects of capital deepening on the factor shares.

However, Proposition 1 suggests that balanced growth with constant factor shares might be possible despite a non-unitary elasticity of substitution between capital and labor and the presence of capital-augmenting technological progress, so long as $\dot{s} \neq 0$ and $\partial(F_s/F_L)/\partial K \neq 0$. Suppose, for example, that $\sigma_{KL} < 1$, as seems most consistent with the evidence. Suppose further that educational attainment grows over time, again in line with observation. Then the existence of a BGP with constant factor shares requires $\partial(F_s/F_L)/\partial K > 0$; i.e., an increase in the capital stock must raise the marginal product of schooling by proportionally more than it does the marginal product of raw labor. In looser parlance, the technology must be characterized by "capital-skill complementarity".[12]

The results in this section use only resource constraints and the assumption that factors are paid their marginal products. We have, as yet, provided no model of savings or of schooling decisions. Moreover, we have shown that a BGP with constant factor shares *might exist*, but not that one *does exist* under some reasonable set of assumptions about individual behavior and a reasonable specification of the aggregate

[11] Our Proposition 1 is predicated on constant and interior factor shares. But, in the Uzawa case, log differentiation of the production function with to respect to time, holding s constant, implies

$$g_Y = \theta_K(g_A + g_K) + (1 - \theta_K)(g_B + g_L)$$

where $\theta_K = KF_K/Y$ is the capital share in national income. In a steady state in which Y and K grow at constant rates in response to constant rates of growth of A, B, L and q, θ_K must be constant as well. Note that Jones and Scrimgeour do not assume constant factor shares in their statement and proof of the Uzawa Growth Theorem.

[12] Some might ask why we interpret s as "schooling," rather than some other variable that evolves over time and affects factor productivity. First, we need s to be endogenous, otherwise it could be subsumed into the technology. Second, we want s to be something that econometricians have used as a control variable when estimating the elasticity of substitution, σ_{KL}, inasmuch as we rely on those estimates when assuming $\sigma_{KL} < 1$. Most recent estimates of the elasticity of substitution use quality-adjusted measures of labor and wages that control for schooling (e.g., Antras, 2004, Klump et al., 2007, Oberfield and Raval, 2014) or focus on cross-sectional variation across industries so that schooling choices do not vary (Chirinko et al., 2011).

production function. In the next section, we study a simple economy in which the level of education can be summarized by a scalar variable that can jump discretely from one moment to the next. In Grossman et al. (2016), we consider a more realistic setting in which individuals' education accumulates slowly over time and the distribution of schooling levels in the economy evolves gradually.

3 Balanced Growth with Short Lifespans

We begin by posing a social planner's problem that incorporates a reduced-form treatment of schooling choice. In Section 3.1, the planner designs a time path for a scalar variable that summarizes the education level in the workforce. The planner faces a trade-off between the level of schooling and the labor available for producing output. The economy experiences both labor-augmenting and capital-augmenting technological progress, and the elasticity of substitution between capital and labor in aggregate production is less than one. Here we show that the planner's allocation converges to a unique BGP for a specified class of production functions and under certain parameter restrictions. Moreover, if the efficient allocation can be characterized by balanced growth after some moment in time, then the technology *must* have a representation with a production function in the specified class. We derive the steady-state growth rate of output for the planner's solution and the associated (and constant) factor shares.

In the succeeding subsections, we develop a pair of models of individual behavior and aggregate production that generate education functions that exhibit the form posited in Section 3.1. At the end of the section, we discuss briefly the results in Grossman et al. (2016) that can be derived from a more realistic model of schooling choice with overlapping generations.[13]

3.1 A Planner's Problem with a Reduced-Form Education Function

The economy comprises a continuum of identical family dynasties of measure one. Each family has a continuum N_t of members alive at time t, where N_t grows at the exogenous rate n. Dynastic utility at some time t_0 is given by

$$u(t_0) = \int_{t_0}^{\infty} N_t e^{-\rho(t-t_0)} \frac{c_t^{1-\eta} - 1}{1-\eta} dt \ , \tag{2}$$

where c_t is consumption per family member at time t and ρ is the subjective discount rate.

Consider the problem facing a social planner who seeks to maximize utility for the representative

[13] In our working paper, we also describe how the model can be extended to include directed technical change, in the manner suggested by Acemoğlu (2003). We show that the equilibirium of such a model generally exhibits both capital-augmenting and labor-augmenting technical change.

dynasty subject to a resource constraint, an evolving technology, and an ongoing trade-off between some measure of the economy's education level and the contemporaneous labor supply. Write this trade-off in reduced form as $L_t = D(s_t) N_t$, with $D'(s_t) < 0$ for all s_t, where L_t measures the "raw labor" that produces output at time t and s_t is a scalar index that summarizes the distribution of schooling levels among those workers. The production function takes the form $Y_t = F(A_t K_t, B_t L_t, s_t)$, where A_t again converts physical capital to "effective capital" in view of the disembodied technology available at time t, and similarly B_t converts raw labor to effective labor. We assume that $F(\cdot)$ has constant returns to scale in its first two arguments, i.e., that doubling the physical inputs doubles output for any education level and any state of technology. The economy can convert one unit of the final good into q_t units of capital at time t. Capital depreciates at the constant rate δ and labor-augmenting technological progress takes place at the constant rate $\gamma_L \equiv \dot{B}_t / B_t$.

We assume that the technology can be represented by a member of a class of aggregate production functions that take the following form.

Assumption 1 *The production function can be written as* $F(AK, BL, s) = \tilde{F}\left[D(s)^a AK, D(s)^{-b} BL\right]$, *with* $a, b > 0$, *where*

(i) $h(z) \equiv \tilde{F}(z, 1)$ *is strictly increasing, twice differentiable, and strictly concave for all* z*; and*

(ii) $\sigma_{KL} \equiv F_L F_K / F F_{LK} < 1$.

Assumption 1 immediately implies that $\partial(F_s/F_L)/\partial K > 0$.[14] Therefore, the technology satisfies the pre-requisites for the existence of a BGP, per Proposition 1, provided that the planner's optimal choice of schooling is rising over time.

We also impose some parameter restrictions. Let $\mathcal{E}_h(z) \equiv z h'(z) / h(z)$ be the elasticity of the $h(\cdot)$ function. Note that $\mathcal{E}_h(z)$ is strictly decreasing under Assumption 1.[15] We adopt

Assumption 2 *(i)* $\lim_{z \to 0} \mathcal{E}_h(z) < \frac{b}{a+b}$*; (ii)* $\lim_{z \to \infty} \mathcal{E}_h(z) < \frac{b-1}{a+b-1} < \lim_{z \to 0} \mathcal{E}_h(z)$*; (iii)* $\rho > n + (1-\eta)\left[\gamma_L + \frac{b-1}{a}\gamma_K\right]$.

Part (i) of Assumption 2 ensures that the marginal product of schooling is non-negative for all levels of K, L, and s.[16] Part (ii) guarantees that the optimal schooling choice is positive, as we will see below. It

[14] See the proof in the online appendix. We also prove that, under Assumption 1, $\sigma_{KL} < 1$ if and only if $F(AK, BL, s)$ is strictly log supermodular in K and s, which is another way of expressing capital-skill complementarity.

[15] To see this, note that $d \ln \mathcal{E}_h(z) / d \ln z = [1 - \mathcal{E}_h(z)](\sigma_{KL} - 1)/\sigma_{KL}$, which is negative when $\sigma_{KL} < 1$.

[16] Assumption 1 implies $F_s(AK, BL, s) = [D'(s)/D(s)][aKF_K(AK, BL, s) - bLF_L(AK, BL, s)]$. The assumption that $F(AK, BL, s)$ is constant returns to scale implies $F(AK, BL, s) = KF_K(AK, BL, s) + LF_L(AK, BL, s)$. Combining these two equations, we see that $F_s > 0$ for all AK, BL and s if and only if $\lim_{z \to 0} \mathcal{E}_h(z) < b/(a+b)$.

also implies, with Assumption 1 and Assumption 2(i), that $b > 1$.[17] Part (iii) ensures that utility in (2) is finite.

The planner's problem has two separable components, one static and one dynamic. The static problem is to choose the education level and the labor force at every moment in time so as to maximize output Y_t, subject to the inverse relationship between the two. The dynamic problem is to allocate consumption over time so as to maximize dynastic utility in (2), subject to the aggregate capital accumulation equation, $\dot{K}_t = q_t (Y_t - N_t c_t) - \delta K_t$. The solution to the dynamic problem is standard and features the familiar Euler equation. We provide the details in the online appendix. Here we focus on the static problem, which captures how the planner's choice of education, s_t, relates to the state of technology, as summarized by $\{A_t, B_t, q_t\}$, and the momentary capital stock, K_t.

In the light of Assumption 1, the planner's static problem boils down to choosing s_t and L_t at every moment in time to maximize $Y_t = \tilde{F}\left[D(s_t)^a A_t K_t, D(s_t)^{-b} B_t L_t\right]$, subject to the resource constraint, $L_t = D(s_t) N_t$. Once we substitute the constraint into the maximand, we have

$$\begin{aligned} Y_t &= \max_{s_t} \tilde{F}\left[D(s_t)^a A_t K_t, D(s_t)^{1-b} B_t N_t\right] \\ &= \max_{s_t} D(s_t)^{1-b} B_t N_t \tilde{F}\left[\frac{D(s_t)^{a+b-1} A_t K_t}{B_t N_t}, 1\right] . \end{aligned}$$

Now, make a change of variables, using $z_t \equiv D(s_t)^{a+b-1} A_t K_t / B_t N_t$, and recall the definition of $h(z) \equiv \tilde{F}(z, 1)$. Then the static problem can be rewritten as

$$Y_t = \max_{z_t} (B_t N_t)^{1-\theta} (A_t K_t)^{\theta} z_t^{-\theta} h(z_t) \ , \tag{3}$$

where $\theta \equiv (b-1)/(a+b-1)$. The first-order condition for this problem implies

$$\mathcal{E}_h(z_t) = \theta \qquad \text{for all } t \geq t_0. \tag{4}$$

In other words, the planner chooses education so that $z_t \equiv D(s_t)^{a+b-1} A_t K_t / B_t N_t$ remains constant over time; $z_t = z^* = \mathcal{E}_h^{-1}(\theta)$. In this sense, the planner offsets capital deepening with increased schooling. Part (ii) of Assumption 2 ensures that there exists a strictly positive solution for z^* and the fact that

[17] Assumption 1(i) implies $\lim_{z\to\infty} \mathcal{E}_h(z) \geq 0$. So, Assumption 2(ii) requires $(b-1)/(a+b-1) > 0$. Thus, if $a+b > 1$, then $b > 1$. Suppose $a+b < 1$ and $b < 1$. Then Assumption 2(i) and Assumption 2(ii) imply $b(a+b-1) < (a+b)(b-1)$ or $b < (b-1)$, which cannot hold. Thus, we must have $b > 1$.

$\mathcal{E}_h(z)$ is strictly decreasing implies that the solution is unique.[18] Once z_t is chosen optimally with $z_t = z^*$, (3) implies that output is a Cobb-Douglas function of effective capital and technology-augmented population, with exponents θ and $1-\theta$, respectively.

We will not rehearse the details of the transition path; these are familiar from neoclassical growth theory. In the appendix, we show that the planner chooses the initial per capita consumption level, c_{t_0}, so as to put the economy on the unique saddle path that converges to a steady state. On the BGP, consumption and output grow at constant rate g_Y and the capital stock grows at constant rate g_K.

We can readily calculate the growth rates of output and consumption along the BGP. From $z_t \equiv D(s_t)^{a+b-1} A_t K_t / B_t N_t$ and the fact that $z_t = z^*$ along an optimal trajectory, we have

$$(a+b-1)\, g_D + g_A + g_K = \gamma_L + n$$

for all $t \geq t_0$. By setting $z_t = z^*$ in (3) and then log differentiating with respect to time, we also find that

$$(a+b-1)\, g_Y = a(\gamma_L + n) + (b-1)(g_A + g_K)$$

along the optimal path. Finally, combining these two equations and using Lemma 1—which requires that $g_Y = g_K - g_q$ along any BGP—we can solve for g_D and g_Y. Proposition 2 reports the results.

Proposition 2 *Suppose there is a trade-off between labor supply and a summary index of economy-wide education given by $L_t = D(s_t) N_t$. Let Assumptions 1 and 2 hold. Then along the optimal trajectory from any initial capital stock, K_{t_0}, the economy converges to a BGP. On the BGP,*

(i) aggregate output and aggregate consumption grow at the common rate $g_Y = n + \gamma_L + \frac{b-1}{a}\gamma_K$;

(ii) the index of education grows according to $\dot{s} = -\frac{\gamma_K D(s)}{a D'(s)}$, so that $g_D = -\frac{\gamma_K}{a}$.

The growth of per capita income is increasing in the rate of labor-augmenting technological progress, just as in the neoclassical growth model without endogenous schooling. But now a BGP exists even when there is ongoing capital-augmenting technological progress or when the price of investment-goods is falling at a constant rate. The fact that $b > 1$ implies that the growth rate of per capita income also is increasing in γ_K, the combined rate of embodied and disembodied capital-augmenting progress.

[18] In the online appendix, we show that the second-order condition is satisfied at $z_t = z^*$ under Assumption 1. Moreover, we show that the second-order condition would be violated if the elasticity of substitution between capital and labor were to exceed one.

We have not as yet introduced any market decentralization, which we will do only for the specific models described in Sections 3.2 and 3.3 below. However, in anticipation that capital will be paid its marginal product in a competitive equilibrium, we can define the capital share in national income at time t as $\theta_{Kt} = (\partial Y_t/\partial K_t)\, K_t/Y_t$. Using (3) with $z_t = z^*$, we see that $\theta_{Kt} = (b-1)\,/\,(a+b-1) \equiv \theta$ for all $t \geq t_0$. The labor share, which includes the return to education, equals $1-\theta$. That is, the planner chooses the trajectories for the capital stock and schooling such that the factor shares remain constant, both along the transition path and in the steady state. Notice that the growth rate and the capital share both are increasing in b and decreasing in a; in this sense, fast growth and a high capital share go hand in hand.

We offer some remarks about the role of Assumption 1 and the intuition for our BGP. With $Y_t = \tilde{F}\left[A_t K_t D\left(s_t\right)^a, B_t L_t D\left(s_t\right)^{-b}\right]$, the effect of schooling on the relationship between inputs and output is akin to that of factor-biased technical progress. Hicks (1932) described the bias in technical progress according to its impact on relative factor demands at given relative factor prices. Technical progress is "labor saving" (or, equivalently, "capital using") if it causes an increased relative demand for capital at the initial wage-to-rental ratio. In our setting, and under Assumption 1, added schooling does exactly that; it tilts the unit isoquants in (K, L) space in such a way that the cost-minimizing technique shifts toward capital.[19] We can say, therefore, that the productivity gains associated with schooling are capital using.

Capital-augmenting technological progress expands the relative supply of effective capital. In our model, it also induces investment in education. This increases the relative demand for capital. With our functional form assumption, the extra demand just absorbs the excess supply. To see that this is so, notice that $D\left(s_t\right)^a A_t q_t$ is constant along the BGP. In short, the optimal schooling choice generates extra demand for equipment that neutralizes the effect of the capital-augmenting progress and the declining investment-good prices on the growth of the effective capital stock.[20]

Effectively, there is a horse race between the effects of capital deepening and of education on the fac-

[19]Following Takayama (1974), define $\varpi\left(k,s,t\right)$ as the ratio of the marginal product of labor to the marginal product of capital, where $k \equiv AK/BL$. Under Assumption 1,

$$\varpi\left(k,s,t\right) = \frac{Bk}{A}\left[\frac{1}{\mathcal{E}_h\left(D\left(s\right)^{a+b}k\right)} - 1\right].$$

Since $D\left(s\right)$ is strictly decreasing in s and $\mathcal{E}_h\left(z\right)$ is strictly decreasing in z, it follows that $\varpi_s < 0$. This means that schooling is Hicks labor-saving in Takayama's terminology.

[20]Violante (2008) defines "skill-biased technical change" as a technology change that, *ceterus paribus*, raises the marginal product of skilled labor relative to that of unskilled labor in the formation of an aggregate labor input. By analogy, we might also say that education under our Assumption 1 is "capital biased"; growth in s raises F_K/F_L at a given input ratio.

tor shares which, with the multiplicative way that $D(s)$ interacts with the two inputs and the constant elasticities on this variable, ends in a dead heat. As capital accumulates and becomes more productive due to technical progress, the less-than-unitary elasticity of substitution between capital and labor exerts downward pressure on the capital share. Meanwhile, complementarity between effective capital and schooling means that capital accumulation raises the return to education. The planner responds by investing more in schooling, which depresses the education-plus-technology augmented capital stock relative to the education-plus-technology augmented labor force. This exerts upward pressure on the capital share. With the functional form specified in Assumption 1, the two forces just balance.

Needless to say, Assumption 1 describes a broad class of technologies. For concreteness, we offer one example. Consider[21]

$$Y_t = (B_t L_t)^{\frac{a}{a+b}} \left\{ (A_t K_t)^{\alpha} + \left[D(s_t)^{-(a+b)} B_t L_t \right]^{\alpha} \right\}^{\frac{b/(a+b)}{\alpha}}. \tag{5}$$

Then output at time t can be expressed as a function of $D(s_t)^a A_t K_t$ and $D(s_t)^{-b} B_t L_t$. With $a > 0, b > 1$, and $\alpha < 0$, Assumptions 1 and 2 are both satisfied. Here, the negative value of α generates the required complementarity between capital and schooling.[22]

One might wonder whether we are able to dispense with the functional-form restriction of Assumption 1. The answer to this question is no. In the appendix, we prove that if $L_t = D(s_t) N_t$ and if the solution to the social planner's problem exhibits balanced growth after some time T with increasing schooling and a constant capital share $\theta_K \in (0,1)$, then either there is no capital-augmenting technological progress ($\gamma_K = 0$) or else the technology can be represented along the equilibrium trajectory by a production function with the form $\tilde{F}\left[A_t K D(s)^a, B_t L D(s)^{-b}\right]$, with $a > 0$ and $b = 1 + a\theta_K/(1-\theta_K) > 1$. In other words, Assumption 1 is not only sufficient for the existence of a BGP with $\gamma_K > 0$ and $\sigma_{KL} < 1$, but it is essentially necessary as well. As with any model that generates balanced growth, knife-edge restrictions

[21] Our example makes use of the fact (shown in the online appendix), that, whenever the marginal product of schooling is positive, Assumption 1 is formally equivalent to assuming that $F(AK, BL, s)$ can be written as

$$F(AK, BL, s) = (BL)^{\frac{a}{a+b}} G\left(AK, D(s)^{-(a+b)} BL\right)^{\frac{b}{a+b}}$$

where $G(\cdot)$ is constant returns to scale, strictly increasing in both its arguments, $G(z,1)$ is twice differentiable and strictly concave for all z and $\sigma_{KL}^G \equiv \frac{G_K G_L}{G G_{KL}} < 1$. Written in this form, the basis for the complementarity between capital and schooling is clear. The example in (5) is the special case of this formulation in which $G(\cdot)$ has a constant elasticity of substitution between its two arguments.

[22] It is possible to interpret (5) in terms of a two-task production process. Suppose each worker contributes a joint input of educated and raw labor ("brains" and "brawn"). The firm combines the educated labor (defined as $D(s_t)^{-(a+b)} L_t$) with effective capital to complete one task. In so doing, the two have a constant elasticity of substitution of $1/(1-\alpha) < 1$. Meanwhile, the input of raw labor addresses the second task. Finally, the two tasks enter the overall production function in Cobb-Douglas form.

are required to maintain the balance; our model is no exception to this rule.

To demonstrate the flexibility of our approach, we next present two examples of market economies that generate the reduced form described above. The discussion of the two models in the main text is brief; details are in the online appendix.

3.2 Balanced Growth in a "Time-in-School" Model

As above, the representative family has a continuum N_t of members at time t. Each life is fleetingly brief; an individual attends school for the first fraction of her momentary existence and then joins the workforce for the remainder of her life. The variable s_t now represents the fraction of life that the representative member of the generation alive at time t devotes to education; she spends the remaining fraction $1 - s_t$ working. In this case, $D(s) = 1 - s$, so that the family's labor supply is $L_t = N_t (1 - s_t)$. Given the brevity of life, there is no discounting of an individual's wages relative to her time in school. But dynasties do discount the earnings (and well being) of subsequent generations relative to those currently alive. Every new cohort starts from scratch with no schooling.

Each individual chooses her consumption, savings, and schooling to maximize total dynastic utility, which at time t_0 is given by (2). Each individual supposes that other family members in her own and subsequent generations will behave similarly. Savings are used to purchase units of physical capital, which are passed on within the family from one generation to the next. The N_t members of the representative dynasty collectively inherit K_t units of capital at time t, considering that the aggregate capital stock is fully owned by the population and there is a unit continuum of dynasties in the economy.

Firms produce output using capital, labor, and the technology available to them at the time. Each firm rents capital on a competitive market and allocates it to its employees, taking into consideration their levels of education. A firm's output is the sum of what is produced by its various workers. As usual, the profit-maximizing choices for the firm equate the marginal product of each unit of capital to the competitive rental rate and the marginal product of each type of worker to her competitive wage. The equilibrium determines a wage schedule, $W_t(s)$, which gives the wage of a worker with schooling s at time t. Even for those schooling options that are not actually chosen in equilibrium at time t, we can calculate a worker's marginal product and thus what the wage would be based on the prevailing technology and the capital that a firm would allocate to such a worker at the prevailing rental rate.

Schooling choices have no persistence for the family. Therefore, an individual alive at time t who seeks to maximize dynastic utility should choose s to maximize her own wage income, $(1 - s) W_t(s)$. The first-order condition for this problem requires $(1 - s_t) W_t'(s_t) = W_t(s_t)$. We show in the appendix that this first–order condition for privately optimal schooling choice implies that in the competitive equilibrium

$$\mathcal{E}_h\left((1 - s_t)^{a+b-1} \frac{A_t K_t}{B_t N_t}\right) = \frac{b-1}{a+b-1}.$$

Evidently, the individual's income-maximizing choice of schooling matches the planner's path for s_t in (4), once we recognize that $D(s) = 1 - s$. Part (ii) of Proposition 2 then implies

$$\dot{s}_t = (1 - s_t) \frac{\gamma_K}{a} \ .$$

On a BGP, schooling rises over time, but at a declining rate; the complementary time spent working, $D(s) = 1 - s$, falls at a constant exponential rate, $\dot{D}(s) / D(s) = -\gamma_K / a$.

It comes as no surprise that the market equilibrium with perfect competition and complete markets mimics the planner's solution. The point we wish to emphasize is that the time-in-school model converges to a BGP and that the wage schedule $W_t(s)$ gives the family members the appropriate incentives to extend their time in school from one generation to the next. The returns to schooling rise with the accumulation of effective capital, thanks to the assumed capital-schooling complementarity, and the extra schooling is exactly what is needed to maintain balanced growth of the two inputs to production.

3.3 Balanced Growth in a "Manager-Worker" Model

Now, we present an entirely different model that yields a similar reduced form. We imagine teams that combine "managers" and "production workers." Firms allocate capital equipment to teams according to their productivity. Only production workers are directly responsible for operating equipment and thus for generating output. But the productivity of a team depends on the ratio of its managers to workers, as in the hierarchical models of management proposed by Beckmann (1977), Rosen (1982), and others.

The family structure, demographics, and preferences are the same as before. Lifespans are short. Each individual decides whether to devote a fixed fraction m of her potential working life to school. If she opts to do so, she will acquire the skills needed to serve as a manager and she will have $1 - m$ units of time remaining to perform this function. Those who do not go for management training are employed as production workers. They will use all of their available time to earn unskilled wages.

Let L_t be the time units supplied by production workers at time t and let M_t be the time units supplied by managers. Since production workers devote all of their time to their jobs, L_t is also the number of production workers. Managers are in school a fraction m of their time, so the number of managers is $M_t/(1-m)$. The population divides between workers and managers, so

$$L_t + \frac{M_t}{1-m} = N_t \ . \tag{6}$$

This time, we take $s_t = M_t/L_t$ to be our index of schooling. This is the ratio of manager hours to worker hours (or of skilled to unskilled labor) and the inverse of the typical manager's "span of control." With this definition, (6) implies $L_t + L_t s_t/(1-m) = N_t$, so that $D(s) = [1 + s/(1-m)]^{-1}$ in this model.

Monitoring makes the workers and their equipment more productive. In particular, we suppose that the production function at time t can be written as $\tilde{F}\left[D(s)^a A_t K, D(s)^{-b} B_t L\right]$, with $\tilde{F}(\cdot)$ homogeneous of degree one in its two arguments. With $s = M/L$, this implies that output is a constant-returns to scale function of the three inputs, $A_t K, B_t L$ and $B_t M$.

In this model, the education decision for the representative individual born at time t is simple: pursue schooling if lifetime earnings of a manager exceed those of a worker and not otherwise. In an equilibrium with $M_t > 0$, every individual must be indifferent between the two occupations, so that $(1-m) W_{Mt} = W_{Lt}$, where W_{Mt} and W_{Lt} are the wages per unit time of managers and workers, respectively. Over time, accumulation of effective capital exerts upward pressure on the skill premium, because the functional form of Assumption 1 ensures that capital is more complementary with managers than it is with production workers. This provides the incentive for a greater fraction of each new generation to gain skills. The expanding relative supply of managers to workers restores the equality in earnings.

In the appendix, we show that equalization of lifetime earnings of workers and managers implies

$$\mathcal{E}_h \left(\left[1 + \frac{s_t}{1-m} \right]^{-(a+b)} \frac{A_t K_t}{B_t L_t} \right) = \frac{b-1}{a+b-1}.$$

This gives the same education index as in the planner's solution (4). It follows that the economy converges to a BGP, with a constant rate of output growth given by part (i) of Proposition 2, and with a constant capital share and an ever increasing ratio of manager hours to worker hours.

3.4 Balanced Growth with Overlapping Generations

The models described in Sections 3.2 and 3.3 are rather stylized, because they assume that an economy's education can be described by a scalar variable that can jump from one moment to the next. In reality, schooling investments take time and an economy's distribution of education levels adjusts slowly. In a companion paper, Grossman et al. (2016), we develop an overlapping-generations (OLG) model that has these features. Here, we describe briefly the additional insights and predictions that emerge from that analysis.

In the OLG model, individuals experience finite but stochastic lifespans. Births and deaths occur with constant hazard rates. An individual devotes the first part of her existence to school. She chooses the target length of time to remain in school before entering the labor force. If the individual survives to adulthood, she spends the second phase of life working, with a productivity that depends on her educational attainment, her experience, and on technology at the time. Firms allocate capital to their workers as a function of these characteristics, and a firm's total output is the sum of what is produced by its various workers. Productivity rises with experience early in a worker's career, but falls with experience subsequently. If a worker survives until her productivity falls to zero, she retires.

Analogous to Assumption 1, we assume in the OLG model that if L workers with s years of schooling and u years of experience are allocated K units of capital, they can produce $\tilde{F}\left(e^{-as}A_tK, e^{bs}B_tL, u\right)$ units of output at time t. In the equilibrium, every birth cohort chooses a different educational target. The labor force comprises workers with different schooling levels and different years of experience who work with different amounts of capital. Despite this richness, the economy-wide distributions of schooling and experience evolve in a relatively simple way that permits aggregation.

Like the short-lifespan models of Sections 3.2 and 3.3, the OLG economy has a unique BGP. Along the BGP, educational attainment increases *linearly* over time, much like the patterns depicted in Figure 2 for long stretches of U.S. history. The wage structure at every moment takes a Mincerian form (see Mincer, 1974), with log wages that vary in the cross-section with schooling and experience. Finally, the model predicts declining labor-force participation, consistent with the post-war evidence for men in the United States.

One important difference between the OLG model and the short-lifespan models is worth emphasizing. In the OLG model of Grossman et al. (2016), factor shares are neither constant along the transition path nor independent of the rates of technological progress in the long run. Our numerical analysis suggests that a permanent slowdown in the rate of capital-augmenting technological progress will induce an increase in the capital share. In fact, with plausible parameter values, a one percentage point decline in the annual rate of investment-specific technical change—such as has been measured by the International Monetary Fund (2014, p.89) for the period after 2000—might account for much or all of the rise in the capital share in U.S. national income that has been witnessed in those years.

4 Relationship to the Literature on Investment-Specific Technical Change

By way of concluding remarks, it might be useful for us to relate our results to the large literature that has studied the long-run implications of investment-specific technological change. In his seminal paper on embodied technical progress, Solow (1960) did not close his model to solve for a steady state, but he indicated how this could be done. However, Solow employed a Cobb-Douglas production function throughout this paper, and his discussion about closing the model relies on this assumption. Sheshinski (1967) demonstrated convergence to a BGP in an extended version of the Johansen (1959) model with both embodied and disembodied technological progress. Although he does not restrict attention to any particular production function, he does insist that both forms of progress are Harrod-neutral, i.e., they augment the productivity of labor. So, the technology gains in Sheshinski's paper, while embodied in vintages of capital, are nonetheless assumed to be labor-augmenting. These findings are echoed in Greenwood et al. (1997), who resurrected the literature on technological improvements that are embodied in new equipment. They studied an economy that has no opportunities for schooling in which two types of capital ("equipment" and "structures") and labor are combined to produce consumption goods. Unlike Sheshinski, they do not assume that embodied progress is Harrod-neutral and, consequently, they are led to conclude that a Cobb-Douglas production function is necessary to generate balanced growth, in keeping with the dictates of the Uzawa Growth Theorem.

Krusell et al. (2000) posit a technology with capital-skill complementarity according to which output is produced with equipment, structures and two types of labor ("skilled" and "unskilled"). Leaving aside their distinction between equipment and structures, their model is one with capital and two types of labor, much like our manager-worker model in Section 3.3 above. Although their production function incorporates capital-skill complementarity, it does not satisfy the dictates of our Assumption 1. Nor do they endogenously determine the supplies of skilled and unskilled workers. They, and much of the substantial literature that has adopted their production function, do not address the prospects for balanced growth with ongoing declines in investment-good prices and endogenous schooling, but instead focus on the transition dynamics that result from a specified sequence of relative price changes and of factor supplies. Two recent papers do try to generate balanced growth in models of investment-specific technological progress that is not Harrod-neutral. He and Liu (2008) introduce endogenous schooling into the Krusell et al. model, so that the relative supplies of skilled and unskilled labor are determined in the general equilibrium. They define a BGP to be an equilibrium trajectory along which equipment, structures and output all grow at constant rates and the fraction of skilled workers converges to a constant. With this definition, they conclude (see their Proposition 1) that balanced growth is consistent with ongoing investment-specific technological change only when the aggregate production function takes a Cobb-Douglas form. Maliar and Maliar (2011) study a similar environment, but assume instead that the stocks of skilled and unskilled labor grow at constant and exogenous rates. They show that balanced growth requires $g_A < 0$ to offset the investment-specific technology gains, such that (in our notation) $\gamma_K = 0$. In contrast to these papers, we have shown that balanced growth is in fact compatible with a falling relative price of capital, non-negative growth in capital productivity, and $\sigma_{KL} \neq 1$, provided that capital and schooling are sufficiently complementary. Our result requires that the aggregate production function falls into the class defined by Assumption 1 and that an appropriate index of the economy's educational outcome is rising over time.

The basic mechanism in our model is straightforward: over time, growing stocks of effective capital raise the returns to schooling, which induces individuals to spend more time in school. Inasmuch as capital and labor are complements, capital accumulation tends to lower capital's share in national income, but this is offset by the subsequent rise in schooling, because capital and schooling are also complements. When capital and schooling are more complementary than capital and labor, the second effect can neutralize the first. Although the presence of these offsetting forces is natural enough, restrictions on how schooling enters the production function are needed to maintain exact balance along an equilibrium trajectory. The restrictions are in a sense analogous to those usually imposed on preferences in a dynamic model in order to generate balanced growth. Specifically, while it may be natural to assume that income and substitution effects offset one another as wages rise, the intratemporal utility function must be specified in a particular away so as to maintain perfect balance along an equilibrium trajectory. Just as balanced-growth preferences are consistent with a range of intertemporal elasticities of substitution and labor-supply elasticities, so too are the restrictions we impose on the production function consistent with a range of elasticities of substitution between capital and labor and between capital and schooling.

References

[1] Acemoğlu, Daron, 2003. "Labor and Capital-Augmenting Technical Change," *Journal of the European Economic Association 1*(1), 1-37.

[2] Acemoğlu, Daron, 2009. *Introduction to Modern Economic Growth,* Princeton, NJ: Princeton University Press.

[3] Antràs, Pol, 2004. "Is the U.S. Aggregate Production Function Cobb-Douglas? New Estimates of the Elasticity of Substitution," *Contributions to Macroeconomics 4(1).*

[4] Beckmann, Martin J, 1977. "Management Production Functions and the Theory of the Firm," *Journal of Economic Theory 14(1)*, 1-18.

[5] Chirinko, Robert S., 2008. "σ: The Long and the Short of It," *Journal of Macroeconomics 30(2)*, 671-86.

[6] Chirinko, Robert S., Fazzari, Steven M., and Meyer, Andrew P., 2011. "A New Approach to Estimating Production Function Parameters: The Elusive Capital-Labor Substitution Elasticity," *Journal of Business & Economic Statistics 29 (4)*, 587-594.

[7] Chirinko. Robert S. and Mallick, Debdulal, 2014. "The Substitution Elasticity, Factor Shares, Long-Run Growth, and the Low-Frequency Panel Model," CESifo Working Paper Series No. 4895, CESifo Group.

[8] Cummins, Jason and Violante, Giovanni L., 2002. "Investment-Specific Technical Change in the US (1947-2000): Measurement and Macroeconomic Consequences," *Review of Economic Dynamics 5(2)*, 243-84.

[9] DiCecio, Riccardo, 2009. "Sticky Wages and Sectoral Labor Co-Movement," *Journal of Economic Dynamics and Control 33(3)*, 538-53.

[10] Elsby, Michael W. L., Hobijn, Bart, and Sahin, Aysegul, 2013. "The Decline of the U.S. Labor Share," *Brookings Papers on Economic Activity 47(2)*, 1-63.

[11] Goldin, Claudia and Katz, Lawrence F., 2007. "Long-Run Changes in the Wage Structure: Narrowing, Widening, and Polarization," *Brookings Papers on Economic Activity 38(2)*, 135-68.

[12] Gordon, Robert J., 1990. *The Measurement of Durable Goods Prices*, Chicago, IL: University of Chicago Press.

[13] Greenwood, Jeremy, Hercovitz, Zvi and Krusell, Per, 1997. "Long-Run Implications of Investment-Specific Technological Change," *American Economic Review 87(3)*, 342-62.

[14] Grossman, Gene M., Helpman, Elhanan, Oberfield, Ezra and Sampson, Thomas A., 2016. "Decline in the Labor Share: A Neoclassical Exploration," in process.

[15] He, Hui and Liu, Zheng, 2008. "Investment-Specific Technological Change, Skill Accumulation, and Wage Inequality," *Review of Economic Dynamics, 11(2)*, 314-34.

[16] Herrendorf, Berthold, Herrington, Christopher and Valentinyi, Akos, 2013. "Sectoral Technology and Structural Transformation," CEPR Discussion Papers No. 9386, London UK.

[17] International Monetary Fund, 2014. *Economic Outlook*, Washington DC: IMF.

[18] Hicks, John R., 1932. *The Theory of Wages*, London: Macmillan and Co., Ltd.

[19] Johansen, Leif, 1959. "Substitution versus Fixed Production Coefficients in the Theory of Economic Growth: A Synthesis," *Econometrica 27(2)*, 157-176.

[20] Jones, Charles I., 2005. "The Shape of Production Functions and the Direction of Technical Change," *Quarterly Journal of Economics 120(2)*, 517-549.

[21] Jones, Charles I., 2015. "The Facts of Economic Growth," in preparation for the *Handbook of Macroeconomics*.

[22] Jones, Charles I. and Romer, Paul M., 2010. "The New Kaldor Facts: Ideas, Institutions, Population, and Human Capital," *American Economic Journal: Macroeconomics 2(1)*, 224-45.

[23] Jones, Charles I. and Scrimgeour, Dean, 2008. "A New Proof of Uzawa's Steady-State Growth Theorem," *Review of Economics and Statistics 90(1)*, 180-2.

[24] Kaldor, Nicholas, 1961. "Capital Accumulation and Economic Growth" in F.A. Lutz and D.C. Hague, eds., *The Theory of Capital*, New York: St. Martins Press.

[25] Karabarbounis, Loukas and Neiman, Brent, 2014. "The Global Decline of the Labor Share," *Quarterly Journal of Economics 129(1)*, 61-103.

[26] Klump, Rainer, McAdam, Peter, and Willman, Alpo, 2007. "Factor Substitution and Factor Augmenting Technical Progress in the U.S.: A Normalized Supply-Side System Approach," *Review of Economics and Statistics 89(1)*, 183-92.

[27] Krusell, Per, Ohanian, Lee E., Rios-Rull, José-Victor, and Violante, Giovanni L., 2000. "Capital-Skill Complementarity and Inequality: A Macroeconomic Analysis," *Econometrica 68(5)*, 1029-63.

[28] Lawrence, Robert Z., "Recent Declines in Labor's Share in US Income: A Preliminary Neoclassical Account," NBER Working Paper No. 21296, National Bureau of Economic Research.

[29] Lucas, Robert E., Jr., 1988. "On the Mechanics of Economic Development," *Journal of Monetary Economics 22(1)*, 3-47.

[30] Mincer, Jacob A., 1974. *Schooling, Experience, and Earnings*, New York: The National Bureau of Economic Research.

[31] Maliar, Lilia and Maliar, Serguei, 2011. "Capital-Skill Complementarity and Balanced Growth," *Economica 78(310)*, 240-59.

[32] Oberfield, Ezra and Raval, Devesh, 2014. "Micro Data and Macro Technology," NBER Working Paper No. 20452, National Bureau of Economic Research.

[33] Piketty, Thomas, 2014. *Capital in the Twenty-First Century*, Cambridge, MA: Harvard University Press.

[34] Rosen, Sherwin, 1982. "Authority, Control, and the Distribution of Earnings," *The Bell Journal of Economics 13(2)*, 311-23.

[35] Schlicht, Ekkehart, 2006. "A Variant of Uzawa's Theorem," *Economics Bulletin 5(6)*, 1–5.

[36] Sheshinski, Eytan, 1967. "Balanced Growth and Stability in the Johansen Vintage Model," *Review of Economic Studies 34(2)*, 239-48.

[37] Solow, Robert M., 1960. "Investment and Technical Progress," in K.J. Arrow, S. Karlin and P. Suppes, eds., *Mathematical Methods in the Social Sciences*, 1959, Stanford, CA: Stanford University Press.

[38] Takayama, Akira, 1974. "On Biased Technological Progress," *American Economic Review 64(4)*, 631-9.

[39] Uzawa, Hirofumi, 1961. "Neutral Inventions and the Stability of Growth Equilibrium," *Review of Economic Studies 28(2)*, 117-24.

[40] Uzawa, Hirofumi, 1965. "Optimal Technical Change in an Aggregate Model of Economic Growth," *International Economic Review 6(1)*, 18-31.

[41] Violante, Giovanni L., 2008. "Skilled-biased Technical Change," in S.N. Durlauf and L.E. Blume, eds., *The New Palgrave Dictionary of Economics, Second Edition*, New York, NY: Palgrave Macmillan.

Online Appendix for "Balanced Growth Despite Uzawa"

by

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Proofs from Section 2

Proof of Lemma 1

By assumption $C_t < Y_t$. Therefore, the resource constraint $Y_t = C_t + I_t/q_t$ ensures $I_t > 0$. The capital accumulation equation is $\dot{K}_t = I_t - \delta K_t$ implying

$$g_K = \frac{\dot{K}_t}{K_t} = \frac{I_t}{K_t} - \delta.$$

On a BGP g_K is constant meaning that since $I_t > 0$ the growth rates of I and K must be the same. Thus, $g_I = g_K$.

Differentiating the resource constraint and rearranging gives

$$\left(g_C - g_Y\right)\frac{C_t}{Y_t} + \left(g_I - g_q - g_Y\right)\frac{I_t/q_t}{Y_t} = 0.$$

Substituting for $\frac{I_t/q_t}{Y_t} = 1 - \frac{C_t}{Y_t}$ in this expression and using $g_I = g_K$ we have

$$\left(g_K - g_q - g_C\right)\frac{C_t}{Y_t} = g_K - g_q - g_Y.$$

If both sides of this expression equals zero we immediately obtain $g_Y = g_C = g_K - g_q$ as claimed in the lemma. Otherwise, since the growth rates are constant on a BGP it must be that C and Y grow at the same rate implying $g_Y = g_C$. But then the resource constraint implies $\frac{I_t/q_t}{Y_t} = 1 - \frac{C_t}{Y_t}$ is constant and, since $g_I = g_K$, this ensures $g_Y = g_K - g_q$. Therefore, the lemma holds.

Proof of Proposition 1

Since factors are paid their marginal products the capital share is $\theta_K = K_t F_K\left(A_t K_t, B_t L_t, s_t\right)/Y_t$. Note also that because F has constant returns to scale in its first two arguments $F_K\left(A_t K_t, B_t L_t, s_t\right) = A_t F_1\left(A_t K_t, B_t L_t, s_t\right) =$

$A_t F_1(k_t, 1, s_t)$ where $k_t = A_t K_t / B_t L_t$.[23] Therefore, on a BGP where the capital share is positive and constant we have[24]

$$0 = \frac{\dot{\theta}_K}{\theta_K} = g_A + g_K - g_Y + \frac{d \log F_1(k_t, 1, s_t)}{dt} = \gamma_K + \frac{d \log F_1(k_t, 1, s_t)}{dt},$$

where the final equality uses Lemma 1 and $\gamma_K = g_A + g_q$.

Taking the derivative of F_1 and using $kF_{11} + F_{12} = 0$ we have

$$\gamma_K = -\frac{F_{11}\dot{k}_t + F_{1s}\dot{s}_t}{F_1} = \frac{F_{12}}{F_1}\frac{\dot{k}_t}{k_t} - \frac{F_{1s}\dot{s}_t}{F_1} = \frac{1}{\sigma_{KL}}\frac{F_2}{F}\frac{\dot{k}_t}{k_t} - \frac{F_{1s}\dot{s}_t}{F_1},$$

where the final equality uses $\sigma_{KL} = (F_1 F_2)/(F F_{12})$. Since $1 - \theta_K = F_2/F$ this can be rearranged to give

$$\sigma_{KL}\gamma_K = (1 - \theta_K)\frac{\dot{k}_t}{k_t} - \sigma_{KL}\frac{F_{1s}\dot{s}_t}{F_1}. \tag{7}$$

To simplify (7) it will be useful to derive an expression for F_{1s}/F_1. Note that

$$\frac{\partial}{\partial K}\left[\frac{F_s(A_t K_t, B_t L_t, s_t)}{F_L(A_t K_t, B_t L_t, s_t)}\right] = \frac{F_{Ks}}{F_L} - \frac{F_{LK}F_s}{F_L^2} = \frac{F_K}{F_L}\left(\frac{F_{Ks}}{F_K} - \frac{1}{\sigma_{KL}}\frac{F_s}{F}\right). \tag{8}$$

Rearranging, we have $\frac{F_{1s}}{F_1} = \frac{F_{Ks}}{F_K} = \frac{F_L}{F_K}\frac{\partial[F_s/F_L]}{\partial K} + \frac{1}{\sigma_{KL}}\frac{F_s}{F}$. Plugging this into (7) gives

$$\sigma_{KL}\gamma_K = (1 - \theta_K)\frac{\dot{k}_t}{k_t} - \sigma_{KL}\frac{F_L}{F_K}\frac{\partial[F_s/F_L]}{\partial K}\dot{s}_t - \frac{F_s\dot{s}_t}{F}. \tag{9}$$

Finally, differentiating the production function $Y_t = F(A_t K_t, B_t L_t, s_t)$ yields

$$\begin{aligned} g_Y &= \theta_K(g_A + g_K) + (1 - \theta_K)(g_B + g_L) + \frac{F_s\dot{s}_t}{F}, \\ &= g_A + g_K - (1 - \theta_K)\frac{\dot{k}_t}{k_t} + \frac{F_s\dot{s}_t}{F}. \end{aligned}$$

[23] To avoid possible confusion, note that we use $F_K(\cdot)$ and $F_L(\cdot)$ to denote the partial derivatives of $F(\cdot)$ with respect to K and L, respectively, while $F_1(\cdot)$ and $F_2(\cdot)$ denote the partial derivatives of $F(\cdot)$ with respect to its first and second arguments, respectively.

[24] Instead of assuming constant factor shares, this expression can also be obtained by assuming the rental price of capital R_t declines at rate g_q. To see this differentiate $R_t = A_t F_1(k_t, 1, s_t)$.

Using Lemma 1 and $\gamma_K = g_A + g_q$ this implies

$$\gamma_K = (1-\theta_K)\frac{\dot{k}_t}{k_t} - \frac{F_s \dot{s}_t}{F}.$$

Substituting this expression into (9) gives equation (1). This completes the proof.

Generalization of Proposition 1

Proposition 1 assumes technical change is factor augmenting, but we can generalize the proposition by relaxing this restriction. Suppose the production function is $Y = \hat{F}(K, L, s; t)$ where technical change is captured by the dependence of $\hat{F}$ on t. We can decompose technical change into a Harrod-neutral component and a non-Harrod-neutral residual. Technical change is Harrod-neutral if, holding the capital-output ratio and schooling fixed, it does not affect the marginal product of capital (Uzawa 1961). Therefore, we can define the non-Harrod-neutral component of technical change as the change in the marginal product of capital for a given capital-output ratio and schooling.

Let φ be the capital-output ratio and define $\hat{\kappa}(\varphi, s; t)$ by

$$\varphi = \frac{\hat{\kappa}(\varphi, s; t)}{\hat{F}(\hat{\kappa}(\varphi, s; t), 1, s; t)}.$$

$\hat{\kappa}(\varphi, s; t)$ is the capital-labor ratio that ensures the capital-output ratio equals φ given s and t. Differentiating this expression with respect to t while holding s and φ constant and using $\theta_K = \hat{\kappa}\hat{F}_1/\hat{F}$ implies

$$\frac{\hat{\kappa}_t}{\hat{\kappa}} = \frac{1}{1-\theta_K}\frac{\hat{F}_t}{\hat{F}}. \tag{10}$$

When technical change is Harrod-neutral $\frac{d}{dt}\log \hat{F}_1(\hat{\kappa}(\varphi, s; t), 1, s; t) = \hat{\kappa}_t \frac{\partial}{\partial \hat{\kappa}} \log \hat{F}_1 + \frac{\partial}{\partial t}\log \hat{F}_1 = 0$. Thus, we define the non-Harrod-neutral component of technical change Ψ by

$$\Psi \equiv -\sigma_{KL}\left[\hat{\kappa}_t \frac{\partial}{\partial \hat{\kappa}} \log \hat{F}_1(\hat{\kappa}(\varphi, s; t), 1, s; t) + \frac{\partial}{\partial t}\log \hat{F}_1(\hat{\kappa}(\varphi, s; t), 1, s; t)\right].$$

From this definition we have

$$\Psi = -\sigma_{KL}\left(\frac{\hat{F}_{11}\hat{\kappa}_t}{\hat{F}_1} + \frac{\hat{F}_{1t}}{\hat{F}_1}\right),$$
$$= -\sigma_{KL}\left(\frac{\hat{F}_{11}}{\hat{F}_1}\frac{\hat{\kappa}}{1-\theta_K}\frac{\hat{F}_t}{\hat{F}} + \frac{\hat{F}_{1t}}{\hat{F}_1}\right),$$
$$= \frac{\hat{F}_t}{\hat{F}} - \sigma_{KL}\frac{\hat{F}_{1t}}{\hat{F}_1}, \quad (11)$$

where the second line follows from (10) and the third line uses $\hat{\kappa}\hat{F}_{11} = -\hat{F}_{12}$, the definition of σ_{KL} and $1-\theta_K = \hat{F}_2/\hat{F}$. Note that in the case where technical change is factor augmenting we have $\hat{F}(K, L, s; t) = F(A_t K, B_t L, s)$ which implies $\Psi = (1-\sigma_{KL})g_A$.

Using the expression for Ψ given in (11) we obtain the following generalization of Proposition 1.

Proposition 3 *Suppose the production function is $Y = \hat{F}(K, L, s; t)$ and that investment-specific technological progress occurs at constant rate g_q. If there exists a BGP along which the income shares of capital and labor are constant and strictly positive when factors are paid their marginal products, then*

$$(1-\sigma_{KL})\, g_q + \Psi = \sigma_{KL}\frac{\hat{F}_L}{\hat{F}_K}\frac{\partial\left[\hat{F}_s/\hat{F}_L\right]}{\partial K}\dot{s}.$$

To avoid repetition, we omit the proof of Proposition 3 since it follows the same series of steps used to prove Proposition 1.

Suppose either s is constant as in Corollary 1 or the production function can be written in terms of a measure of human capital $H(L, s, t)$ implying $\frac{\partial[\hat{F}_s/\hat{F}_L]}{\partial K} = 0$ as in Corollary 2. Then Proposition 3 implies a BGP with constant and strictly positive factor shares can exist only if $(1-\sigma_{KL})g_q + \Psi = 0$. Thus, a BGP with $\sigma_{KL} \leq 1$, $g_q \geq 0$ and $\Psi \geq 0$ is possible only if technical change that affects the production function is Harrod-neutral and either the elasticity of substitution between capital and labor equals one or there is no investment-specific technological change.

Proofs from Section 3

Implications of Assumption 1

Taking the partial derivative of the production function with respect to s gives

$$F_s = -\frac{D'(s)}{D(s)}\left[bLF_L - aKF_K\right],$$

and from this we obtain

$$\frac{\partial}{\partial K}\left(\frac{F_s}{F_L}\right) = -\frac{D'(s)}{D(s)} a \left[-\frac{F_K}{F_L} - \frac{KF_{KK}}{F_L} + \frac{KF_K F_{LK}}{F_L^2} \right].$$

Since F exhibits constant returns to scale in K and L we have $F = KF_K + LF_L$ and $KF_{KK} = -LF_{LK}$. Using these results in the expression above we have

$$\frac{\partial}{\partial K}\left(\frac{F_s}{F_L}\right) = -\frac{D'(s)}{D(s)} a \frac{FF_{LK}}{F_L^2} \left(1 - \sigma_{KL}\right),$$

which is strictly positive under Assumption 1 since $a > 0$, $\sigma_{KL} < 1$ and $D'(s) < 0$.

F is strictly log supermodular in K and s if and only if $F_{Ks}F - F_K F_s > 0$. Using Assumption 1 to compute these derivatives gives

$$F_{Ks}F - F_K F_s = -\frac{D'(s)}{D(s)} (a+b) LFF_{LK} \left(1 - \sigma_{KL}\right).$$

Since $a + b > 0$ and $D'(s) < 0$ it follows that under the functional form restriction in Assumption 1 the production function F is strictly log supermodular in K and s if and only if $\sigma_{KL} < 1$.

Second Order Condition of the Planner's Problem

The planner chooses z_t to maximize Y_t which is equivalent to choosing z_t to maximize $z_t^{-\theta} h(z_t)$. The first order condition is

$$-\theta z_t^{-\theta-1} h(z_t) + z_t^{-\theta} h'(z_t) = 0,$$

and the second order condition is

$$(z^*)^{-\theta-1} h(z^*) \frac{d}{dz} \mathcal{E}_h(z^*) < 0.$$

Since $\mathcal{E}_h(z)$ is strictly decreasing in z if and only if $\sigma_{KL} < 1$ it follows that the second order condition is satisfied if and only if $\sigma_{KL} < 1$.

Transition Dynamics of the Planner's Problem

After solving for optimal schooling we can write the planner's problem as

$$\max_{\{c_t\}} \int_{t_0}^{\infty} N_t e^{-\rho(t-t_0)} \frac{c_t^{1-\eta} - 1}{1-\eta} dt$$

subject to

$$\dot{K}_t = q_t \left[Y_t(K_t) - N_t c_t \right] - \delta K_t.$$

where $Y_t(K_t)$ is given by (3) with $z_t = z^*$.

Solving this problem we find the planner chooses a consumption path that satisfies

$$\frac{\dot{c}_t}{c_t} = -\frac{\rho + \delta + g_q}{\eta} + \frac{\theta q_t}{\eta} \frac{Y_t(K_t)}{K_t}. \tag{12}$$

Now let $\tilde{Y}_t = e^{-g_Y(t-t_0)} Y_t(K_t)$, $\tilde{C}_t = e^{-g_Y(t-t_0)} N_t c_t$ and $\tilde{K}_t = e^{-g_K(t-t_0)} K_t$ where g_Y is given by part (i) of Proposition 2 and $g_K = g_Y + g_q$. Using (12) and the capital accumulation equation together with the fact that q_t, A_t, B_t and N_t grow at constant rates g_q, g_A, γ_L and n, respectively, we have

$$\tilde{Y}_t = \tilde{Y}\left(\tilde{K}_t\right) = A_{t_0}^{\theta} \left(B_{t_0} N_{t_0}\right)^{1-\theta} \left(z^*\right)^{-\theta} h\left(z^*\right) \tilde{K}_t^{\theta},$$

$$\dot{\tilde{C}}_t = \left[-g_Y + n - \frac{\rho + \delta + g_q}{\eta} + \frac{\theta q_{t_0}}{\eta} \frac{\tilde{Y}(\tilde{K}_t)}{\tilde{K}_t} \right] \tilde{C}_t, \tag{13}$$

$$\dot{\tilde{K}}_t = -(g_Y + g_q + \delta)\tilde{K}_t + q_{t_0} \left[\tilde{Y}\left(\tilde{K}_t\right) - \tilde{C}_t \right]. \tag{14}$$

Since consumption and schooling can jump, K_t (or, equivalently $\tilde{K}_t$) is the economy's only state variable. The pair of differential equations (13) and (14) govern the evolution of the economy from any initial condition K_{t_0}.

Figure 3 depicts a familiar phase diagram. The vertical line labeled CC has $\tilde{K} = \tilde{K}^*$ such that

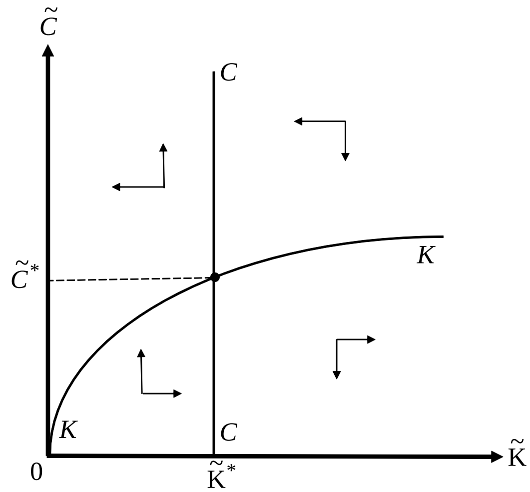

Figure 3: Transitional dynamics and stability of the balanced growth path

$$\frac{\tilde{Y}(\tilde{K}^*)}{\tilde{K}^*} = \frac{1}{\theta q_{t_0}} \left[\eta \left(g_Y - n \right) + \rho + \delta + g_q \right].$$

From (13), we see that $\dot{\tilde{C}}_t = 0$ along this line. The curve labeled KK has $\tilde{C} = \tilde{Y}(\tilde{K}) - (g_Y + g_q + \delta)\tilde{K}/q_{t_0}$. This curve, which from (14) depicts combinations of $\tilde{C}$ and $\tilde{K}$ such that $\dot{\tilde{K}}_t = 0$, can be upward sloping (as drawn) or hump-shaped. In either case, the two curves intersect on the upward sloping part of KK.[25] The intersection gives the unique steady-state values of $\tilde{K} = \tilde{K}^*$ and $\tilde{C} = \tilde{C}^*$, which in turn identify the unique BGP. As is clear from the figure, the BGP is reached by a unique equilibrium trajectory that is saddle-path stable.

Alternative Formulation of Assumption 1

Proposition 4 provides an alternative formulation of Assumption 1 that can be used whenever the marginal product of schooling is positive as guaranteed by part (i) of Assumption 2.

Proposition 4 *Assumption 1 holds with $F_s(AK, BL, s) > 0$ if and only if the production function can be written as $F(AK, BL, s) = (BL)^{\frac{a}{a+b}} G\left[AK, D(s)^{-(a+b)}BL\right]^{\frac{b}{a+b}}$ with $a, b > 0$, where $G(\cdot)$ is constant returns to scale, strictly increasing in both its arguments and*

(i) $G(z,1)$ is twice differentiable, and strictly concave for all z;

(ii) $\sigma^G_{KL} \equiv G_L G_K / G G_{KL} < 1$.

[25] To see this, note that $\tilde{Y}'\left(\tilde{K}_t\right) = \theta \frac{\tilde{Y}(\tilde{K}_t)}{\tilde{K}_t}$. Consequently, the slope of the KK curve is $\theta \frac{\tilde{Y}(\tilde{K}_t)}{\tilde{K}_t} - \frac{g_Y + g_q + \delta}{q_{t_0}}$ which is positive when $\tilde{K} = \tilde{K}^*$ by part (iii) of Assumption 2.

Proof. Suppose Assumption 1 holds with $F_s > 0$ and define

$$G\left[AK, D(s)^{-(a+b)}BL\right] = \left[D(s)^{-(a+b)}BL\right]^{\frac{-a}{b}} \tilde{F}\left[AK, D(s)^{-(a+b)}BL\right]^{\frac{a+b}{b}}.$$

This definition implies $G(\cdot)$ is constant returns to scale and

$$F\left(AK, BL, s\right) = \tilde{F}\left[D(s)^{a}AK, D(s)^{-b}BL\right] = (BL)^{\frac{a}{a+b}}\, G\left[AK, D(s)^{-(a+b)}BL\right]^{\frac{b}{a+b}}.$$

Differentiating $G(\cdot)$ yields

$$\begin{aligned} G_K &= \left[D(s)^{-(a+b)}BL\right]^{\frac{-a}{b}} \frac{a+b}{b}\tilde{F}^{\frac{a}{b}}\tilde{F}_K > 0, \\ G_L &= \left[D(s)^{-(a+b)}BL\right]^{\frac{-a}{b}} \frac{1}{bL}\tilde{F}^{\frac{a}{b}}\left[(a+b)L\tilde{F}_L - a\tilde{F}\right]. \end{aligned}$$

$F_s > 0$ implies $bL\tilde{F}_L - aK\tilde{F}_K > 0$. Using this result together with $\tilde{F} = K\tilde{F}_K + L\tilde{F}_L$ gives $G_L > 0$.

Next, observe that $G(z,1) = \tilde{F}(z,1)^{\frac{a+b}{b}}$. Therefore

$$G_{zz}(z,1) = \frac{a+b}{b}\tilde{F}(z,1)^{\frac{a}{b}-1}\left[\tilde{F}(z,1)\tilde{F}_{zz}(z,1) + \frac{a}{b}\tilde{F}_z(z,1)^2\right].$$

This expression is negative since $z\tilde{F}_{zz}(z,1) = -\tilde{F}_{z2}(z,1), b\tilde{F}_2(z,1) - az\tilde{F}_z(z,1) > 0$ because $F_s > 0$ and $\sigma_{KL} < 1$. It follows that $G(z,1)$ is twice differentiable, and strictly concave for all z.

Finally, we have

$$G_{KL} = \left[D(s)^{-(a+b)}BL\right]^{\frac{-a}{b}} \frac{a+b}{b}\tilde{F}^{\frac{a}{b}}\left[\tilde{F}_{KL} + \frac{a}{b}\frac{\tilde{F}_K\tilde{F}_L}{\tilde{F}} - \frac{a}{b}\frac{\tilde{F}_K}{L}\right],$$

meaning

$$\sigma^G_{KL} = \frac{\tilde{F}_K\tilde{F}_L + \frac{a}{b}\tilde{F}_K\tilde{F}_L - \frac{a}{b}\frac{\tilde{F}\tilde{F}_K}{L}}{\tilde{F}\tilde{F}_{KL} + \frac{a}{b}\tilde{F}_K\tilde{F}_L - \frac{a}{b}\frac{\tilde{F}\tilde{F}_K}{L}},$$

which is less than one since $\sigma_{KL} < 1$.

The converse can be proved in the same manner after defining

$$\tilde{F}\left[D(s)^a AK, D(s)^{-b} BL\right] = \left[D(s)^{-b} BL\right]^{\frac{a}{a+b}} G\left[D(s)^a AK, D(s)^{-b} BL\right]^{\frac{b}{a+b}}.$$

■

Necessity of Functional Form

Consider an economy that satisfies the assumptions required for Lemma 1 to hold and has production function $F(K, L, s; t)$ which is constant returns to scale in its first two arguments. Suppose factors are paid their marginal products and schooling is chosen to satisfy

$$s_t = \arg\max_s F(K_t, L_t, s; t) \text{ subject to } L_t = D(s) N_t.$$

We assume this optimization problem has a unique interior maximum.

Suppose the economy is on a BGP from time T onwards with a constant capital share $\theta_K \in (0, 1)$. With a slight abuse of notation define $\tilde{F}$by

$$\tilde{F}(K, L, s; t) = \tilde{F}\left[A_t K D(s)^a, B_t L D(s)^{-b}\right] \equiv F\left[A_t K D(s)^a, B_t L D(s)^{-b}, s_T; T\right],$$

where $b = 1 + a\theta_K / (1 - \theta_K)$, while A_t and B_t are defined by

$$A_t \equiv e^{g_Y(t-T)} D(s_t)^{-a} \frac{K_T}{K_t},$$
$$B_t \equiv e^{g_Y(t-T)} D(s_t)^{b} \frac{L_T}{L_t}.$$

Since a and b jointly satisfy a single restriction, $\tilde{F}$ defines a one dimensional family of functions.

Differentiating the definitions of A_t and B_t together with the constraint $L_t = D(s_t) N_t$ and using Lemma 1 we obtain

$$\gamma_K \equiv \frac{\dot{A}_t}{A_t} + g_q = a(n - g_L),$$

$$\gamma_L \equiv \frac{\dot{B}_t}{B_t} = g_Y - n - \frac{\theta_K}{1-\theta_K}\gamma_K.$$

γ_K is the total rate of capital-augmenting technical change, while γ_L is the rate of labor-augmenting technical change. When both n and the labor force growth rate g_L are constant then γ_K and γ_L are also constant. Also, provided schooling is increasing over time $n > g_L$ implying that $a > 0$ if and only if γ_K is strictly positive.

We can now prove the following proposition. Part (i) shows that on the BGP F has a one dimensional family of representations of the form $\tilde{F}\left[A_t K D(s)^a, B_t L D(s)^{-b}\right]$. From the expressions for γ_K and γ_L above we see that each member of this family has a different combination of capital-augmenting and labor-augmenting technical change. When we say the production function can be represented by $\tilde{F}$ we mean that the equilibrium allocation and the marginal products of capital, labor and schooling on the BGP are the same under $\tilde{F}$ as under F. However, this does not imply that counterfactual experiments using $\tilde{F}$ will necessarily coincide with counterfactuals under F. The first order impact of some policy changes (e.g., schooling subsidies, capital taxation) depends on σ_{KL} and $\sigma_{Ks} \equiv (F_K F_s)/(F_{Ks} F)$. Therefore, in part (ii) of the proposition we show that if σ_{KL} is constant on the BGP then $\sigma_{KL} = \tilde{\sigma}_{KL} \equiv (\tilde{F}_K \tilde{F}_L)/(\tilde{F}_{KL}\tilde{F})$ and that $\tilde{\sigma}_{Ks} \equiv (\tilde{F}_K \tilde{F}_s)/(\tilde{F}_{Ks}\tilde{F})$ can be written as a function of $\tilde{\sigma}_{KL}$, a and b. Consequently, if σ_{KL} and σ_{Ks} are constant on the BGP then there exist unique values of a and b such that $\tilde{\sigma}_{KL} = \sigma_{KL}$ and $\tilde{\sigma}_{Ks} = \sigma_{Ks}$. Thus, knowing σ_{KL} and σ_{Ks} is sufficient to separate the roles played by capital-augmenting and labor-augmenting technical change. Moreover, when a and b are chosen appropriately counterfactual analysis using $\tilde{F}$ instead of F will, to a first order, give accurate predictions.

Proposition 5 *Suppose for all $t \geq T$ the economy's equilibrium trajectory $\{Y_t, K_t, L_t, s_t\}$ is a BGP with constant and strictly positive factor shares. On the BGP,*

(i) The production function F can be represented by $\tilde{F}$ in the sense that for all $t \geq T$

$$\tilde{F}(K_t, L_t, s_t; t) = F(K_t, L_t, s_t; t),$$

$$\tilde{F}_K(K_t, L_t, s_t; t) = F_K(K_t, L_t, s_t; t),$$

$$\tilde{F}_L(K_t, L_t, s_t; t) = F_L(K_t, L_t, s_t; t),$$

$$\tilde{F}_s(K_t, L_t, s_t; t) = F_s(K_t, L_t, s_t; t);$$

(ii) $\tilde{\sigma}_{KL}$ *and* $\tilde{\sigma}_{Ks}$ *satisfy*

$$\frac{1}{\tilde{\sigma}_{Ks}} - 1 = (a+b)\left(\frac{1}{\tilde{\sigma}_{KL}} - 1\right),$$

and if σ_{KL} *is constant then* $\tilde{\sigma}_{KL} = \sigma_{KL}$.

Proof. Without loss of generality let $T = 0$. Output at $t \geq 0$ is given by

$$\begin{aligned} F(K_t, L_t, s_t; t) = Y_t = e^{g_Y t} Y_0 = e^{g_Y t} F(K_0, L_0, s_0; 0) &= F\left(e^{g_Y t} K_0, e^{g_Y t} L_0, s_0; 0\right), \\ &= F\left(A_t K_t D(s_t)^a, B_t L_t D(s_t)^{-b}, s_0; 0\right), \\ &= \tilde{F}(K_t, L_t, s_t; t). \end{aligned}$$

To show the marginal products of capital are equal, we use the facts that the capital share is constant over time and capital is paid its marginal product. Therefore

$$\begin{aligned} \frac{K_t F_K(K_t, L_t, s_t; t)}{Y_t} = \theta_K = \frac{K_0 F_1(K_0, L_0, s_0; 0)}{Y_0} &= \frac{e^{g_Y t} K_0 F_1(e^{g_Y t} K_0, e^{g_Y t} L_0, s_0; 0)}{e^{g_Y t} Y_0}, \\ &= \frac{A_t K_t D(s_t)^a F_1\left(A_t K_t D(s_t)^a, B_t L_t D(s_t)^{-b}, s_0; 0\right)}{Y_t}, \\ &= \frac{K_t \tilde{F}_K(K_t, L_t, s_t; t)}{Y_t}. \end{aligned}$$

Dividing each side by K_t/Y_t gives $F_K(K_t, L_t, s_t; t) = \tilde{F}_K(K_t, L_t, s_t; t)$. Identical logic using the labor share gives $F_L(K_t, L_t, s_t; t) = \tilde{F}_L(K_t, L_t, s_t; t)$.

To complete the proof of part (i) we show equality of the marginal products of schooling. Optimal schooling choice implies

$$\frac{D'(s_t) L_t}{D(s_t)} = -\frac{F_s(K_t, L_t, s_t; t)}{F_L(K_t, L_t, s_t; t)}.$$

This means the ratio of the marginal product of schooling to output can be written as

$$\frac{F_s(K_t, L_t, s_t; t)}{Y_t} = -(1-\theta_K)\frac{D'(s_t)}{D(s_t)}.$$

We now show that same equation holds for $\tilde{F}$. Differentiating $\tilde{F}$ with respect to s and dividing by output gives

$$\begin{aligned}\frac{\tilde{F}_s\left(K_t, L_t, s_t; t\right)}{Y_t} &= \frac{1}{Y_t}\frac{D'\left(s_t\right)}{D\left(s_t\right)}\Big[aA_tK_tD\left(s_t\right)^a F_1\left(A_tK_tD\left(s_t\right)^a, B_tL_tD\left(s\right)^{-b}, s_0; 0\right) \\ &\qquad -bB_tL_tD\left(s_t\right)^{-b} F_2\left(A_tK_tD\left(s_t\right)^a, B_tL_tD\left(s_t\right)^{-b}, s_0; 0\right)\Big], \\ &= \left[a\theta_K - b\left(1-\theta_K\right)\right]\frac{D'\left(s_t\right)}{D\left(s_t\right)}, \\ &= -(1-\theta_K)\frac{D'\left(s_t\right)}{D\left(s_t\right)}.\end{aligned}$$

To prove part (ii) we start by noting that when σ_{KL} is constant on the BGP, the homogeneity of F implies

$$\begin{aligned}\sigma_{KL} &= \frac{F_1\left(K_0, L_0, s_0; 0\right)F_2\left(K_0, L_0, s_0; 0\right)}{F_{12}\left(K_0, L_0, s_0; 0\right)F\left(K_0, L_0, s_0; 0\right)}, \\ &= \frac{F_1\left(e^{g_Y t}K_0, e^{g_Y t}L_0, s_0; 0\right)F_2\left(e^{g_Y t}K_0, e^{g_Y t}L_0, s_0; 0\right)}{F_{12}\left(e^{g_Y t}K_0, e^{g_Y t}L_0, s_0; 0\right)F\left(e^{g_Y t}K_0, e^{g_Y t}L_0, s_0; 0\right)}, \\ &= \frac{F_1\left(A_tK_tD\left(s_t\right)^a, B_tL_tD\left(s_t\right)^{-b}, s_0; 0\right)F_2\left(A_tK_tD\left(s_t\right)^a, B_tL_tD\left(s_t\right)^{-b}, s_0; 0\right)}{F_{12}\left(A_tK_tD\left(s_t\right)^a, B_tL_tD\left(s_t\right)^{-b}, s_0; 0\right)F\left(A_tK_tD\left(s_t\right)^a, B_tL_tD\left(s_t\right)^{-b}, s_0; 0\right)}, \\ &= \frac{\tilde{F}_K\left(K_t, L_t, s_t; t\right)\tilde{F}_L\left(K_t, L_t, s_t; t\right)}{\tilde{F}_{KL}\left(K_t, L_t, s_t; t\right)\tilde{F}\left(K_t, L_t, s_t; t\right)}, \\ &= \tilde{\sigma}_{KL}.\end{aligned}$$

Next define $\hat{h}(z) \equiv F(z, 1, s_0; 0)$. Then we have

$$\tilde{F}\left(K, L, s; t\right) = B_tLD(s)^{-b}\hat{h}\left[\frac{A_tK}{B_tL}D(s)^{a+b}\right].$$

Taking derivatives of this expression implies

$$\tilde{\sigma}_{KL} = \frac{\mathcal{E}_{\hat{h}}\left[\frac{A_t K}{B_t L} D(s)^{a+b}\right] - 1}{\mathcal{E}_{\hat{h}'}\left[\frac{A_t K}{B_t L} D(s)^{a+b}\right]},$$

$$\tilde{\sigma}_{Ks} = \frac{\frac{b}{a+b} - \mathcal{E}_{\hat{h}}\left[\frac{A_t K}{B_t L} D(s)^{a+b}\right]}{\frac{b}{a+b} - 1 - \mathcal{E}_{\hat{h}'}\left[\frac{A_t K}{B_t L} D(s)^{a+b}\right]}.$$

On the BGP we also have

$$\theta_K = \frac{K_t \tilde{F}_K\left(K_t, L_t, s_t; t\right)}{Y_t} = \mathcal{E}_{\hat{h}}\left[\frac{A_t K_t}{B_t L_t} D(s_t)^{a+b}\right].$$

Combining these expressions and using $b = 1 + a\theta_K / \left(1 - \theta_K\right)$ we have that on the BGP

$$\frac{1}{\tilde{\sigma}_{Ks}} - 1 = (a+b)\left(\frac{1}{\tilde{\sigma}_{KL}} - 1\right).$$

This completes the proof. ■

"Time-in-School" Model

A firm that employs K_t units of physical capital and hires L_t time units from workers with schooling s_t at time t produces $F\left(A_t K_t, B_t L_t, s_t\right) = \tilde{F}\left[A_t K_t \left(1 - s_t\right)^a, B_t L_t \left(1 - s_t\right)^{-b}\right]$ units of output. The production technology satisfies Assumption 1 and the parameter restrictions in Assumption 2 also apply. Aggregate output is simply the sum of the outputs produced by all firms.

Since $F\left(\cdot\right)$ has constant returns to scale in its first two arguments we can define the intensive form production function by $f(k, s) \equiv F(k, 1, s)$ where $f\left(\cdot\right)$ is output per effective unit of labor and $k = A_t K / B_t L$ is the ratio of effective capital to effective labor. Using Assumption 1 the intensive form production function can be written as $f(k, s) = (1 - s)^{-b} h\left[k(1 - s)^{a+b}\right]$.

The competitive firms take the rental rate per unit of capital, R_t, and the wage schedule per unit of time, $W_t\left(s\right)$, as given. A firm that hires workers with education s_t chooses L_t and k_t to maximize $B_t L_t\left[f\left(k_t, s_t\right) - r_t k_t - w_t\left(s_t\right)\right]$, where $r_t \equiv R_t / A_t$ is the rental rate per effective unit of capital and $w_t\left(s_t\right) \equiv W_t\left(s_t\right) / B_t$ is the wage per effective unit of labor. Profit maximization implies, as usual, that

$$f_k(k_t, s_t) = r_t \tag{15}$$

and[26]

$$f(k_t, s_t) - r_t k_t = w_t(s_t). \tag{16}$$

We define the functions $\kappa(s,r)$ and $\omega(s,r)$ such that $f_k[\kappa(s,r), s] \equiv r$ and $\omega(s,r) \equiv f[\kappa(s,r), s] - r\kappa(s,r)$. Then, in equilibrium, $k_t = \kappa(s_t, r_t)$ and $w_t(s_t) = \omega(s_t, r_t)$.

An individual alive at time t who seeks to maximize dynastic utility should choose s to maximize her own wage income, $B_t(1-s)\omega(s, r_t)$, taking the rental rate per unit of effective capital as given. The rental rate will determine, via (15), how much capital the individual will be allocated by her employer as a reflection of her schooling choice. The individual's education decision is separable from her choice of consumption. The first-order condition for income maximization at time t requires

$$(1-s_t)\,\omega_s(s_t, r_t) = \omega(s_t, r_t).$$

But using $\omega(s, r_t) \equiv f[\kappa(s, r_t), s] - r_t\kappa(s, r_t)$ and noting (15), we have $\omega_s(s_t, r_t) = f_s[\kappa(s_t, r_t), s_t]$. In other words, the marginal effect of schooling on the wage reflects only the direct effect of schooling on per capita output; the extra output that comes from a greater capital allocation to more highly educated workers, $f_k\kappa_s$, just offsets the extra part of revenue that the firm must pay for that capital, $r\kappa_s$. Consequently, we can rewrite the first-order condition as

$$(1-s_t)\, f_s[\kappa(s_t, r_t), s_t] = f[\kappa(s_t, r_t), s] - f_k[\kappa(s_t, r_t), s_t]\,\kappa(s_t, r_t)\ .$$

Now replace $f(k,s)$ by $(1-s)^{-b}h\left[k(1-s)^{a+b}\right]$ and use this representation to calculate $f_s(\cdot)$ and $f_k(\cdot)$. After rearranging terms, this yields

$$(b-1)\,h\left[\kappa(s_t, r_t)(1-s_t)^{a+b}\right] = (a+b-1)\,h'\left[\kappa(s_t, r_t)(1-s_t)^{a+b}\right]\kappa(s_t, r_t)(1-s_t)^{a+b}$$

[26] Equation (16) is the zero-profit condition, which is implied by the optimal choice of L_t in an equilibrium with positive output.

or

$$\mathcal{E}_h\left[\kappa\left(s_t, r_t\right)\left(1-s_t\right)^{a+b}\right]=\frac{b-1}{a+b-1}\ .$$

Since $\kappa(s_t, r_t) = k_t = A_t K_t / B_t L_t$ and $L_t = N_t(1 - s_t)$ this expression is identical to the first order condition for optimal schooling choice given in the paper.

Dynasties' intertemporal optimization decisions yield the same consumption and savings choices as in the planner's problem. To see this, start from the no arbitrage condition $\iota_t = R_t/p_t + g_p - \delta$ where ι_t denotes the real interest rate and $p_t = 1/q_t$ is the equilibrium price of a unit of capital.[27] Combining this with $r_t = R_t/A_t$ gives

$$r_t=\frac{1}{q_t A_t}\left(\iota_t+g_q+\delta\right). \tag{17}$$

Individuals' optimal schooling choices imply $\kappa(s_t, r_t)(1 - s_t)^{a+b} = z^*$ for all $t \geq t_0$ where z^* takes the same value as in the planner's problem. Therefore, aggregate output is given by (3) with $z_t = z^*$, just as in the planner's problem.

Using $f(k, s) = (1 - s)^{-b} h\left[k(1 - s)^{a+b}\right]$ the first order condition for profit maximization (15) yields

$$r_t = (1 - s_t)^a h'(z^*).$$

Substituting this expression into the capital market clearing condition $k_t = \kappa(s_t, r_t)$ and using (17) implies the real interest rate satisfies

$$\iota_t=-g_q-\delta+q_t A_t^{\theta}\left(\frac{B_t N_t}{K_t} z^*\right)^{1-\theta} h'(z^*).$$

Combining this equation with the representative dynasty's Euler equation $\dot{c}_t/c_t = (\iota_t - \rho)/\eta$ and using $\mathcal{E}_h(z^*) = \theta$ and (3) gives

$$\frac{\dot{c}_t}{c_t}=-\frac{\rho+\delta+g_q}{\eta}+\frac{\theta q_t}{\eta} \frac{Y_t(K_t)}{K_t}.$$

[27] The no-arbitrage condition states that the real interest rate on a short-term bond equals the dividend rate on a unit of physical capital plus the rate of capital gain on capital equipment (positive or negative), minus depreciation.

Noting that this equation is identical to equation (12) we see that consumption per capita satisfies the same differential equation as in the planner's problem. Since the capital accumulation equation is also the same in both cases we conclude that consumption and the aggregate capital stock follow the same equilibrium trajectory as in the planner's problem.

Schooling Choice in the "Manager-Worker" Model

Recall that the production function can be written as $\tilde{F}\left[A_t K D(s)^a, B_t L D(s)^{-b}\right] = B_t L D(s)^{-b} h\left[k D(s)^{a+b}\right]$ where $s = M/L$, $k = A_t K / B_t L$ and $D(s) = [1 + s/(1-m)]^{-1}$. Since $W_{Mt} = \tilde{F}_M$ and $W_{Lt} = \tilde{F}_L$, differentiating yields

$$W_{Mt} = (a+b) B_t D(s_t)^{-b} \frac{D'(s_t)}{D(s_t)} h\left[k_t D(s_t)^{a+b}\right] \left\{ -\frac{b}{a+b} + \mathcal{E}_h\left[k_t D(s_t)^{a+b}\right] \right\},$$

$$W_{Lt} = B_t D(s_t)^{-b} h\left[k_t D(s_t)^{a+b}\right] \left(1 - \mathcal{E}_h\left[k_t D(s_t)^{a+b}\right] + (a+b)\frac{s_t D'(s_t)}{D(s_t)} \left\{ \frac{b}{a+b} - \mathcal{E}_h\left[k_t D(s_t)^{a+b}\right] \right\} \right).$$

Substituting these expressions into $(1-m)W_{Mt} = W_{Lt}$ and using $D'(s) = -D(s)^2/(1-m)$ implies that, in equilibrium,

$$\mathcal{E}_h\left[\left(1 + \frac{s_t}{1-m}\right)^{-(a+b)} k_t\right] = \frac{b-1}{a+b-1}.$$

The fact that $\mathcal{E}_h(z)$ is declining in z ensures stability of the equilibrium.