AMBIGUOUS MARKET MAKING

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ABSTRACT. This paper investigates the impact of informational ambiguity and attitude to it on security prices. Attention is focused on equilibria in which a market maker has an ambiguity in his probability assessment. We show that the equilibrium ambiguous bid-ask spread can be decomposed into the probabilistic spread and an "ambiguity premium/discount" component characterizing the ambiguity aversion/seeking of the market maker. In particular, for a sufficiently ambiguity averse market maker, the bid-ask spread widens with the informational ambiguity of the market maker, which provides an explanation to drying liquidity and price inefficiency during financial turmoils. An extension to different trade sizes shows that informed traders are more likely to trade large orders when there is a major ambiguous shock to the economy.

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"The action which follows upon an opinion depends as much upon the amount of confidence in that opinion as it does upon the favorableness of the opinion itself."

Frank Knight (1921)

1. Introduction

The main objective of this paper is to understand how informational ambiguity and, naturally, ambiguity attitude affect security prices in an otherwise Glosten and Milgrom (1985) (GM) type sequential trading models. By incorporating ambiguity relevant uncertainty (i.e. uncertainty that can not be explained by probability theory) on the market maker side, we examine the question "how perceived ambiguity or confidence of the market maker in his probability assessment affects market microstructure".

We obtain two main results. First, we derive equilibrium bid/ask prices and spread under an informationally ambiguous market maker in closed form. It turns out that the bid-ask spread with an informational ambiguity can be wider or narrower than the standard probabilistic GM spread, depending on the combination of ambiguity and ambiguity attitude. To be more precisely, we introduce a concept of *"bid-ask spread neutrality"* in which the equilibrium spread under ambiguity is the same as the one in GM model without ambiguity. We provide a neutrality condition that connects informational ambiguity to ambiguity attitude. Consequently, we show that the equilibrium ambiguous spread can be decomposed into the standard GM spread and an "ambiguity premium/discount" component characterizing the ambiguity aversion/seeking of the market maker. For a sufficiently ambiguity averse market maker, the "*ambiguity premium effect*" on the bid-ask spread provides a potential explanation to drying liquidity and price inefficiency during the periods of extreme market stress. On the other hand, "*ambiguity discount effect*" prevails when the market maker is not sufficiently ambiguity averse. Second, we extend our analysis to different order sizes and examine the joint impact of order size and ambiguity.[1]

[1] We mainly focus on the static aspects with two trade sizes as Easley and O'Hara (1987). Ozsoylev and Takayama (2010) extend their framework to n trade sizes in a multi-period setting. We believe our primary results would remain true with n trade size in a dynamic set-up. There is also an argument of irrelevance of trade sizes in today's financial markets since the large orders are "sliced up" into small orders. To this end, Seppi (1990) theoretically shows that even when a block can be "sliced up" into a sequence of small trades, blocks may still be traded as part of both informed and uninformed investors' optimal trading strategies. Using transactions data for a sample of NYSE firms, Barclay and Warner (1993) show that most of the cumulative stock-price change is due to medium-size traders and Chakravarty (2001), Anand and Chakravarty (2007) find that the source of this disproportionately large cumulative price impact of medium-size trades is institutional trades. Alexander and Peterson (2007) further find evidence of trade-size clustering on multiples of 500, 1,000, and 5,000 shares in NYSE and Nasdaq consistent with the stealth trading.

We obtain necessary and sufficient conditions for the existence of pooling and separating equilibria. It turns out that for a sufficiently ambiguity averse market maker, Easley and O'Hara (1987) bounds (the large buy (sell) to small buy (sell) order ratio) for the existence of pooling and separating equilibria are pulled down with informational ambiguity if the large orders are more likely. In extreme informational ambiguity, the market maker always considers the market as a separating equilibrium.

This paper makes economic and methodological advances over previous research on sequential trading models. Specifically, we propose a GM type model such that a decision maker does not depend on a purely probabilistic information. We make three additions to the original model (one economic and two methodological in nature). First, we use non-additive probabilities to capture the perceived ambiguity of the market maker. Second, instead of Bayesian updating (BU) we use generalized Bayesian updating (GBU) which is an appropriate updating rule for non-additive probabilities. Third, we use the Choquet expectation w.r.t. (with respect to) a non-additive probability instead of the classic Lebesgue expectation w.r.t. a probability measure. There are several motivations of this extension.

First, the standard market microstructure theory models are designed for well-defined gambles where a single probability distribution captures the total uncertainty of the decision maker. This is sharply in contrast with real life decision situation where ambiguity (i.e. unmeasurable uncertainty) plays an important role. We know from a growing body of research in decision theories that ambiguity matters for the decision-making purposes.[2] An ambiguity-sensitive decision maker does not act as if there is a single probability distribution of states of nature; instead, the decision maker's confidence in the probability assessment is relevant for the decision-making. In the case of uncertainty, one rule in the realm of probability theory is that if the sample space is partitioned into k symmetric events, then the probability of each event is $1/k$. Consider Schmeidler (1989) coin example; if each of the symmetric and complementary uncertain events is assigned an index of 3/7, the number 1/7 (1 - (3/7 + 3 /7)) would then indicate

[2] The idea of unmeasurable uncertainty dates back to Knight (1921) and Keynes (1921), where they distinguish between risk (when relative odds of the events are known) and uncertainty (when the degree of knowledge only allows us to work with estimates). Ellsberg (1961) provides experimental evidence to the tentative ideas of Knight and Keynes. The behavior of uncertainty-aversion documented by Ellsberg (1961) has been first axiomatized in the decision making context by Choquet expected utility of Schmeidler (1989) and Maxmin expected utility of Gilboa and Schmeidler (1989). Since then, different approaches such as unanimity preferences of Bewley (2002), smooth preferences of Klibanoff, Marinacci and Mukerji (2005), variational preferences of Maccheroni, Marinacci and Rustichini (2006) have been taken to model ambiguity. We refer to Gilboa and Marinacci (2016) and Epstein and Schneider (2008) for extensive surveys of the literature.

the decision maker's loss of confidence in the probability assessment. This simple example shows the account of uncertainty that is captured by non-additive probabilities (i.e. Choquet capacities) cannot be represented by additive probabilities.

Second, modeling ambiguity in this context pushes the market microstructure theory beyond the Bayesian paradigm and arguably, explores the intersection between market microstructure theory and behavioral finance. Perhaps, the most relevant cognitive and behavioral bias used in behavioral finance for our purposes is probability matching.[3] Consider a repeated coin toss guessing game in which the coin is biased 60% head and 40% tail: if you are correct, you win $1 and otherwise lose $1. A typical Bayesian (*homo economicus*) would argue to bet on the head all the time. However, humans tend to randomize between heads and tails where the randomization matches the probability of the biased coin. Pushing the boundaries of market microstructure theory beyond the Bayesian paradigm allows us to adapt the market microstructure literature to the phenomena like probability matching.

Our third motivation takes its roots from the empirical facts about the behavior of financial markets during financial turmoils. Financial crisis is often associated with a decrease in liquidity and extreme market inefficiency. It is adequate to provide an excerpt from Scholes (2000) to delineate our third motivation: *"In periods of extreme market stress...many statistically uncorrelated activities using historical data exhibited high degrees of association. For example, in 1998 the spreads over treasuries widened on U.S. AAA bonds, AAA commercial mortgage pools, credit instruments, country risks, and swap contracts. On 21 August 1998, one week after Russia defaulted on its debt, swap spreads shot up from 60 basis points to 80 basis points in one day."*

Lastly, the "invariance principle" of Kyle and Obizhaeva (2016) suggests the trade size is an endogenous factor that depends on price volatility. Gradojevic, Erdemlioglu and Gençay (2017) find that large orders in an electronic spot foreign exchange market are likely to be placed by informed traders during increased price volatility episodes, the finding which is consistent with the "invariance principle". However, little is known about any specific quantitative association

[3] One of the earliest papers to document this phenomenon is Grant, Hake and Hornseth (1951). Recent literature in experimental psychology suggests that the probability matching is not a strategy per se, but rather another outcome of people's misperception of randomness. The literature also suggests that there might be a smart potential behind probability matching by documenting that probability matchers actually have a higher chance of finding a pattern if one exists. In this paper we refrain from addressing normative vs. descriptive discussion. See Aliyev and He (2016) for the discussion.

between the size of individual trades and uncertainty in the financial system. An extension to different trade sizes is motivated by the lack of theoretical analysis on the response of trade size to an additional uncertainty in financial markets.

There is a respectable body of the literature dealing with axiomatic foundations of the non-Bayesian decision theory and behavioral finance. However, our aim in this paper is to examine their relevance in the context of market microstructure. In that regard, this paper mainly contributes to the literature that studies the implications of informational ambiguity on market microstructure. Closest to us are Routledge and Zin (2009), Ozsoylev and Werner (2011) and Xia and Zhou (2014) that study ambiguity-averse market makers and liquidity. There are some differences in the settings as well as the implications. By assuming a monopolist market maker with an uncertainty-averse utility function, Routledge and Zin (2009) follow Epstein and Wang (1994) to capture model uncertainty. They show that non-competitive market making and relatively large discrete trades, rather than ambiguity and ambiguity aversion, are necessary to generate illiquidity that are significantly different from the standard Savage expected utility models. Ozsoylev and Werner (2011) extend the framework of Vives (1995*a*, 1995*b*) to the case of ambiguous information. However, they rely on the non-participation of market makers. Xia and Zhou (2014) adopt the smooth ambiguity model of Klibanoff et al. (2005), $U = E[\phi(E[u(w)])]$, with exponential-power specification (u, ϕ) for tractability. All of these papers focus on ambiguity-averse market maker through utility specification. The use of neo-additive capacity in this paper allows us to look at ambiguity-attitude without utility specification. Besides the simplicity and generality of our approach, some of the results obtained in this setting are different from the one found through utility specification. For example, Xia and Zhou (2014) find a negative relation between ambiguity aversion and the bid-ask spread while we show that it can be opposite for a given level of informational ambiguity. There are some related papers on theoretical microstructure literature that study ambiguity of traders and its effect on market microstructure as oppose to the informational ambiguity of the market maker. Ford et al. (2013) consider a similar and simplified sequential trading model in which informed traders have informational ambiguity and show that ambiguity and attitudes to it can lead to herd and contrarian behavior causing the market to break down. When the extreme uncertainty of traders can be characterized by incomplete preferences over portfolios, Easley and O'Hara (2010*a*) use Bewley's decision making under uncertainty to explain market illiquidity and freeze during the

GFC. Easley and O'Hara (2010*b*) focus on market design to reduce ambiguity and hence increase market participation. With two types of traders in their model, sophisticated traders with rational expectations and unsophisticated traders with informational ambiguity, they show that how specific design of the market can benefit investors.

In this paper, we focus on the effects of informational ambiguity and ambiguity attitude of the market maker on security prices. To this end, we simplify the traders' side to easily convey the main message. Since we use the quote driven framework of Glosten and Milgrom (1985) and Easley and O'Hara (1987), our theoretical findings can be tested in NASDAQ Stock Market, London Stock Exchnage's SEAQ (Stock Exchange Automated Quotation system) and CBOE.

The paper proceeds as follows. In the next section, we provide a preliminary introduction on non-additive probabilities, GBU under ambiguity and Choquet expectation. We then present the model and the equilibrium concept. In section 4, we first present our equilibrium results on a simple illustrative example, then generalize and provide economic explanations. In section 5, we extend our analysis to different order sizes. In section 6, we briefly discuss the implications and conclude. Proofs are collected in the Appendix.

2. Preliminaries

We assume that the uncertainty of the decision maker can be described by a non-empty set of finite states, denoted by $\mathcal{S}$. A non-additive probability is a real-valued set function defined on the set of events $\mathcal{E}$ of the sample space $\mathcal{S}$ that is normalized ($v(\varnothing) = 0$, $v(\mathcal{S}) = 1$) and monotonic (for all A, B in $\mathcal{E}$, $A \subseteq B \Rightarrow v(A) \leq v(B)$). In our analysis, we specifically use neo-additive capacities of Chateauneuf, Eichberger and Grant (2007) to capture the ambiguity of the environment and ambiguity attitude of the market maker. Given an additive probability π on $\mathcal{E}$, a neo-additive capacity is defined to be $v(A) = (1-\delta) \cdot \pi(A) + \delta \cdot \alpha$ for $\varnothing \subsetneq A \subsetneq \mathcal{S}$ and $(\alpha, \delta) \in [0, 1]$. The parameter δ is a measure of ambiguity and the parameter α measures the individuals attitude to it. With this formulation, $\delta = 1$ corresponds to fully ambiguous information and $\alpha = 0$ to fully ambiguity-averse attitude. The GBU rule of updating neo-additive capacities is defined to be[4]

[4] The term GBU rule is due to Walley (1991). The rule is also called the Dempster-Fagin-Halpern rule (Dempster (1967), Fagin and Halpern (1991)). Eichberger, Grant and Kelsey (2007) use the decision theoretic framework of Pires (2002) to axiomatize GBU rule. Horie (2013) provides a relaxation on one of their axioms. Eichberger, Grant and Kelsey (2010) apply different updating rules to neo-additive capacities and find that GBU updated neo-additive capacity stays the same capacity with the same ambiguity attitude.

$$v(A|B) = \frac{v(A \cap B)}{v(A \cap B) + 1 - v(A \cup B^c)}. \tag{1}$$

It is straightforward to check that Eq. (1) reduces to the original BU when v is additive. Additivity implies $1 - v(A \cup B^c) = v(A^c \cap B)$, and applying additivity again yields $v(A \cap B) + v(A^c \cap B) = v(B)$, and hence BU. The last addition to our model is Choquet expectation w.r.t. a non-additive probability instead of the classic Lebesgue expectation w.r.t. a probability measure. Without loss of generality, we rank a non-negative function f on $\mathcal{S}$ as $f(s_1) \geq f(s_2) \geq ... \geq f(s_n)$ and $f(s_{n+1}) = 0$. From Choquet (1953), the Choquet expectation (i.e. integral) of the non-negative function f on $\mathcal{S}$ w.r.t. a non-additive probability v is given by

$$E_v[f] := \int_{\mathcal{S}} f \cdot dv = \sum_{k=1}^{n} \big(f(s_k) - f(s_{k+1})\big) \cdot v\big(\{s_1, s_2, ..., s_k\}\big). \tag{2}$$

To provide an intuition, we consider the following example.

Example 2.1. *Consider an asset (a) that pays either \$2 in low state (l) or \$5 in high state (h) with non-additive probabilities of $v_l = 0.2$ and $v_h = 0.4$ respectively. The (Choquet) expected payoff of buying a unit of this asset is given by $E_v[a_b] = 0.6 \cdot 2 + 0.4 \cdot 5 = 3.2$. The (Choquet) expected payoff of selling a unit of this asset is given by $E_v[a_s] = 0.2 \cdot (-2) + 0.8 \cdot (-5) = -4.4$.*

3. The Model of Ambiguous Market Making

We consider a simplification of the sequential trade GM model presented in O'Hara (1995) under informational ambiguity. There are three classes of risk-neutral market participants: informed (I) traders, uninformed (U) traders, and a competitive and ambiguity-sensitive market maker (M). There is one security with a value V that is either low (V_l) with probability of π_l or high (V_h) with probability of $\pi_h = 1 - \pi_l$. For brevity, we omit the time subscripts throughout the paper. Informed traders know the value of outcome and the proportion of informed traders' population is μ. The neo-additive capacities (i.e. beliefs) of the low and high outcomes for the market maker are $v_l = (1-\delta)\cdot\pi_l + \delta\cdot\alpha$ and $v_h = (1-\delta)\cdot(1-\pi_l) + \delta\cdot\alpha$ respectively. The amount of probability "lost" by the presence of ambiguity is $1 - v_l - v_h = \delta \cdot (1 - 2\alpha)$, representing the perceived ambiguity or the confidence in probability assessment. It essentially measures the deviation of the measure $v = \{v_l, v_h\}$ from the additivity of probability measure $\pi = \{\pi_l, \pi_h\}$. One can also think of the capacity v as a "squeeze" of the original probability measure π when

$\alpha = 0$. By characterizing the uncertainty of the market maker with neo-additive capacities, we adopt Choquet Expected Utility framework of Schmeidler (1989). The value of the security is determined by a random draw of nature at the opening of the market and revealed after the market closes without being affected by trade. The market maker posts bid $(B_{\alpha,\delta})$ and ask $(A_{\alpha,\delta})$ prices that equal to the expected value of the asset, conditional on the type of the order. Then a trader is drawn at random from the population. If the trader is informed, she buys (b) if $V = V_h$ and sells (s) if $V = V_l$ with certainty. If the trader is uninformed, he buys and sells with equal probabilities. The event tree for the trading decision is given in Fig. 1.

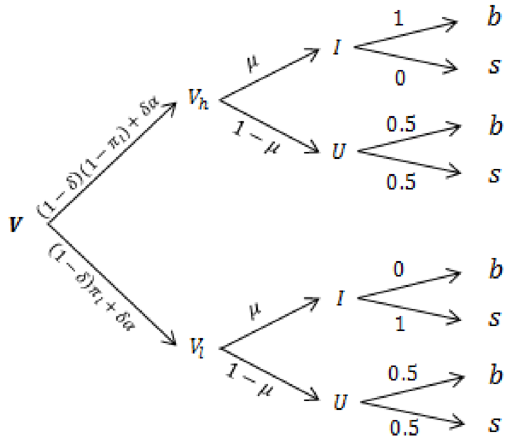


FIGURE 1. The event tree for the basic sequential trade model with ambiguity

In the figure, I and U denote the arrivals of informed and uninformed traders. A buy is a purchase of one unit at the ask price $A_{\alpha,\delta}$ and a sell is a sale of one unit at the bid price $B_{\alpha,\delta}$. The values we attach to the first arrow are the Choquet belief (v) of the indicated transition as opposed to the probability (π). Then for the market maker, the unconditional beliefs of buy and sell orders are $v_b = (1-\delta)\cdot\pi_b + \delta\cdot\alpha$ and $v_s = (1-\delta)\cdot\pi_s + \delta\cdot\alpha$ respectively. The structure of the economy is a common knowledge.

Consider a competitive market maker with zero expected profit (due to Bertrand competition) who does not know whom he is trading with. He, therefore, sets ask price as $E_v[V|b]$ and bid price as $E_v[V|s]$ by revising his beliefs $v(\cdot|\cdot)$ with GBU. Given the submitted buy (sell) order, he revises his beliefs of V_l accordingly.

Lemma 3.1. *[Eichberger et al. (2010)] Fix a conditioning event $a \in \mathcal{S} = \{b, s\}$ and unconditional neo-additive capacity, v_l characterized by $< \pi_l, \delta, \alpha >$. Assume $\pi_a > 0$. Then the GBU updated capacity v_l^a is also a neo-additive capacity characterized by $< \pi_l^a, \Delta_a, \alpha >$, where $\Delta_a = \delta / \big((1-\delta)\cdot\pi_a + \delta\big)$ and π_l^a is a Bayesian update of π_l conditional on a.*

Lemma 3.1 tells us that the market maker's updated belief of V_l conditional on the order type, $a \in \{b, s\}$, is given as $v_l^a = \big(1 - \Delta_a\big) \cdot \pi_l^a + \Delta_a \cdot \alpha$. Similar results hold for the revised beliefs of V_h. The updated ambiguity measure Δ_a is an increasing function of original ambiguity measure δ. Note that the revised beliefs reduce to original posterior probabilities updated by BU when $\delta = 0$. Also note that $\frac{\partial v_l^a}{\partial \delta} \leq 0$ for $\alpha \leq \pi_l^a$, implying the revision in beliefs is stronger when the market maker is becoming less confident about his probability assessment. Unless $v_l^a = 1$, the market maker will tilt his belief revisions downward or upward depending on his ambiguity attitude. The tilt will be downward (upward) when the market maker is more (less) ambiguity averse. This type of tilting the updated measure is characterized as a behavioral bias in behavioral finance literature. There is still no consensus on the normativeness or descriptiveness of the Ellsberg type behavior. Next we define the equilibrium for our economy.

Definition 3.2. *An equilibrium consists of the the marker maker's prices, informed traders' trading strategies, and posterior beliefs such that:*

(C1) *the bid and ask prices satisfy the zero-profit condition, given the market maker's posterior beliefs;*

(C2) *Informed traders' optimal trading strategies correspond to the buy order when they have a high signal and sell order when they have a low signal.*

(C3) *The market maker's belief satisfies GBU.*

4. Ambiguity, Ambiguity Aversion, and Bid-Ask Spread

Our analysis makes economic and methodological advances over previous research on sequential trade models. We take the intrinsic ambiguity of the real world information into account on the market maker side. This allows us to see the impact of ambiguity and ambiguity attitude of the market maker on security prices.

In this framework, the market maker with informational ambiguity must decide what to do when asked for a quote. At the beginning of the trading, the market maker must determine his initial prices of the traded asset. In equilibrium, these quotes must yield the market maker zero expected profit on each trade. However, the expectation is not solely probabilistic since the market maker has informational ambiguity. We focus on the case of $0 \leq \alpha \leq 1/2$, in which $\alpha = 0$ correspond to fully ambiguity-aversion. This essentially means the missing probability measure $\delta \cdot (1 - 2\alpha) \geq 0$ and $v(V_l) + v(V_h) \leq 1$. Inventory cost is not considered. The ambiguity-sensitive market maker with $0 \leq \alpha \leq 1/2$ and zero profit, sets the bid/ask prices and spread, respectively, as

$$\begin{aligned} B_{\alpha,\delta} = E_v[V|s] =& \big[v_l^s + \Delta_s \cdot (1 - 2 \cdot \alpha)\big] \cdot V_l + v_h^s \cdot V_h \\ =& V_l + v_h^s \cdot (V_h - V_l), \end{aligned} \tag{3}$$

$$\begin{aligned} A_{\alpha,\delta} = E_v[V|b] =& v_l^b \cdot V_l + \big[v_h^b + \Delta_b \cdot (1 - 2 \cdot \alpha)\big] \cdot V_h \\ =& V_h - v_l^b \cdot (V_h - V_l), \end{aligned} \tag{4}$$

$$S_{\alpha,\delta} = (1 - v_l^b - v_h^s) \cdot (V_h - V_l). \tag{5}$$

Eqs. (3) and (4) directly follow from the definition of Choquet expectation. The intuition is provided by example 2.1. Since $V_h > V_l$ for buying and $(-V_l) > (-V_h)$ for selling purposes, the minimum probability when evaluating an expectation is the probability in the core of (v) that puts the most possible weight on V_l and $(-V_h)$ respectively. Therefore, the missing probability measures $\Delta_s \cdot (1 - 2\alpha)$ and $\Delta_b \cdot (1 - 2\alpha)$ are added to v_l^s and v_h^b in the calculation of $B_{\alpha,\delta}$ and $A_{\alpha,\delta}$ respectively. The original probabilistic bid/ask prices and spread are given, respectively, as

$$B = V_l + \pi_h^s \cdot (V_h - V_l) = V_h - \pi_l^s \cdot (V_h - V_l), \tag{6}$$

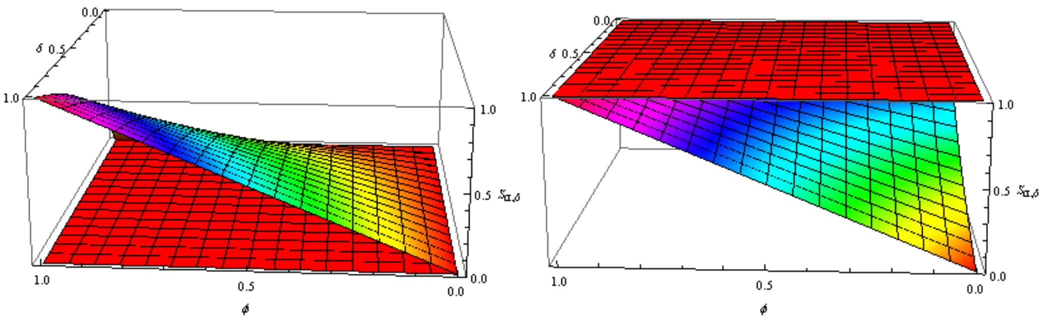
$$A = V_h - \pi_l^b \cdot (V_h - V_l) = V_l + \pi_h^b \cdot (V_h - V_l), \tag{7}$$

$$S = (1 - \pi_l^b - \pi_h^s) \cdot (V_h - V_l) = (\pi_h^b + \pi_l^s - 1) \cdot (V_h - V_l). \tag{8}$$

4.1. **An illustrative example.** Before we look at the general comparison of ambiguous and probabilistic bid-ask spreads, it may be a source of insight to study our economy in a simple example in which we set $\pi_l = 1/2$ and $V_h - V_l = 1$. It is well-known that the bid-ask spread of GM model corresponds to μ in this scenario. In the classic framework, the spread stems from the probability of informed trading due to adverse selection costs. In our framework, the spread is characterized by $1 - v_l^b - v_h^s$. Substituting the values of v_l^b and v_l^s yields the ambiguous bid-ask spread as $(1 - \Delta) \cdot \mu + \Delta \cdot (1 - 2 \cdot \alpha)$, where $\Delta = \Delta_b = \Delta_s = \frac{2\delta}{1+\delta}$. This formulation has an intuitive explanation. Let $\phi \in [0, 1]$ be a normalized ambiguity aversion characterized by $\phi = (1 - 2 \cdot \alpha)$ where 0 and 1 characterize no ambiguity aversion and full ambiguity aversion respectively. Then the ambiguous bid-ask spread is characterized by the *"effective"* probability of informed trading

$$S_{\alpha,\delta} = \mu_{\alpha,\delta} = \mu + \Delta \cdot (\phi - \mu). \tag{9}$$

When the market maker has no informational ambiguity the *"effective"* probability of informed trading, $\mu_{\alpha,\delta}$, is the same as the probability of informed trading, μ (i.e. the ambiguous bid-ask spread is the same as its classical counterpart). The ambiguous bid-ask spread is fully characterized by the normalized degree of ambiguity aversion of the market maker when he has full informational ambiguity. When the degree of ambiguity aversion, ϕ, exceeds the information share of the market, μ, then $\mu_{\alpha,\delta} > \mu$ holds, and hence a widened bid-ask spread. However, when the degree of ambiguity aversion is less than the information share of the market, the market maker sets the spread lower than the probabilistic spread. We provide the extreme cases of informed trading in Fig. 2 to see how the *"effective"* probability of informed trading evolves over time w.r.t. to the changes in informational ambiguity, δ, and the normalized degree of ambiguity aversion, ϕ.



(A) $\mu = 0$ (B) $\mu = 1$

FIGURE 2. Ambiguous and probabilistic bid-ask spreads

Fig. 2 (A) shows the ambiguous and probabilistic bid-ask spreads when the market maker believes that there is no informed trading in the market (i.e. $S = 0$), while Fig. 2 (B) shows the case of full informed trading (i.e. $S = 1$). We observe that the ambiguous bid-ask spread changes due to the changes in informational ambiguity and ambiguity aversion of the market maker, while probabilistic spread stays constant. Particularly, in Fig. 2 (A) as the informational ambiguity increases for the small degree of ambiguity aversion, $\phi > 0$, the market maker's *"effective"* probability of informed trading increases with his informational ambiguity. Only in extreme informational ambiguity, $\delta = 1$, irrespective of the probability of informed trading, the *"effective"* probability is solely determined by his degree of ambiguity aversion ϕ. He, therefore, sets the maximum spread when he is fully ambiguity averse, $\phi = 1$, under full informational ambiguity. In Fig 2 (B) the *"effective"* probability of informed trading, $\mu_{\alpha,\delta}$, only matches to the probability of informed trading, μ, when the market maker is fully ambiguity averse, $\phi = 1$, or there is no informational ambiguity, $\delta = 0$, since the probability of informed trading takes its maximum value. Fig. 2 (A) and (B) also elucidate the fact that for a given level of informational ambiguity, the degree of ambiguity aversion required for the market maker to have equal ambiguous and probabilistic bid-ask spreads is dependent on the market parameters.

Next we turn to the general case. Intuition of the simple illustrative example is also applicable to the general case. In this market, only one form of equilibria can occur where the bid/ask prices and the spread are given as Eqs. (3), (4) and (5). Equilibrium bid/ask prices and the spread of the ambiguity-sensitive market maker is summarized in the following proposition.

Proposition 4.1. *[Equilibrium Prices] Suppose equilibrium bid and ask prices exist satisfying the zero expected profit conditions in both ambiguous and unambiguous cases;*

$$B_{\alpha,\delta} = E_v[V|s], \qquad A_{\alpha,\delta} = E_v[V|b],$$
$$B = E_\pi[V|s], \qquad A = E_\pi[V|b].$$

Then the competitive and ambiguity-sensitive market maker sets bid and ask prices as a convex combination of original probabilistic bid and ask prices, and a Hurwicz criterion,

$$B_{\alpha,\delta} = [1 - \Delta_s] \cdot B + \Delta_s \cdot \big[\alpha \cdot V_h + (1-\alpha) \cdot V_l\big], \tag{10}$$

$$A_{\alpha,\delta} = [1 - \Delta_b] \cdot A + \Delta_b \cdot [(1-\alpha) \cdot V_h + \alpha \cdot V_l], \tag{11}$$

and the bid-ask spread takes the form of

$$S_{\alpha,\delta} = S + \Bigg[\Delta_b \cdot \big(\pi_l^b - \alpha\big) - \Delta_s \cdot \big(\alpha - \pi_h^s\big)\Bigg] \cdot \big(V_h - V_l\big), \tag{12}$$

where S denotes the original probabilistic bid-ask spread.

We stress the importance of Proposition 4.1 by rewriting Eqs. (10) and (11) in different forms and comparing to the original Glosten and Milgrom (1985);

$$B_{\alpha,\delta} = B + \Delta_s \cdot \big[(\alpha - \pi_h^s) \cdot (V_h - V_l)\big], \tag{13}$$

$$A_{\alpha,\delta} = A + \Delta_b \cdot [(\pi_l^b - \alpha) \cdot (V_h - V_l)] \tag{14}$$

In the classic case, it is necessarily the case that expectations of V are revised upward in response to specialist sales, and revised downward in response to specialist purchases. In the case of ambiguous market making, there is an *"ambiguity premium effect"* on this revisions when $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$. This is so because the ask and bid prices are the revised Choquet expectations. The revision is linear with updated ambiguity of the environment and ambiguity attitude of the market maker. This is why, for the fully ambiguity-averse market maker case, when the conditional probabilities are non-zero, the bid price is lower than original probabilistic bid price, and the ask price is higher than original probabilistic ask price, and hence a widened bid-ask spread.

Fig. 3 illustrates the equilibrium bid-ask spread with informational ambiguity given in Proposition 4.1 and its probabilistic counterpart for the given values of V_l, V_h, π_l and μ. For comparison, we cut through the bid-ask spread with informational ambiguity by the probabilistic bid-ask spread and obtain two distinct areas of the ambiguous bid-ask spread.

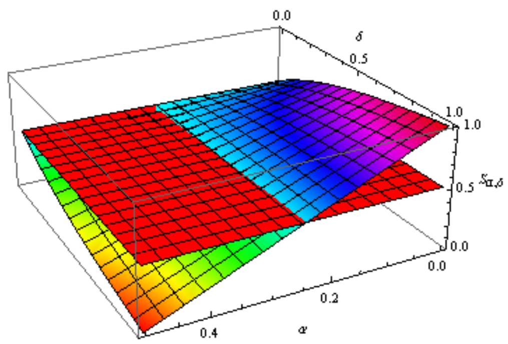


FIGURE 3. Ambiguous and probabilistic bid-ask spreads for $V_l = 0, V_h = 1, \pi_l = 0.35$ and $\mu = 0.55$.

In the figure, the highest and lowest spreads correspond to fully ambiguity-averse ($\alpha = 0$) and no ambiguity-averse ($\alpha = 1/2$) cases when the market maker is in the environment of full informational ambiguity. There is also a combination of ambiguity (δ) and ambiguity attitude (α) that generates a bid-ask spread that is equal to the original probabilistic bid-ask spread of the GM model. Let us formalize this observation.

Definition 4.2. *A "bid-ask spread neutrality curve" is a combination of informational ambiguity, δ^*, and ambiguity attitude, α^*, that makes an equilibrium ambiguous bid-ask spread, $S_{\alpha,\delta}$, the same as the standard probabilistic bid-ask spread, S.*

The curve separates the ambiguous bid-ask spread into *"ambiguity premium"* and *"ambiguity discount"* areas. An *"ambiguity premium"* area adds premium over the original probabilistic bid-ask spread, while an *"ambiguity discount"* area reduces the probabilistic bid-ask spread due to the optimistic behavior of the market maker. The combination of α^* and δ^* that generates a *"bid-ask spread neutrality curve"* is given in the following lemma. This curve essentially sets an upper (lower) bound for α that yields an *"ambiguity premium"* (*"ambiguity discount"*) over the probabilistic spread.[5]

Lemma 4.3. *In equilibrium, there is an ambiguity attitude, α^*, dependent on the market parameters which equalizes the ambiguous and probabilistic bid-ask spreads and divides the ambiguous spread into ambiguity premium, $\alpha < \alpha^*$ and ambiguity discount, $\alpha > \alpha^*$, areas.*

$$\alpha^* = w \cdot \pi_l^b + (1 - w) \cdot \pi_h^s, \tag{15}$$

where $w = (\pi_s + \delta^* \cdot \pi_b)/(1 + \delta^*)$.

Lemma 4.3 is similar in nature with the illustrative example. Recall that in the illustrative example a *"bid-ask spread neutrality curve"* is obtained when the normalized ambiguity aversion equals the information share in the market (i.e. $\phi = \mu$), since it makes the *"effective"* probability of informed trading, $\mu_{\alpha,\delta}$, the same as the probability of informed trading, μ. The intuition goes similarly: Suppose the probabilistic and ambiguous bid-ask spreads in Eq. (12) are given as $S_{\alpha,\delta} = \beta \cdot \mu_{\alpha,\delta} \cdot (V_h - V_l)$ and $S = \beta \cdot \mu \cdot (V_h - V_l)$ respectively for a given market parameter β. The condition (15) is necessary and sufficient condition that equalizes the *"effective"* probability of informed trading, $\mu_{\alpha,\delta}$, and the information share of the market, μ. Under full informational ambiguity, the condition is the half of the sum of probabilities of low outcome conditional on a buy order and high outcome conditional on a sell order. Otherwise, the condition corresponds to the weighted average of the same sum. When there is no informational ambiguity, the weights are simply probabilities of sell and buy orders respectively. There is also an inverse relation between α^* and δ^*. For the bid-ask spread under informational ambiguity to be greater than the probabilistic bid-ask spread, the market maker must be more

[5] One should note that our definition of neutrality only equalizes the ambiguous and probabilistic bid-ask spreads, not the bid/ask prices. In that sense, we can differentiate between the *"strict form neutrality"* where the ambiguous bid/ask prices (naturally the spread) are equalized to the probabilistic bid/ask prices and the *"weak form neutrality"* where only the bid-ask spreads are equalized. The strict form bid-ask spread neutrality for the ambiguity averse market maker is only attained when $\alpha = \pi_h^s = \pi_l^b = (1-\mu)/2$. This is only the case when $\pi_l = 1/2$. In general, we refer to a bid-ask spread neutrality in the weak sense. The market maker can be a *"weak form neutral"* by increasing or decreasing the bid/ask midpoint. It turns out that the necessary and sufficient condition to be a *"weak form of bid-ask spread neutral"* by increasing (decreasing) the midpoint is $\pi_l > 1/2$ ($\pi_l < 1/2$). That means, if the ambiguity-averse market maker wants to maintain the ambiguous bid-ask spread the same as the probabilistic bid-ask spread, for not to expose himself to adverse selection, he increases (decreases) the bid-ask midpoint when the fundamentals suggest otherwise.

ambiguity averse as the informational ambiguity increases. As discussed, when $\pi_l = 1/2$ the condition corresponds to $\alpha^* = (1-\mu)/2$.

Next we elaborate on the most important special cases of our model.

Case 1 [$\delta = 0$]: It is clear from the illustrative example that when $\pi_l = 1/2$ the ambiguous bid-ask spread without informational ambiguity collapses to its original probabilistic counterpart. This result holds for the other values of π_l when the market maker has full confidence (i.e. no perceived ambiguity) in his probability assessment (and naturally no attitude to it). In this case, the marker maker assigns additive probability of π_l and $1-\pi_l$ to V_l and V_h respectively. Then $A \geq B$ directly follows from Glosten and Milgrom (1985) and Eq. (10), (11), and (12) collapse to their probabilistic counterparts, respectively, as

$$B = \frac{[(1+\mu)\cdot\pi_l]\cdot V_l + [(1-\mu)\cdot(1-\pi_l)]\cdot V_h}{1-(1-2\cdot\pi_l)\cdot\mu}, \tag{16}$$

$$A = \frac{[(1-\mu)\cdot\pi_l]\cdot V_l + [(1+\mu)\cdot(1-\pi_l)]\cdot V_h}{1+(1-2\cdot\pi_l)\cdot\mu}, \tag{17}$$

$$S = \frac{4\cdot(1-\pi_l)\cdot\pi_l\cdot\mu\cdot(V_h - V_l)}{1-(1-2\cdot\pi_l)^2\cdot\mu^2}. \tag{18}$$

In the symmetric case, $\pi_l = \pi_h = 1/2$, $A - B = \mu\cdot(V_h - V_l)$. The spread is determined by the difference between V_h and V_l weighted by the proportion of the informed traders in the market.

Case 2 [$\alpha = 0$]: Suppose the market maker is fully ambiguity-averse in the presence of informational ambiguity. It follows from Eqs. (13) and (14) that $V_l \leq B_{\alpha,\delta} \leq B$ and $A \leq A_{\alpha,\delta} \leq V_h$. The value of $B_{\alpha,\delta}$ ($A_{\alpha,\delta}$) fluctuates between B and V_l (A and V_h) depending on the ambiguity parameter δ. An additional informational ambiguity adds an additional premium over the bid-ask spread by reducing the bid and increasing the ask prices.

Case 3 [$\delta = 1$]: Suppose the ambiguity-sensitive market maker has full informational ambiguity. Then his bid and ask prices correspond to $B_{\alpha,\delta} = \alpha V_h + (1-\alpha)V_l$ and $A_{\alpha,\delta} = (1-\alpha)V_h + \alpha V_l$. The market maker makes decisions fully based on his optimism and pessimism irrespective of the proportion of informed traders in the market.

Case 4 [$\delta = 1$ and $\alpha = 0$]: This case illustrates a fully ambiguity averse market maker with full informational ambiguity. In this case, the market maker has no belief in his probability assessment (no evidence in the sense of Dempster-Shafer theory[6]), $v(V_l) = v(V_h) = 0$. Therefore, the bid and ask prices correspond to V_l and V_h respectively and the spread takes its maximum

[6] See Dempster (1967) and Shafer (1976).

value. Case 4 essentially suggests some intuition about the widening spread, drying liquidity and price inefficiency when there is a major ambiguous shock to the economy with the very ambiguity averse market maker.

Case 5 [$\alpha = 1/2$]: Suppose the market maker has no ambiguity aversion with $\alpha = 1/2$. Then he acts as if he has a probability measure since the missing probability measure, $\delta \cdot (1 - 2 \cdot \alpha)$, is zero. Depending on the value of conditional probabilities, the market maker can either increase or decrease the bid-ask spread. For example, when $\pi_h^s \leq 1/2$ and $\pi_l^b \leq 1/2$, it follows that $v_h^s \geq \pi_h^s$ and $v_l^b \geq \pi_l^b$. Hence, $B_{\alpha,\delta} \geq B$ and $A_{\alpha,\delta} \leq A$. The opposite is also true. By Lemma 4.3, one can obtain the lower bound for the informational ambiguity for the bid-ask spread under informational ambiguity to be greater than the bid-ask spread under the probabilistic information and the bound takes the form of

$$\delta^* = \frac{1/2 - (\pi_l^b \cdot \pi_s + \pi_h^s \cdot \pi_b)}{(\pi_l^b \cdot \pi_b + \pi_h^s \cdot \pi_s) - 1/2}. \tag{19}$$

Case 6 [$\delta = 1$ and $\alpha = 1/2$]: This case always yields conditional beliefs of $v_h^s = 1/2$ and $v_l^b = 1/2$. Therefore, the bid and ask prices of the market maker converge to $(V_l + V_h)/2$ as if $\pi_h^s = 1/2$ and $\pi_l^b = 1/2$ with principle of insufficient reason and the spread converges to zero.

Proposition 4.1 and Lemma 3.1 give rise to several important corollaries.

Corollary 4.4. *For any non-zero level of informational ambiguity,*
(i) higher ambiguity-aversion of the market maker always leads to higher bid-ask spread and
(ii) the bid-ask spread of sufficiently ambiguity-averse market maker is always higher than original probabilistic bid-ask spread.

Corollary 4.4 *(i)* is intuitive. As the market maker becomes more pessimistic about the uncertainty that he can not quantify by probability theory, he charges an extra spread. This is related to our second motivation in the sense that the market maker tries to look for patterns in the value of the security. But when he can not distinguish the pattern (or at least pessimistic about distinguishing the pattern), he sets an extra spread. The opposite is true for the optimistic case. Corollary 4.4 *(ii)* is a natural extension of *(i)*. Since, higher ambiguity aversion monotonically increases the bid-ask spread, a sufficiently ambiguity averse market maker (i.e. $\alpha < \alpha^*$) sets the ambiguous bid-ask spread higher than the original probabilistic bid-ask spread.

Corollary 4.5. *For a sufficiently ambiguity-averse market maker,*
(i) incremental informational ambiguity adds "ambiguity premium" to the bid-ask spread, and
(ii) the magnitude of "ambiguity premium" is decreasing by the incremental informational ambiguity.

Corollary 4.5 (i) is directly related to our third motivation. Historical flight-to-liquidity episodes such as shorting ban, transaction taxes or in extreme, market crashes are often associated with substantial liquidity drop. Corollary 4.5 (i) explains a possible channel of the drop in market liquidity measured by the bid-ask spread. In extreme informational ambiguity, a fully ambiguity averse market maker sets the maximum spread between two possible values of V_l and V_h. Since there is no room for exploiting private information, in that case, informed traders stop trading and the trading volume collapses. This is indeed what has happened during the recent financial crisis. Although the market makers continued to post bid and ask prices on mortgage-backed securities and collateralized debt obligations, the trading volume of these securities decreased substantially (Easley and O'Hara (2010a)). Corollary 4.5 (ii) shows the concavity of ambiguity premium w.r.t. the informational ambiguity.

5. Different Order Sizes

In this section we examine the question "how informational ambiguity and ambiguity attitude affect security prices with different order sizes". Easley and O'Hara (1987) argue that the size of the trade affects the prices by revealing the type of the market participants to the market maker.[7] Their model results in two different type of equilibria: A *separating equilibrium* prevails, if informed traders trade only large quantities and separate themselves from small uninformed traders. A *pooling equilibrium* prevails, if informed traders trade either small or large quantities. We adopt the same methodology and analyze the *separating* and *pooling equilibria* under the case of an informationally ambiguous market maker. We keep the assumptions of our ambiguous market making model in section 4. However, order can be two different sizes with $0 < b_1 < b_2$ and $0 < s_1 < s_2$. We add an ambiguity to the probability assessment of the market maker in the same way we added in section 4. The market maker has to set prices taking into account that informed traders can trade different quantities depending on their private information. Informed's trading strategy is also dependent on the market maker's pricing rule.

[7] The same argument follows in Ozsoylev and Takayama (2010).

5.1. **The Separating Equilibrium.** In this market the market maker must calculate the conditional values of v_l and v_h given the type and size of the trade. He updates his beliefs by GBU. The market maker's probabilistic expectations of informed and uninformed traders in the market stay as μ and $1-\mu$. That means information event occurs with the probability of μ. We let γ denote the market maker's expectation of the fraction of informed traders who submit large orders. In the separating equilibrium, the orders submitted by the informed can only be large orders. Precisely, γ proportion of informed submit b_2 if they have a high signal and s_2 if they have a low signal, and $1-\gamma$ proportion do not trade. The market maker expects uninformed to submit b_2, s_2 with equal probabilities of $\theta/2$ and b_1, s_1 with equal probabilities of $(1-\theta)/2$. The event tree for the first trade in the separating equilibrium case is given in Fig. 4

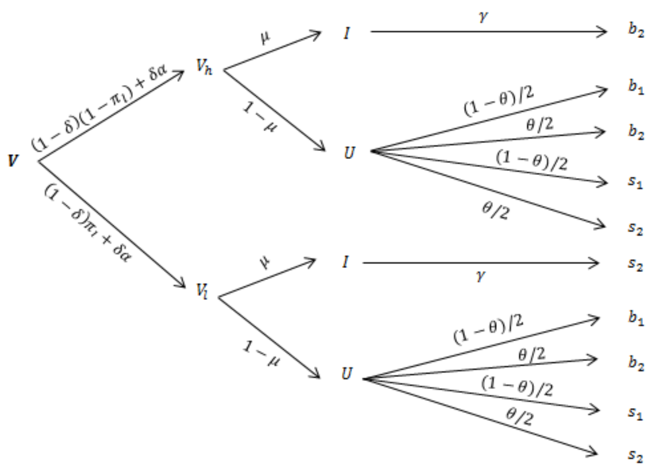


FIGURE 4. A separating event tree with ambiguity and different order sizes

The intuition of Easley and O'Hara (1987) loosely applies in our setting as well, though addition of an informational ambiguity shows that the analysis is not straightforward as it seems. Without informational ambiguity, the market maker does not update his probabilistic belief when there is small buy or sell orders in the separating equilibrium case ($\pi_l^{b_1} = \pi_l^{s_1} = \pi_l$). That means if the market is characterized by a separating equilibrium, small trades can not be information based. However, with an informational ambiguity the market maker's beliefs conditional on the small trades are

$$v_l^{b_1} = (1-\Delta_{b_1}) \cdot \pi_l + \Delta_{b_1} \cdot \alpha = \pi_l - \Delta_{b_1} \cdot (\pi_l - \alpha), \tag{20}$$

$$v_l^{s_1} = (1-\Delta_{s_1}) \cdot \pi_l + \Delta_{s_1} \cdot \alpha = \pi_l - \Delta_{s_1} \cdot (\pi_l - \alpha) \tag{21}$$

where Δ_{b_1} and Δ_{s_1} follow from Lemma 3.1. Eqs. (20) and (21) elucidate that although the probability is not updated with small trades, the informational ambiguity of the market maker is updated with small trades. So, in our setting small trades are only information based to the extent that they affect informational ambiguity. However, this is not the case for large trades. For large trades the conditional probabalities as well as informational ambiguity are updated as

$$v_l^{b_2} = \left(\frac{(1-\mu)\theta}{2\mu\gamma(1-\pi_l) + (1-\mu)\theta} \right) \cdot \pi_l - \Delta_{b_2} \cdot \left[\frac{(1-\mu)\theta}{2\mu\gamma(1-\pi_l) + (1-\mu)\theta} \cdot \pi_l - \alpha \right], \tag{22}$$

and

$$v_l^{s_2} = \left(\frac{2\mu\gamma + (1-\mu)\theta}{2\pi_l\mu\gamma + (1-\mu)\theta} \right) \cdot \pi_l - \Delta_{s_2} \cdot \left[\frac{2\mu\gamma + (1-\mu)\theta}{2\pi_l\mu\gamma + (1-\mu)\theta} \cdot \pi_l - \alpha \right]. \tag{23}$$

Given the conditional beliefs for both small and large trades, the market maker sets the bid and ask prices for each trading quantity. Let $B^i_{\alpha,\delta}$, $A^i_{\alpha,\delta}$ and B^i, A^i denote the bid and ask prices of the market maker given the trade size $i = 1, 2$ with and without informational ambiguity respectively. By Eqs. (13) and (14),

$$B^i_{\alpha,\delta} = E_v[V|s_i] = B^i + \Delta_{s_i} \cdot (\alpha - \pi_h^{s_i}) \cdot (V_h - V_l), \tag{24}$$

$$A^i_{\alpha,\delta} = E_v[V|b_i] = A^i + \Delta_{b_i} \cdot (\pi_l^{b_i} - \alpha) \cdot (V_h - V_l). \tag{25}$$

Eqs. (24) and (25) indicate the effect of an informational ambiguity on the bid-ask spread when the sufficiently ambiguity-averse market makers also learn the occurrence of information from the quantity traded. The learning is perfect without informational ambiguity, and it yields a result of widened bid-ask spread with increased trade size. There is an additional *"ambiguity premium"* on top of the informational block trade effect when the market maker has a sufficient level of ambiguity aversion. However, Eqs. (24) and (25) also indicate that without a sufficient level of ambiguity aversion (i.e. if he is sufficiently optimistic about the unquantifiable uncertainty), the market maker could offset the informational block trade effect on the bid-ask spread. Optimistic probability matching behavior would justify this type of behavior.

In the separating equilibrium, the market maker sets the prices by assuming that informed traders only choose to trade large quantities. On the existence of a separating equilibrium, the profit maximization condition of informed must always correspond to large trades, since informed's trading strategy is also dependent on the pricing rule. For informed traders with high and low signal a separating equilibrium exists if and only if

$$b_2 \cdot [V_h - A^2_{\alpha,\delta}] \geq b_1 \cdot [V_h - A^1_{\alpha,\delta}], \tag{26}$$

$$s_2 \cdot [B^2_{\alpha,\delta} - V_l] \geq s_1 \cdot [B^1_{\alpha,\delta} - V_l] \tag{27}$$

are satisfied. Let ξ^b and ξ^s denote the lower bounds of large buy to small buy order ratio and large sell to small sell order ratio for the existence of a separating equilibrium without informational ambiguity. Easley and O'Hara (1987) show that separating equilibria on both side of the market exist if and only if

$$\xi^b = \frac{\pi_l}{\pi_l^{b_2}} \qquad \text{and} \qquad \xi^s = \frac{\pi_h}{\pi_h^{s_2}}, \tag{28}$$

where $\pi_l^{b_2} = \big(\frac{(1-\mu)\theta}{2\mu\gamma(1-\pi_l)+(1-\mu)\theta}\big)\pi_l$ and $\pi_h^{s_2} = \big(\frac{(1-\mu)\theta}{2\mu\gamma\pi_l+(1-\mu)\theta}\big)\pi_h$. Similarly, let $\xi^b_{\alpha,\delta}$ and $\xi^s_{\alpha,\delta}$ denote the lower bounds of large buy to small buy order ratio and large sell to small sell order ratio with informational ambiguity. The existence of a separating equilibrium with informational ambiguity is given in the following proposition.

Proposition 5.1. *[Ambiguous separating equilibrium] There is an equilibrium on the ask side, if and only if*

$$\frac{b_2}{b_1} \geq \xi^b_{\alpha,\delta} = \frac{\pi_l - \Delta_{b_1} \cdot \big(\pi_l - \alpha\big)}{\pi_l^{b_2} - \Delta_{b_2} \cdot \big(\pi_l^{b_2} - \alpha\big)}, \tag{29}$$

and there is an equilibrium on the bid side if and only if

$$\frac{s_2}{s_1} \geq \xi^s_{\alpha,\delta} = \frac{\pi_h - \Delta_{s_1} \cdot \big(\pi_h - \alpha\big)}{\pi_h^{s_2} - \Delta_{s_2}\big(\pi_h^{s_2} - \alpha\big)\big)}. \tag{30}$$

In what follows we mainly focus on the ask side. The bid side is symmetric. Proposition 5.1 clarifies that the bounds for the existence of the separating equilibrium are tilted due to the ambiguity and ambiguity attitude of the market maker. For a sufficiently ambiguity averse market maker, the direction of the tilt is determined by the probabilities of large and small buy orders. If the probability of large buy order is greater than the probability of small buy order, then the lower bound of large buy to small buy order ratio, $\xi^b_{\alpha,\delta}$, for the existence of the separating equilibrium is pulled further down. When the market maker has full informational ambiguity, the lower bound of the ratio becomes 1. That means under full informational ambiguity, the market maker will post his quotes as if there is a separating equilibrium no matter what the ratio is and the informed with a high signal will only buy large quantities. This is also true on the bid side. This finding is also consistent with the empirical evidence of Gradojevic et al. (2017) who find that large orders are likely to be placed by informed traders during increased price volatility episodes. High informational ambiguity is often associated with the increased price volatility. The results are summarized in the following corollary.

Corollary 5.2. *(i) For a sufficiently ambiguity-averse market maker, the necessary and sufficient conditions for the existence of a separating equilibrium is lower than its unambiguous counterparts, if $\pi_{b_2} \geq \pi_{b_1}$ and $\pi_{s_2} \geq \pi_{s_1}$ respectively. (ii) Under full informational ambiguity, the market maker always considers the market as a separating equilibrium.*

5.2. **The Pooling Equilibrium.** Similar to the unambiguous case, if the necessary and sufficient conditions of ambiguous separating equilibrium are violated on either side of the market, then there can be no separating equilibrium on that side of the market. Analogously, there will be an ambiguous pooling equilibrium. For the pooling equilibrium, the market maker's expectations of the fractions of uninformed trades are the same as the separating equilibrium. However, informed trading appears to be in both large and small quantities with probabilities of γ and $1 - \gamma$ respectively. The event tree for the first trade in the pooling equilibrium case is given in Fig. 5.

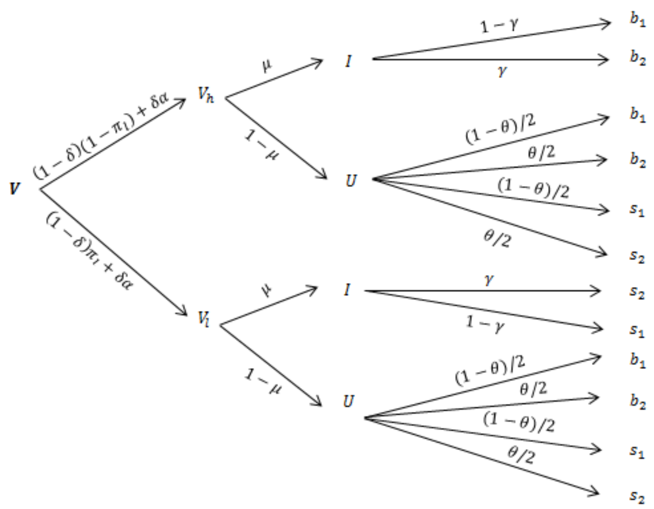


FIGURE 5. A pooling event tree with ambiguity and different order sizes

Three conditions must also satisfy to have an ambiguous pooling equilibrium.

(i) An informed trader must be indifferent between trading a large and a small quantities. Assume that an informed trader has received a high signal. Then she must expect equal profits from buying small and large quantities with the ask prices of $A^1_{\alpha,\delta}$ and $A^2_{\alpha,\delta}$.

$$b_2 \cdot [V_h - A^2_{\alpha,\delta}] = b_1 \cdot [V_h - A^1_{\alpha,\delta}] \tag{31}$$

(ii) The market maker must satisfy zero expected profit condition. This is equivalent to Eqs. (24) and (25) where B^i, A^i are calculated in a standard way, and conditional probabilities are Bayesian.

(iii) It must be possible to choose γ and simultaneously satisfy the equal profit condition for any informed traders and the zero-profit condition for the market maker.

Let $\xi^b_{\alpha,\delta}$ and $\xi^s_{\alpha,\delta}$ (with a slight abuse of notation) denote the upper bounds of the large buy to small buy order ratio and large sell to small sell order ratio for the existence of ambiguous pooling equilibrium respectively. Then it is not surprising that the upper bounds of ambiguous pooling equilibrium correspond to the lower bounds of ambiguous separating equilibrium on either side of the market as in the unambiguous case.

Proposition 5.3. *[Ambiguous Pooling equilibrium] There is an equilibrium on the ask side if and only if*

$$\frac{b_2}{b_1} < \xi^b_{\alpha,\delta} = \frac{\pi_l - \Delta_{b_1} \cdot \big(\pi_l - \alpha\big)}{\pi_l^{b_2} - \Delta_{b_2} \cdot \big(\pi_l^{b_2} - \alpha\big)}, \tag{32}$$

and there is an equilibrium on the bid side if and only if

$$\frac{s_2}{s_1} < \xi^s_{\alpha,\delta} = \frac{\pi_h - \Delta_{s_1} \cdot \big(\pi_h - \alpha\big)}{\pi_h^{s_2} - \Delta_{s_2} \cdot \big(\pi_h^{s_2} - \alpha\big)}. \tag{33}$$

Similar logic applies to Proposition 5.3. When the probability of large buy order is greater than the small buy order, the market maker lowers the upper bound of the large buy to small buy order ratio for the existence of a pooling equilibrium on the ask side. Similarly, under full informational ambiguity, the ratio corresponds to 1 and the ambiguity averse market maker assume the market as a separating equilibrium. We summarize the corollaries of the proposition as follows.

Corollary 5.4. *(i) For a sufficiently ambiguity-averse market maker, the necessary and sufficient conditions for the existence of a pooling equilibrium is lower than its unambiguous counterparts, if $\pi_{b_2} \geq \pi_{b_1}$ and $\pi_{s_2} \geq \pi_{s_1}$ respectively. (ii) Under full informational ambiguity, there is no pooling equilibrium on either side of the market.*

6. DISCUSSION

"Clarice, does this random scattering of sites seem overdone to you? Doesn't it seem desperately random? Random past all possible convenience? Does it suggest to you the elaborations of a bad liar? In Thomas Harris's "The Silence of the Lambs", an imprisoned murderous psychiatrist Dr. Hannibal Lecter helps FBI agent Clarice Starling to solve a serial killer Buffalo Bill case. Buffalo Bill by being too random unintentionally helps Clarice to discover a pattern. The example is in sharp contrast with the tongue-in-cheek assumption of a purely random Bayesian decision making of market makers in the standard market microstructure theory. We don't find it amusing to compare Buffalo Bill to high skilled modern specialists. Nevertheless, we believe behaviors like probability matching are intrinsic (evolutionary if you like to call it) nature of human decision making. The example shows that it is difficult to find a pure randomness on the phenomena related to human decision making and what is important is how we react to this non-randomness. The degree of ambiguity aversion and ambiguity seeking (i.e. optimism and pessimism) is an indispensable part of human decision making. Therefore, pushing the boundaries of the market microstructure theory beyond the Bayesian paradigm clearly has a potential to explain certain phenomena that we otherwise label as anomalies.

Conventionally, a trading cost is attributed to the order-processing cost, inventory control cost, and adverse selection components. However, an informational ambiguity of the market makers in our setting can make trading more or less costly for market participants depending on the ambiguity attitude of the market makers. The questions like *"what is the average level of ambiguity attitude?"*, *"does the ambiguity attitude changes in different situations?*, and *"is there a way to disentangle between informational ambiguity and ambiguity attitude in the market data?"* still stay challenges for future researchers. These questions have affirmative answers in an experimental setting and there has been some progress lately. For example, Kilka and Weber (2001) examine the properties of the probability weighting functions. Du and Budescu (2005) confirm that individuals' ambiguity attitudes are malleable, contingent on the dimension salience and the reference domain. Trautmann, Vieider and Wakker (2008) find that being observed by others leads to stronger ambiguity aversion, questioning the normative appeal of ambiguity neutrality in social settings. Chakravarty and Roy (2009) find that subjects are ambiguity neutral over gains and mildly ambiguity seeking over losses. Baillon and Bleichrodt

(2015) observe a fourfold pattern of ambiguity attitudes: ambiguity aversion for likely gains and unlikely losses and ambiguity seeking for unlikely gains and likely losses. König-Kersting and Trautmann (2016) find that in the agency conditions, ambiguity attitudes of participants are same as in decisions for their own account. In contrast, Voorhoeve et. al. (2016) find that a switch from a gain to a loss frame does not lead to a switch from ambiguity aversion to ambiguity neutrality or ambiguity seeking. The list is not exhaustive. However, the experimental literature has not still decided the ambiguity attitude patterns for general human subjects, let alone their small subsets such as high-skilled individuals.

We consider the market makers who have Choquet preferences with neo-additive capacities and use the GBU updating rule. The usual *"money pump"* argument in this line of research is that such individuals will persistently lose money in financial markets and will not survive in the market for a long time. This argument does not apply to Choquet preferences. CEU preferences satisfy standard rationality conditions (i.e. they are complete, reflexive, transitive and respect state-wise dominance). Our equilibrium is also such that the expectation of the market maker's profit is zero. So, our market maker is not disadvantaged in a static context.

However, the violations of dynamic consistency are possible due to using neo-additive capacities in a dynamic set up. Eichberger, Grant and Kelsey (2005) show that the necessary and sufficient condition for CEU preferences to be dynamically consistent is additive beliefs over the final stage in the filtration, the property which neo-additive capacities do not satisfy. Also, early work by Epstein and Le Breton (1993) shows that dynamically consistent beliefs must be Bayesian. GBU updated beliefs can not be dynamically consistent in the classic economics sense. Therefore, we do not claim that our analysis rationalizes (in the classic sense) the behaviors outlined in the paper. Nevertheless, our analysis has straightforward implications for the price (in)efficiency and market (il)liquidity due to informational ambiguity and ambiguity attitude.

The ambiguity of the participants is not out of the picture in the standard sequential trading models. It implicitly takes the value of "zero" when beliefs are additive. Hence, it seems intuitive to incorporate "non-zero" valued ambiguity of the market makers into the standard microstructure models. In the worst case scenario, this extension provides a comparative static purpose if someone does not agree with the parametric ambiguity approach. It also seems intuitive to extend the existing static setting to a dynamic set-up. However, the issues like *“dynamic inconsistency”* would be more severe in the dynamic setup. We believe that the primary results of our paper would remain true under more complex scenarios.

APPENDIX

Proof of Lemma 3.1.

Proof. We provide the proof conditional on a buy order to make the paper self-contained. It is similar conditioning on a sell order.

$$\begin{aligned}
v_l^b &= \frac{v(V_l\cap b)}{v(V_l\cap b)+1-v(V_l\cup s)} \\
&= \frac{(1-\delta)\pi\big(V_l\cap b\big)+\delta\alpha}{\big[(1-\delta)\pi\big(V_l\cap b\big)+\delta\alpha\big]+1-\big[(1-\delta)\pi\big(V_l\cup s\big)+\delta\alpha\big]} \\
&= \frac{(1-\delta)\pi\big(V_l\cap b\big)+\delta\alpha}{(1-\delta)\pi\big(V_l\cap b\big)+1-(1-\delta)\big(\pi_l+\pi_s-\pi(V_l\cap s\big)} \qquad \text{(A-1)}\\
&= \frac{(1-\delta)\pi\big(V_l\cap b\big)+\delta\alpha}{1-(1-\delta)\pi_s} = \frac{(1-\delta)\pi\big(V_l\cap b\big)+\delta\alpha}{(1-\delta)\pi_b+\delta} \\
&= \frac{(1-\delta)\pi_b}{\big(1-\delta\big)\pi_b+\delta}\cdot\frac{\pi\big(V_l\cap b\big)}{\pi_b}+\frac{\delta}{\big(1-\delta\big)\pi_b+\delta}\cdot\alpha = \big(1-\Delta_b\big)\cdot\pi_l^b+\Delta_b\cdot\alpha
\end{aligned}$$

■

Proof of Proposition 4.1.

The proof follows from the definition of Choquet expectation and zero profit conditions.

$$\begin{aligned}
B_{\alpha,\delta} &= E_v[V|s] = \big[v_l^s+\Delta_s\cdot(1-2\alpha)\big]\cdot V_l+v_h^s\cdot V_h = V_l+v_h^s\cdot(V_h-V_l) \\
&= V_l+\big[\big(1-\Delta_s\big)\cdot\pi_h^s+\Delta_s\cdot\alpha\big]\cdot(V_h-V_l) \\
&= V_l+\pi_h^s\cdot V_h-\Delta_s\cdot\pi_h^s\cdot V_h+\Delta_s\cdot\alpha\cdot V_h-\pi_h^s\cdot V_l+\Delta_s\cdot\pi_h^s\cdot V_l-\Delta_s\cdot\alpha\cdot V_l \qquad \text{(A-2)}\\
&= [1-\Delta_s]\cdot[\pi_h^s\cdot V_h+\pi_l^s\cdot V_l]+\Delta_s\cdot[\alpha\cdot V_h+(1-\alpha)\cdot V_l] \\
&= [1-\Delta_s]\cdot B+\Delta_s\cdot[\alpha\cdot V_h+(1-\alpha)\cdot V_l]
\end{aligned}$$

Ask price follows a similar calculation. The only difference in the calculation of Choquet expectation for the ask price is to recognize the relevant minimizing probability when evaluating expectation as the probability in the core of (v) that puts most weight on V_h as oppose to V_l.

$$\begin{aligned}
A_{\alpha,\delta} &= E_v[V|b] = v_l^b\cdot V_l+\big[v_h^b+\Delta_b\cdot(1-2\alpha)\big]\cdot V_h = V_h-v_l^b\cdot(V_h-V_l) \\
&= V_h-\big[\big(1-\Delta_b\big)\cdot\pi_l^b+\Delta_b\cdot\alpha\big]\cdot(V_h-V_l) \\
&= V_h-\pi_l^b\cdot V_h+\Delta_b\cdot\pi_l^b\cdot V_h-\Delta_b\cdot\alpha\cdot V_h+\pi_l^b\cdot V_l-\Delta_b\pi_l^b\cdot V_l+\Delta_b\cdot\alpha\cdot V_l \qquad \text{(A-3)}\\
&= [1-\Delta_b]\cdot[\pi_h^b\cdot V_h+\pi_l^b\cdot V_l]+\Delta_b\cdot[(1-\alpha)\cdot V_h+\alpha\cdot V_l] \\
&= [1-\Delta_b]\cdot A+\Delta_b\cdot[(1-\alpha)\cdot V_h+\alpha\cdot V_l],
\end{aligned}$$

The bid-ask spread follows from the difference of Eqs. (A-3) and (A-2).

$$\begin{aligned}S_{\alpha,\delta} &= A_{\alpha,\delta} - B_{\alpha,\delta} = [1-\Delta_b]\cdot A + \Delta_b\cdot[(1-\alpha)\cdot V_h + \alpha\cdot V_l] - [1-\Delta_s]\cdot B \\ &\quad - \Delta_s\cdot[\alpha\cdot V_h + (1-\alpha)\cdot V_l] \\ &= \underbrace{A-B}_{S} + \Big[\Delta_b\cdot\big(\pi_l^b-\alpha\big) - \Delta_s\cdot\big(\alpha-\pi_h^s\big)\Big]\cdot\big(V_h-V_l\big)\end{aligned} \tag{A-4}$$

■

Proof of Lemma 4.3.

The proof directly follows from Proposition 4.1. Since $V_h > V_l$, for $S_{\alpha,\delta} = S$ to hold, $\Delta_b\cdot\big(\pi_l^b-\alpha\big) - \Delta_s\cdot\big(\alpha-\pi_h^s\big) = 0$ must hold. We now substitute the values of

$$\Delta_b = \delta/\big((1-\delta)\pi_b+\delta\big) \qquad \text{and} \qquad \Delta_s = \delta/\big((1-\delta)\cdot\pi_s+\delta\big).$$

Then,

$$\frac{\delta^*}{\big((1-\delta^*)\pi_b+\delta^*\big)}\cdot\big(\pi_l^b-\alpha^*\big) - \frac{\delta^*}{\big((1-\delta^*)\cdot\pi_s+\delta^*\big)}\cdot\big(\alpha^*-\pi_h^s\big) = 0. \tag{A-5}$$

After rearranging Eq. (A-5), we obtain

$$\begin{aligned}\alpha^* &= \frac{\frac{\pi_l^b}{\big((1-\delta^*)\pi_b+\delta^*\big)} + \frac{\pi_h^s}{\big((1-\delta^*)\pi_s+\delta^*\big)}}{\frac{1}{\big((1-\delta^*)\pi_s+\delta^*\big)} + \frac{1}{\big((1-\delta^*)\pi_b+\delta^*\big)}} = \frac{\pi_l^b\cdot\big[(1-\delta^*)\cdot\pi_s+\delta^*\big] + \pi_h^s\cdot\big[(1-\delta^*)\cdot\pi_b+\delta^*\big]}{(1-\delta^*)\cdot\pi_b+\delta^*+(1-\delta^*)\cdot\pi_s+\delta^*} \\ &= \frac{\pi_l^b\cdot\big[\pi_s+\delta^*\cdot\pi_b\big] + \pi_h^s\cdot\big[\pi_b+\delta^*\cdot\pi_s\big]}{(1-\delta^*)\cdot\pi_b+\delta^*+(1-\delta^*)\cdot\pi_s+\delta^*} = \frac{\pi_l^b\cdot\pi_s+\pi_h^s\cdot\pi_b+\delta^*\cdot(\pi_l^b\cdot\pi_b+\pi_h^s\cdot\pi_s)}{(\pi_b+\pi_s)-\delta^*\cdot(\pi_b+\pi_s)+2\delta^*} \\ &= \frac{(\pi_s+\delta^*\cdot\pi_b)}{(1+\delta^*)}\cdot\pi_l^b + \frac{(\pi_b+\delta^*\cdot\pi_s)}{(1+\delta^*)}\cdot\pi_h^s.\end{aligned} \tag{A-6}$$

Now we denote $w = \big(\pi_s+\delta^*\cdot\pi_b\big)/\big(1+\delta^*\big)$ and exploit the fact that $\pi_b+\pi_s = 1$ to obtain

$$\alpha^* = w\cdot\pi_l^b + (1-w)\cdot\pi_h^s. \tag{A-7}$$

■

Proof of Corollary 4.4.

(i) *higher ambiguity-aversion of the market maker always leads to higher bid-ask spread* means that $\frac{\partial S_{\alpha,\delta}}{\partial\alpha} \leq 0$. We complete the proof by differentiating Eq. (A-4) w.r.t α,

$$\frac{\partial S_{\alpha,\delta}}{\partial\alpha} = [-\Delta_b-\Delta_s]\cdot(V_h-V_l) \leq 0, \tag{A-8}$$

since since $V_h > V_l$ and $0 < \Delta_b \leq 1$ and $0 < \Delta_s \leq 1$.

(ii) *the bid-ask spread of a sufficiently ambiguity-averse market maker is always higher than original probabilistic bid-ask spread* means that there is a threshold value of α, that is below this value the ambiguous bid-ask spread is always (irrespective of informational ambiguity) wider than the probabilistic bid-ask spread. Since from (i) increasing ambiguity-aversion of the market maker monotonically increases the bid-ask spread in equilibrium, there exists a threshold level α^*, that is below this level the bid-ask spread is always higher than original probabilistic bid-ask spread. From Eq. (A-4), the threshold $\alpha^* \leq \min\{\pi_h^s, \pi_l^b\}$ satisfies $S_{\alpha,\delta} \geq S$.

■

Proof of Corollary 4.5.

(i) *For a sufficiently ambiguity-averse market maker, incremental informational ambiguity adds "ambiguity premium" to the bid-ask spread* means $\frac{\partial S_{\alpha,\delta}}{\partial \delta} \geq 0$, where the sufficiency of ambiguity aversion is determined by $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$. We first substitute the values of

$$\Delta_b = \delta / \big((1-\delta) \cdot \pi_b + \delta \big) \qquad \text{and} \qquad \Delta_s = \delta / \big((1-\delta) \cdot \pi_s + \delta \big).$$

into Eq. (A-4) and then differentiate $S_{\alpha,\delta}$ w.r.t δ as

$$\begin{aligned}\frac{\partial S_{\alpha,\delta}}{\partial \delta} &= \frac{\partial}{\partial \delta}\left[S + \left(\frac{\delta}{(1-\delta)\cdot\pi_b + \delta} \cdot (\pi_l^b - \alpha) - \frac{\delta}{(1-\delta)\cdot\pi_s + \delta} \cdot (\alpha - \pi_h^s) \right) \cdot (V_h - V_l) \right] \\ &= \Bigg[(\pi_l^b - \alpha) \cdot \big(\frac{1}{(1-\delta)\cdot\pi_b + \delta} \cdot (1 - \frac{\delta \cdot \pi_s}{\pi_b + \delta\cdot\pi_s}) \big) \\ &+ (\pi_h^s - \alpha) \cdot \big(\frac{1}{(1-\delta)\cdot\pi_s + \delta} \cdot (1 - \frac{\delta \cdot \pi_b}{\pi_s + \delta\cdot\pi_b}) \big) \Bigg] \cdot (V_h - V_l) \geq 0 \\ &\text{for} \quad \alpha \leq \min\{\pi_h^s, \pi_l^b\}.\end{aligned} \tag{A-9}$$

(ii) *For a sufficiently ambiguity-averse market maker, the magnitude of "ambiguity premium" is decreasing by the incremental informational ambiguity* means $\frac{\partial^2 S_{\alpha,\delta}}{\partial \delta^2} \leq 0$ with the same sufficiency condition. Differentiationg Eq. (A-9) w.r.t δ again

$$\begin{aligned}\frac{\partial^2 S_{\alpha,\delta}}{\partial \delta^2} &= \frac{2\cdot\pi_s\cdot(\pi_l^b - \alpha)}{\big((1-\delta)\cdot\pi_b + \delta\big)^2} \cdot \big(\frac{\delta\cdot\pi_s}{\pi_b + \delta\cdot\pi_s} - 1 \big) \\ &+ \frac{2\cdot\pi_b\cdot(\pi_h^s - \alpha)}{\big((1-\delta)\cdot\pi_s + \delta\big)^2} \cdot \big(\frac{\delta\cdot\pi_b}{\pi_s + \delta\cdot\pi_b} - 1 \big) \leq 0\end{aligned} \tag{A-10}$$

for $\alpha \leq \min\{\pi_h^s, \pi_l^b\}$.

■

Proof of Proposition 5.1.

We prove the separating equilibrium on the ask side. That means we assume informed traders with high signals. A similar calculation follows on the bid side if informed have low signals.

From Eq. (26), we know

$$\frac{b_2}{b_1} \geq \frac{[V_h - A^1_{\alpha,\delta}]}{[V_h - A^2_{\alpha,\delta}]} = \xi^b_{\alpha,\delta}, \tag{A-11}$$

We now substitute the values of $A^1_{\alpha,\delta}$ and $A^2_{\alpha,\delta}$ from Eq. (25) to obtain

$$\xi^b_{\alpha,\delta} = \frac{V_h - A_1 - \Delta_{b_1} \cdot [\pi_l^{b_1} - \alpha](V_h - V_l)}{V_h - A_2 - \Delta_{b_2} \cdot [\pi_l^{b_2} - \alpha] \cdot (V_h - V_l)}. \tag{A-12}$$

A_1 and A_2 are the ask prices conditional on small buy and large buy orders respectively. They can be obtained by the zero expected profit condition of Glosten and Milgrom (1985) with Lebesgue expectations. Substituting the values of A_1 and A_2 from Eq. (7),

$$\begin{aligned} \xi^b_{\alpha,\delta} &= \frac{\pi_l^{b_1} \cdot (V_h - V_l) - \Delta_{b_1} \cdot [\pi_l^{b_1} - \alpha] \cdot (V_h - V_l)}{\pi_l^{b_2} \cdot (V_h - V_l) - \Delta_{b_2}[\pi_l^{b_2} - \alpha] \cdot (V_h - V_l)} \\ &= \frac{\pi_l^{b_1} - \Delta_{b_1} \cdot [\pi_l^{b_1} - \alpha]}{\pi_l^{b_2} - \Delta_{b_2} \cdot [\pi_l^{b_2} - \alpha]} = \frac{\pi_l - \Delta_{b_1} \cdot [\pi_l - \alpha]}{\pi_l^{b_2} - \Delta_{b_2} \cdot [\pi_l^{b_2} - \alpha]} \end{aligned} \tag{A-13}$$

completes the proof.

■

Proof of Corollary 5.2.

(i) *For a sufficiently ambiguity-averse market maker,*

$$\xi^b_{\alpha,\delta} \leq \xi^b \textit{ if } \pi_{b_2} \geq \pi_{b_1} \quad \textit{and} \quad \xi^s_{\alpha,\delta} \leq \xi^s \textit{ if } \pi_{s_2} \geq \pi_{s_1}.$$

This means that with informational ambiguity, the large buy (sell) to small buy (sell) order ratio needed for the market maker to assume that the market is in the separating equilibrium is less than its unambiguous counterpart, if the probability of large buy (sell) order is greater than the small buy (sell) order. To prove the ask side, we first substitute the values of

$$\Delta_{b_1} = \delta / \big((1-\delta) \cdot \pi_{b_1} + \delta\big) \qquad \text{and} \qquad \Delta_{b_2} = \delta / \big((1-\delta) \cdot \pi_{b_2} + \delta\big),$$

into Eq. (A-13) and obtain

$$\xi^b_{\alpha,\delta} = \frac{\pi_l - \Delta_{b_1} \cdot [\pi_l - \alpha]}{\pi_l^{b_2} - \Delta_{b_2}[\pi_l^{b_2} - \alpha]} = \frac{\pi_l - \frac{\delta}{(1-\delta)\cdot\pi_{b_1}+\delta} \cdot [\pi_l - \alpha]}{\pi_l^{b_2} - \frac{\delta}{(1-\delta)\cdot\pi_{b_2}+\delta} \cdot [\pi_l^{b_2} - \alpha]}. \tag{A-14}$$

We know in Eq. (A-14), $\pi_l > \pi_l^{b_2} = (\frac{(1-\mu)\theta}{2\mu\gamma(1-\pi_l)+(1-\mu)\theta}) \cdot \pi_l$. Therefore, it is sufficient to assume $\pi_{b_2} \geq \pi_{b_1}$ for $\xi^b_{\alpha,\delta}$ to be less than $\xi^b = \frac{\pi_l}{\pi_l^{b_2}}$, if $\alpha \leq \pi_l^{b_2}$. The proof for the bid side follows similar steps.

(ii) *Under full informational ambiguity, the market maker always considers the market as a separating equilibrium* means that (M) will assume that informed traders separate themselves, if the large buy (sell) order is greater than small buy (sell) order which is always true. The result of the ask side immediately follows from Eq. (A-14) after substituting $\delta = 1$.

■

Proof of Proposition 5.3.

Suppose the necessary and sufficient condition for the existence of the separating equilibrium is violated on the ask side of the market. That means $\frac{b_2}{b_1} < \xi^b_{\alpha,\delta}$. Since informed traders are indifferent between buying small and large quantities by condition (i) (Eq. (31)) in the pooling equilibrium, by condition (iii) the modeler can choose γ that satisfies both the profit maximization of informed traders and zero expected profit condition of the market maker and hence a pooling equilibrium with the necessary and sufficient condition of

$$\frac{b_2}{b_1} < \frac{\pi_l - \Delta_{b_1} \cdot [\pi_l - \alpha]}{\pi_l^{b_2} - \Delta_{b_2} \cdot [\pi_l^{b_2} - \alpha]} = \xi^b_{\alpha,\delta}. \tag{A-15}$$

The analysis for the bid side is symmetric.

■

Proof of Corollary 5.4

(i) *For a sufficiently ambiguity-averse market maker,*

$$\xi^b_{\alpha,\delta} \leq \xi^b \textit{ if } \pi_{b_2} \geq \pi_{b_1} \quad \textit{and} \quad \xi^s_{\alpha,\delta} \leq \xi^s \textit{ if } \pi_{s_2} \geq \pi_{s_1}.$$

The proof is same as the proof of Corollary 5.2. We first substitute the values of Δ_{b_1} and Δ_{b_2} into Eq. (A-15), exploit the fact that $\pi_l > \pi_l^{b_2}$ and assume a sufficiently ambiguity-averse market maker with $\alpha \leq \pi_l^{b_2}$ to obtain a sufficiency condition of $\pi_{b_2} \geq \pi_{b_1}$ for $\xi^b_{\alpha,\delta} \leq \xi^b$.

(ii) *Under full informational ambiguity, there is no pooling equilibrium on either side of the market.* We substitute $\Delta_{b_1} = \Delta_{b_2} = 1$ in Eq. (A-15) and obtain $\xi_{\alpha,\delta} = 1$. Since the large buy to small buy order ratio can not be less than 1, the pooling equilibrium can not prevail under full informational ambiguity.

■

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