

A Note on Stability in One-to-One, Multi-period Matching Markets*

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Abstract

We introduce a new stability concept for multi-period matching markets. Robust prescient stability asserts that agents exhibit foresight concerning how a market can develop in the future, but they retain ambiguity concerning how the market will develop. We show that a robustly presciently stable matching exists for any configuration of agents' preferences.

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In a matching market, there are two groups of agents. In classic terminology, these groups may be men and women, workers and firms, students and schools, or doctors and hospitals. An agent from one group must match from an agent from the other group to realized economic value. Agents have preferences over their potential partners and, of course, more preferred partners are associated with superior outcomes. As preferences may conflict, the emergent pattern of pairings or matchings is far from trivial at the outset.

In their seminal analysis, Gale and Shapley (1962) argue that *stable* matchings are likely to form in a two-sided market. Using their terminology of a matching between men and women, a matching is stable if (i) each agent’s assignment is individually rational and (ii) no pair of unmatched agents prefers to be together in lieu of their assigned partner. When either condition fails, a matching is said to be “blocked” by the agent or the pair. As Roth (2002) explains, a market’s durability is intimately tied to its ability to consistently coordinate upon a stable outcome.

A key feature of the canonical examples of matching markets—the marriage market, the labor market, etc.—is that they involve long-term interactions. Marriages are rarely ephemeral and jobs may last months, if not years. Men and women care about their relationship’s long-term durability, workers care about their entire career, and a firm may be concerned about the evolution of its personnel during a project. Maintaining employee engagement and retaining star talent are perennial challenges. Reluctantly or enthusiastically, long-term relationships may be revised and agents often re-match with others as time passes.

Once we approach the matching problem with a long-term perspective, several complications emerge. Chief among these is the absence of an immediate analogue to Gale and Shapley’s (1962) stability concept. In a multi-period market, agents encounter a sequence of pairings. If commitment is limited or imperfect, an agent may object to a proposed plan at any moment in time conditional on the market’s history. The anticipated gains from an objection, however, depend critically on how the market might evolve in the future, conditional on that objection. Will agents be able to maintain future relationships conditional on a single-period absence? How do others respond? Are coordinated, long-term deviations with groups of others credible options or are short-term pairwise deviations the only realistic option?

Many plausible answers lie behind each of the preceding questions. Thus, “stability” in a multi-period market comes in a variety of flavors. Damiano and Lam (2005) and Kurino (2009), for example, emphasize the importance of credible group deviations. Both identify drawbacks with typical definitions of the market’s core. Kadam and Kotowski (2015a) pro-

pose a stability concept emphasizing agents’ uncertainty concerning future developments in the wider market, though they downplay the role of commitment in the formulation of deviating plans. Kennes et al. (2014a), who study the assignment of children to Danish daycares, have defined stability tailored to their particular multi-period application.

Contributing to this debate, we propose a new stability concept—robust prescient stability (RPS)—that both generalizes Gale and Shapley’s (1962) classic idea while accommodating two salient features of a multi-period economy. First, in a RPS matching agents have foresight concerning how the market may develop in the future. Specifically, they foresee that “stable” or RPS continuation plans will define the market’s future development. Thus, agents’ conjectures concerning the future are limited to credible future outcomes. Crucially, however, foresight is not perfect. Generally many RPS continuation plans may exist at any moment in time and an agent evaluates the virtue of a particular blocking action by focusing on the worst-case RPS continuation plan. Thus, agents assess the future robustly. Together, these ingredients yield a solution concept balancing the credibility concerns identified by Damiano and Lam (2005) while accommodating uncertainty concerning future developments.

In the following section we relate our analysis to the prior literature. In Section 2 we introduce the model with notation similar to that employed by Kadam and Kotowski (2015a) and Kadam and Kotowski (2015b). In Section 3 we define a RPS matching and we verify its existence for *any* configuration of agents’ preferences over life-time partnership plans. In Section 4 we provide a refinement of robust prescient stability. This refinement reduces the stable set by eliminating certain incredible counterfactual outcomes. We highlight some directions for future research before concluding.

1 Related Literature

From a thematic point of view, our study contributes the growing literature on two-sided, dynamic or multi-period matching economies (Damiano and Lam, 2005; Kurino, 2009; Pereyra, 2013; Kennes et al., 2014a,b; Kadam and Kotowski, 2015a,b). In such economies, agents are long-lived and experience a sequence of matchings. Assignments may change from period-to-period. In relation to this literature, we define a new stability concept distinct from prior proposals. Our proposal’s broad applicability is its main distinguishing quality. We do not demand agents’ preferences to be additively separable across time periods as Damiano and Lam (2005) do. History-independence or time-invariance are also unnecessary (Kurino, 2009, 2014; Bando, 2012). Nor do we restrict the nature of inter-temporal complementarities, as

imposed by or Kadam and Kotowski (2015a). Our solution concept is also independent of institutional context. Thus, it does not rely on the features of a particular application, such as daycare assignment (Kennedy et al., 2014a,b) or school choice (Dur, 2012; Pereyra, 2013).

Doval (2015) proposes a stability concept, termed dynamic stability, that can be applied to one- and two-side multi-period matching markets, including those that we study.¹ The stability concepts that we investigate share the backward-inductive reasoning found in Doval’s (2015) dynamic stability. Agents who “block” a matching in period t anticipate that future assignments will be “stable” given their deviation.² However, several important differences distinguish our study from Doval’s analysis. First, Doval follows Corbae et al. (2003) and Kurino (2009) by studying contingent matchings, which specify an assignment conditional on all histories of the economy. In our study, a (multi-period) matching refers only to the observed sequence of assignments. Second, as matchings are economy-wide contingent plans, Doval’s dynamic stability assumes that agents are in agreement concerning the future consequences of their behavior. Such agreement is not presumed in our analysis. Doval (2015) shows that dynamically stable matchings may not exist in her model of a two-sided, one-to-one matching market and she provides sufficient conditions to ensure existence. Several aspects of her model differ from our setting. Among notable differences, she assumes that matchings are irrevocable, i.e. if two agents match in period t they remain matched thereafter, agents’ preferences over partners are time invariant and feature discounting, and agents may arrive to the market at different times. Our model differs in these respects.

From a technical point of view, Sasaki and Toda’s (1996) study of a one-to-one matching market with externalities is related to our analysis. The conceptual parallel is the following. In Sasaki and Toda (1996), agents impose externalities on third parties when they match together. In our setting, cross-agent externalities are absent. However, an agent’s matching in an early period imposes an externality on his future self. It does so in two ways. First, since preferences may be path- or history-dependent, an initial matching may materially affect the agent’s future interests or preferences. Second, an early matching may change others’ future interests or preferences precluding or facilitating future pairings. Both effects interact to

¹This stability concept was first introduced in the working paper Doval (2014), which preceded this study. The “dynamic stability” proposed contemporaneously and independently by Kadam and Kotowski (2015a) is distinct. We relate our analysis to that of Kadam and Kotowski (2015a) in Section 3.

²Though formulated in an infinite horizon model, the equilibrium concept proposed by Corbae et al. (2003) shares this feature. In their model, a matching is an equilibrium if it specifies an assignment that cannot be blocked by any agent or pair after any history, including histories following a deviation from the prescribed matching rule. Not all solution concepts proposed for multi-period matching models impose this requirement.

determine the desirability of a particular matching from an agent’s point of view. While our main solution concept draws on Sasaki and Toda’s (1996) intuition by stressing worst-case outcomes, the existence of a RPS matching is not a corollary of their model. Damiano and Lam (2005) also point to the importance of cross-period externalities by defining a multi-period matching problem’s “agent form.” We do not employ this formulation in our analysis.

In Section 4 we propose a refinement of robust prescient stability. Borrowing Sasaki and Toda’s (1996) terminology, this refinement rests on a modification of agents’ “estimation functions.” In a static matching market with externalities, Hafalir (2008) has proposed a different preference-based refinement of agents’ estimation functions. He constructs a “sophisticated estimation function” through an algorithm that expands the set of conjectured outcomes. The algorithm that we propose differs from his construction and it is analogous to the elimination of strictly dominated strategies in a normal-form game. As noted by Hafalir (2008), there are many other behaviorally-plausible ways in which an estimation function can be refined. Each of these can be applied to our model yielding somewhat different conclusions.

2 The Model

Let M and W be finite disjoint sets of agents, whom we will call men and women, respectively. We let i, j , and k represent generic agents. When needed, we use m ’s and w ’s to identify an agent’s membership in M and W , respectively. Agents interact over $T < \infty$ periods. In every period, each man can be matched to at most one woman or not matched at all. When an agent is not matched, we adopt the common convention that he is “matched to himself.” Thus, the set of potential partners for $m \in M$ in period t is $W \cup \{m\}$. Symmetrically, the set of potential partners for $w \in W$ is $M \cup \{w\}$.

Each agent i has a strict and complete preference \succ_i over *partnership plans*.³ He is never indifferent among alternative plans. A partnership plan is a sequence of assignments, one for each period. For example, $(w_1, w_2, m_1, w_2, \dots) \in (W_{m_1})^T$ is a partnership plan for m_1 where he is matched to w_1 in period 1 and to w_2 in period 2. He is not matched in period 3 and matched again to w_2 in period 2. And so on. Unless confusion is a risk, we will suppress commas and brackets, i.e. $(w_1, w_2, m_1, w_2, \dots) = w_1 w_2 m_1 w_2 \dots$.

As standard, we call the function $\mu_t: M \cup W \rightarrow M \cup W$ an *one-period matching* (for

³The weak preference \succeq_i is defined in the usual manner.

period t) if (i) $\forall m \in M, \mu_t(m) \in W_m$; (ii) $\forall w \in W, \mu_t(w) \in M_w$; and, (iii) $\forall i \in M \cup W, \mu_t(i) = j \implies \mu_t(j) = i$. Thus a matching assigns each agent a partner or leaves him unmatched. Of course, if i is matched to j , then j is matched to i . Paralleling Sasaki and Toda's (1996) notation, let \mathbb{A} be the set of all one-period matchings among agents in M and W . Let $\mathbb{A}(i, j) = \{\mu_t \in \mathbb{A} \mid \mu_t(i) = j\}$ be the set of matchings where agent i is matched to agent j . It follows that $\mathbb{A}(i, j) = \mathbb{A}(j, i)$. A (*multi-period*) *matching*, $\mu: M \cup W \rightarrow (M \cup W)^T$, is a sequence of one-period matchings: $\mu = (\mu_1, \dots, \mu_T)$. Unless emphasis is needed, we henceforth refer to a multi-period matching simply as a *matching*.

3 Robust Prescient Stability

In this section we introduce our first solution concept, which we term robust prescient stability. Its definition is inductive, starting with the final period and some specialized notation simplifies its introduction. Given a matching μ , define the partial matchings $\mu_{\leq t} = (\mu_1, \dots, \mu_t)$ and $\mu_{> t} = (\mu_{t+1}, \dots, \mu_T)$. When convenient, we call the partial matching $\mu_{> t}$ a *continuation plan*. Of course, we can write $\mu(i) = (\mu_{\leq t}(i), \mu_{> t}(i))$. Similarly, if $\mu(i) = (\mu_{< t}(i), j, \mu_{> t}(i))$ then j is the period- t assignment of i under the matching μ .

We will call a matching robustly presciently stable if it cannot be “blocked” in any period. Blocking, however, is defined inductively, starting with the final period. After defining the concept formally, we discuss its key assumptions and interpretations.

Definition 1 (Period- T Blocking).

1. Agent i can block the matching μ in period T if $(\mu_{< T}(i), i) \succ_i \mu(i)$.
2. Agents $m \in M$ and $w \in W$ can block the matching μ in period T if

$$(\mu_{< T}(m), w) \succ_m \mu(m) \quad \text{and} \quad (\mu_{< T}(w), m) \succ_w \mu(w).$$

Suppose blocking has been defined for all periods $t' > t$. Given a partial matching $\mu_{\leq t}$, let $\mathbb{S}(\mu_{\leq t}) \subset \mathbb{A}^{T-t}$ be the set of continuation plans $\tilde{\mu}_{> t} = (\tilde{\mu}_{t+1}, \dots, \tilde{\mu}_T)$ such that the matching $(\mu_{\leq t}, \tilde{\mu}_{> t})$ cannot be blocked by any agent or by any pair in any period $t' > t$.

Definition 2 (Period- t Blocking).

1. Agent i can block the matching μ in period t if for all $\tilde{\mu}_t \in \mathbb{A}(i, i)$,

$$(\mu_{< t}, i, \tilde{\mu}'_{> t}(i)) \succ_i \mu(i)$$

for all $\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{<t}, \tilde{\mu}_t))$.

2. Agents $m \in M$ and $w \in W$ can block the matching μ in period t if for all $\tilde{\mu}_t \in \mathbb{A}(m, w)$,

$$(\mu_{<t}(m), w, \tilde{\mu}'_{>t}(m)) \succ_m \mu(m) \quad \text{and} \quad (\mu_{<t}(w), m, \tilde{\mu}'_{>t}(w)) \succ_w \mu(w)$$

for all $\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{<t}, \tilde{\mu}_t))$.

Definition 3. A matching is *robustly presciently stable (RPS)* if it cannot be blocked in any period t by any agent or by any pair.

It follows immediately from the definition that when $T = 1$ a matching is robustly presciently stable if and only if it is stable in the sense of Gale and Shapley (1962). Hence, an RPS matching exists when the market lasts a single period. Of course, we are concerned with a multi-period market. Thus, our main result can be formulated as follows.

Theorem 1. *If $T < \infty$, there exists a robustly presciently stable (RPS) matching.*

We prove Theorem 1 in the following section. Here we elaborate upon the definition of robust prescient stability and we link it to other stability concepts.

To provide an interpretation of RPS matchings, it is natural to start with the nomenclature. An RPS matching is *prescient* as it demands that agents are farsighted. A prescient individual has foresight or “knowledge of things or events before they exist or happen.”⁴ In our application, that foresight rests on the ability to know which matchings can occur without being blocked in future periods. Thus, the agents’ farsightedness is distinct from that studied by Chwe (1994). RPS matchings demand that agents know others’ interests sufficiently well to rule out future incredible outcomes with confidence.

Though RPS demands foresight, perfect foresight is not presumed. While an agent knows what continuation plans *can* occur in the future, he does not know which plan *will* occur. To resolve this ambiguity, RPS imposes a robustness requirement inductively by asking the agent to hone in on the worst-case possibility in each period. In evaluating the implications of blocking a matching, an agent conjectures that the market will evolve in such a manner that the worst-case, not-blockable continuation plan will result.

It is interesting to contrast the form of robustness encountered in an RPS matching with that seen in a dynamically stable matching, as defined by Kadam and Kotowski (2015a).

⁴“Prescient | Define Prescient at Dictionary.com.” Dictionary.com Unabridged. Random House, Inc. Accessed March 29, 2015. <<http://dictionary.reference.com/browse/prescient>>.

Dynamic stability also bears the flavor of robustness as agents believe the market will evolve in the most unfavorable manner. This conjecture downplays the outcome’s credibility or future stability. Thus, the solution concepts differ.

Example 1. Consider a market with one man and one woman.⁵ Their preferences are:

$$\succsim_m : wm, ww, mm \quad \succsim_w : mm, ww$$

In this case, wm is m ’s most preferred outcome. ww is second best, and so on.

This economy does not have a dynamically stable matching in the sense of Kadam and Kotowski (2015a). It has a unique RPS matching: $\mu^*(m) = mm$ and $\mu^*(w) = ww$. The matching where both agents are together in both periods is not stable as m will renege on the plan in period 2. Recognizing this fact, both agents agree to remain unmatched since they cannot credibly embark on the two-period partnership.

The following example, adapted from Kadam and Kotowski (2015b), shows that there may exist dynamically stable matchings that are not RPS matchings.

Example 2. There are three men and women. Their preferences are:

$$\begin{aligned} \succsim_{m_1} &: w_2w_2, w_2w_1, w_2w_3, w_1w_1, w_3w_3, m_1m_1, \dots \\ \succsim_{m_2} &: w_3w_3, w_3w_2, w_3w_1, w_2w_2, w_1w_1, m_2m_2, \dots \\ \succsim_{m_3} &: w_1w_1, w_1w_3, w_3w_3, w_1w_2, w_2w_2, m_3m_3, \dots \\ \\ \succsim_{w_1} &: m_2m_2, m_1m_1, m_2m_1, m_1m_2, w_1m_2, m_3m_2, w_1w_1, m_3m_3, \dots \\ \succsim_{w_2} &: m_3m_3, m_2m_2, m_3m_2, m_2m_3, w_2m_3, m_1m_3, w_2w_2, m_1m_1, \dots \\ \succsim_{w_3} &: m_1m_1, m_1m_3, m_3m_1, w_3m_1, m_2m_1, m_3m_3, w_3w_3, m_2m_2, \dots \end{aligned}$$

Appendix A provides the full preference ranking of the agents in this example. In this market there are three dynamically stable matchings as outlined in Table 1 (for the details see Kadam and Kotowski (2015b)). Of the three, only μ^1 and μ^2 are RPS matchings.

The following example shows that some properties enjoyed by the set of stable matchings in a one-period economy are not necessarily inherited by the set of RPS matchings.

⁵Kadam and Kotowski (2015a) employ this same example to show the non-existence of a “dynamically-stable” matching. Hatfield and Kominers (2012) consider a similar, more elaborate, example of a doctor and a hospital contracting morning and afternoon shifts.

Table 1: All dynamically stable matchings in Example 2.

Matching	m_1	m_2	m_3	w_1	w_2	w_3	Dynamically Stable	RPS
μ^1	w_1w_1	w_2w_2	w_3w_3	m_1m_1	m_2m_2	m_3m_3	✓	✓
μ^2	w_3w_3	w_1w_1	w_2w_2	m_2m_2	m_3m_3	m_1m_1	✓	✓
μ^3	w_2w_3	w_3w_1	w_1w_2	m_3m_2	m_1m_3	m_2m_1	✓	

Example 3. Consider a two-period economy with two men and two women. The agents' preferences are the following:

$$\succ_{m_1} : \underline{w_1m_1}, w_1w_2, \boxed{w_2w_2}, w_2w_1, w_1w_1, m_1m_1, m_1w_2, m_1w_1, w_2m_1$$

$$\succ_{m_2} : \underline{m_2w_2}, \boxed{w_1w_1}, w_1w_2, w_2w_1, w_1m_2, m_2w_1, w_2w_2, w_2m_2, m_2m_2$$

$$\succ_{w_1} : m_1m_1, m_1m_2, \boxed{m_2m_2}, m_2m_1, \underline{m_1w_1}, w_1w_1, w_1m_2, w_1m_1, m_2w_1$$

$$\succ_{w_2} : \underline{w_2m_2}, \boxed{m_1m_1}, m_1m_2, m_2m_1, m_1w_2, w_2m_1, w_2w_2, m_2w_2, m_2m_2$$

There are two RPS matchings in this economy. In the above preference list, μ^* is underlined while μ^{**} is boxed. In this example, there does not exist a “ W -optimal” RPS matching. When it exists, such a matching is preferred by all women to all other stable matchings. For w_1 , $\mu^{**}(w_1) \succ_{w_1} \mu^*(w_1)$ while w_2 holds the opposite opinion. An implication of this outcome is that the set of RPS matchings need not form a lattice under the usual common-preference partial order.⁶

While robustness and prescience are the main features of an RPS matching, several other observations are worthwhile. First, RPS follows Gale and Shapley (1962) and maintains a pairwise definition of blocking. Others have proposed group-wise definitions of stability in multi-period markets.⁷ Kadam and Kotowski (2015a), whose dynamic stability is also a pairwise concept, discuss and defend the use of pairwise definitions of blocking in multi-period economies.

Second, RPS posits that agents have essentially no commitment ability. All blocking

⁶In a one-period market, the set of stable matchings forms a lattice. For more on this property of stable matchings, we refer the reader to the survey by Roth and Sotomayor (1990).

⁷For example, see Damiano and Lam (2005), Kurino (2009), or Doval (2015). In contrast, Corbae et al. (2003) focus on pairwise blocking as we do.

actions are one-period affairs. A pair of agents cannot commit to coordinate their assignments for two periods, say, so as to block a proposed matching. Unless such a conjectured plan yields an improved worst-case outcome, one party party will not trust the other to follow through on the agreed-to arrangement. Many of the aforementioned alternative stability concepts assume that groups of agents can formulate and credibly implement elaborate continuation arrangements only among themselves. RPS does not bestow agents with this degree of trust.

3.1 Proof of Theorem 1

To prove Theorem 1, we proceed by induction. Lemma 1 proves the base-case. The inductive step is considered in the main proof to follow below. Throughout, let $T \geq 2$.

Lemma 1. *Let $\mu_{\leq T-1}$ be a $(T-1)$ -period partial matching. $\mathbb{S}(\mu_{\leq T-1}) \neq \emptyset$.*

Proof. Fix $\mu_{\leq T-1}$. For each agent i , define a one-period preference conditional on $\mu_{\leq T-1}$, denoted by $P_i^{\mu_{\leq T-1}(i)}$, as follows:

$$jP_i^{\mu_{\leq T-1}(i)}k \iff (\mu_{\leq T-1}(i), j) \succ_i (\mu_{\leq T-1}(i), k).$$

Now consider the one-period economy where each agent's preference over potential partners is given by $P_i^{\mu_{\leq T-1}(i)}$. By Gale and Shapley (1962), there exists a stable one-period matching in this economy. Denote this stable one-period matching by μ_T^* .

Next, consider the T period matching $(\mu_{\leq T-1}, \mu_T^*)$. Suppose that agent i can block this matching in period T in the sense of Definition 1. Thus, $(\mu_{\leq T-1}(i), i) \succ_i (\mu_{\leq T-1}(i), \mu_T^*(i)) \implies iP_i^{\mu_{\leq T-1}(i)}\mu_T^*(i)$, which is a contradiction since $\mu_T^*(i)$ is a stable matching in a one-period economy.

Similarly, if $m \in M$ and $w \in W$ can block the matching $(\mu_{\leq T-1}, \mu_T^*)$ in period T then $(\mu_{\leq T-1}(m), w) \succ_m (\mu_{\leq T-1}(m), \mu_T^*(m)) \implies wP_m^{\mu_{\leq T-1}(m)}\mu_T^*(m)$ and $(\mu_{\leq T-1}(w), m) \succ_w (\mu_{\leq T-1}(w), \mu_T^*(w)) \implies mP_w^{\mu_{\leq T-1}(w)}\mu_T^*(w)$. Hence, m and w can block the matching μ_T^* in the one-period economy, which is a contradiction. Therefore, $\mu_T^* \in \mathbb{S}(\mu_{\leq T-1})$. \square

Proof of Theorem 1. To verify that there exists a RPS matching it is sufficient to show that for all $t = 0, \dots, T-1$ and any partial matching $\mu_{\leq t}$, $\mathbb{S}(\mu_{\leq t}) \neq \emptyset$. Existence is implied by the $t = 0$ case.

Lemma 1 has confirmed that $\mathbb{S}(\mu_{\leq T-1}) \neq \emptyset$ for all partial matchings $\mu_{\leq T-1}$. Thus, proceeding by induction, suppose $\mathbb{S}(\mu_{\leq t}) \neq \emptyset$ for all partial matchings $\mu_{\leq t}$. We will verify

that $\mathbb{S}(\mu_{\leq t-1}) \neq \emptyset$.

Fix a $(t-1)$ -period partial matching, μ_{t-1} . For each agent i , define a one-period preference conditional on μ_{t-1} , denoted $P_i^{\mu_{\leq t-1}(i)}$, as follows:

$$jP_i^{\mu_{\leq t-1}(i)}k \iff \min_{\tilde{\mu}_t \in \mathbb{A}(i,j)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{\leq t-1}, \tilde{\mu}_t))} (\mu_{\leq t-1}(i), j, \tilde{\mu}'_{>t}(i)) \right) \succ_i \min_{\tilde{\mu}_t \in \mathbb{A}(i,k)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{\leq t-1}, \tilde{\mu}_t))} (\mu_{\leq t-1}(i), k, \tilde{\mu}'_{>t}(i)) \right).$$

The minimizations in the above expression are taken with respect to the (total) order \succ_i over finite sets. By the induction hypothesis, for each $\tilde{\mu}_t$, $\mathbb{S}((\mu_{\leq t-1}, \tilde{\mu}_t)) \neq \emptyset$. Therefore, $P_i^{\mu_{\leq t-1}(i)}$ is well defined.

Now consider the one-period economy where each agent's preference over potential partners is given by $P_i^{\mu_{\leq t-1}(i)}$. By Gale and Shapley (1962), there exists a stable one-period matching in this economy. Denote this stable one-period matching by μ_t^* . Now, choose some $\mu'_{>t} \in \mathbb{S}((\mu_{\leq t-1}, \mu_t^*))$. We will verify that the matching $(\mu_{<t}, \mu_t^*, \mu'_{>t})$ cannot be blocked in period t . There are two cases.

First, suppose $(\mu_{<t}, \mu_t^*, \mu'_{>t})$ can be blocked in period t by agent i . Thus, for all $\tilde{\mu}_t \in \mathbb{A}(i, i)$,

$$(\mu_{<t}(i), i, \tilde{\mu}'_{>t}(i)) \succ_i (\mu_{<t}(i), \mu_t^*(i), \mu'_{>t}(i))$$

for all $\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{<t}, \tilde{\mu}_t))$. In particular, this implies,

$$\begin{aligned} & \min_{\tilde{\mu}_t \in \mathbb{A}(i,i)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{\leq t-1}, \tilde{\mu}_t))} (\mu_{\leq t-1}(i), i, \tilde{\mu}'_{>t}(i)) \right) \\ & \succ_i (\mu_{<t}(i), \mu_t^*(i), \mu'_{>t}(i)) \\ & \succsim_i \min_{\tilde{\mu}_t \in \mathbb{A}(i, \mu_t^*(i))} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{\leq t-1}, \tilde{\mu}_t))} (\mu_{\leq t-1}(i), \mu_t^*(i), \tilde{\mu}'_{>t}(i)) \right). \end{aligned}$$

Thus, $iP_i^{\mu_{\leq t-1}(i)}\mu_t^*(i)$. Hence, i can block the matching μ_t^* in the one-period economy defined above. But this contradicts the stability of the one-period matching $\mu_t^*(i)$.

Instead, suppose agents $m \in M$ and $w \in W$ can block $(\mu_{<t}, \mu_t^*, \mu'_{>t})$ in period t . Thus, for all $\tilde{\mu}_t \in \mathbb{A}(m, w)$,

$$(\mu_{<t}(m), w, \tilde{\mu}'_{>t}(m)) \succ_m (\mu_{<t}(m), \mu_t^*(m), \mu'_{>t}(m))$$

for all $\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{<t}, \tilde{\mu}_t))$. In particular, this implies

$$\begin{aligned} & \min_{\tilde{\mu}_t \in \mathbb{A}(m, w)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{\leq t-1}, \tilde{\mu}_t))} (\mu_{\leq t-1}(m), w, \tilde{\mu}'_{>t}(m)) \right) \\ & \succ_m (\mu_{<t}(m), \mu_t^*(m), \mu_{>t}^*(m)) \\ & \approx_m \min_{\tilde{\mu}_t \in \mathbb{A}(m, \mu_t^*(m))} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{\leq t-1}, \tilde{\mu}_t))} (\mu_{\leq t-1}(m), \mu_t^*(m), \tilde{\mu}'_{>t}(m)) \right). \end{aligned}$$

Thus, $w P_m^{\mu_{\leq t-1}(m)} \mu_t^*(m)$. Similarly, we conclude that $m P_w^{\mu_{\leq t-1}(w)} \mu_t^*(w)$. Thus, m and w can block μ_t^* in the one-period economy, which is a contradiction. \square

4 Prudent Prescient Stability

Situations may arise where it may be unreasonable for an agent to anticipate that others will form a particular matching when evaluating the consequences of his or her blocking behavior. For example, a strong intuition suggests that an agent should not believe others will enter into a matching that unambiguously leads to a not individually rational outcome. Even if such behavior serves as a forceful “punishment” for a “deviation” from a particular matching, it lacks credibility—at least in a very basic sense. Thus, when contemplating a blocking action, an agent may be sufficiently comfortable to disregard such contingent contemporaneous assignments. A result of this more refined model of others’ plausible behavior is that blocking *ought* to become easier, thereby reducing the stable set’s size.

In this section we refine robust prescient stability accounting for this more sophisticated model of agent blocking behavior. We operationalize the preceding intuition with Algorithm 1, which defines an iterative process closely resembling the iterated elimination of strictly dominated actions common to game theory. In each period, this process is applied to winnow-down the set $\mathbb{A}(\cdot, \cdot)$ that features in the definition of blocking introduced above. The resulting modified definition of blocking, which we term prudent blocking, allows an agent to disregard incredible outcomes when contemplating a blocking action. As noted above, there are many other behaviorally-plausible ways in which $\mathbb{A}(\cdot, \cdot)$ may be reduced in size yielding slightly different definitions of blocking.

Mirroring the discussion above, prudent blocking is defined recursively, starting with the final period. In period T it coincides with the usual definition of blocking as there is but one period remaining.

Definition 4 (Prudent Period- T Blocking).

1. Agent i can credibly block the matching μ in period T if $(\mu_{<T}(i), i) \succ_i \mu(i)$.
2. Agents $m \in M$ and $w \in W$ can credibly block the matching μ in period T if $(\mu_{<T}(m), w) \succ_m \mu(m)$ and $(\mu_{<T}(w), m) \succ_w \mu(w)$.

Proceeding inductively, suppose that prudent blocking has been defined for all periods $t' > t$. Given the partial matching $\mu_{\leq t}$, let $\mathbb{S}^*(\mu_{\leq t})$ be the set of continuation plans $\tilde{\mu}_{>t} = (\tilde{\mu}_{t+1}, \dots, \tilde{\mu}_T)$ such that the matching $(\mu_{\leq t}, \tilde{\mu}_{>t})$ cannot be prudently blocked by any agent or pair in any period $t' > t$.

To define prudent blocking for an arbitrary period t requires some additional notation. As before, let \mathbb{A} be the set of all one-period assignments. For any $\tilde{\mathbb{A}} \subset \mathbb{A}$, let $\tilde{\mathbb{A}}(i, j) \subset \tilde{\mathbb{A}}$ be the subset of assignments where i is assigned to j . That is, $\tilde{\mathbb{A}}(i, j) = \{\mu_t \in \tilde{\mathbb{A}} : \mu_t(i) = j\}$.

Algorithm 1 (Iterated Elimination of Individually Incredible Contemporaneous Assignments). Fix t and a partial matching $\mu_{<t}$. For each pair $\{i, j\}$ define $\mathbb{A}_{\mu_{<t}}^*(i, j)$ inductively as follows.⁸

1. Let $\mathbb{A}_{\mu_{<t}}^0 := \mathbb{A}$ be the set of one-period matchings.
2. Given $\tau \geq 1$ and $\mu_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}$, $\mu_t \in \mathbb{A}_{\mu_{<t}}^\tau$ if and only if μ_t *does not* mandate an individually incredible assignment at step τ . An assignment of i to j is *individually incredible at step τ* if for all $\hat{\mu}_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, i)$ and for all $\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, j)$,

$$(\mu_{<t}(i), \hat{\mu}_t(i), \hat{\mu}_{>t}(i)) \succ_i (\mu_{<t}(i), \tilde{\mu}_t(i), \tilde{\mu}_{>t}(i))$$

for all $\hat{\mu}_{>t} \in \mathbb{S}^*((\mu_{<t}, \hat{\mu}_t))$ and all $\tilde{\mu}_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))$.

Let $\mathbb{A}_{\mu_{<t}}^* := \bigcap_{\tau \geq 0} \mathbb{A}_{\mu_{<t}}^\tau$.

If $\mu_t \in \mathbb{A}_{\mu_{<t}}^*$ then we say that μ_t *survives the iterated elimination of individually incredible contemporaneous assignments*.

The definition of prudent blocking (intentionally) bears close resemblance to the definition of blocking, introduced above. The difference is that $\mathbb{A}(\cdot, \cdot)$ and $\mathbb{S}(\cdot)$ are replaced by $\mathbb{A}_{\mu_{<t}}^*(\cdot, \cdot)$ and $\mathbb{S}^*(\cdot)$, respectively.

Definition 5 (Prudent Period- t Blocking).

⁸The case where $i = j$ is admissible.

1. Agent i can prudently block the matching μ in period t if $\mathbb{A}_{\mu_{<t}}^*(i, i) \neq \emptyset$ and for all $\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(i, i)$,

$$(\mu_{<t}, i, \tilde{\mu}'_{>t}(i)) \succ_i \mu(i)$$

for all $\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))$.

2. Agents $m \in M$ and $w \in W$ can prudently block the matching μ in period t if $\mathbb{A}_{\mu_{<t}}^*(m, w) \neq \emptyset$ and for all $\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(m, w)$,

$$(\mu_{<t}(m), w, \tilde{\mu}'_{>t}(m)) \succ_m \mu(m) \quad \text{and} \quad (\mu_{<t}(w), m, \tilde{\mu}'_{>t}(w)) \succ_w \mu(w)$$

for all $\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))$.

Definition 6. A matching is *prudently presciently stable (PPS)* if it cannot be prudently blocked in any period t by any agent or by any pair.

Theorem 2. *If $T < \infty$, there exists a PPS matching.*

Theorem 3. *If μ^* is a PPS matching, then it is a RPS matching.*

Theorem 3 identifies a natural logical relationship among PPS and RPS matchings. The theorem's conclusion is not an immediate consequence of the definition of prudent blocking for two reasons. First, inductive reasoning is required to accommodate the recursive appearances of $\mathbb{S}(\cdot)$ and $\mathbb{S}^*(\cdot)$ throughout the definitions of regular and prudent blocking. Second, the definition of prudent blocking constrains the conjectured contemporaneous assignments of others and it reduces the set of available blocking actions available to agent i . (Recall that agent i cannot block with j if $\mathbb{A}_{\mu_{<t}}^*(i, j) = \emptyset$.) These two features of the definition carry divergent implications for the ease with which a matching can be blocked.

Before proving theorems 2 and 3 below, we present an example that illustrates the difference between RPS and PPS matchings. It additionally confirms that an RPS matching need not be a PPS matching.

Example 4. Consider a two-period economy with two men and two women. The agents' preferences are the following:

$$\begin{aligned} \succ_{m_1} : & \underbrace{w_1 w_1}_{\mu^*, \mu^{**}}, m_1 m_1, \dots & \succ_{w_1} : & \underbrace{m_1 m_1}_{\mu^*, \mu^{**}}, w_1 w_1, m_2 m_2, \dots \\ \succ_{m_2} : & \underbrace{w_2 w_2}_{\mu^{**}}, \underbrace{m_2 w_2}_{\mu^*}, m_2 m_2, w_1 w_1, \dots & \succ_{w_2} : & \underbrace{w_2 m_2}_{\mu^*}, \underbrace{m_2 m_2}_{\mu^{**}}, w_2 w_2, \dots \end{aligned}$$

Two matchings are highlighted in the above preference lists. Call the boxed matching μ^* and the underlined matching μ^{**} . Both are RPS matchings and there do not exist any other RPS matchings in this economy. Only μ^{**} is a PPS matching. The (RPS) stability of μ^* depends on the proposition that w_2 is partnered with m_2 in period 2 despite preferring to remain unmatched in period 1. This arrangement is rationalized in w_2 's mind by the belief that if she does not match with m_2 in period 1, m_2 will match with w_1 in both periods. (Conditional on m_1 matching with w_1 in period 1, maintaining that assignment for the next period is a stable continuation.) Thus, the best w_2 can do is to remain unmatched in both periods. This outcome is worse than the $\mu^*(w_2) = m_2m_2$ plan.

The tenuous aspect of the preceding situation is that m_2 views a two-period matching with w_1 as a *less desirable* outcome than remaining unmatched. Likewise, w_1 would prefer to remain unmatched rather than match with m_2 for two periods. Neither fact lends support to w_2 's rationalization described above. Consequently, μ^* is not a PPS matching.

Remark 1. Though we have assumed that the set of agents is fixed, one can model agent arrivals and departures within our setting as follows. Conditional on any history, an agent who is not “present” in period t ranks plans where he is unmatched in period t as superior to all plans that posit otherwise. For example, in a three period economy where agent i arrives in period 3, his (top) preferences may read as follows

$$ij \succ_i ik \succ_i \dots \succ_i iil \succ_i iii \succ_i \dots$$

In such applications, PPS would be a preferable solution concept to RPS as it precludes incredible counterfactual assignments involving agents who are not present in the economy.

4.1 Proofs of Theorems 2 and 3

To prove Theorem 2, we first show that $\mathbb{A}_{\mu_{<t}}^* \neq \emptyset$.

Lemma 2. *Fix t and a partial matching $\mu_{<t}$. Suppose $\mathbb{S}^*((\mu_{<t}, \mu_t)) \neq \emptyset$ for all one-period matchings μ_t . Define $\mathbb{A}_{\mu_{<t}}^*$ as in Algorithm 1. Then $\mathbb{A}_{\mu_{<t}}^* \neq \emptyset$. Moreover, $\mathbb{A}_{\mu_{<t}}^*(i, i) \neq \emptyset$ for all i .*

Proof. Consider the one-period matching μ_t such that $\mu_t(i) = i$ for all i . If $\mu_t \notin \mathbb{A}_{\mu_{<t}}^*$, it must have involved an assignment that was individually incredible at some step τ of Algorithm 1. Without loss of generality, suppose agent i 's assignment is deemed incredible. Hence, for all

$\hat{\mu}_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, i)$ and for all $\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, \mu_t(i))$,

$$(\mu_{<t}(i), \hat{\mu}_t(i), \hat{\mu}_{>t}(i)) \succ_i (\mu_{<t}(i), \tilde{\mu}_t(i), \tilde{\mu}_{>t}(i))$$

for all $\hat{\mu}_{>t} \in \mathbb{S}^*((\mu_{<t}, \hat{\mu}_t))$ and all $\tilde{\mu}_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))$. But, $\mu_t(i) = i$ and thus $\mathbb{A}_{\mu_{<t}}^{\tau-1}(i, \mu_t(i)) = \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, i)$. The strict preference in the criterion above implies the contradiction. As $\mu_t \in \mathbb{A}_{\mu_{<t}}^*$ and $\mu_t(i) = i$, then $\mu_t \in \mathbb{A}_{\mu_{<t}}^*(i, i)$ as well. \square

Proof of Theorem 2. To prove Theorem 2 it is sufficient to verify that $\mathbb{S}^*(\mu_{\leq t})$ is not empty for all partial matchings $\mu_{\leq t}$, where $0 \leq t \leq T - 1$. Given an arbitrary $\mu_{\leq T-1}$, the same argument as presented in Lemma 1 confirms that $\mathbb{S}^*(\mu_{\leq T-1}) \neq \emptyset$. (The definitions of prudent blocking and regular blocking coincide for period T .)

Proceeding by induction, suppose $\mathbb{S}^*(\mu_{\leq t'}) \neq \emptyset$ for all $t' \geq t$. We will show that $\mathbb{S}^*(\mu_{\leq t-1}) \neq \emptyset$. Fix $\mu_{\leq t-1} = \mu_{<t}$. In a manner similar to the proof of Theorem 1, define a one-period preference for each agent i , denoted $P_i^{*\mu_{<t}(i)}$, as follows:

- If $\mathbb{A}_{\mu_{<t}}^*(i, j) \neq \emptyset$ and $\mathbb{A}_{\mu_{<t}}^*(i, k) \neq \emptyset$,

$$jP_i^{*\mu_{<t}(i)}k \iff \min_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(i, j)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(i), j, \tilde{\mu}'_{>t}(i)) \right) \succ_i \min_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(i, k)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(i), k, \tilde{\mu}'_{>t}(i)) \right).$$

In the preceding expression, $\mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t)) \neq \emptyset$ by the induction hypothesis.

- If $\mathbb{A}_{\mu_{<t}}^*(i, j) = \emptyset$, then set $iP_i^{*\mu_{<t}(i)}j$. (The ordering of assignments ranked below i by $P_i^{*\mu_{<t}(i)}$ can be arbitrary.)

From Lemma 2, $\mathbb{A}_{\mu_{<t}}^*(i, i) \neq \emptyset$. Hence $P_i^{*\mu_{<t}(i)}$ is well-defined.

Now considered a one-period matching market where each agent's preference is $P_i^{*\mu_{<t}(i)}$. From Gale and Shapley (1962), we know that this market has at least one stable matching. Let μ_i^* be a stable matching in this one-period economy. As above, the induction hypothesis implies that $\mathbb{S}^*((\mu_{<t}, \mu_i^*)) \neq \emptyset$. Fix $\mu_{>t}^* \in \mathbb{S}^*((\mu_{<t}, \mu_i^*))$ and define $\mu_{\geq t}^* = (\mu_i^*, \mu_{>t}^*)$. We will confirm that $\mu_{\geq t}^* \in \mathbb{S}^*(\mu_{<t})$.

As $\mu_{>t}^* \in \mathbb{S}^*((\mu_{<t}, \mu_i^*))$, $\mu^* := (\mu_{<t}, \mu_i^*, \mu_{>t}^*)$ cannot be blocked in any period $t' > t$. Therefore, to confirm that $\mu_{\geq t}^* \in \mathbb{S}^*(\mu_{<t})$ it is sufficient to verify that μ^* cannot be blocked in period t . To do so, we first confirm the following claim.

Claim 1. For each agent i , $\mu_i^* \in \mathbb{A}_{\mu_{<t}}^*(i, \mu_t^*(i))$.

Proof of Claim 1. As μ_t^* is a one-period matching that is stable in a one-period economy when each agent's preference is $P_i^{*\mu_{<t}(i)}$, necessarily it must be the case that $\mu_t^*(i)R_i^{*\mu_{<t}(i)}i$.⁹ Hence, by definition of $P_i^{*\mu_{<t}(i)}$, $\mathbb{A}_{\mu_{<t}}^*(i, \mu_t^*(i)) \neq \emptyset$ for each i .

Suppose $\mu_t^* \notin \mathbb{A}_{\mu_{<t}}^*(i, \mu_t^*(i))$ for some i . Thus, there exists some step τ in Algorithm 1 such that $\mu_t^* \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, \mu_t^*(i))$ but $\mu_t^* \notin \mathbb{A}_{\mu_{<t}}^\tau(i, \mu_t^*(i))$. Hence, there exists some j for whom the assignment of j to $\mu_t^*(j)$ is individually incredible at step τ . But this implies that *all* assignments where j is assigned to $\mu_t^*(j)$ cannot belong to $\mathbb{A}_{\mu_{<t}}^*$ and, therefore, $\mathbb{A}_{\mu_{<t}}^*(j, \mu_t^*(j)) = \emptyset$. By definition of $P_j^{*\mu_{<t}(j)}$, it follows that $jP_j^{*\mu_{<t}(j)}\mu_t^*(j)$. However, we have arrived at a contradiction. As μ_t^* is a stable one-period matching in the sense of Gale and Shapley (1962) it cannot assign j to a partner who is less preferred than remaining unmatched. \diamond

The remainder of the proof of Theorem 2 proceeds analogously to the proof of Theorem 1. Again, there are two cases. First, suppose $\mu^* = (\mu_{<t}, \mu_t^*, \mu_{>t}^*)$ can be prudently blocked in period t by agent i . Thus, for all $\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(i, i)$,

$$(\mu_{<t}(i), i, \tilde{\mu}'_{>t}(i)) \succ_i (\mu_{<t}(i), \mu_t^*(i), \mu_{>t}^*(i))$$

for all $\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))$. In particular, this implies,

$$\begin{aligned} & \min_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(i, i)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(i), i, \tilde{\mu}'_{>t}(i)) \right) \\ & \succ_i (\mu_{<t}(i), \mu_t^*(i), \mu_{>t}^*(i)) \\ & \succeq_i \min_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(i, \mu_t^*(i))} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(i), \mu_t^*(i), \tilde{\mu}'_{>t}(i)) \right). \end{aligned}$$

The third line follows from Claim 1. Thus, $iP_i^{*\mu_{<t-1}^{(i)}}\mu_t^*(i)$. Hence, i can block μ_t^* in the one-period economy used to derive μ_t^* . But this contradicts the stability of the one-period matching μ_t^* .

Instead, and second, suppose agents $m \in M$ and $w \in W$ can block $\mu^* = (\mu_{<t}, \mu_t^*, \mu_{>t}^*)$ in period t . Thus, $\mathbb{A}_{\mu_{<t}}^*(m, w) \neq \emptyset$ and for all $\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(m, w)$,

$$(\mu_{<t}(m), w, \tilde{\mu}'_{>t}(m)) \succ_m (\mu_{<t}(m), \mu_t^*(m), \mu_{>t}^*(m))$$

⁹As usual, $jR_i^{*\mu_{<t}(i)}k$ if $jP_i^{*\mu_{<t}(i)}k$ or $j = k$.

for all $\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t)$. In particular, this implies

$$\begin{aligned} & \min_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(m, w)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(m), w, \tilde{\mu}'_{>t}(m)) \right) \\ & \succ_m (\mu_{<t}(m), \mu_t^*(m), \mu_{>t}^*(m)) \\ & \succ^m \min_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}^*}(m, \mu_t^*(m))} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(m), \mu_t^*(m), \tilde{\mu}'_{>t}(m)) \right). \end{aligned}$$

Thus, $wP_m^{*\mu_{<t}(m)} \mu_t^*(m)$. Similarly, we conclude that $mP_w^{*\mu_{<t}(w)} \mu_t^*(w)$. Thus, m and w can block μ_t^* in the one-period economy, which is a contradiction. Thus, we conclude that μ^* cannot be blocked in any period $t' \geq t$. Hence, $\mu_{\geq t}^* \in \mathbb{S}^*(\mu_{<t})$. \square

The following lemma is used in the proof of Theorem 3 below.

Lemma 3. *Fix $\mu_{<t}$ and suppose $\mathbb{A}_{\mu_{<t}}^*(i, j) = \emptyset$. Then there exists some τ at which an assignment of i to j is deemed individually incredible.*

Proof. Consider a one-period matching μ_t such that $\mu_t(i) = j$ and $\mu_t(k) = k$ for all $k \neq i, j$. Since $\mathbb{A}_{\mu_{<t}}^*(i, j) = \emptyset$ and $\mu_t \in \mathbb{A}(i, j)$, μ_t must involve an individually incredible assignment. From Lemma 2, the matching where all agents are unmatched belongs to $\mathbb{A}_{\mu_{<t}}^*$. Hence, all the assignments in μ_t where k is assigned to k cannot be individually incredible. Therefore, the assignment of i to j must be individually incredible at some step τ . \square

Proof of Theorem 3. For any partial matching $\mu_{\leq t}$, let $\mathbb{S}(\mu_{\leq t})$ be the set of continuation plans $\mu_{>t}$ that cannot be blocked in any period $t' > t$. Likewise, let $\mathbb{S}^*(\mu_{\leq t})$ be the set of continuation plans $\mu_{>t}$ that cannot be prudently blocked in any period $t' > t$. To prove the theorem, it is sufficient to show that $\mathbb{S}^*(\mu_{\leq t}) \subset \mathbb{S}(\mu_{\leq t})$ for all $\mu_{\leq t}$, $0 \leq t \leq T-1$.

As the definitions of blocking and prudent blocking coincide in period T , $\mathbb{S}^*(\mu_{\leq T-1}) = \mathbb{S}(\mu_{\leq T-1})$ for all $\mu_{\leq T-1}$. Proceeding by induction, suppose that for all $t' \geq t$, $\mathbb{S}^*(\mu_{\leq t'}) \subset \mathbb{S}(\mu_{\leq t'})$ for all $\mu_{\leq t'}$. Now fix $\mu_{\leq t-1}$ and consider $\mu_{\geq t}^* \in \mathbb{S}^*(\mu_{\leq t-1})$. That is, $(\mu_{<t}, \mu_{\geq t}^*)$ cannot be prudently blocked in any period $t' \geq t$. Thus, $\mu_{\geq t}^* \in \mathbb{S}^*((\mu_{<t}, \mu_t^*))$. By the induction hypothesis, $\mathbb{S}^*((\mu_{<t}, \mu_t^*)) \subset \mathbb{S}((\mu_{<t}, \mu_t^*))$. Hence, $\mu_{\geq t}^* \in \mathbb{S}((\mu_{<t}, \mu_t^*))$. Thus, $(\mu_{<t}, \mu_{\geq t}^*)$ cannot be blocked in any period $t' > t$.

To show that $(\mu_{<t}, \mu_{\geq t}^*)$ cannot be blocked in period t , assume the contrary. Suppose that i and j can block $(\mu_{<t}, \mu_{\geq t}^*)$. Thus,

$$\min_{\tilde{\mu}_t \in \mathbb{A}(i, j)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(k), \tilde{\mu}(k), \tilde{\mu}'_{>t}(k)) \right) \succ_k (\mu_{<t}(k), \mu_t^*(k), \mu_{>t}^*(k))$$

for each $k \in \{i, j\}$. By the induction hypothesis, $\mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t)) \subset \mathbb{S}((\mu_{<t}, \tilde{\mu}_t))$ and $\mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t)) \neq \emptyset$. Thus, since a minimum is evaluated over a smaller set,

$$\min_{\tilde{\mu}_t \in \mathbb{A}(i, j)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(k), \tilde{\mu}(k), \tilde{\mu}'_{>t}(k)) \right) \succ_k (\mu_{<t}(k), \mu_t^*(k), \mu_{>t}^*(k)).$$

There are two cases.

1. Suppose $\mathbb{A}_{\mu_{<t}}^*(i, j) \neq \emptyset$. As $\mathbb{A}_{\mu_{<t}}^*(i, j) \subset \mathbb{A}(i, j)$,

$$\min_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(i, j)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(k), \tilde{\mu}(k), \tilde{\mu}'_{>t}(k)) \right) \succ_k (\mu_{<t}(k), \mu_t^*(k), \mu_{>t}^*(k)).$$

But this implies that i and j can prudently block $(\mu_{<t}, \mu_{\geq t}^*)$ in period t , which is a contradiction.

2. Suppose $\mathbb{A}_{\mu_{<t}}^*(i, j) = \emptyset$.

By Lemma 3, the assignment of i to j must be deemed incredible at some step τ of Algorithm 1. Thus, without loss of generality focusing on agent i ,

$$\begin{aligned} & \min_{\hat{\mu}_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, i)} \left(\min_{\hat{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \hat{\mu}_t))} (\mu_{<t}(i), \hat{\mu}_t(i), \hat{\mu}'_{>t}(i)) \right) \\ & \succ_i \min_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, j)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(i), \tilde{\mu}_t(i), \tilde{\mu}'_{>t}(i)) \right) \end{aligned} \quad (1)$$

As $\mathbb{A}_{\mu_{<t}}^*(i, i) \subset \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, i)$,

$$\begin{aligned} & \min_{\hat{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(i, i)} \left(\min_{\hat{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \hat{\mu}_t))} (\mu_{<t}(i), \hat{\mu}_t(i), \hat{\mu}'_{>t}(i)) \right) \\ & \succsim_i \max_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, i)} \left(\max_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(i), \tilde{\mu}_t(i), \tilde{\mu}'_{>t}(i)) \right) \end{aligned} \quad (2)$$

As $\mathbb{A}_{\mu_{<t}}^{\tau-1}(i, j) \subset \mathbb{A}(i, j)$ and $\mathbb{S}^*(\mu_{\leq t'}) \subset \mathbb{S}(\mu_{\leq t'})$ for all $t' \geq t$ and $\mu_{\leq t'}$,

$$\begin{aligned} & \max_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, j)} \left(\max_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(i), \tilde{\mu}_t(i), \tilde{\mu}'_{>t}(i)) \right) \\ & \succsim_i \min_{\tilde{\mu}_t \in \mathbb{A}_{\mu_{<t}}^{\tau-1}(i, j)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}^*((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(i), \tilde{\mu}_t(i), \tilde{\mu}'_{>t}(i)) \right) \\ & \succsim_i \min_{\tilde{\mu}_t \in \mathbb{A}(i, j)} \left(\min_{\tilde{\mu}'_{>t} \in \mathbb{S}((\mu_{<t}, \tilde{\mu}_t))} (\mu_{<t}(i), \tilde{\mu}_t(i), \tilde{\mu}'_{>t}(i)) \right) \succ_i (\mu_{<t}(i), \mu_t^*(i), \mu_{>t}^*(i)) \end{aligned} \quad (3)$$

Combining (2) and (3) via (1), we conclude that

$$\min_{\hat{\mu}_t \in \mathbb{A}_{\mu_{<t}}^*(i,i)} \left(\min_{\hat{\mu}'_{\geq t} \in \mathbb{S}^*((\mu_{<t}, \hat{\mu}_t))} (\mu_{<t}(i), \hat{\mu}_t(i), \hat{\mu}'_{\geq t}(i)) \right) \succ_i (\mu_{<t}(i), \mu_t^*(i), \mu_{\geq t}^*(i)).$$

But this implies that i can prudently block $(\mu_{<t}(i), \mu_{\geq t}^*(i))$ in period t , which is a contradiction.

As each of the preceding cases led us to a contradiction, we conclude that $(\mu_{<t}, \mu_{\geq t}^*)$ cannot be blocked in period t . Therefore, $\mu_{\geq t}^* \in \mathbb{S}(\mu_{\leq t-1})$, which proves the theorem. \square

5 Concluding Remarks

We have proposed a new stability concept for multi-period matching economies. Robust prescient stability combines foresight concerning what *can* happen while maintaining ambiguity concerning what *will* happen. It balances these two competing themes, which recur in models of multi-period matching. Prudent prescient stability offers a plausible refinement of RPS. It reduces the stable set's size by precluding some implausible contemporaneous outcomes given a considered blocking action.

Noting the brevity of our analysis, many extensions are possible. Applications of RPS or PPS to many-to-one and to many-to-many matching economies seem promising as does a further extension incorporating monetary transfers. There is also a literature on one-sided, multi-period matching economies generalizing Shapley and Scarf's (1974) model of a housing market. Kurino (2014) is a recent example. Adapting RPS or PPS to this class of problems may also yield new insights. Further refinements of RPS, similar in spirit to PPS but employing different definitions of the sets \mathbb{A}^* or \mathbb{S}^* , are an additional possible direction.

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A Preferences in Example 2

While dynamically stable matchings do not depend on agents' preferences over assignments that are not individually rational, i.e. worse than being unmatched in all periods, robust pre-scient stability depends on the full preference specification. Here we provide the completion of the preferences that support the conclusion of Example 2:

$$\succ_{m_1}: w_2w_2, w_2w_1, w_2w_3w_1, w_1, w_3w_3, m_1m_1, \\ w_3m_1, w_3w_2, w_3w_1, w_2m_1, w_1m_1, w_1w_3, w_1w_2, m_1w_3, m_1w_2, m_1w_1$$

$$\succ_{m_2}: w_3w_3, w_2w_2, w_3w_1, w_1w_1, m_2m_2, \\ w_3m_2, w_3w_2, w_2m_2, w_2w_3, w_2w_1, w_1m_2, w_1w_3, w_1w_2, m_2w_3, m_2w_2, m_2w_1$$

$$\succ_{m_3}: w_1w_1, w_1w_3, w_3w_3, w_1w_2, w_2w_2, m_3, m_3, \\ w_3m_3, w_3w_2, w_3w_1, w_2m_3, w_2w_3, w_2w_1, w_1m_3, m_3w_3, m_3w_2, m_3w_1$$

$$\succ_{w_1}: m_2m_2, m_1m_1, m_2m_1, m_1m_2, w_1m_2, m_3m_2, w_1w_1, \\ m_3m_3, m_1w_1, m_1m_3, m_3w_1, m_3m_1, m_2w_1, m_2m_3, w_1m_1, w_1m_3$$

$$\succ_{w_2}: m_3m_3, m_2m_2, m_3m_2, m_2m_3, w_2m_3, m_1m_3, w_2w_2, \\ m_1m_1, m_1w_2, m_1m_2, m_3w_2, m_3m_1, m_2w_2, m_2m_1, w_2m_1, w_2m_2$$

$$\succ_{w_3}: m_1m_1, m_1m_3, m_3m_1, w_3m_1, m_2m_1, m_3, m_3, w_3w_3, \\ m_2m_2, m_1w_3, m_1m_2, m_3w_3, m_3m_2, m_2w_3, m_2m_3, w_3m_3, w_3m_2$$