# Mortgage Risk and the Yield Curve*

Aytek Malkhozov
McGill†

Philippe Mueller
LSE‡

Andrea Vedolin
LSE§

Gyuri Venter
CBS¶

**Abstract**

We find strong empirical evidence that MBS dollar duration predicts bond excess returns with longer-maturity bonds being affected the most. Negative MBS dollar convexity increases bond yield volatility, and its effect is hump shaped, with maturities around two years being affected the most. Moreover, the predictive power is not subsumed by either yield factors or macroeconomic variables. We rationalize these empirical findings within a parsimonious dynamic equilibrium term structure model in which the net supply of long-term bonds is endogenously driven by the interest rate risk of mortgages. A calibration of our model confirms the quantitative relevance of the mechanism.

First Version: August 2012
This Version: November 2014

*We would like to thank Tobias Adrian, Caio Almeida, Michael Bauer, Ruslan Bikbov, Mike Chernov, John Cochrane, Pierre Collin-Dufresne, Jefferson Duarte, Greg Duffee, Mathieu Fournier, András Fülöp, Christian Julliard, Anh Le, Neil Pearson, Douglas McManus, Marcel Rindisbacher, Dimitri Vayanos, Paul Whelan, and seminar and conference participants at the Bank of England, Banque de France, CBS, Deutsche Bank, Duke University, Erasmus University, Exeter University, Imperial Business School, LSE, Manchester Business School, the Oxford-MAN Institute, Toulouse School of Economics, Luxembourg School of Finance, University of Nottingham, SAC Capital Advisors, University of Bern, the University of Bonn, the 2012 Arne Ryde Workshop for Financial Economics, the 2012 IFSID Conference on Structured Products and Derivatives, the 2012 Young Scholars Nordic Finance Workshop, the 2012 Junior Faculty Research Roundtable at UNC, the 2012 Conference on Advances in the Analysis of Hedge Fund Strategies, the 2013 Mathematical Finance Days, the 2013 Bank of Canada Fixed Income conference, the 2013 Financial Econometrics conference in Toulouse, the 2013 SFS Cavalcade, the 2013 Western Finance Meeting, the UNC/Atlanta Fed Conference on Housing, and the NBER SI AP for thoughtful comments. Malkhozov gratefully acknowledges financial support from the Institut de Finance Mathématique de Montréal. Mueller and Vedolin gratefully acknowledge financial support from the LSE Research Committee Seed Fund. Venter gratefully acknowledges financial support from the Center for Financial Frictions (FRIC), grant no. DNRF102. All authors thank the Paul Woolley Centre at the LSE and the Dauphine-Amundi Chair in Asset Management for financial support.

†Desautels Faculty of Management, Email: aytek.malkhozov@mcgill.ca

‡Department of Finance, Email: p.mueller@lse.ac.uk

§Department of Finance, Email: a.vedolin@lse.ac.uk

¶Department of Finance, Email: gv.fi@cbs.dk

Mortgage-backed securities (MBS) and, more generally, mortgage loans expose investors to interest rate risk. Unlike with regular bonds, this exposure can change considerably with interest rate conditions.[1] This is the case because mortgages typically feature an embedded prepayment option that makes their convexity negative: Lower interest rates increase the probability that outstanding mortgages will be prepaid in the future and thereby substantially decrease their duration. This leaves financial institutions who invest in MBS short of duration exposure, until either interest rates revert back to higher levels, or the prepayment option becomes sufficiently in the money for a large number of households to refinance and take on new mortgages. Because households do not play an active role in bond markets and do not hedge their time-varying interest rate risk exposure, it is the position of financial institutions that determines the pricing of interest rate risk (see Gabaix, Krishnamurthy, and Vigneron (2007)). In other words, a fall in mortgage duration is equivalent to a negative transitory shock to the supply of long-term bonds and therefore can have an effect on their prices.[2] Moreover, mortgage investors who want to keep the duration of their portfolios constant after a drop in MBS duration (for hedging or portfolio rebalancing reasons) induce additional buying pressure on Treasuries and push interest rates further down. Thus, negative convexity due to the prepayment option creates an amplification channel for interest rate shocks.

The negative convexity channel described above has attracted the attention of practitioners, policy makers, and empirical researchers alike.[3] Our paper makes four contributions to the existing literature. First, we document the effect of mortgage risk on bond risk premia and bond yield volatilities for the entire term structure of interest rates, and present novel findings in this regard. Second, we explore the relationship between yield, macroeconomic, and MBS factors. Third, we provide evidence on the role played by interest rate risk hedging by government-sponsored enterprises (GSEs) and its

[1]In contrast with Treasuries, MBS duration can drop by more than 50%, e.g., from 5 to 2.5 years in a short period (see also Figure 1). Taking into account the value of outstanding mortgage debt, we calculate that one monthly standard deviation shock to MBS duration is a dollar duration equivalent of a USD 368bn shock to the supply of 10-year Treasuries.

[2]See Lou, Yan, and Zhang (2013) on how financial intermediaries' supply shocks can affect Treasury bond prices.

[3]See Perli and Sack (2003), Chang, McManus, and Ramagopal (2005), Duarte (2008), Li and Wei (2013) and Hanson (2014) among others.

effect on bond risk premia and volatility. Finally, we build a parsimonious equilibrium dynamic term structure model that allows us to rationalize our findings. To the best of our knowledge, it is the first model to formalize the intuition behind the negative convexity channel.

We start our paper with an in-depth empirical analysis of the predictive power of MBS dollar duration for bond excess returns. We find a highly significant positive link between duration and bond risk premia that becomes stronger as bond maturity increases. The relationship is also economically significant: A one standard deviation change in MBS duration implies a change of 316 basis points in the expected one-year excess return on a 10-year bond.[4] Both the statistical significance and the magnitude of the coefficients are robust to other standard predictors of bond risk premia.

For second moments of bond yields, we find that more negative MBS convexity significantly increases bond yield and swaption implied volatilities. Interestingly, the effect is most pronounced for intermediate maturities between two and three years. For example, any one standard deviation change in MBS dollar convexity increases bond yield volatility for these maturities by approximately 30 basis points. Moreover, this strong link remains if we add other determinants of interest rate volatility.

Since MBS duration depends on interest rate conditions, it is natural to ask whether duration contains any information above the one encoded in yields. We find that MBS duration is not fully spanned by the cross section of Treasury yields. Regressing duration on the first three principal components of yields results in $R^2$ of a mere 22%. This implies that 78% of the variation in MBS duration is not captured by information in the cross-section of yields. Using the residual from this regression, we then study whether MBS duration is still a significant predictor of bond excess returns. We find no quantitative difference in terms of both economic and statistical significance, whether we use MBS duration which has been orthogonalized with respect to the yield factors or not. Similarly, one might wonder whether MBS duration is simply a proxy for macroeconomic factors as the probability of mortgage refinancing could also depends on the state of the economy. Using a proxy for economic activity and inflation, we find that MBS duration

[4] We find similar results when we use swap rather than Treasury data.

remains a strong predictor of bond risk premia, even if we orthogonalize the series with respect to these two variables.

Among the largest holders of MBS are the two GSEs, Fannie Mae and Freddie Mac. The interest rate risk on their combined retained portfolio is known to be large and both GSEs actively hedge part of their exposure. Using accounting data from Fannie Mae and Freddie Mac, we assess whether the importance of the MBS duration factor changes over time as the GSEs' portfolio increases. To this end, we gather data on the size of the retained portfolio and the amount of derivatives used to hedge their interest rate risk exposure. We find that both variables are excellent predictors of bond excess returns, moreover, interacting these variables with MBS duration, we find a strong economic effect on bond returns: For example, any one standard deviation change in the interaction variable between the size of the retained portfolio and MBS dollar duration leads to a 349 bp change in the 10-year bond excess return.

We rationalize our findings within a stylized dynamic equilibrium model of the term structure. In our model the term structure of interest rates is driven by the interaction between (i) exogenous shocks to the short rate and (ii) changes in the net supply of long-term bonds. The latter is endogenously determined by the aggregate interest rate risk exposure of mortgages that itself depends on the path taken by long-term interest rates. The equilibrium takes the form of a standard Vasicek (1977) short rate model augmented by an additional affine factor that captures the duration of outstanding MBS. This factor drives the market price of interest rate risk and affects the risk premia of long-term bonds but not the dynamics of the risk-free rate itself. Its contribution tends to zero if the risk-bearing capacity of financial institutions who invest in bonds is high.

Our key empirical findings come out naturally from the model. First, the dollar duration of outstanding MBS predicts bond excess returns. Moreover, this effect is stronger for longer maturity bonds. This happens because in the model the market price of interest rate risk is proportional to the quantity of duration risk that investors have to hold. Longer maturity bonds with a higher exposure to interest rate risk are more strongly affected through this channel.

Second, the average volatility of all yields is increasing in the dollar convexity of outstanding MBS because of the positive feedback between interest rates and mortgage duration that is due to negative convexity. This effect has a hump-shaped pattern across maturities, with intermediate maturities being most strongly affected. In the model, supply shocks create transitory variations in the market price of interest rate risk. Short-maturity bonds are close substitutes to the short rate and thus are not very sensitive to variations in the market price of risk, while for long maturities, the market price of risk is expected to mean revert. As a result, the effect of negative convexity on yield volatility is strongest for intermediate maturities. The same intuition applies to swaption implied volatilities.

While our stylized model is not designed to address the possibility that MBS duration is unspanned, two features of the model nevertheless speak to the empirical facts. First, aggregate MBS duration depends on the refinancing incentive, i.e., the difference between the average fixed rate paid on outstanding mortgages and the current mortgage rate. This means that duration is not a function of the current level of interest rates alone, but also depends on past levels. Thus, the short rate factor can only explain a fraction of the variation in duration even though they share the same shocks. Second, mortgage duration decreases after a negative shock to interest rates. This results in a lower term premium and a flatter yield curve. At the same time, a drop in the short rate has a smaller effect on the long end of the curve, leading to a steeper yield curve. Because these two effects of the same shock on the slope partially offset each other, the correlation between duration and the slope of the yield curve is low. In our two factor model, the short rate factor explains most of the variation in the cross section of yields while duration accounts for all the predictability in excess returns.

Lastly, we calibrate our model to the data. We first estimate the parameters for the short-rate process and the MBS duration series. Using these parameters, we find that the model-implied loadings of bond excess returns on MBS dollar duration and the effect of MBS dollar convexity on bond yield volatilities are quantitatively similar to their empirical counterparts.

Our work is related to a series of empirical papers on the negative convexity channel. Perli and Sack (2003), Chang, McManus, and Ramagopal (2005), and Duarte (2008) test the presence of a linkage between various proxies for MBS hedging activity and interest rate volatility, without considering the simultaneous effect that hedging activity can have on bond pricing. Unlike those papers we look at the effect that MBS convexity has on the entire term structure of yield volatilities and find that it is strongest for intermediate (and not long, as previously assumed) maturities. Our model provides an explanation for this finding.

In contemporaneous work, Hanson (2014) reports results similar to ours regarding the predictability of bond excess returns by MBS duration. The author's focus is mainly on documenting how MBS duration affects term premia and forward rates. In contrast to the theoretical framework that guides the author's analysis, our dynamic term structure model allows us to jointly explain the effect of mortgage risk on bond risk premia and bond yield volatilities across different maturities. We provide novel predictions and empirical evidence in this regard. Our analysis also sheds new light on the relationship between MBS duration and the information encoded in the yield curve. In this respect our paper is related to Li and Wei (2013) who study a no arbitrage model of the term structure that includes an unspanned MBS supply factor.

Our work is also related to several strands in the asset pricing literature. We make use of the framework developed by Vayanos and Vila (2009). In their model, the term structure of interest rates is determined by the interaction of preferred habitat investors and risk-averse arbitrageurs, who demand higher risk premia as their exposure to long-term bonds increases. Thus, the net supply of bonds matters. Greenwood and Vayanos (2014) use this theoretical framework to study the implications of a change in the maturity structure of government debt supply, similar to the one undertaken in 2011 by the Federal Reserve during "Operation Twist". Our paper is different in at least three respects. First, in our model the variation in the net supply of bonds is driven endogenously by changing MBS duration and not exogenously by the government. Second, the supply factor in Greenwood and Vayanos (2014) explains low frequency variation in risk premia, because movements in maturity-weighted government debt to GDP occur at a lower frequency than movements in the short rate. Our duration factor, on the other hand, explains variations in risk premia at a higher frequency than movements in the level of interest rates. Finally, Greenwood and Vayanos (2014) posit that the government adjusts the maturity structure of its debt in a way that stabilizes bond markets. For instance, when interest rates are high, the government will finance itself with shorter maturity debt and thereby reduce the quantity of interest rate risk held by agents. Our mechanism goes exactly in the opposite direction: Because of the negative convexity in MBS, the supply effect amplifies interest rates shocks.

Corporate debt constitutes another important class of fixed income instruments, which is shown to be negatively correlated with the supply of Government debt (see, e.g., Greenwood, Hanson, and Stein (2010) and Bansal, Coleman, and Lundblad (2011)). For example, Greenwood, Hanson, and Stein (2010) show that firms choose their debt maturity in a way that tends to offset the variations in the supply and maturity of Government debt. However, the authors find no relationship between corporate debt and MBS supply, which provides us with an additional motivation to focus on the latter.

This paper contributes to the literature on equilibrium term structure models, e.g., Le, Singleton, and Dai (2010), Xiong and Yan (2010), Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2013), Bansal and Shaliastovich (2013), Hong, Sraer, and Yu (2013), and Le and Singleton (2013), among others. Contrary to these papers, in which bond risk premia are determined by macroeconomic fundamentals or differences in beliefs about these fundamentals, we focus on the aggregate demand and supply of bonds as the main driver of risk premia. Le and Singleton (2013) also show that time-varying market prices of risk, in addition to time-varying quantities of risk, are required to explain bond risk premia. In our model, duration drives the variation in the market price of risk and, hence, bond risk premia. Recent empirical work by Duffee (2011), Le and Singleton (2013) and Joslin, Priebsch, and Singleton (2014) documents the existence of unspanned factors that are unrelated to the cross section of yields but have a significant impact on risk premia. Empirically, we find that MBS duration is not spanned by the first three principal components that drive virtually all variation in the cross section of yields. In our model, duration is spanned by the cross section of yields by construction, but not by the short rate factor that accounts for most of the variation in the shape of the yield curve. Finally, Joslin, Le, and Singleton (2013) and Fenou and Fontaine (2013) develop term structure models that include additional lags in the dynamics of yield factors. Similarly, in our paper, MBS duration depends on both current and past yields.

There is an important literature that studies the optimal prepayment in the MBS market (see, e.g., Schwartz and Torous (1989), Stanton (1995), Stanton and Wallace (1998), Longstaff (2005) and, more recently, Agarwal, Driscoll, and Laibson (2013) for examples of prepayment models and optimal prepayment decisions). Furthermore, there is evidence that households' prepayment is too sluggish and that this non-optimal prepayment can be best explained using micro-level data (see, e.g., Campbell's 2006 AFA Presidential Address for an overview). In this paper we are interested in how the aggregate properties of prepayment affect the risk of mortgage-related portfolios of financial institutions. Boyarchenko, Fuster, and Lucca (2014) study the spread between MBS rates and Treasuries and find that prepayment risk explains well the cross-section but non-prepayment risk related factors explain the time-series. Closest to us are Gabaix, Krishnamurthy, and Vigneron (2007) who study the effect of limits to arbitrage in the MBS market. The authors show that, while mortgage prepayment risk resembles a wash on an aggregate level, it nevertheless carries a positive risk premium because it is the risk exposure of financial intermediaries that matters. Our paper is based on a similar premise. Different from these authors, however, we do not study prepayment risk, but changes in interest rate risk of MBS that are driven by the prepayment probability, and their effect on the term structure of interest rates.

Finally, our paper is related to the literature that studies the effect of the Fed's recent purchase of long-term assets on interest rates (see, e.g., Gagnon, Raskin, Remache, and Sack (2010), Krishnamurthy and Vissing-Jorgensen (2011), D'Amico, English, Lopez-Salido, and Nelson (2012), D'Amico and King (2013)). While our model mainly focusses on the relationship between long-term yields and volatilities and MBS duration, other papers present evidence for alternative channels. For example, Krishnamurthy and Vissing-Jorgensen (2011) report that QE1, which involved major purchases of agency

MBS, led not only to large reductions in mortgage rates, but also helped drive down Treasury yields and caused a drop in the default risk premium of corporate bonds. They interpret their findings as evidence for a long-term safety channel. D'Amico and King (2013) emphasize a scarcity channel, i.e. a localized effect of supply shocks on yields of nearby maturities.

The remainder of the paper is organized as follows. Section 1 describes the data used and Section 2 empirically tests our hypotheses. Section 3 sets up a model of the term structure of interest rates and MBS duration. Section 4 discusses the equilibrium of the model and calibrates it to the data to confirm our empirical predictions. Finally, Section 5 concludes. Proofs are deferred to the Online Appendix.

## 1 Data

We use data from several sources. We use Treasury data from the Federal Reserve Board and agency bond index data from Datastream. Furthermore, from Bloomberg, we collect data on interest rate swaps. Data are weekly and span the time period from January 1997 through December 2011 for a total of 783 observations.

We use estimates of MBS duration, convexity, index, and average coupon from Barclays. The Barclays US MBS index covers mortgage-backed pass-through securities guaranteed by Ginnie Mae, Fannie Mae, and Freddie Mac. The index is comprised of pass-throughs backed by conventional fixed rate mortgages and is formed by grouping the universe of over one million agency MBS pools into generic pools based on agency, program (i.e., 30-year, 15-year, etc.), coupon (e.g., 6.0%, 6.5%, etc.), and vintage year (e.g., 2011, 2012, etc.). A generic pool is included in the index if it has a weighted-average contractual maturity greater than one-year and more than USD 250m outstanding. We construct measures of dollar duration and dollar convexity by multiplying the duration and convexity time-series with the index level.

The middle panel of Figure 1 depicts MBS duration. The average duration in our sample is around 4.5 years which is mainly due to the prepayment option in fixed rate mortgages which reduces the duration of MBS. There are two noteworthy periods when

MBS duration is particularly low, namely during 2003 and in 2008. The lower panel plots MBS convexity which is negative throughout the whole sample and the average convexity is around -1.5. In addition to MBS duration, we also use Treasury duration and corporate bond duration for two different rating categories: AAA and BBB. Again, we use Barclays bond indices to get duration for a Treasury and corporate bond portfolio, respectively. The Barclays US Treasury index includes notes and bonds with at least one year to maturity. The included securities can be puttable or callable, but TIPS, STRIPS, and T-bills are excluded. The corporate bond indices include fixed rate bullet, puttable or callable bonds within the respective rating classes. We use the subindices that include longer maturities above ten years. The average Treasury duration is about 5.5 with a standard deviation of only 0.3 (whereas MBS duration has a similar mean but a standard deviation that is more than twice as large). The average corporate bond duration is around 10 years with a standard deviation of between 1 and 1.6, i.e., a similar order of magnitude expressed in percentage terms as MBS duration.

[Insert Figure 1 here.]

We use the Gürkaynak, Sack, and Wright (2007, GSW henceforth), zero coupon yield data available from the Federal Reserve Board. Unlike the Fama and Bliss discount bond database from CRSP, the GSW data is available at the weekly frequency. We use the raw data to calculate annual Treasury bond excess returns for 2- to 10-year bonds. We also download interest rate swap data from Bloomberg from which we bootstrap a zero-coupon yield curve.

We denote the return between year $t$ and $t+1$ on a $\tau$-year bond with price $\Lambda_t^\tau$ by $r_{t+1}^\tau = \log \Lambda_{t+1}^{\tau-1} - \log \Lambda_t^\tau$. The annual excess bond return is then defined as $rx_{t+1}^\tau \equiv r_{t+1}^\tau - y_t^1$, where $y_t^1 = -\log \Lambda_t^1$ is the one-year yield. As we have weekly data, the annual excess returns are overlapping by 51 weeks. From the same data, we also construct a tent-shaped factor from forward rates, the Cochrane and Piazzesi (2005) factor (CP factor, labeled $\text{cp}_t$).

We calculate the slope of the term structure as the difference between the 10-year and the one-year zero coupon yield (labeled $\text{slope}_t$).

Using the GSW yields ranging from six months to ten years, we estimate a time-varying term structure of yield volatility. We sample the data at the weekly frequency and take log yield changes. We then construct rolling window measures of realized volatility which represent the conditional bond yield volatility. The resulting term structure of unconditional volatility exhibits a hump shape consistent with the stylized facts reported in Dai and Singleton (2010), with the volatility peak being at the two-year maturity.

Mueller, Vedolin, and Choi (2014) calculate measures of model-free implied bond market volatilities for a one-month horizon using Treasury futures and options data from the Chicago Mercantile Exchange (CME). We use their data for the 30-year Treasury bond and henceforth label this measure $\text{tiv}_t$.

From Bloomberg, we also get implied volatility for at-the-money swaptions for different maturities ranging from one to ten years and we fix the tenor to ten years. We label these swaption implied volatilities by $\text{iv}^{\tau y 10y}$, where $\tau = 1, \ldots, 10$.

Motivated by Hu, Pan, and Wang (2013) who report a link between swaption implied volatility and measures of liquidity in bond markets, we use their proxy of noise illiquidity, which measures an average yield pricing error from the Nelson, Siegel and Svensson model (see Svensson (2004)). To this end, we construct a daily measure of noise illiquidity from bond data available from Datastream. As macro variables we chose economic growth proxied by the three-month moving average of the Chicago Fed National Activity Index. Negative (positive) values indicate a below (above) average growth. We also use a measure of inflation proxied by the consensus estimate of professional forecasts available from Blue Chip Economic Forecasts. We denote these two factors by $\text{gro}_t$ and $\text{infl}_t$ respectively.

## 2 Empirical analysis

In this section we study the predictive power of MBS dollar duration and convexity for bond excess returns, bond yield volatility, and swaption implied volatility, respectively. We start with simple univariate regressions to document the role of our main explanatory variables. Then, for robustness and to address a potential omitted variable bias, we also control for other well-known predictors of bond risk premia and interest rate volatility. We find that not only MBS duration and convexity remain statistically significant, but also the economic size of the coefficients stays stable across different specifications.

The start date for all regressions is dictated by the availability of the MBS convexity time series which starts in January 1997. All regressions are standardized, meaning that we first de-mean and then divide each variable by its standard deviation to make slope coefficients comparable across different regressors. With each estimated coefficient, we report t-statistics adjusted for Newey and West (1987) standard errors. The lag length is determined using the Stock and Watson (2007) rule.

### *2.1 Bond risk premia*

We first assess the predictive power of MBS dollar duration for bond risk premia with different maturities. We run linear regressions of annual excess returns on the dollar duration factor. The regression is as follows:

$$rx_{t+1}^{\tau} = \beta_1^{\tau}\text{duration}_t + \beta_2^{\tau}\text{level}_t + \epsilon_{t+1}^{\tau},$$

where $\text{duration}_t$ is MBS dollar duration and $\text{level}_t$ is the first principal component of the yield curve at time $t$. The univariate results are depicted in the upper two panels of Figure 2, which plot the estimated slope coefficients of duration, i.e., $\hat{\beta}_1$ (upper left panel), and the associated adjusted $R^2$ (upper right panel). Both univariate and multivariate results are presented in Table 1.

[Insert Figure 2 and Table 1 here.]

The univariate regression results indicate that MBS duration is a significant predictor of bond excess returns at all maturities except for the shortest maturity (2y). The coefficient has a positive sign and is increasing with maturity.[5] The estimated coefficients

[5] We can also test whether the estimated slope coefficients are monotonically increasing using the monotonicity test developed by Patton and Timmermann (2010), i.e., we can test whether

$$\mathcal{H}_0 : \hat{\beta}_1^{10y} \leq \hat{\beta}_1^{9y} \leq \cdots \leq \hat{\beta}_1^{2y}$$

are also economically significant, especially for longer maturities. For example, for any one standard deviation increase in duration, there is a $0.442 \times 7.15\% = 316$ basis point (slope coefficient times the volatility of the 10-year bond excess return) increase in expected 10-year bond excess returns. Adjusted $R^2$s range from 2% for the shortest maturity to almost 20% for the longest maturity.

Because duration could in part be determined by the current level of interest rates, we include it as a control in our multivariate test. The results indicate that the slope coefficient on duration remains positive and increasing with maturity, while the slope coefficient on the level of interest rates has a negative sign and is decreasing with maturity. The economic size of MBS dollar duration remains quantitatively unchanged and we conclude that the predictive power of MBS dollar duration is not subsumed by the level of interest rates.

### *2.2 Bond yield volatility*

Lower interest rates increase the probability that outstanding mortgages will be prepaid in the future and thereby considerably decrease their duration due to the negative convexity in MBS. Investors in the mortgage market who want to keep the duration of their portfolios constant after a drop in MBS duration (for hedging or portfolio rebalancing reasons) induce buying pressure on Treasuries and push interest rates further down. Thus, negative convexity due to the prepayment option creates an amplification channel for interest rate shocks and can increase volatility. In the following, we test how bond yield volatility is related to MBS dollar convexity. Due to the amplification channel described before, we expect a more negative convexity of MBS to result in larger bond yield volatility. As hedging can potentially take place both in the bond and in

versus

$$\mathcal{H}_1 : \hat{\beta}_1^{10y} > \hat{\beta}_1^{9y} > \cdots > \hat{\beta}_1^{2y},$$

where $\hat{\beta}_1^j, j = 2y, \cdots, 10y$ are estimated slope coefficients from an univariate regression from bond excess returns with maturity $j$ onto MBS dollar duration. First, note that the spread between the coefficients on the 2y and 10y bond excess returns is 0.360. The associated t-statistic is 3.90 and is therefore statistically significant. Turning to the tests for monotonicity, using 10,000 bootstrap iterations, we find that the p-value is almost zero and we hence strongly reject the null hypothesis of no monotonic relationship between estimated coefficients.

the fixed-income derivative market, we also test the impact of MBS dollar convexity on measures of implied volatility from swaptions. For example, Wooldridge (2001) notes that non-government securities were routinely hedged in the Treasury market until the financial crisis of 1998 when investors started hedging their interest rate exposure in the swaptions market. This point is also made in Perli and Sack (2003), Duarte (2008), or Feldhütter and Lando (2008).[6]

We run the following two univariate regressions from conditional bond yield volatility and swaption implied volatility onto convexity:

$$\mathrm{vol}_t^\tau / \mathrm{iv}_t^{\tau y 10 y} = \beta_1^\tau \mathrm{convexity}_t + \epsilon_t^\tau,$$

where $\mathrm{vol}_t^\tau$ is the conditional bond yield volatility at time $t$ of a bond with maturity $\tau = 1, \ldots, 10$ years and $\mathrm{iv}_t^{\tau y 10 y}$ is the $\tau$-year maturity implied volatility from swaptions on the 10-year swap rate.

The univariate results are presented in the lower two panels of Figure 2 and Table 2. In line with our intuition, we find a significant effect from convexity onto bond yield volatility and the effect is most pronounced for intermediate maturities.[7] The estimated slope coefficients produce the hump shaped feature similar to the one observed in the unconditional averages of yield volatility. Adjusted $R^2$s range from 8% for the shortest maturities, increase to 9% for the two and three year maturities and decrease again to 3% for longer maturities. Estimated coefficients are not only statistically but also economically significant: For the two-year maturity, any one standard deviation change in MBS convexity is associated with a 30 basis point increase in bond yield volatility.

[Insert Table 2 here.]

[6] More recently, Begenau, Piazzesi, and Schneider (2013) and Landier, Sraer, and Thesmar (2013) also report that financial institutions hold large positions in interest rate derivatives to hedge their interest rate risk exposure.

[7] While there are no procedures which specifically test for a hump-shape, we can can test whether the estimated coefficient on the 2y bond yield volatility is statistically different from the 3y volatility. Indeed, the difference which is 0.0177 has a t-statistic of 2.45 and is hence different from zero. We can then again test for monotonicity between the 3y and 10y coefficients. Using the procedure from Patton and Timmermann (2010), we strongly reject the null of no relationship as the p-value is 0.002.

The same picture emerges for implied volatilities from swaptions reported in Panel B: Higher convexity induces higher volatility on swaptions. All estimation coefficients are statistically significant with t-statistics ranging from 3.41 for the longest maturity to 4.38 for the intermediate maturities.

An obvious concern with our regression results is that negative convexity could itself depend on volatility. Note, that it is a priori unclear in which direction volatility affects convexity as this depends on whether a particular MBS is in-, out-, or at-the-money.[8] For an at-the-money MBS, an increase in volatility will lead to an increase in negative convexity. Discount (i.e., small negative to positive convexity) and premium (negative convexity) mortgages will in general have a much lower sensitivity to changes in volatility, and the effect could go in the opposite direction.[9]

To address causality, we first run Granger tests between MBS dollar convexity and volatility and present the results in Figure 3. In the left panel, we plot p-values from F-tests that assess the null hypothesis whether negative convexity does not Granger cause volatility. On the right panel, we plot the p-values of the reversed Granger regression, i.e., we test the null hypothesis whether volatility does not Granger cause negative convexity. We note that for standard confidence levels, we can reject the null of no Granger causality from convexity to volatilities up to a maturity of seven years. On the other hand, volatility does not seem to Granger cause convexity, as we cannot reject the null hypothesis up to a maturity of five years.

[Insert Figure 3 here.]

As an additional robustness check we use the lagged values of convexity as instruments in an Instrumental Variable (IV) estimation. Running two-stage-least-square regressions, we find that the standard errors are in fact smaller if we lag the instrument further. In line with the empirical and theoretical findings in Stock, Wright, and Yogo (2002), we conclude that IV estimation performs worse than OLS and therefore report OLS results only (see also Krishnamurthy and Vissing-Jorgensen (2012)).

[8] This is analogous to the Zomma (sensitivity of an option's Gamma with respect to changes in the implied volatility) for equity options.

[9] We thank Bruce Phelps at Barclays Capital for insightful discussions on this.

### *2.3 What does MBS duration capture beyond information in yields?*

Households refinance mortgages when interest rates drop and it is therefore natural to assume that MBS duration is a mere reflection of information already contained in the yield curve. In the following, we hence study the relation of MBS dollar duration to factors borne out by yields themselves and study the predictive power of MBS duration for bond risk premia beyond these factors.

Most models in fixed income decompose movements in yields into principal components and argue that the first three factors explain most of the variation in yields. Table 3 (Panel A) reports the unconditional correlations between MBS dollar duration and the first three principal components (PCs) from the cross section of yields, commonly known to represent its level, slope, and curvature. We note that the correlation between MBS duration and the first PC is relatively high 40% (with t-statistic of 5.66) but that the correlation becomes lower for the second and third PC (the unconditional correlations being 16% (t-statistic of 1.88) and $-20\%$ (t-statistic of -2.78), respectively). To test more formally whether duration is spanned by these yield factors we run the following regression similar to Joslin, Priebsch, and Singleton (2014):

$$\text{duration}_t = \alpha + \beta_1 \text{level}_t + \beta_2 \text{slope}_t + \beta_3 \text{curvature}_t + \epsilon_t,$$

where $\text{level}_t, \text{slope}_t$ and $\text{curvature}_t$ are the first three PCs. For our sample period, the adjusted $R^2$ of this regression is equal to 22%. This means that 78% of the variation in MBS duration arises from risks which are distinct from these PCs. Moreover, the AR(1) coefficient of residuals is equal to 0.78 and the associated Durbin and Watson statistic to 0.1, which clearly rejects the null of zero autocorrelation. In the following, we are going to use the residual from this regression, denoted by $\overline{\text{duration}^y}_t = \text{duration}_t - \left(\hat{\alpha} + \hat{\beta}_1 \text{level}_t + \hat{\beta}_2 \text{slope}_t + \hat{\beta}_3 \text{curvature}_t\right)$.

This now begs the question of whether duration contains any information beyond these principal components to predict bond returns. We tackle this question by regressing bond excess returns on the orthogonalized duration series, $\overline{\text{duration}^y}_t$. Table 3, Panel B reports the results. The economic and statistical significance of the duration factor remains very close to the results reported in Table 1. We still find that duration is a highly significant predictor of bond risk premia and that the effect becomes stronger, the longer the maturity.

[Insert Table 3 here.]

### *2.4 MBS duration and macro factors*

One might also suspect that MBS duration is related to the business cycle as empirical evidence shows that the refinancing incentive of mortgage holders depends on the economic state. For example, Chen, Michaux, and Roussanov (2013) find a strong link between households' refinancing incentive and macro variables such as industrial production. In the following, we explore how much of MBS dollar duration is indeed driven by macro variables. To proxy for the economic state, we use two measures: First, economic growth is proxied by the three-month moving average of the Chicago Fed National Activity Index and inflation is calculated as the consensus estimate from surveys on future inflation from Blue Chip Financial Forecasts. These two macro variables have previously been used in the term structure literature and have been shown to have a significant bearing on bond returns (see Joslin, Priebsch, and Singleton (2014)).[10]

The unconditional correlation between MBS dollar duration and economic growth and inflation is given in Panel A of Table 3. We note that the correlation with both inflation and economic growth is relatively high around 44% and 37%, respectively. To assess the predictive power of MBS duration beyond the one contained in these two macro variables, we run the following regression:

$$\text{duration}_t = \alpha + \beta_1 \text{infl}_t + \beta_2 \text{gro}_t + \epsilon_t,$$

and use the residual from this regression, denoted by $\overline{\text{duration}^m}_t = \text{duration}_t - \left(\hat{\alpha} + \hat{\beta}_1 \text{infl}_t + \hat{\beta}_2 \text{gro}_t\right)$.

We now run regressions of bond risk premia onto this orthogonalized duration factor and the results are presented in Panel C of Table 3. Again, we find that estimated

[10]We also use the eight principal components from macro variables as in Ludvigson and Ng (2009). The results do not change qualitatively.

coefficients remain very similar to the baseline regression results presented in Table 1. Coefficients are positive, increasing with maturity and highly significant.

As a last robustness check, we orthogonalize the MBS duration series with respect to both macro and yield factors, denoted by $\overline{\text{duration}^{y+m}}$. The results are reported in Panel D of Table 3. We still find that MBS dollar duration is highly significant and the effect becomes stronger the longer the maturity of bonds. For 10-year bond excess returns, we find that the slope coefficient has a t-statistic of 3.51 with an associated $R^2$ of 13%.

One key difference of MBS duration from the above factors is its relatively low persistence. Figure 4 plots the MBS dollar duration together with the three yield PCs (upper panel) and the macro variables, inflation and growth (lower panel). We note that shocks to MBS duration are much more transient than shocks to either yield PCs or the macro variables. This is manifested in a much lower half-life of the MBS duration series compared to the other factors, notably the level of interest rates. We find that MBS dollar duration has a half-life of around 3 months, whereas the yield PCs have a half-life of 68, 23 and 4 months, respectively, and the macro variables have a half-life of 34 and 13 months for the inflation and growth series.

[Insert Figure 4 here.]

### *2.5 MBS duration and other predictors of bond returns*

In the following, we include two other well-known predictors of bond risk premia, the CP factor and the slope of the term structure (defined as the difference between the 10-year and 2-year yield); see Cochrane and Piazzesi (2005). We run the following regression:

$$rx^{\tau}_{t+1} = \beta^{\tau}_1 \text{duration}_t + \beta^{\tau}_2 \text{slope}_t + \beta^{\tau}_3 \text{cp}_t + \epsilon_{t+1}.$$

Results are reported in Table 4. We find that including these additional regressors does not deteriorate the significance of MBS dollar duration, moreover, estimated coefficients on MBS duration remain remarkably stable. When we add the slope of the term structure and the CP factor to the regressions, the estimated coefficients on duration remain highly significant for maturities of five years and beyond. All three regressors combined explain between 21% and 46% of the time variation of annual bond excess returns.

[Insert Table 4 here.]

We conclude that there is a strong link between bond risk premia and MBS duration. The effect is more pronounced for longer maturity bonds and remains significant when we add other predictors to the regressions.

### *2.6 MBS convexity and other determinants of yield volatility*

We now control for additional determinants of yield and swaption implied volatility that have been documented in the literature. For example, it is well-known that volatility tends to increase in periods of high illiquidity (see, e.g., Hu, Pan, and Wang (2013)). In our multivariate specification, we therefore add a proxy for illiquidity and a proxy of fixed-income implied volatility, similar to the VIX in equity markets.[11] We run the following regression from conditional bond yield volatility onto MBS dollar convexity and a set of other predictors:

$$\text{vol}_t^\tau/\text{iv}_t^{\tau y10y} = \beta_1^\tau \text{convexity}_t + \beta_2^\tau \text{illiq}_t + \beta_3^\tau \text{tiv}_t + \epsilon_t,$$

where $\text{vol}_t^\tau$ is the conditional bond yield volatility at time $t$ of a bond with maturity $\tau = 1, \dots, 10$ years, $\text{iv}_t^{\tau y10y}$ is the $\tau$-year maturity implied volatility from swaptions on the 10-year swap rate, $\text{illiq}_t$ is the illiquidity factor at time $t$, and $\text{tiv}_t$ is the Treasury-implied volatility at time $t$. Results are reported in Table 5. We find that when we add illiquidity and tiv into the regression, convexity still remains highly statistically significant. The estimated coefficients reveal that the effect is the largest for the intermediate

[11] Hu, Pan, and Wang (2013) document a strong link between their illiquidity proxy and a fixed-income implied volatility index, the Bank of America/Merrill Lynch MOVE index. Note that the MOVE is calculate from rather illiquid over-the-counter Treasury options while our proxy $\text{tiv}_t$ is calculated using extremely liquid Treasury future options.

maturities of two-three years as indicated by the size of the coefficient. Illiquidity has the expected positive sign in the swaption implied volatility regressions as bond volatility tends to be high when markets are illiquid. However, the effect is insignificant. All three factors together explain between 50% and 63% of the time variation in bond yield volatility across different maturities. The same picture emerges for implied volatilities from swaptions: Estimated slope coefficients on convexity are robust to the inclusion of other regressors.

[Insert Tables 5 here.]

### *2.7 The interest rate risk of Government-Sponsored Enterprises*

The US mortgage market is dominated by the two government-sponsored enterprises (GSE), Fannie Mae and Freddie Mac. The importance of the GSEs can be illustrated in Figure 5 where we plot the total mortgage debt held by Fannie Mae and Freddie Mac. We note that between 1990 and 2011, the holdings of the GSEs have increased from USD 1.1 trillion to USD 6.2 trillion in 2011. As a percentage of all US mortgages, they currently hold almost 50%, which corresponds to 40% of GDP. The business of GSEs can be summarized as follows: GSEs buy mortgage loans on the secondary market. These purchases result in two different portfolios: The retained and the guarantee portfolio. The retained portfolio consists of loans and MBS that are owned outright by the GSEs. The capital to invest in retained portfolios is in large parts funded by issuing notes and bonds. The guarantee portfolio, on the other hand, consists of various MBS that the GSEs guarantee the credit risk of, but they do not invest their capital in. The capital to fund these securities is provided by the MBS investors themselves.

The interest rate risk of the two portfolios from the viewpoint of the GSEs is very different. For the guarantee portfolio, all the interest rate risk lies with the investors. For the retained portfolio on the other hand, the interest rate risk lies with the GSEs. Figure 5 (middle panel) plots the size of the retained portfolio of Fannie Mae and Freddie Mac. The figure shows the steady growth of the retained portfolio that started at around USD 200 billion in the 1990s and reached a level of almost USD 1.6 trillion in 2003 and

2008. The eight-fold increase over this period occurred mainly between 1997 to 2003, when their portfolio grew from USD 481 billion to USD 1.6 trillion; after a slight decline and rebound to the previous peak in 2008, the value has decreased to USD 1.36 trillion by 2011.

[Insert Figure 5 here.]

The interest rate risk embedded in the retained portfolio comes from two sources: (i) the maturity mismatch between mortgage assets and the bond liabilities and (ii) the prepayment option on the mortgage assets. When interest rates rise, long-term fixed-rate mortgages lose substantial value. When interest rates fall, mortgages are prepaid by mortgage borrowers and consequently the GSEs have to replace the mortgages at lower interest rates. Taken together, the GSEs lose money independently of the direction of interest rate changes. As a consequence, the GSEs hedge some parts of their interest rate exposure. Note, however, that the GSEs do not hedge all their interest rate risk on the retained portfolios as otherwise the retained mortgage portfolios and the MBS business line would have the same impact on the mortgage market (see Jaffee (2003) for a discussion). Hedging is done through interest rate swaps under which they trade the fixed-rate interest payments of mortgage loans for floating-rate interest rate payments that correspond more closely to their short-term borrowing costs. To hedge prepayment risk, the GSEs issue callable debt and buy swaptions.[12] If interest rates fall, the GSEs can redeem their callable debt at lower rates or similarly, exercise their swaptions. Historically, the GSEs have started hedging during the 1990s (see Howard (2013)).

Figure 5 (lower panel) plots the notional value of the GSEs' derivative contracts.[13] While in the beginning it was below USD 300 billion, it has been around USD 1.3 trillion

[12] Fannie Mae specifically stress the fact that they hedge both duration and prepayment risk in their 10K filings. They write "Risk management derivative instruments are an integral part of our management of interest rate risk. We supplement our issuance of debt securities with derivative instruments to further reduce duration risk, which includes prepayment risk. We purchase option-based risk management derivatives to economically hedge prepayment risk."

[13] According to the Financial Accounting Standard (FAS) 133, any firm is required to publish the fair value of derivatives designated as hedging instrument.

recently. During this period, it peaked twice: first it increased rapidly to USD 2.2 trillion by 2003, then again to USD 2.6 trillion in 2008. The two peaks coincide with the large drops in MBS duration discussed earlier.

Given the dominating role of the GSEs in the mortgage market and their hedging activity, in the following we study the importance of GSEs to explain the predictive power of the MBS duration measure for bond excess returns. Our premise is that as the size of the retained portfolio becomes larger, the GSEs' hedging need should increase, and hence, the effect from MBS duration interacted with the retained portfolio should increase. A similar intuition applies to the notional size of derivatives used. As a first test, we run univariate regressions from bond excess returns onto the size of the retained portfolio and the notional amount of derivatives held. The results are summarized in Table 6.

[Insert Table 6 here]

Both the size of the retained portfolio and the notional amount of derivatives held are excellent predictors of bond returns. In line with our intuition, we find that all coefficients are positive and significant for longer maturity bonds. For example, the t-statistic from regressing the 10-year bond excess return on the size of the retained portfolio (derivatives) is 3.86 (4.75).

Next, in Panel B, we test the hypothesis that a larger size of the retained portfolio and derivatives held will lead to a stronger impact of MBS dollar duration onto bond excess returns. To this end, we interact these measures with MBS duration. When we compare the results to the benchmark results in Table 1, we find that the economic size has increased: For example, for the 10-year bond risk premium, we find that any one standard deviation change in MBS dollar duration interacted with the retained portfolio (derivatives), leads to a 349 bp (369 bp) increase in bond risk premium.

### *2.8 The impact of MBS duration and convexity over time*

Figure 6 plots the ratio of outstanding mortgages and GDP from 1990 to 2012 together with the ratio of the amount outstanding in mortgages and Treasuries. As one can see, the importance of the mortgage market vis-à-vis both GDP and Treasuries has increased over the past 20 years, although both ratios peak in 2010 and since then have somewhat declined. If mortgage markets have become more important over time, one would expect the effect of MBS duration and convexity to increase over time as well. To control for the variation in mortgage volume, we run similar regressions as before but interact the dollar duration and convexity measures with the mortgage to GDP ratio. The results are reported in Table 7.

[Insert Figure 6 and Table 7 here.]

The results indicate that MBS duration and convexity are still highly significant predictors of bond excess returns and bond yield volatility. For example, we find that the t-statistic of the slope coefficient on the 10-year risk premium is 3.43, and for the 2-year bond yield volatility regression the associated t-statistic is 6.91.

### *2.9 Other duration measures*

One might wonder how and whether MBS dollar duration is related to measures of duration for other fixed-income instruments. For example, Greenwood, Hanson, and Stein (2010) show that there is no relationship between the maturity structure of corporate debt and MBS supply. On the other hand, using an affine term structure model, Li and Wei (2013) show that both MBS and Treasury supply potentially matter to explain term premia during the Large Scale Asset Purchase Program undertaken by the Federal Reserve in 2008.

In the following, we make use of different measures of fixed-income duration, namely dollar duration from Treasuries and corporate bonds for two different rating categories. Table 8 reports unconditional correlations for MBS, Treasury, corporate AAA and BBB dollar duration. We note that MBS dollar duration is most highly (negatively) correlated with Treasury duration but the correlation drops to insignificant levels for corporate AAA duration (-7%) and corporate BBB duration (13%). In the following, we want to study the predictive power of each of these duration measures. To this end, we regress annual bond risk premia onto the different duration measures. The results are found in Table 9.

[Insert Tables 8 and 9 here.]

The predictive power of MBS dollar duration is not significantly different from the results reported in Table 1. The size and significance of the estimated coefficient is virtually the same, moreover, as before, longer maturity bonds are more affected by MBS dollar duration than shorter maturity bonds. Interestingly, we find that corporate bond dollar duration loads significantly on bond risk premia but with opposite signs. Treasury dollar duration is not significant at any maturity.

### *2.10 Robustness*

*Interest rate swaps:* Interest rate risk is primarily hedged in either the Treasury or interest rate swap market and the main focus in the previous section has been on Treasury data. The reason for this is twofold. First, interest rate swap data contain a considerable credit risk component (see Feldhütter and Lando (2008)) which is outside the scope of our paper to explain. Second, after the Lehman default in 2008, prices of interest rate swaps (especially at longer maturities) got possibly distorted due to a decline in arbitrage capital (see Krishnamurthy (2010)). In particular, our data sample also covers the time period where the swap spread, defined as the difference between the fixed rate on a fixed-for-floating 10-year swap and 10-year Treasury rate, turned negative. Nevertheless, for robustness reasons, we also run bond risk premia regressions using swap rather than Treasury data and we report estimated coefficients in Table 10.[14] We note that the size and significance of the estimated coefficients are almost identical to those reported for Treasuries. Adding explanatory factors such as the level of the term structure does not deteriorate the significance of MBS dollar duration.

[Insert Table 10 here.]

[14] We bootstrap a zero-coupon curve from swap rates and calculate excess returns that are directly comparable to the Treasury excess returns we use in the benchmark results.

*Bond portfolios:* One issue with using annual bond excess returns is the short sample period. In our data of 16 years, we have a maximum of 16 independent observations. To address this issue, we re-run our regressions using actual bond returns for different maturity bins available from CRSP. Data is monthly and represents an equally-weighted average of holding period returns for each bond in the portfolio. We calculate excess returns by subtracting the T-bill rate. Because of the large impact of monetary policy on T-bills in the past couple of years, we also use the one-month Eurodollar deposit rate as an alternative as suggested by Duffee (1996). Results are reported in Table 11.

[Insert Table 11 here.]

We note that while the adjusted $R^2$s are almost halved compared to the previous results, the estimated coefficients for duration are still highly significant, even in the multivariate regressions. Moreover, these results hold whether we use the T-bill or the Eurodollar rate to construct the excess returns.

## 3 Model

In this section we propose a parsimonious dynamic equilibrium term structure model that can explain our empirical findings. In the model the net supply of fixed income securities is driven by MBS dollar duration. For instance, we interpret the shortening of MBS duration due to the increased probability of future refinancing as a negative shock to the supply of long-term bonds: Before refinancing happens and investors have access to new mortgage pools, the interest rate risk profile of mortgage-related securities available to investors resembles that of relatively short maturity bonds. This induces additional buying pressure on Treasuries if investors want to keep their interest rate risk exposure constant.[15]

We focus on modeling the MBS channel only. In doing so, we abstract from several other important drivers of bond prices. In particular, we do not consider any other

[15] Market participants can invest in new mortgage loans by buying corresponding MBS. Up to 90 days before those MBS are issued, investors have access to them through the "to-be-announced" (TBA) market. We thank Douglas McManus for an insightful discussion; see also Vickery and Wright (2010).

sources of variation in bond risk premia coming from yield, macroeconomic, or any additional supply factors. The motivation for this narrow emphasis is twofold. First, negative convexity is a rather peculiar feature of the MBS market. In order to facilitate interpretation and to be able to tease out its isolated effect, we write the most simple model possible. Second, as discussed in the previous section, we do not find any significant overlap between the explanatory power of MBS dollar duration/convexity and that of other factors.

### *3.1 Bond market*

Time is continuous and goes from zero to infinity. We denote the time $t$ price of a zero coupon bond paying one dollar at maturity $t+\tau$ by $\Lambda_t^\tau$, and its yield by $y_t^\tau = -\frac{1}{\tau}\log \Lambda_t^\tau$. The short rate $r_t$ is the limit of $y_t^\tau$ when $\tau \to 0$. We take $r_t$ as exogenous and assume that its dynamics under the physical probability measure are given by

$$dr_t = \kappa\left(\theta - r_t\right)dt + \sigma dB_t, \tag{1}$$

where $\theta$ is the long run mean of $r_t$, $\kappa$ is the speed of mean reversion, and $\sigma$ is the volatility of the short rate.[16]

At each date $t$, there exists a continuum of zero-coupon bonds with time to maturity $\tau \in (0, T]$ in total supply of $s_t^\tau$. Bonds are held by financial institutions who are competitive and have mean-variance preferences over the instantaneous change in the value of their bond portfolio. If $x_t^\tau$ denotes the quantity they hold in maturity-$\tau$ bonds at time $t$, investors' budget constraint becomes

$$dW_t = \left(W_t - \int_0^T x_t^\tau \Lambda_t^\tau d\tau\right) r_t dt + \int_0^T x_t^\tau \Lambda_t^\tau \frac{d\Lambda_t^\tau}{\Lambda_t^\tau} d\tau, \tag{2}$$

and their optimization problem is given by

$$\max_{\{x_t^\tau\}_{\tau\in(0,T]}} \mathrm{E}_t\left[dW_t\right] - \frac{\alpha}{2}\mathrm{Var}_t\left[dW_t\right], \tag{3}$$

[16] Similar to Collin-Dufresne and Harding (1999), we use a Vasicek process because of its simplicity.

where $\alpha$ is their absolute risk aversion. Since financial institutions have to take the other side of the trade in the bond market, the market clearing condition is given by

$$x_t^\tau = s_t^\tau, \quad \forall t \text{ and } \tau. \tag{4}$$

### *3.2 MBS duration*

The supply of bonds is driven by households' mortgage liabilities. Without explicitly modeling them, we think about a continuum of households who do not themselves invest in bonds but take fixed-rate mortgage loans that are then sold on the market as MBS. The aggregate duration of outstanding MBS is driven by two forces: (i) changes in the level of interest rates that affect the prepayment probability of each outstanding mortgage, and (ii) actual prepayment that changes the composition of the aggregate mortgage pool. In the following, we posit a reduced-form model of aggregate prepayment in the spirit of Gabaix, Krishnamurthy, and Vigneron (2007) that captures these two effects.[17]

Households refinance their mortgages when the incentive to do so is sufficiently high. Prepaying a mortgage is equivalent to exercising an American option. As shown in Richard and Roll (1989), the difference between the fixed rate paid on a mortgage and the current mortgage rate is a good measure of the moneyness of this prepayment option.

[17]The literature has adopted different ways to describe the prepayment behavior. Dunn and McConnell (1981) and Brennan and Schwartz (1985) pioneered the application of contingent claim techniques to the problem by modeling prepayment as an optimal decision by borrowers who minimize the value of their loans. This approach is used more recently in Longstaff (2005). However, micro-level evidence suggests that individual household prepayment is often non-optimal relative to the option pricing approach and prepayment depends both on observable factors such as the incentive to refinance, seasonality or the level of house prices in general as well as on non-observable factors such as the media effect (for example prepayment rates can directly react to specific news stories relating to interest rates and the mortgage market, see e.g., Soo (2014)). Non-optimality has been a major topic in the literature on MBS valuation (see, e.g., Campbell (2006) or Amromin, Huang, and Sialm (2007)). To address this in the context of a contingent claim analysis, Stanton and Wallace (1998) add an exogenous delay to refinancing, while Schwartz and Torous (1989) and Stanton (1995) are examples for purely reduced-form econometric models that aim to capture the empirical behavior. Our motivation for using a reduced-form approach is twofold. First, we avoid making strong assumptions regarding the optimal prepayment. Second, the incentive to prepay on aggregate is well explained by interest rates themselves. Boudoukh, Whitelaw, Richardson, and Stanton (1997) for example show that for a given MBS coupon the level of long-term interest rates is a very good proxy for the likelihood that a mortgage will be prepaid.

Because households can have mortgages with different characteristics, we focus on the average mortgage coupon (interest payment) on outstanding mortgages, $c_t$. Following Schwartz and Torous (1989) we approximate the current mortgage rate by the long-term interest rate $y_t^{\bar{\tau}}$ with reference maturity $\bar{\tau}$.[18,19] In sum, we define the refinancing incentive as $c_t - y_t^{\bar{\tau}}$.

On aggregate, refinancing activity does not change the size of the mortgage pool: when a mortgage is prepaid, another mortgage is issued. However, the average coupon $c_t$ is affected by prepayment, because the coupon of the newly issued mortgages depends on the current level of mortgage rates. We assume that the evolution of the average coupon is a function of the refinancing incentive:

$$dc_t = -\kappa_c \left(c_t - y_t^{\bar{\tau}}\right) dt, \tag{5}$$

with $\kappa_c > 0$. This means that a lower interest rate $y_t^{\bar{\tau}}$, i.e., a higher refinancing incentive, leads to more prepayments, and new mortgages issued at this low rate decrease the average coupon more. Because our focus is the feedback between the MBS market and interest rates, we also assume that on aggregate, there is no additional uncertainty about refinancing. The upper panel of Figure 1 provides empirical motivation for (5). We plot the difference between long-term interest rates and the average MBS coupon, together with the subsequent change in the average coupon. The two series are closely aligned with the coupon reacting with a slight delay to a change in the refinancing incentive.[20]

The distinctive feature of mortgage-related securities is that their duration depends primarily on the likelihood that they will be refinanced in the future. The MBS coupon and the level of interest rates proxy for the expected level of prepayments and the

[18] An argument against this choice is mentioned in Krishnamurthy (2010) who studies the spread between mortgage rates and interest rate swaps. He notes that especially during autumn 2008 there was a large disconnect between the two which can be attributed to a flight-to-liquidity episode from relatively illiquid mortgages to more liquid government bonds. Since these considerations are outside the scope of the model, we leave this to future research.

[19] We use $\bar{\tau} = 10$ years. According to Hancock and Passmore (2010), it is common industry practice to use either the 5- or 10-year swap rate as a proxy for MBS duration.

[20] The correlation between the two series is 48% for the period 1990-2011 and 47% for 1997-2011. If we use a 3-month lag for the 10-year yield minus average MBS coupon, the correlation becomes 71% for the period 1990-2011 and 73% for 1997-2011. Employing a 4-month lag we find that the correlation is 75% for the period 1990-2011 and 78% for 1997-2011.

moneyness of the option (see Boudoukh, Whitelaw, Richardson, and Stanton (1997)). We thus assume that the aggregate dollar duration of outstanding mortgages is a function of the refinancing incentive:

$$D_t = \theta_D - \eta_y \left(c_t - y_t^{\bar{\tau}}\right), \tag{6}$$

where duration, $D_t \equiv -dMBS_t/dy_t^{\bar{\tau}}$, is the observable sensitivity of the aggregate mortgage portfolio value ($MBS_t$) to the changes in the reference long-maturity rate $y_t^{\bar{\tau}}$, and $\theta_D$, $\eta_y > 0$ are constants. The middle panel of Figure 1 provides empirical motivation for (6). We plot the difference between long-term interest rates and the average MBS coupon, together with aggregate MBS duration. The two series are again very closely aligned.[21] Overall, we note that our model captures well the key stylized properties of aggregate refinancing activity.

[Insert Figure 1 here.]

Combining (5) and (6) gives us the dynamics of $D_t$:

$$dD_t = \kappa_D \left(\theta_D - D_t\right) dt + \eta_y dy_t^{\bar{\tau}}, \tag{7}$$

where $\kappa_D = \kappa_c$. Note that dollar duration is driven both by changes in long-term interest rates and refinancing activity. The parameter $\eta_y = dD_t/dy_t^{\bar{\tau}}$ is the negative of the dollar convexity: When $\eta_y > 0$, lower interest rates increase the probability of borrowers prepaying their mortgages in the future, leading to a lower duration. The lower panel of Figure 1 plots the MBS convexity series showing that in our sample it always stays negative. Comparative statics with respect to $\eta_y$ allow us to derive predictions regarding the effect of negative convexity on interest rate volatility.[22]

[21]The correlation between dollar duration and the difference between the 10-year yield and average MBS coupon is 67% for the period 1990-2011, and 72% for 1997-2011.

[22]A model where $\eta_y$ itself follows a stochastic process would not fall into one of the standard tractable classes of models. The Online Appendix presents a version of the model that accommodates time-varying convexity. While this model implies a quadratic instead of an affine term structure, it leads to identical qualitative predictions.

### 3.3 Discussion

In this section we discuss our interpretation of the supply of bonds and the identity of bond investors in the model. We understand the former as the net supply of bonds coming from the re-balancing of fixed income portfolios in response to fluctuations in MBS duration. For instance, the hedging positions of the GSEs mentioned in Section 2.7 would be one of its components.[23] In turn, rather than modeling all bond market investors, we abstract from buy-and-hold investors and focus directly on those who end up absorbing this additional net supply. In particular, we have in mind financial institutions such as investment banks, hedge funds, and fund managers, who trade actively in fixed income markets and act as marginal investors there in the short to medium run.[24]

The risk-bearing capacity of financial institutions is key to why shocks to MBS duration matter. The mortgage choice of households affects the supply of fixed income securities, $s_t^\tau$, through the duration of mortgages, $D_t$, but in addition to this channel households are not present on either side of the market-clearing condition (4). In other words, except for having a constant amount of mortgage debt, households do not take part in fixed income markets. We note that, to the best of our knowledge, there is no evidence suggesting that households actively manage the duration of their liabilities by trading fixed income instruments. In the model the supply of bonds is held by financial institutions who are independent from households and who in practice are active participants in the bond market. As a result, the variation in the supply of bonds induced by changes in MBS duration is not washed out and matters for bond prices.[25]

[23]Among the largest holders of US Treasuries are foreign investors such as the Bank of China. However, we do not have any evidence that these investors actively hedge the interest rate risk of their portfolio and we hence think of the supply of bonds net of the positions of these investors.

[24]Financial intermediaries and institutional investors hold approximately 25% of the total amount outstanding in Treasuries, and daily trading volume is almost 10% of the total amount outstanding. In addition, these financial intermediaries hold around 30% of the total amount outstanding in MBS, and daily trading is almost 25% of the total amount outstanding. GSEs hold on average around 13% of all outstanding MBS. Data are for the period 1997 to 2012; see Securities Industry and Financial Markets Association (2013) and Flow of Funds Tables of the Federal Reserve.

[25]Gabaix, Krishnamurthy, and Vigneron (2007) make a related point that from the perspective of financial intermediaries who are the marginal investors in MBS, mortgage prepayment risk cannot be hedged and therefore is priced. Note that the prepayment risk of MBS is different from their interest rate risk.

More precisely, while lower interest rates trigger a certain amount of refinancing of the most in-the-money mortgages, they also increase the probability of future prepayment and thus decrease the duration of all outstanding mortgages. In fact, there is ample empirical evidence that shows that households' refinancing is gradual and sluggish (see Campbell (2006)). The progressive nature of refinancing ($\kappa_c < \infty$) leaves financial institutions who invest in MBS on aggregate short of duration exposure after a negative shock to interest rates. The opposite happens when interest rates increase and MBS duration lengthens.

### *3.4 Equilibrium term structure*

Before solving for equilibrium yields, we determine the market price of interest rate risk.

**Lemma 1.** *Given (1)-(4), the unique market price of interest rate risk is proportional to the dollar duration of the total supply of bonds:*

$$\lambda_t = \alpha\sigma \frac{d\left(\int_0^T s_t^\tau \Lambda_t^\tau d\tau\right)}{dr_t}. \tag{8}$$

Lemma 1 follows from the absence of arbitrage and implies that, regardless of the specific maturity composition of the supply of bonds, the market price of interest rate risk is determined by its total quantity. This means that in order to derive the equilibrium term structure, it is not necessary to explicitly model the full dynamics of the supply of bonds, but it is sufficient to capture its duration.

In this paper we are interested in only one source of variation in the duration of bond supply, namely the changes in MBS dollar duration. To this end, we replace the dollar value of bond net supply with the aggregate mortgage portfolio value $MBS_t$ to obtain

$$\lambda_t = \alpha\sigma \frac{dMBS_t}{dr_t}. \tag{9}$$

Using a simple chain rule, $\frac{dMBS_t}{dr_t} = \frac{dMBS_t}{dy_t^{\bar{\tau}}}\frac{dy_t^{\bar{\tau}}}{dr_t}$, we rewrite (9) in terms of the sensitivity to the reference long-maturity rate $y_t^{\bar{\tau}}$:

$$\lambda_t = -\alpha\sigma_y^{\bar{\tau}} D_t, \tag{10}$$

where $\sigma_y^{\bar{\tau}} \equiv \frac{dy_t^{\bar{\tau}}}{dr_t}\sigma$, the volatility of $y_t^{\bar{\tau}}$, is a constant to be determined in equilibrium.

We look for an equilibrium in which yields are affine in the short rate and the duration factor. Under the conjectured affine term structure, the physical dynamics of MBS duration (7) can be written as

$$dD_t = (\delta_0 - \delta_r r_t - \delta_D D_t)\, dt + \eta_y \sigma_y^{\bar{\tau}} dB_t, \tag{11}$$

where $\delta_0$, $\delta_r$ and $\delta_D$ are constants to be determined in equilibrium. In turn, equations (1), (10) and (11) together imply that the dynamics of the short rate and the MBS duration factor under the risk-neutral measure are

$$dr_t = \left(\kappa\theta - \kappa r_t + \alpha\sigma\sigma_y^{\bar{\tau}} D_t\right) dt + \sigma dB_t^{\mathbb{Q}} \text{ and} \tag{12}$$

$$dD_t = \left(\delta_0 - \delta_r r_t - \delta_D^{\mathbb{Q}} D_t\right) dt + \eta_y \sigma_y^{\bar{\tau}} dB_t^{\mathbb{Q}}, \tag{13}$$

where $\delta_D^{\mathbb{Q}} \equiv \delta_D - \alpha\eta_y \left(\sigma_y^{\bar{\tau}}\right)^2$.

We now have all the ingredients to solve for the equilibrium term structure.

**Theorem 1.** *In the term structure model described by (12) and (13), equilibrium yields are affine and given by*

$$y_t^{\tau} = \mathcal{A}(\tau) + \mathcal{B}(\tau)\, r_t + \mathcal{C}(\tau)\, D_t, \tag{14}$$

*where the functional forms of $\mathcal{A}(\tau)$, $\mathcal{B}(\tau)$, and $\mathcal{C}(\tau)$ are given in the Online Appendix, and the parameters $\sigma_y^{\bar{\tau}}$, $\delta_r$, $\delta_D$, and $\delta_0$ satisfy*

$$\sigma_y^{\bar{\tau}} = \frac{\sigma\mathcal{B}(\bar{\tau})}{1-\eta_y\mathcal{C}(\bar{\tau})},\ \delta_r = \frac{\kappa\eta_y\mathcal{B}(\bar{\tau})}{1-\eta_y\mathcal{C}(\bar{\tau})},\ \delta_D = \frac{\kappa_D}{1-\eta_y\mathcal{C}(\bar{\tau})},\ \textit{and}\ \delta_0 = \delta_r\theta + \delta_D\theta_D. \tag{15}$$

*Equations (15) have a solution whenever $\alpha$ is below a threshold $\bar{\alpha} > 0$.*

## 4 Inspecting the mechanism

The key empirical facts reported in Section 2 are naturally reproduced by our model, despite its stylized nature. We summarize our results as follows: (i) The dollar duration of MBS positively predicts excess bond returns for all maturities and the effect is stronger for longer maturities; (ii) The volatility of all yields is increasing in the negative dollar convexity of MBS and the effect is strongest for intermediary maturities, i.e., is hump-shaped; (iii) Endogenously determined MBS duration is not spanned by the factor that accounts for most of the variation in the cross-section the yields. In this section we analyze the mechanism behind these results. In addition, we confirm the quantitative relevance of the mechanism in a calibration of the model.

### *4.1 Calibration*

We estimate the parameters of the short rate process (1) and the dollar duration process (7) on the data between 1997 and 2011.[26] In order to fully characterize the theoretical effect of MBS duration and convexity on bond returns and yield volatility, we set the risk aversion of financial institutions to $\alpha = 80$. This value allows the model to match the $R^2$ of the predictive regression of 10-year bond excess returns on duration reported in Table 1. Note that $\alpha$ is the coefficient of absolute risk aversion. In order to interpret this value, we multiply it by financial institutions' wealth to obtain a coefficient of relative risk aversion. In a setting similar to ours, Greenwood and Vayanos (2014) use financial institutions' capital to GDP ratio of 13.3%. Because we use the dollar duration of the MBS index to calibrate the model and the average value of the index itself is standardized to one dollar, we also need to adjust for the size of the MBS market relative to GDP. Between 1997 and 2011 the average value of outstanding mortgage-related debt was equal to 53% of the GDP. This implies a coefficient of relative risk aversion of approximately $20 \approx 80 \times 13.3\%/53\%$.

We summarize all calibrated parameter values in Table 12.

[26]We note that equation (7) provides a very good description of the monthly series of MBS dollar duration as the associated $R^2$ is 84%.

[Insert Table 12 here.]

### 4.2 Predictability of bond risk premia

The predictability of excess bond returns by the dollar duration of MBS is a natural outcome of our model. The market price of interest rate risk depends on the quantity of the risk that financial institutions hold to clear the supply. In turn, bonds with higher exposure to interest rate risk are more affected. As a result, MBS duration should predict excess bond returns and the effect is stronger for longer maturity bonds.[27]

Formally, running a univariate regression of excess returns over horizon $h$ of bonds with maturity $\tau$ on the MBS duration factor,

$$rx_{t,t+h,\tau} = \alpha + \beta^{\tau,h} D_t + \epsilon_{t+h},$$

we obtain the following result on the theoretical slope coefficient:

**Proposition 1.** *We have* $\lim_{\tau \to h} \beta^{\tau,h} = 0$ *and* $d\beta^{\tau,h}/d\tau > 0$ *for all* $\tau > 0$. *Hence,* $\beta^{\tau,h}$ *is positive and increasing across maturities.*

In addition to the theoretical result, the left panel of Figure 7 reports the term structure of theoretical $\beta^{\tau,h}$ together with our empirical estimates. We note that the calibrated model is able to quantitatively match the overall magnitude of the slope coefficients across maturities, although it underpredicts them at the long end.

[Insert Figure 7 here.]

### 4.3 Bond yield volatility

Next, our model rationalizes the hump-shaped effect of MBS convexity on yield volatilities. Higher MBS convexity implies that the quantity of duration risk and therefore the

[27] Note that the effect of MBS dollar duration on the level of yields is not necessarily monotonic in maturity. A yield depends on the average of risk premia over the life of the bond. Higher risk premia increase yields. However, because of mean reversion in interest rates and duration, we expect risk premia at longer horizons to be lower. We are not testing this implication empirically, because duration itself depends on yields, thus causing an endogeneity problem for identification.

market price of risk are more sensitive to changes in interest rates. Because MBS convexity is negative, portfolio rebalancing by investors amplifies, rather than offsets, the effect that the initial shock to the short rate has on long-term interest rates. As a result, interest rate volatility is higher. Moreover, the link between convexity and volatility has a term structure dimension. Short-maturity yields are close to the short rate and therefore are not significantly affected by the variations in the market price of risk. For long maturities, we expect the duration of MBS to revert to its long-term mean. At the limit, yields at the infinite horizon should not be affected by current changes in the short rate and MBS duration at all.[28] As a result, the effect of MBS convexity on yield volatilities has a hump-shaped term structure.

From our model we obtain the following comparative statics result regarding the effect of negative convexity $\eta_y$ on bond yield volatilities $\sigma_y^\tau$:

**Proposition 2.** *We have $d\sigma_y^\tau / d\eta_y > 0$ for all $\tau > 0$. In addition, $\lim_{\tau \to 0} \sigma_y^\tau = \sigma$ and $\lim_{\tau \to \infty} \sigma_y^\tau = 0$, where neither limit depends on $\eta_y$. Hence, $d\sigma_y^\tau / d\eta_y$ is hump-shaped across maturities.*

An intuitive way to understand the effect of negative convexity on volatility within the model is to consider an approximation of the results in Theorem 1 where we replace $\mathcal{B}(\bar{\tau})$ and $\frac{\mathcal{C}(\bar{\tau})}{\alpha}$ that are non-trivial functions of yield volatility with $\mathcal{B}(\bar{\tau})|_{\alpha=0} = b$ and $\frac{\mathcal{C}(\bar{\tau})}{\alpha}|_{\alpha=0} = c$. When $c\alpha\eta_y < 1$, i.e. the risk aversion is below the threshold $\bar{\alpha} = \frac{1}{c\eta_y}$, we have an affine equilibrium where the volatility of the reference maturity yield solves $\sigma_y^{\bar{\tau}} = b\sigma + c\alpha\eta_y\sigma_y^{\bar{\tau}}$. This fixed point problem is the result of a feedback mechanism between long rates and duration: lower interest rates decrease duration, which in turn decreases the term premium and further lowers long rates, etc. Through this mechanism the negative convexity increases the volatility by a factor $\frac{1}{1-c\alpha\eta_y} = 1 + c\alpha\eta_y + (c\alpha\eta_y)^2 + ... > 1$ that captures the combined effect of the successive iterations of the feedback loop. The feedback explains why negative convexity can cause potentially significant interest rate volatility even for moderate level of risk aversion.

[28] This is just an application of a more general argument by Dybvig, Ingersoll, and Ross (1996) on why the yield volatility curve for long maturities should be downward sloping.

The right panel of Figure 7 reports the change in yield volatility across maturities that can be attributed to negative convexity. For instance, the model implies a 40 basis point increase in the two-year yield volatility compared to a 93 basis point increase implied by our linear regression results. The respective numbers are 16 and 25 basis points for the ten-year yield volatility.[29] In line with our empirical findings the calibrated model implies that the effect of negative convexity is hump-shaped and strongest for maturities around 2 and 3 years. As a result, the negative convexity channel could provide one possible explanation for a well-documented phenomenon of the hump-shaped term structure of yield volatilities.

### *4.4 MBS duration and yield factors*

While our stylized model is not designed to address the possibility that MBS duration is unspanned, it nevertheless speaks to the empirical facts regarding the information in MBS duration and its relation to the information encoded in the yields.

Theorem 1 motivates the inclusion of MBS duration in the term structure analysis. Even though the model has only one shock, long-term yields depend on two separate factors: the short rate and the aggregate dollar duration of MBS. This is the case because duration depends not only on the current mortgage rate, but also on the history of past mortgage rates that determine the coupon of outstanding mortgages.[30]

In our model, running a bivariate regression of excess returns over horizon $h$ of bonds with maturity $\tau$ on the MBS duration factor while controlling for the short rate,

$$rx_{t,t+h,\tau} = \alpha + \beta_1^{\tau,h} D_t + \beta_2^{\tau,h} r_t + \epsilon_{t+h},$$

we obtain the following result on the theoretical slope coefficients:

[29] One reason the model underpredicts the basis point effect of convexity is that it produces a lower level of interest rate volatility compared to the data, and hence the volatility amplification mechanism is applied to a lower base level of volatility.

[30] Formally, when $\kappa_D \neq 0$, interest rates in our model are non-Markovian with respect to the short rate $r_t$ alone. However, their history dependence can be summarized by an additional Markovian factor, namely the duration $D_t$.

**Proposition 3.** *We have* $\lim_{\tau \to h} \beta_1^{\tau,h} = \lim_{\tau \to h} \beta_2^{\tau,h} = 0$ *and* $d\beta_1^{\tau,h}/d\tau > 0 > d\beta_2^{\tau,h}/d\tau$ *for all* $\tau > 0$. *Hence, the slope coefficient on duration,* $\beta_1^{\tau,h}$, *is positive and increasing in maturity, while the slope coefficient on the short rate,* $\beta_2^{\tau,h}$, *is negative and decreasing (i.e., becoming more negative) in maturity.*

The model implies that the two slope coefficients on the two factors should have opposite signs. This is the case because the level of interest rates does not contain any information about the current market price of risk beyond that already encoded in duration. However, including the short rate allows to control for the mean reversion in interest rates, and therefore to better predict the mean reversion in duration over the return horizon $h$; hence the negative sign on the level of interest rates. Note that the result in Proposition 3 corresponds exactly to our empirical findings in Table 1.

Turning to the whole cross-section of yields, the spanning properties of the model can be illustrated within our calibration exercise. The short rate factor explains over 97% of the variation in yields across maturities, but only around 7% of the variation in MBS duration and only around 1% of the one-year excess returns on the 10-year bond. At the same time duration accounts for all the return predictability and explains the same proportion of 10-year bond returns ($R^2 = 19.58\%$) in the model as in the data. In other words, in our model the factor that accounts for all the predictive power is not spanned by the factor that accounts for a dominant fraction of the cross sectional variation in yields.

To better understand the above results, consider the relationship between duration and the level of long-maturity rates. The calibrated model implies a correlation of 0.39 between MBS dollar duration and the 10-year yield (compared to 0.33 in the data). Despite the fact that duration and interest rates are buffeted by the same shock, the time series properties of the duration factor are different. As can be seen from (7), aggregate duration mean reverts not only because of the mean reversion in interest rates, but also through the renewal in the aggregate pool of mortgages. As a result, duration is a much less persistent factor compared to the level of interest rates, and their unconditional correlation is not perfect. Similarly, dollar duration is not strongly correlated with the slope of the yield curve defined as the difference between the 10-year yield and the short rate. The model implies a correlation of -0.15 (compared to -0.14 in the data). The low magnitude of this correlation is due to the effect that duration has on the term premium. To see this note that a negative shock to the short rate steepens the slope, because interest rates are expected to mean revert. At the same time, the corresponding drop in duration reduces the term premium. As these two effects on the slope tend to cancel each other out, the correlation between duration and the slope of the yield curve is pushed closer to 0.

## 5 Conclusion

This paper studies the predictive power of MBS dollar duration and convexity for bond excess returns and bond yield volatility. We find a strong positive link between MBS duration and bond excess returns. The relationship is not only statistically significant but also economically relevant. This relationship remains highly significant and stable when we add other standard regressors. We then proceed to study the relationship between MBS hedging and bond yield volatilities. We find that MBS convexity significantly affects bond yield volatilities and the relationship is hump shaped across maturities, similar to the hump-shaped pattern found in the unconditional volatilities of bond yields.

We then propose an equilibrium model of bond supply shocks driven by changes in MBS duration and embed it into an otherwise standard one factor term structure model. Despite its simple structure, our model has interesting implications for first and second moments of interest rates: Our model is able to replicate the predictive power of MBS duration for bond excess returns and can accommodate a hump shaped term structure of bond yield volatilities. A calibrated version of our model replicates the empirical patterns.

While MBS duration and convexity are naturally related to information in the term structure of bond yields, we provide novel evidence that duration is not spanned by the usual bond yield factors (principal components), as well as a theoretical motivation why this could be the case. An investigation of supply factors within a multi-factor reduced form term structure model and their relation to higher order yield factors is an exciting avenue which we leave to future research.

## References


AGARWAL, S., J. C. DRISCOLL, AND D. I. LAIBSON (2013): "Optimal Mortgage Refinancing: A Closed Form Solution," *Journal of Money, Credit, and Banking*, 45, 591–622.

AMROMIN, G., J. HUANG, AND C. SIALM (2007): "The Tradeoff Between Mortgage Prepayments and Tax-Deferred Retirement Savings," *Journal of Public Economics*, 91, 2014–2040.

BANSAL, R., W. J. COLEMAN, AND C. T. LUNDBLAD (2011): "Endogenous Liquidity Supply," Working Paper, Duke University.

BANSAL, R., AND I. SHALIASTOVICH (2013): "A Long-Run Risk Explanation of Predictability Puzzles in Bond and Currency Markets," *Review of Financial Studies*, 26, 1–33.

BEGENAU, J., M. PIAZZESI, AND M. SCHNEIDER (2013): "Banks' Risk Exposure," Working Paper, Stanford University.

BOUDOUKH, J., R. F. WHITELAW, M. RICHARDSON, AND R. STANTON (1997): "Pricing Mortgage-Backed Securities in a Multifactor Interest Rate Environment: A Multivariate Density Estimation Approach," *Review of Financial Studies*, 10, 405–446.

BOYARCHENKO, N., A. FUSTER, AND D. O. LUCCA (2014): "Understanding Mortgage Spreads," Working Paper, Federal Reserve Bank of New York.

BRENNAN, M. J., AND E. S. SCHWARTZ (1985): "Determinants of GNMA Mortgage Prices," *Journal of the American Real Estate and Urban Economics Association*, 13, 209–228.

CAMPBELL, J. Y. (2006): "Household Finance," *Journal of Finance*, 61, 1553–1604.

CHANG, Y., D. MCMANUS, AND B. RAMAGOPAL (2005): "Does Mortgage Hedging Raise Long-Term Interest Rate Volatility?," *Journal of Fixed Income*, 14, 57–66.

CHEN, H., M. MICHAUX, AND N. ROUSSANOV (2013): "Houses as ATMs? Mortgage Refinancing and Macroeconomic Uncertainty," Working Paper, MIT.

CHENG, P., AND O. SCAILLET (2007): "Linear-Quadratic Jump Diffusion Modelling," *Mathematical Finance*, 17, p. 575 – 598.

COCHRANE, J. H., AND M. PIAZZESI (2005): "Bond Risk Premia," *American Economic Review*, 95, 138–160.

COLLIN-DUFRESNE, P., AND J. P. HARDING (1999): "A Closed-Form Formula for Valuing Mortgages," *Journal of Real Estate Finance and Economics*, 19, 133–146.

DAI, Q., AND K. SINGLETON (2003): "Term Structure Dynamics in Theory and Reality," *Review of Financial Studies*, 16, 631–678.

D'AMICO, S., W. ENGLISH, D. LOPEZ-SALIDO, AND E. NELSON (2012): "The Federal Rerseves Large Scale Asset Purchase Programs: Rationale and Effects," *Economic Journal*, 122, 415–446.

D'AMICO, S., AND T. B. KING (2013): "Flow and Stock Effects of Large-Scale Treasury Purchases: Evidence on the Importance of Local Supply," *Journal of Financial Economics*, 108, 425–448.

Duarte, J. (2008): "The Causal Effect of Mortgage Refinancing on Interest Rate Volatility: Empirical Evidence and Theoretical Implications," *Review of Financial Studies*, 21, 1698–1731.

Duffee, G. R. (1996): "Idiosyncratic Variation of Treasury Bill Yields," *Journal of Finance*, 51(2), 527–551.

Duffee, G. R. (2011): "Information in (and not in) the term structure," *Review of Financial Studies*, 24(9), 2895–2934.

Dunn, K. B., and J. J. McConnell (1981): "Valuation of GNMA Mortgage-Backed Securities," *Journal of Finance*, 36, 599–616.

Dybvig, P. H., J. E. Ingersoll, and S. A. Ross (1996): "Long Forward and Zero-Coupon Rates Can Never Fall," *Journal of Business*, 69, 1–25.

Ehling, P., M. F. Gallmeyer, C. Heyerdahl-Larsen, and P. K. Illeditsch (2013): "Disagreement about Inflation and the Yield Curve," Working Paper, Norwegian Business School.

Feldhütter, P., and D. Lando (2008): "Decomposing Swap Spreads," *Journal of Financial Economics*, 88, 375–405.

Fenou, B., and J.-S. Fontaine (2013): "Gaussian Term Structure Models and Bond Risk Premia," Working Paper, Bank of Canada.

Gabaix, X., A. Krishnamurthy, and O. Vigneron (2007): "Limits of Arbitrage: Theory and Evidence from Mortgage-Backed Securities Market," *Journal of Finance*, 62, 557–595.

Gagnon, J. E., M. Raskin, J. Remache, and B. P. Sack (2010): "Large-Scale Asset Purchases by the Federal Reserve: Did They Work?," Working Paper, Board of Governors of the Federal Reserve System.

Greenwood, R., S. G. Hanson, and J. C. Stein (2010): "A Gap-Filling Theory of Corporate Debt Maturity Choice," *Journal of Finance*, 65, 993–1028.

Greenwood, R., and D. Vayanos (2014): "Bond Supply and Excess Bond Returns," *Review of Financial Studies*, 27, 663–713.

Gürkaynak, R., B. Sack, and J. Wright (2007): "The U.S. Treasury Yield Curve: 1961 to Present," *Journal of Monetary Economics*, 54, 2291–2304.

Hancock, D., and W. Passmore (2010): "Did the Federal Reserve's MBS Purchase Program Lower Mortgage Rates?," Working Paper, Board of Governors of the Federal Reserve System.

Hanson, S. G. (2014): "Mortgage Convexity," *Journal of Financial Economics*, 113, 270–299.

Hong, H., D. Sraer, and J. Yu (2013): "Reaching for Maturity," Working Paper, Princeton University.

Howard, T. (2013): *The Mortgage Wars: Inside Fannie-Mae Big-Money Politics, and the Collapse of the American Dream*. McGraw Hill.

Hu, G. X., J. Pan, and J. Wang (2013): "Noise as Information for Illiquidity," *Journal of Finance*, 68, 2223–2772.

Jaffee, D. (2003): "The Interest Rate Risk of Fannie Mae and Freddie Mac," *Journal of Financial Services Research*, 24, p. 5 – 29.

Joslin, S., A. Le, and K. J. Singleton (2013): "Gaussian Macro-Finance Term Structure Models with Lags," *Journal of Financial Econometrics*, 11, 581–609.

Joslin, S., M. Priebsch, and K. J. Singleton (2014): "Risk Premiums in Dynamic Term Structure Models with Unspanned Macro Risks," *forthcoming, Journal of Finance*.

Krishnamurthy, A. (2010): "How Debt Markets Have Malfunctioned in the Crisis," *Journal of Economic Perspectives*, 24, 3–28.

Krishnamurthy, A., and A. Vissing-Jorgensen (2011): "The Effects of Quantitative Easing on Interest Rates: Channels and Implications for Policy," *Brookings Papers on Economic Activity*, 2, 215–287.

——— (2012): "The Aggregate Demand for Treasury Debt," *Journal of Political Economy*, 120, 233–267.

Landier, A., D. Sraer, and D. Thesmar (2013): "Banks' Exposure to Interest Rate Risk and the Transmission of Monetary Policy," Working Paper, Toulouse School of Economics.

Le, A., and K. J. Singleton (2013): "The Structure of Risks in Equilibrium Affine Models of Bond Yields," Working Paper, UNC.

Le, A., K. J. Singleton, and Q. Dai (2010): "Discrete-Time Affine Q Term Structure Models with Generalized Market Prices of Risk," *Review of Financial Studies*, 23, 2184 – 2227.

Li, C., and M. Wei (2013): "Term Structure Modeling with Supply Factors and the Federal Reserve's Large Scale Asset Purchase Programs," Working Paper, Board of Governors of the Federal Reserve System.

Longstaff, F. (2005): "Borrower Credit and the Valuation of Mortgage-Backed Securities," *Real Estate Economics*, 33, 619–661.

Lou, D., H. Yan, and J. Zhang (2013): "Anticipted and Repeated Shocks in Liquid Markets," *Review of Financial Studies*, 26, 1891–1912.

Ludvigson, S. C., and S. Ng (2009): "Macro Factors in Bond Risk Premia," *Review of Financial Studies*, 22, 5027–5067.

Mueller, P., A. Vedolin, and H. Choi (2014): "Bond Variance Risk Premiums," Working Paper, London School of Economics.

Newey, W., and K. West (1987): "A Simple, Positive Semi-Definite, Heteroscedasticity and Autocorrelation Consistent Covariance Matrix," *Econometrica*, 55, 703–708.

Patton, A. J., and A. Timmermann (2010): "Monotonicity in Asset Returns: New Tests with Applications to the Term Structure, the CAPM, and Portfolio Sorts," *Journal of Financial Economics*, 98, 605–625.

Perli, R., and B. P. Sack (2003): "Does mortgage hedging amplify movements in long-term interest rates?," *Journal of Fixed Income*, 13, 7–17.

Richard, S. F., and R. Roll (1989): "Prepayments on Fixed-rate Mortgage-Backed Securities," *Journal of Portfolio Management*, 15, 73–82.

Schwartz, E. S., and W. Torous (1989): "Prepayment and the Valuation of Mortgage-Backed Securities," *Journal of Finance*, 44, 375–392.

Securities Industry and Financial Markets Association (2013): "Data Statistics," http://www.sifma.org/research/statistics.aspx.

Soo, C. K. (2014): "Quantifying Animal Spirits: News Media and Sentiment in the Housing Market," Working Paper, University of Michigan.

Stanton, R. S. (1995): "Rational Prepayment and the Valuation of Mortgage-Backed Securities," *Review of Financial Studies*, 8, 677–708.

Stanton, R. S., and N. Wallace (1998): "Mortgage Choice: What's the Point?," *Real Estate Economics*, 26, 173–205.

Stock, J. H., and M. W. Watson (2007): "Introduction to Econometrics," Addison Wesley.

Stock, J. H., J. Wright, and M. Yogo (2002): "A Survey of Weak Instruments and Weak Identification in Generalized Method of Moments," *Journal of Business and Economic Statistics*, 20, 518–529.

Svensson, L. (2004): "Estimating and Interpreting Forward Rates: Sweden 1992–1994," National Bureau of Economic Research Working Paper No. 4871.

Vasicek, O. (1977): "An Equilibrium Characterization of the Term Structure," *Journal of Financial Economics*, 5, 177–188.

Vayanos, D., and J.-L. Vila (2009): "A Preferred-Habitat Model of the Term Structure of Interest Rates," Working Paper, London School of Economics.

Vickery, J., and J. Wright (2010): "TBA Trading and Liquidity in the Agency MBS Market," Working Paper, Federal Reserve Bank of New York.

Wooldridge, P. D. (2001): "The Emergence of New Benchmark Yield Curves," BIS Quarterly Review.

Xiong, W., and H. Yan (2010): "Heterogeneous Expectations and Bond Markets," *Review of Financial Studies*, 23, 1433–1466.

## Appendix A Tables

**Table 1**
**Bond risk premia regressions: Treasuries**

This table reports estimated coefficients from regressing annual bond excess returns constructed from Treasuries, $rx^{\tau}_{t+1}$, onto a set of variables:

$$rx^{\tau}_{t+1} = \beta^{\tau}_1 \text{duration}_t + \beta^{\tau}_2 \text{level}_t + \epsilon^{\tau}_{t+1},$$

where $\text{duration}_t$ is MBS dollar duration and $\text{level}_t$ is the first principal component from bond yields. t-Statistics presented in parentheses are calculated using Newey and West (1987). All variables are standardized to have mean zero and a standard deviation of one. Data is weekly and runs from 1997 through 2011.

| | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|
| duration | 0.082 | 0.152 | 0.214 | 0.268 | 0.315 | 0.355 | 0.389 | 0.418 | 0.442 |
| | (1.21) | (2.13) | (2.94) | (3.63) | (4.20) | (4.65) | (5.01) | (5.30) | (5.56) |
| Adj. $R^2$ | 0.68% | 2.30% | 4.56% | 7.17% | 9.90% | 12.59% | 15.12% | 17.45% | 19.58% |
| duration | 0.048 | 0.142 | 0.224 | 0.296 | 0.356 | 0.407 | 0.450 | 0.485 | 0.514 |
| | (0.66) | (1.84) | (2.90) | (3.87) | (4.74) | (5.50) | (6.15) | (6.70) | (7.17) |
| level | 0.147 | 0.040 | -0.047 | -0.118 | -0.177 | -0.223 | -0.259 | -0.285 | -0.304 |
| | (1.71) | (0.48) | (-0.57) | (-1.46) | (-2.18) | (-2.74) | (-3.16) | (-3.47) | (-3.67) |
| Adj. $R^2$ | 2.58% | 2.33% | 4.64% | 8.37% | 12.74% | 17.18% | 21.33% | 25.02% | 28.19% |

**Table 2**
**Bond volatility regressions**

Panel A and B report estimated coefficients from regressing bond yield volatility, $\mathrm{vol}_t^{\tau}$, and $\tau$-year maturity implied volatility of swaptions written on a 10-year swap rate, $\mathrm{iv}_t^{\tau y10y}$, onto MBS dollar convexity:

$$\mathrm{vol}_t^{\tau}/\mathrm{iv}_t^{\tau y10y} = \beta_1^{\tau}\mathrm{convexity}_t + \epsilon_t^{\tau},$$

where $\tau = 1, \ldots, 10$. t-Statistics presented in parentheses are calculated using Newey and West (1987). All variables are standardized to have mean zero and a standard deviation of one. Data is weekly and runs from 1997 through 2011.

Panel A: Bond Yield Volatility

| | 1y | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|---|
| convexity | 0.284 | 0.301 | 0.294 | 0.281 | 0.264 | 0.244 | 0.225 | 0.209 | 0.194 | 0.182 |
| | (3.66) | (3.89) | (3.82) | (3.68) | (3.47) | (3.24) | (3.02) | (2.83) | (2.67) | (2.53) |
| Adj. $R^2$ | 8.05% | 9.08% | 8.64% | 7.90% | 6.95% | 5.96% | 5.07% | 4.35% | 3.77% | 3.32% |

Panel B: Swaption Implied Volatility

| | 1y | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|---|
| convexity | 0.397 | 0.407 | 0.391 | 0.380 | 0.365 | 0.363 | 0.348 | 0.368 | 0.338 | 0.323 |
| | (4.17) | (4.38) | (4.26) | (4.20) | (4.06) | (4.02) | (3.88) | (4.06) | (3.58) | (3.41) |
| Adj. $R^2$ | 15.74% | 16.60% | 15.27% | 14.41% | 13.32% | 13.16% | 12.09% | 13.55% | 11.45% | 10.43% |

**Table 3**
**Duration and yield factors**

Panel A reports the unconditional correlation between MBS dollar duration and the first three principal components (PCs) of bond yields and two macro factors: inflation and economic growth. Inflation is measured as the consensus estimate from survey forecasts and growth is the three-month moving average of the Chicago Fed National Activity Index. Numbers in parentheses are t-Statistics. Panel B reports estimated coefficients from regressing bond excess returns onto MBS dollar duration which has been orthogonalized with respect to the three yield PCs. Panel C reports coefficients from regressing bond excess returns onto MBS dollar duration which has been orthogonalized with respect to the two macro variables and finally Panel D reports bond excess return regression on a dollar duration measure which has been orthogonalized with respect to yield and macro factors. t-Statistics presented in parentheses are calculated using Newey and West (1987). All variables are standardized to have mean zero and a standard deviation of one. Data is monthly and runs from 1997 through 2011.

Panel A: Unconditional Correlations

| PC1 | PC2 | PC3 | infl | gro |
|---|---|---|---|---|
| 0.40 | 0.16 | -0.20 | 0.44 | 0.37 |
| (5.66) | (1.88) | (-2.78) | (5.23) | (5.76) |

| | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|
| **Panel B: Duration orthogonalized wrt yield PCs** | | | | | | | | | |
| $\overline{\text{duration}}^y$ | 0.131 | 0.193 | 0.246 | 0.292 | 0.330 | 0.361 | 0.386 | 0.406 | 0.422 |
| | (1.38) | (2.07) | (2.70) | (3.23) | (3.63) | (3.90) | (4.08) | (4.21) | (4.30) |
| Adj. $R^2$ | 1.70% | 3.71% | 6.06% | 8.51% | 10.88% | 13.02% | 14.88% | 16.46% | 17.77% |
| **Panel C: Duration orthogonalized wrt macro variables** | | | | | | | | | |
| $\overline{\text{duration}}^m$ | 0.260 | 0.313 | 0.353 | 0.385 | 0.411 | 0.434 | 0.451 | 0.466 | 0.476 |
| | (2.28) | (2.64) | (2.98) | (3.27) | (3.51) | (3.72) | (3.88) | (4.02) | (4.13) |
| Adj. $R^2$ | 6.74% | 9.78% | 12.43% | 14.80% | 16.93% | 18.79% | 20.38% | 21.68% | 22.69% |
| **Panel D: Duration orthogonalized wrt yield and macro variables** | | | | | | | | | |
| $\overline{\text{duration}}^{y+m}$ | 0.181 | 0.225 | 0.261 | 0.289 | 0.313 | 0.332 | 0.346 | 0.358 | 0.366 |
| | (1.53) | (1.93) | (2.29) | (2.61) | (2.89) | (3.11) | (3.29) | (3.42) | (3.51) |
| Adj. $R^2$ | 3.27% | 5.07% | 6.79% | 8.37% | 9.78% | 11.00% | 12.00% | 12.79% | 13.41% |

**Table 4**
**Bond risk premia regressions with controls**

This table reports estimated coefficients from regressing bond excess returns, $rx^{\tau}_{t+1}$, onto a set of variables:

$$rx^{\tau}_{t+1} = \beta^{\tau}_1 \text{duration}_t + \beta^{\tau}_2 \text{slope}_t + \beta^{\tau}_3 \text{cp}_t + \epsilon_t,$$

where $\text{slope}_t$ is the slope at time $t$ and $\text{cp}_t$ is the Cochrane and Piazzesi factor at time $t$. t-Statistics presented in parentheses are calculated using Newey and West (1987). All variables are standardized to have mean zero and a standard deviation of one. Data is monthly and runs from 1997 through 2011.

| | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|
| duration | -0.025 | 0.061 | 0.135 | 0.200 | 0.257 | 0.306 | 0.348 | 0.385 | 0.416 |
| | (-0.33) | (0.76) | (1.71) | (2.59) | (3.43) | (4.23) | (4.96) | (5.64) | (6.26) |
| slope | -0.189 | -0.071 | 0.022 | 0.100 | 0.166 | 0.221 | 0.266 | 0.302 | 0.330 |
| | (-2.02) | (-0.78) | (0.24) | (1.15) | (1.94) | (2.63) | (3.20) | (3.66) | (4.03) |
| cp factor | 0.506 | 0.472 | 0.447 | 0.424 | 0.401 | 0.377 | 0.352 | 0.327 | 0.303 |
| | (6.66) | (6.43) | (6.09) | (5.73) | (5.35) | (4.96) | (4.57) | (4.20) | (3.85) |
| Adj. $R^2$ | 21.62% | 21.61% | 24.43% | 28.55% | 33.03% | 37.29% | 41.02% | 44.12% | 46.60% |

**Table 5**
**Bond volatility regressions with controls**

Panel A and B report estimated coefficients from regressing bond yield volatility, $\text{vol}_t^\tau$, and $\tau$-year maturity implied volatility of swaptions written on the 10-year swap rate, $\text{iv}_t^{\tau y10y}$, onto MBS dollar convexity and illiquidity:

$$\text{vol}_t^\tau/\text{iv}_t^{\tau y10y} = \beta_1^\tau \text{convexity}_t + \beta_2^\tau \text{illiq}_t + \beta_3^\tau \text{tiv}_t + \epsilon_t^\tau,$$

where $\tau = 1, \ldots, 10$, $\text{illiq}_t$ is the illiquidity factor and $\text{tiv}_t$ is an implied volatility index from Treasury options at time $t$. t-Statistics presented in parentheses are calculated using Newey and West (1987). All variables are standardized to have mean zero and a standard deviation of one. Data is weekly and runs from 1997 through 2011.

Panel A: Bond Yield Volatility

| | 1y | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|---|
| convexity | 0.201 | 0.185 | 0.175 | 0.165 | 0.153 | 0.140 | 0.127 | 0.116 | 0.106 | 0.098 |
| | (3.67) | (3.58) | (3.46) | (3.33) | (3.16) | (2.98) | (2.79) | (2.62) | (2.47) | (2.32) |
| illiq | -0.023 | -0.179 | -0.183 | -0.156 | -0.117 | -0.074 | -0.034 | -0.001 | 0.027 | 0.047 |
| | (-0.29) | (-2.04) | (-1.94) | (-1.58) | (-1.16) | (-0.73) | (-0.33) | (-0.01) | (0.24) | (0.42) |
| tiv | 0.669 | 0.824 | 0.844 | 0.841 | 0.829 | 0.812 | 0.793 | 0.773 | 0.755 | 0.739 |
| | (8.50) | (10.21) | (10.10) | (9.73) | (9.26) | (8.81) | (8.41) | (8.07) | (7.79) | (7.56) |
| Adj. $R^2$ | 49.77% | 57.21% | 59.23% | 60.46% | 61.59% | 62.49% | 63.00% | 63.13% | 62.94% | 62.50% |

Panel B: Swaption Implied Volatility

| | 1y | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|---|
| convexity | 0.197 | 0.213 | 0.202 | 0.197 | 0.186 | 0.183 | 0.172 | 0.187 | 0.162 | 0.155 |
| | (6.10) | (5.88) | (5.61) | (5.35) | (4.90) | (4.43) | (4.24) | (4.07) | (3.31) | (3.06) |
| illiq | -0.029 | -0.032 | 0.001 | 0.027 | 0.045 | 0.028 | 0.061 | 0.006 | 0.043 | 0.093 |
| | (-0.31) | (-0.30) | (0.01) | (0.23) | (0.37) | (0.22) | (0.47) | (0.04) | (0.31) | (0.65) |
| tiv | 0.895 | 0.871 | 0.842 | 0.815 | 0.794 | 0.801 | 0.779 | 0.807 | 0.785 | 0.743 |
| | (12.12) | (10.64) | (9.82) | (8.93) | (8.16) | (7.93) | (7.56) | (7.46) | (7.18) | (6.74) |
| Adj. $R^2$ | 87.30% | 83.95% | 82.69% | 81.18% | 79.38% | 77.87% | 77.94% | 76.05% | 75.75% | 75.24% |

**Table 6**
**Bond risk premia regressions: GSEs**

Panel A reports estimated coefficients from regressing annual bond excess returns constructed from Treasuries, $rx^{\tau}_{t+1}$, onto the size of the retained portfolio and notional amount of derivatives held by GSEs:

$$rx^{\tau}_{t+1} = \beta^{\tau}_1 \text{retained}_t/\text{derivatives}_t + \epsilon^{\tau}_{t+1},$$

where $\text{retained}_t$ is the size of the retained portfolio of Fannie Mae and Freddie Mac, and $\text{derivatives}_t$ is the notional amount of derivatives held. Panel B reports estimated coefficients from regressing annual bond returns onto an interaction term between MBS dollar duration and the retained portfolio size or derivatives. All variables are standardized to have mean zero and a standard deviation of one.

| | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Panel A: Retained portfolio and derivatives | | | | | |
| retained | 0.035 | 0.121 | 0.178 | 0.219 | 0.248 | 0.270 | 0.285 | 0.296 | 0.303 |
| | (0.44) | (1.50) | (2.23) | (2.75) | (3.14) | (3.42) | (3.62) | (3.76) | (3.86) |
| Adj. $R^2$ | 0.12% | 1.45% | 3.16% | 4.78% | 6.16% | 7.28% | 8.13% | 8.74% | 9.16% |
| derivatives | 0.314 | 0.374 | 0.403 | 0.413 | 0.412 | 0.402 | 0.388 | 0.372 | 0.354 |
| | (4.22) | (5.13) | (5.58) | (5.72) | (5.68) | (5.51) | (5.28) | (5.02) | (4.75) |
| Adj. $R^2$ | 9.89% | 14.00% | 16.25% | 17.08% | 16.93% | 16.17% | 15.06% | 13.82% | 12.56% |
| | | | | Panel B: Interaction term | | | | | |
| dur × retained | 0.066 | 0.171 | 0.252 | 0.316 | 0.367 | 0.408 | 0.440 | 0.466 | 0.486 |
| | (0.88) | (2.26) | (3.37) | (4.29) | (5.05) | (5.67) | (6.17) | (6.59) | (6.93) |
| Adj. $R^2$ | 0.44% | 2.93% | 6.35% | 10.00% | 13.50% | 16.66% | 19.38% | 21.67% | 23.58% |
| dur × derivatives | 0.345 | 0.427 | 0.478 | 0.508 | 0.524 | 0.529 | 0.528 | 0.522 | 0.514 |
| | (4.91) | (6.24) | (7.10) | (7.57) | (7.77) | (7.78) | (7.68) | (7.52) | (7.35) |
| Adj. $R^2$ | 11.91% | 18.25% | 22.87% | 25.83% | 27.44% | 28.01% | 27.85% | 27.26% | 26.43% |

## Table 7
## Regressions with interaction terms

This table reports estimated coefficients from regressing annual bond excess returns (Panel A) and bond yield volatility (Panel B) onto MBS dollar duration and dollar convexity interacted with the ratio between total mortgages outstanding and GDP. t-Statistics presented in parentheses are calculated using Newey and West (1987). All variables are standardized to have mean zero and a standard deviation of one. Data is quarterly and runs from 1997 through 2011.

**Panel A: Bond Excess Return Regression**

| | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|
| duration × ratio | -0.011 | 0.070 | 0.129 | 0.171 | 0.201 | 0.222 | 0.237 | 0.247 | 0.254 |
| | (-0.17) | (1.08) | (1.97) | (2.56) | (2.95) | (3.19) | (3.33) | (3.40) | (3.43) |
| Adj. $R^2$ | 0.01% | 0.49% | 1.67% | 2.93% | 4.03% | 4.93% | 5.61% | 6.10% | 6.43% |
| duration × ratio | 0.115 | 0.152 | 0.172 | 0.177 | 0.175 | 0.169 | 0.163 | 0.156 | 0.149 |
| | (1.56) | (2.03) | (2.24) | (2.27) | (2.21) | (2.10) | (1.99) | (1.88) | (1.78) |
| level | 0.234 | 0.152 | 0.079 | 0.011 | -0.048 | -0.098 | -0.139 | -0.170 | -0.195 |
| | (3.13) | (2.07) | (1.06) | (0.15) | (-0.63) | (-1.27) | (-1.77) | (-2.16) | (-2.44) |
| Adj. $R^2$ | 3.35% | 1.56% | 1.52% | 2.35% | 3.62% | 5.04% | 6.42% | 7.62% | 8.58% |

**Panel B: Bond Yield Volatility Regression**

| | 1y | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|---|
| convexity × ratio | 0.572 | 0.591 | 0.608 | 0.616 | 0.621 | 0.625 | 0.628 | 0.630 | 0.632 | 0.634 |
| | (5.89) | (6.91) | (7.60) | (7.70) | (7.38) | (6.93) | (6.53) | (6.23) | (6.02) | (5.89) |
| Adj. $R^2$ | 32.70% | 34.92% | 36.91% | 37.93% | 38.61% | 39.07% | 39.40% | 39.67% | 39.95% | 40.25% |
| convexity × ratio | 0.239 | 0.240 | 0.250 | 0.256 | 0.262 | 0.266 | 0.270 | 0.274 | 0.277 | 0.281 |
| | (2.99) | (3.69) | (4.26) | (4.37) | (4.28) | (4.13) | (4.00) | (3.90) | (3.85) | (3.84) |
| illiq | -0.126 | -0.162 | -0.126 | -0.114 | -0.117 | -0.125 | -0.133 | -0.139 | -0.144 | -0.146 |
| | (-1.29) | (-1.69) | (-1.39) | (-1.24) | (-1.22) | (-1.26) | (-1.30) | (-1.34) | (-1.38) | (-1.40) |
| tiv | 0.710 | 0.764 | 0.757 | 0.755 | 0.756 | 0.759 | 0.761 | 0.762 | 0.762 | 0.760 |
| | (6.67) | (6.50) | (6.85) | (7.15) | (7.30) | (7.33) | (7.30) | (7.27) | (7.24) | (7.22) |
| Adj. $R^2$ | 61.31% | 66.89% | 70.04% | 71.53% | 72.22% | 72.53% | 72.63% | 72.66% | 72.65% | 72.65% |

**Table 8**
**Unconditional correlations duration measures**

This table reports unconditional correlations between MBS, Treasury, Corporate AAA and Corporate BBB dollar duration. Data is weekly and runs from 1997 through 2011.

| | MBS | Treasury | AAA | BBB |
|---|---|---|---|---|
| MBS | 100.00% | -38.25% | -7.14% | 13.78% |
| Treasury | -38.25% | 100.00% | -19.47% | -30.67% |
| AAA | -7.14% | -19.47% | 100.00% | 86.00% |
| BBB | 13.78% | -30.67% | 86.00% | 100.00% |

**Table 9**
**Bond risk premia regressions: Other durations**

This table reports estimated coefficients from regressing annual bond excess returns constructed from Treasuries, $rx^{\tau}_{t+1}$, onto different measures of duration:

$$rx^{\tau}_{t+1} = \beta^{\tau}_1 \text{MBS duration}_t + \beta^{\tau}_2 \text{Treasury duration}_t + \beta^{\tau}_3 \text{AAA duration}_t + \beta^{\tau}_4 \text{BBB duration}_t + \epsilon^{\tau}_{t+1},$$

where MBS duration$_t$ is MBS dollar duration, Treasury duration$_t$ is Treasury dollar duration, AAA duration$_t$ and BBB duration$_t$ are corporate bond dollar duration measures for bonds rated AAA and BBB, respectively. t-Statistics presented in parentheses are calculated using Newey and West (1987). All variables are standardized to have mean zero and a standard deviation of one. Data is weekly and runs from 1997 through 2011.

| | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|
| MBS dur | 0.133 | 0.189 | 0.245 | 0.298 | 0.346 | 0.389 | 0.426 | 0.456 | 0.481 |
| | (1.39) | (1.97) | (2.57) | (3.20) | (3.79) | (4.33) | (4.80) | (5.20) | (5.54) |
| Treasury dur | -0.029 | -0.084 | -0.117 | -0.131 | -0.135 | -0.134 | -0.131 | -0.127 | -0.123 |
| | (-0.28) | (-0.79) | (-1.10) | (-1.26) | (-1.34) | (-1.38) | (-1.39) | (-1.40) | (-1.39) |
| AAA dur | 0.052 | 0.145 | 0.221 | 0.282 | 0.331 | 0.369 | 0.396 | 0.413 | 0.422 |
| | (0.36) | (0.98) | (1.50) | (1.92) | (2.27) | (2.55) | (2.77) | (2.94) | (3.05) |
| BBB dur | -0.423 | -0.433 | -0.438 | -0.438 | -0.436 | -0.432 | -0.426 | -0.417 | -0.406 |
| | (-2.32) | (-2.38) | (-2.43) | (-2.48) | (-2.54) | (-2.59) | (-2.64) | (-2.67) | (-2.68) |
| Adj. $R^2$ | 13.88% | 10.85% | 10.77% | 12.27% | 14.56% | 17.17% | 19.79% | 22.24% | 24.43% |

**Table 10**
**Bond risk premia regressions: swaps**

This table reports estimated coefficients from regressing annual bond excess returns constructed from interest rate swaps, $rxsw^{\tau}_{t+1}$, onto a set of variables:

$$rxsw^{\tau}_{t+1} = \beta^{\tau}_1 \text{duration}_t + \beta^{\tau}_2 \text{level}_t + \epsilon^{\tau}_{t+1},$$

where $\text{duration}_t$ is MBS dollar duration and $\text{level}_t$ is the first principal component from bond yields. t-Statistics presented in parentheses are calculated using Newey and West (1987). All variables are standardized to have mean zero and a standard deviation of one. Data is weekly and runs from 1997 through 2011.

| | 2y | 3y | 4y | 5y | 6y | 7y | 8y | 9y | 10y |
|---|---|---|---|---|---|---|---|---|---|
| duration | 0.053 | 0.125 | 0.202 | 0.266 | 0.321 | 0.366 | 0.402 | 0.432 | 0.459 |
| | (0.65) | (1.55) | (2.55) | (3.49) | (4.33) | (5.06) | (5.68) | (6.23) | (6.72) |
| Adj. $R^2$ | 0.28% | 1.57% | 4.06% | 7.07% | 10.28% | 13.38% | 16.12% | 18.64% | 21.03% |
| duration | 0.018 | 0.126 | 0.223 | 0.302 | 0.366 | 0.415 | 0.453 | 0.485 | 0.511 |
| | (0.21) | (1.50) | (2.77) | (3.98) | (5.07) | (6.01) | (6.78) | (7.45) | (8.01) |
| level | 0.149 | -0.002 | -0.093 | -0.154 | -0.191 | -0.210 | -0.220 | -0.224 | -0.223 |
| | (1.58) | (-0.03) | (-1.07) | (-1.82) | (-2.29) | (-2.56) | (-2.71) | (-2.79) | (-2.79) |
| Adj. $R^2$ | 2.24% | 1.44% | 4.76% | 9.20% | 13.60% | 17.42% | 20.58% | 23.28% | 25.63% |

**Table 11**
**Bond portfolio regressions**

This table reports estimated coefficients from regressing bond portfolio excess returns onto duration and level.

$$rxpf^{\tau}_{t+1} = \beta^{\tau}_1 \text{duration}_t + \beta^{\tau}_2 \text{level}_t + \epsilon^{\tau}_{t+1},$$

where $\text{duration}_t$ is MBS dollar duration, $\text{level}_t$ is the first principal component from bond yields and $rxpf^{\tau}_{t+1}$ are monthly excess returns on the CRSP bond portfolios with maturities between 5 and 10 years and larger than 10 years. Returns are in excess of either the 1-month T-bill or Eurodollar deposit rate. All variables are standardized to have mean zero and a standard deviation of one. Data is monthly and runs from 1990 through 2011.

| | T-bill | | ED rate | |
|---|---|---|---|---|
| | $\geq$ 5y $<$ 10y | $>$ 10y | $\geq$ 5y $<$ 10y | $\geq$ 5y $<$ 10y |
| duration | 0.215 | 0.263 | 0.216 | 0.264 |
| | (3.63) | (3.88) | (3.57) | (3.84) |
| Adj. $R^2$ | 4.62% | 6.93% | 4.66% | 6.95% |
| duration | 0.235 | 0.286 | 0.238 | 0.288 |
| | (3.28) | (3.44) | (3.33) | (3.46) |
| level | -0.042 | -0.050 | -0.047 | -0.053 |
| | (-0.50) | (-0.59) | (-0.58) | (-0.64) |
| Adj. $R^2$ | 4.36% | 6.74% | 4.43% | 6.78% |

**Table 12**
**Calibrated parameters**

This table reports parameters used for the calibration exercise. The mean reversion and the volatility of the short rate process are estimated directly from the short rate series between 1997 and 2011. The sensitivity of mortgage refinancing to the incentive to refinance and the negative dollar convexity are set to match the aggregate MBS duration dynamics. The absolute risk aversion of financial institutions is chosen to match the predictive $R^2$ of the duration factor on the 10-year bond excess returns. The long-run means of the short rate ($\theta$) and duration ($\theta_D$) are not presented here because they have no bearing on the effect that duration and convexity have on excess returns and volatilities. Data is monthly.

| | | |
|---|---|---|
| $\kappa$ | Short rate mean reversion | 0.13 |
| $\sigma$ | Short rate volatility | 1.33% |
| $\kappa_D$ | Sensitivity of refinancing to the incentive | 1.13 |
| $\eta_y$ | Negative dollar convexity | 1.04 |
| $\alpha$ | Absolute risk aversion | 80 |

## Appendix B Figures

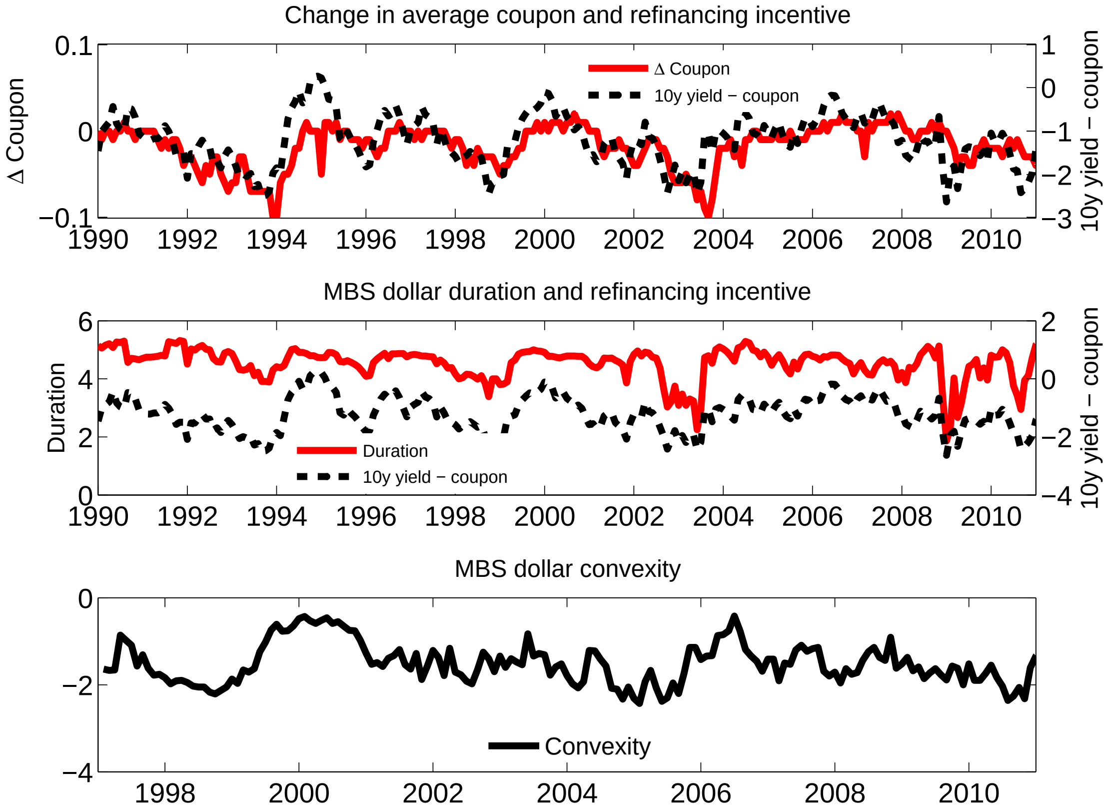


**Figure 1. Average coupon, dollar duration, and dollar convexity**

The upper panel plots the difference between the 10-year yield and the average MBS coupon together with the subsequent change in the average MBS coupon. The middle panel plots the difference between the 10-year yield and the average MBS coupon together with MBS dollar duration. The lower panel plots MBS dollar convexity. For MBS dollar duration and dollar convexity the average value of the MBS index is normalized to one dollar. Data is monthly and runs from January 1990 to December 2011 (upper and middle panels) and from January 1997 to December 2011 (lower panel), respectively.

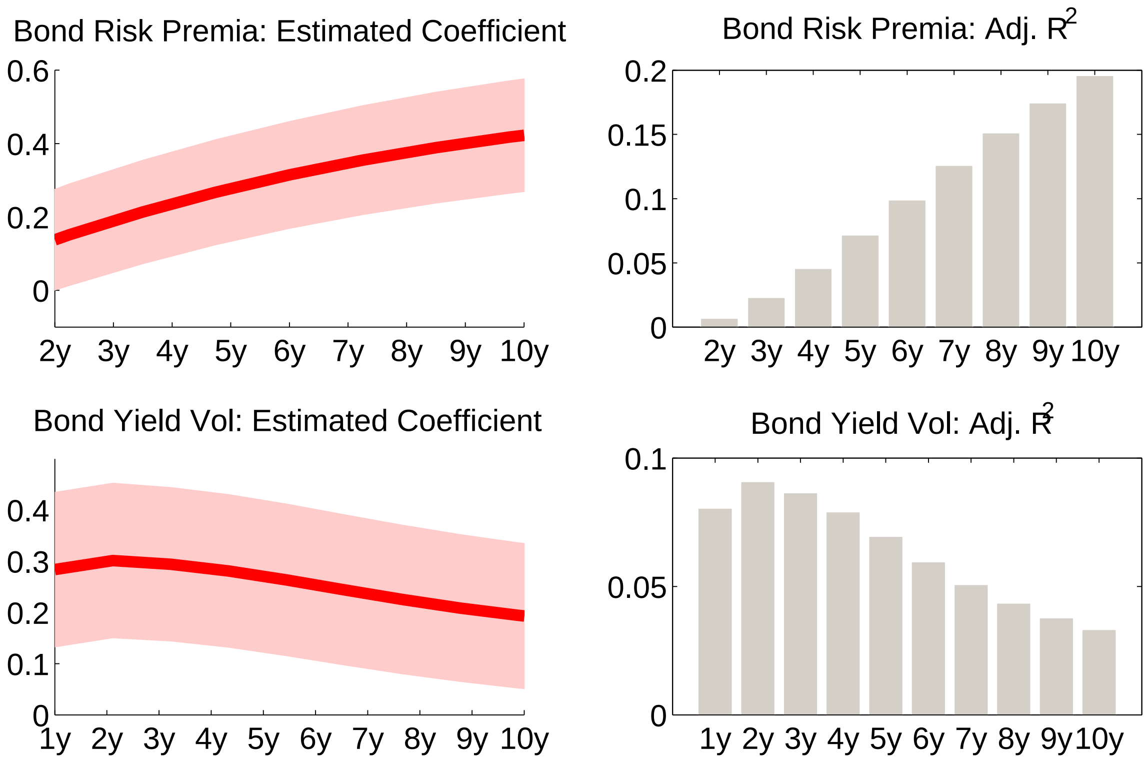


**Figure 2. Univariate regression coefficients**

The figure plots estimated coefficients and adjusted $R^2$ from univariate regressions of bond excess returns (upper panels) and bond yield volatilities (lower panels) onto MBS dollar duration (bond excess returns) and MBS dollar convexity (bond yield volatilities), respectively. All variables are standardized, i.e., they have a mean of zero and a standard deviation of one. Data is weekly and runs from 1997 through 2011. Shaded areas represent confidence levels on the 95% level.

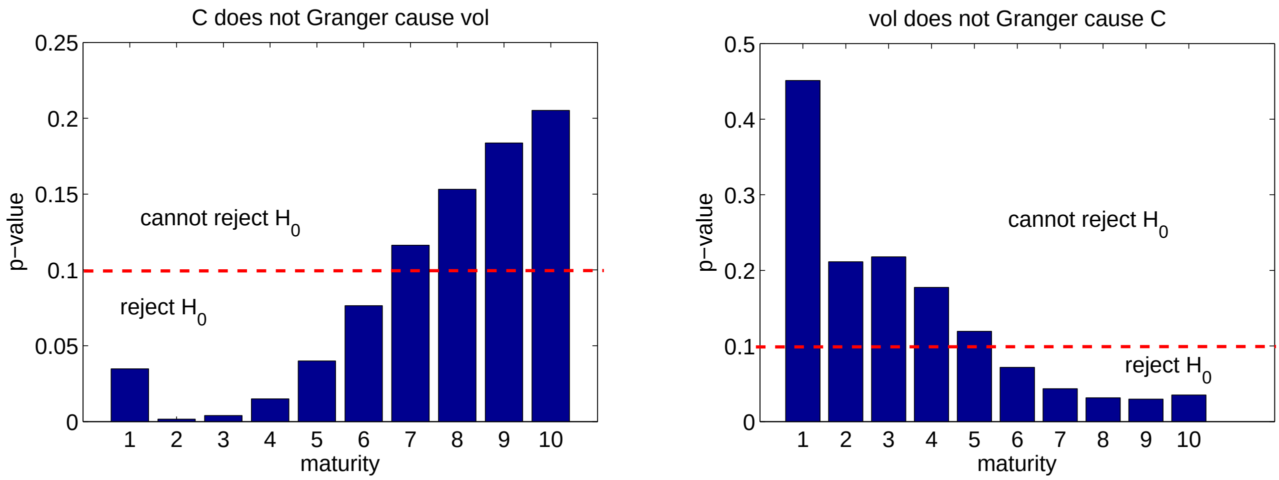


**Figure 3. p-Values of Granger causality tests**

These figures present results for Granger causality tests. The null hypothesis for the left (right) panel is that negative convexity (volatility) does not Granger cause bond yield volatility (convexity). The regressions are estimated on weekly data from 1997 through 2011.

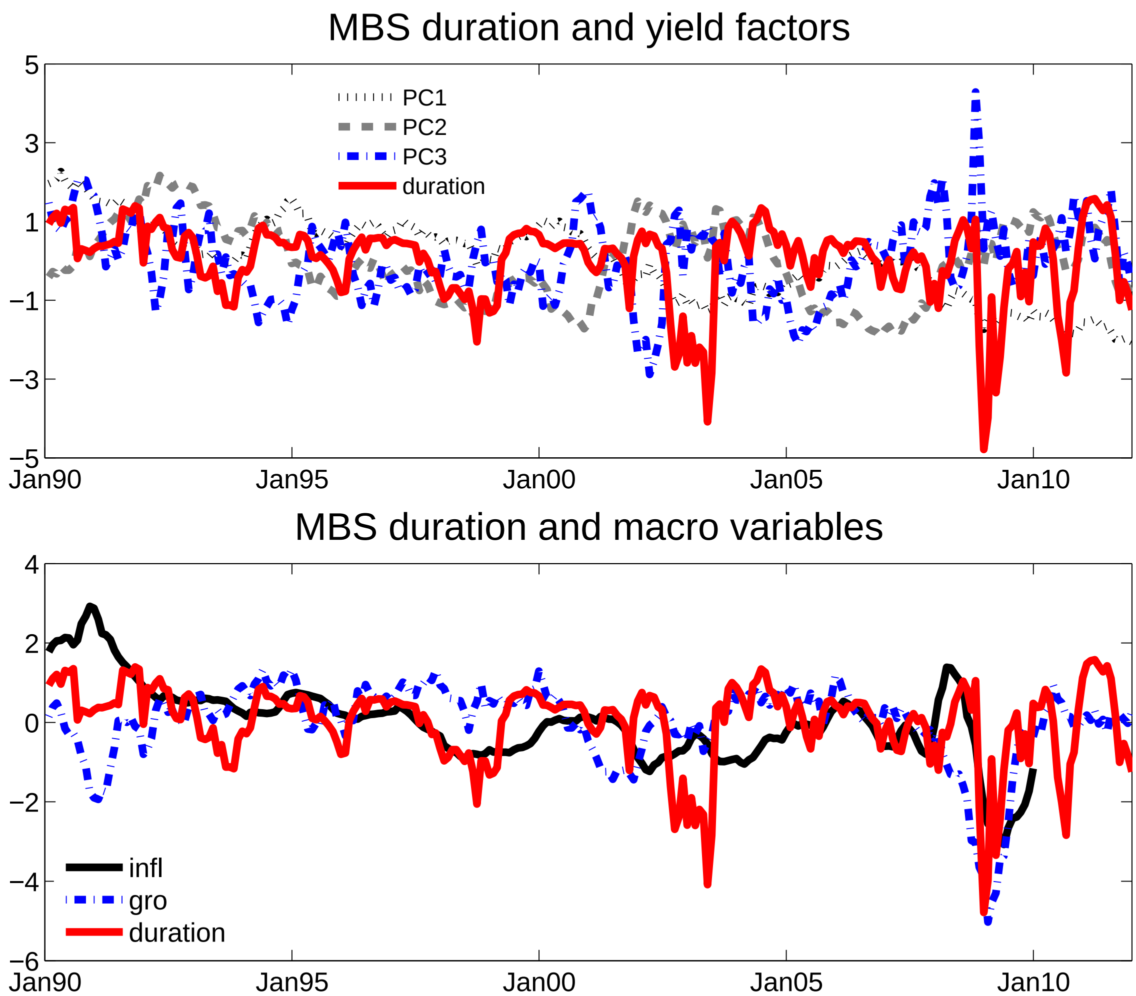


**Figure 4. MBS duration and other factors**

The figure plots MBS dollar duration together with the first three principal components estimated from yields (upper panel) and expected inflation and economic growth (lower panel). Economic growth is measured by the three-month moving average of the Chicago Fed National Activity Index and expected inflation is the consensus estimate computed from surveys of professional forecasters by Blue Chip Financial Forecasts. All variables have been standardized to have a mean of zero and a standard deviation of one. The data is monthly and runs from 1990 through 2011.

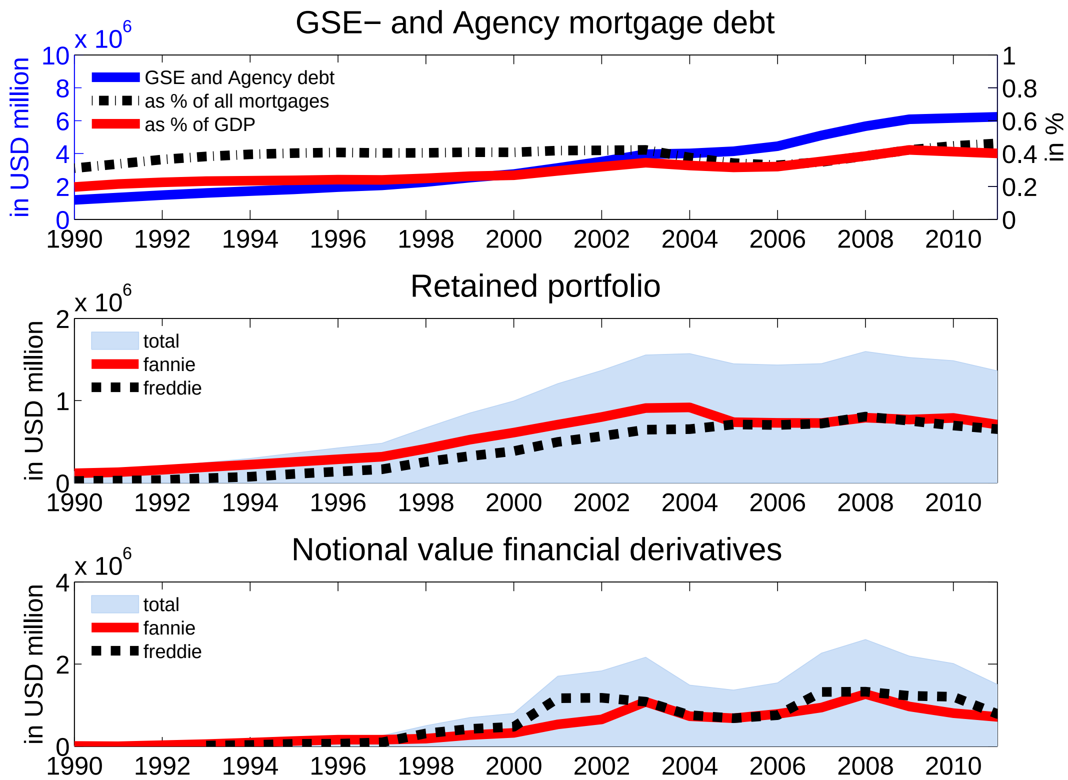


**Figure 5. GSEs size of retained portfolio and derivative holdings**

The upper panel plots GSE- and agency mortgage debt in USD millions together as a percentage of all mortgages and as a percentage of GDP. Source is the Flow of Funds Table of the Federal Reserve (Table L217). The middle panel plots the size of the retained portfolio of Fannie Mae and Freddie Mac in USD millions. The lower panel plots the notional value of derivatives held by the two GSEs in USD millions. Data is annual from 1990 to 2011. Source is the Federal Housing Finance Agency Annual Report to Congress.

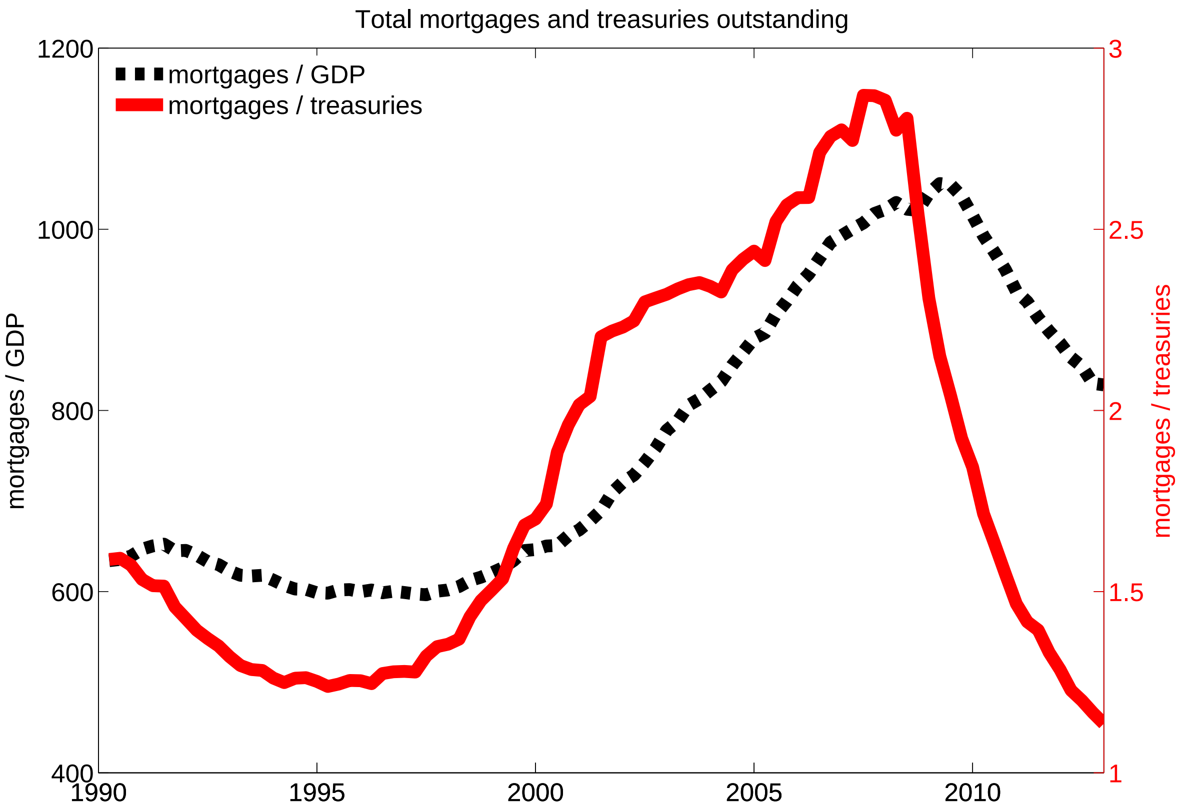


**Figure 6. Total mortgages and treasuries outstanding**

This figure plots total nominal value outstanding of all mortgages divided by GDP (left axis) and the total amount outstanding of mortgages divided by amount outstanding in Treasuries (right axis). Data is from the webpage of the Board of Governors of the Federal Reserve and its frequency is quarterly from 1990 through 2011.

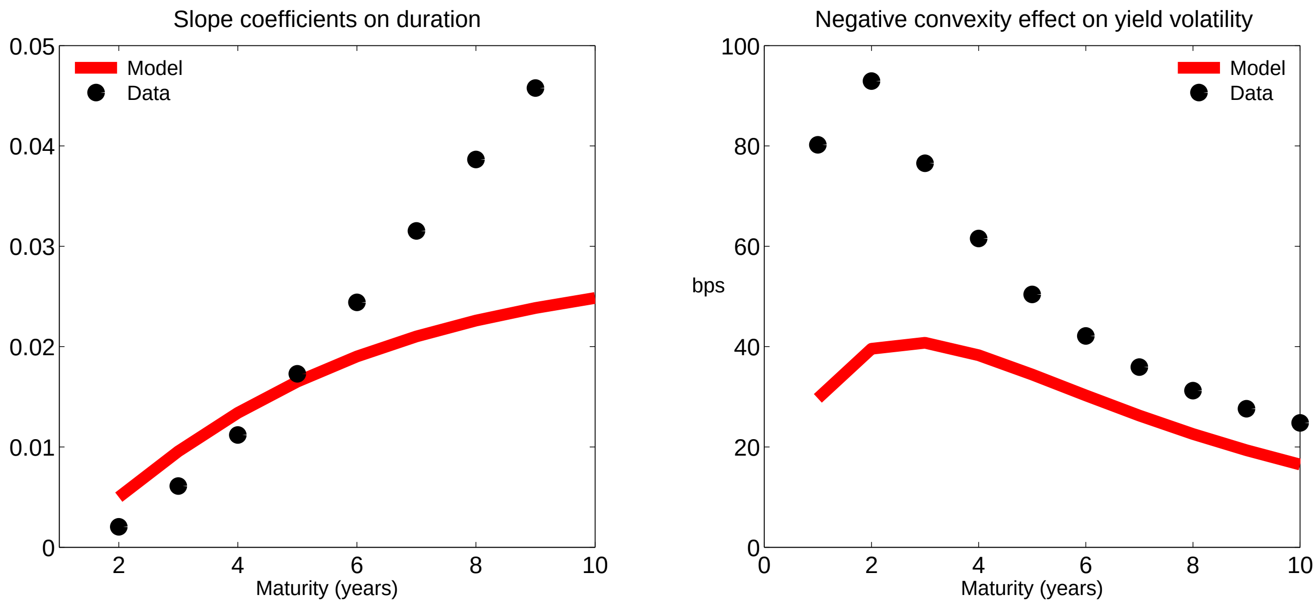


**Figure 7. Calibration results**

The left panel plots the theoretical slope coefficient of the regression of excess bond returns on the duration factor together with the estimated (non-standardized) values. The right panel plots the increase in yield volatility that can be attributed to negative convexity. In the model this effect is calculated as the difference between yield volatility in the benchmark calibration and an otherwise identical calibration where $\alpha$ is set to 0 and thus the negative convexity channel is shut down. Its empirical counterpart is based on our linear regression results. The data is between 1997 and 2011.

## Appendix C Online Appendix to "Mortgage Risk and the Yield Curve"

This online appendix contains all the proofs omitted in the main part of the paper. Section C.1 provides some preliminary results needed for the proofs given in Section C.2. Section C.3 outlines a model with time-varying convexity.

### *Appendix C.1 Preliminary results*

#### *Appendix C.1.1 Properties of useful functions*

This appendix introduces five functions necessary to derive our main results and studies their properties.

**Lemma 2.** *For any $\tau > 0$, the function $g_1(x) \equiv \frac{1-e^{-x\tau}}{x\tau}$ for all $x \neq 0$ and $g_1(0) \equiv 1$ is positive, decreasing, and convex for all $x \in \mathbb{R}$. Moreover, for arbitrary $x, y \in \mathbb{R}$, $x \neq y$,*

$$\frac{1-e^{-x\tau}}{x\tau} - \frac{1-e^{-x\tau}-x\tau e^{-x\tau}}{x^2\tau}(y-x) < \frac{1-e^{-y\tau}}{y\tau}. \tag{A-1}$$

*Proof.* The derivative of $g_1$ is given by

$$g_1'(x) = -\frac{1-e^{-x\tau}-x\tau e^{-x\tau}}{x^2\tau}, \tag{A-2}$$

which has the opposite sign as $F(x) \equiv 1-e^{-x\tau}-x\tau e^{-x\tau}$. But $\lim_{x\to 0} F(x) = 0$, and $F'(x) = x\tau^2 e^{-x\tau}$, which is negative for $x < 0$ and positive for $x > 0$. Hence, $F(x) \geq 0$ for all $x \in \mathbb{R}$, and $g_1'(x) \leq 0$ for all $x \in \mathbb{R}$; $g_1$ is a decreasing function. However, the limit of $g_1$ when $x \to \infty$ is $\lim_{x\to\infty} g_1(x) = 0$. Since $g_1$ is a decreasing function and it converges to zero when $x \to \infty$, it must be that $g_1(x) > 0$ for all $x \in \mathbb{R}$.

Regarding convexity, (A-2) implies

$$g_1''(x) = \frac{2e^{-x\tau}}{x^3\tau}\left[e^{x\tau} - \left(1 + x\tau + \frac{x^2\tau^2}{2}\right)\right],$$

but $1 + z + \frac{z^2}{2}$ are the first three terms of the power series of $e^z$, and it is well-known that $1 + z + \frac{z^2}{2} < e^z$ for $z > 0$ and $1 + z + \frac{x^2}{2} > e^z$ for $z < 0$. Therefore, $g_1''(x) > 0$ and thus $g_1$ is convex.

Finally, convexity of $g_1$ is equivalent to the function lying above all of its tangents. From (A-2), (A-1) is describing exactly this inequality for the point of tangency $x$ and an arbitrary $y$: $g_1(x) + g_1'(x)(y-x) < g_1(y)$. □

**Lemma 3.** *For any $h > 0$ and $\kappa \in \mathbb{R}$, the function*

$$g_2(x) \equiv \frac{e^{-\kappa h} - e^{-xh}}{\kappa - x} \tag{A-3}$$

*is negative and increasing for all $x \in \mathbb{R}$.*

*Proof.* We have $e^{-\kappa h} > e^{-xh}$ iff $\kappa < x$, which implies negativity. Also, differentiation gives

$$g_2'(x) = \frac{e^{-(\kappa-x)h} - [1-(\kappa-x)h]}{(\kappa-x)^2} e^{-xh}.$$

but since $e^z > 1+z$ for all $z \in \mathbb{R}$, we have $e^{-(\kappa-x)h} \geq 1-(\kappa-x)h$ and thus $g_2'(x) \geq 0$ for all $x$, which concludes the proof. □

**Lemma 4.** *Suppose $\kappa, h > 0$ constants that satisfy $\kappa h < 1$. The function*

$$g_3(x) \equiv \frac{\kappa e^{-\kappa h} - x e^{-xh}}{\kappa - x} \tag{A-4}$$

*is positive and decreasing for all $x < \frac{1}{h}$.*

*Proof.* It is easy to confirm that the function $G(x) = xe^{-xh}$ satisfies (i) $G(0) = 0$ and $G(x) > 0$ iff $x > 0$, and (ii) $G'(x) \geq 0$ for $x \leq \frac{1}{h}$ and $G'(x) < 0$ otherwise. Therefore, as long as $\kappa, x < \frac{1}{h}$, the numerator and the denominator of (A-4) have the same sign and thus $g_3(x) > 0$. Next we differentiate (A-4) to obtain

$$\frac{dg_3}{dx} = \frac{\left(\kappa e^{-\kappa h} - x e^{-xh}\right) - (1-xh)(\kappa-x) e^{-xh}}{(\kappa-x)^2}. \tag{A-5}$$

Denoting the numerator of (A-5) by $H(x)$ and differentiating, we obtain

$$H'(x) = (\kappa - x) h (2 - xh) e^{-xh}$$

while $H(\kappa) = 0$. Hence, $H(x)$ increases for $x < \kappa$ where it reaches zero, and afterwards it decreases. That is, the numerator of (A-5) is negative for all $x < \frac{1}{h}$ and so is $\frac{dg_3}{dx}$; $g_3$ is decreasing. □

**Lemma 5.** *For any $\tau > 0$, the function*

$$g_4(x,y) \equiv \frac{g_1(x) - g_1(y)}{x-y} = \frac{\frac{1-e^{-x\tau}}{x\tau} - \frac{1-e^{-y\tau}}{y\tau}}{x-y}, \tag{A-6}$$

*$x, y \in \mathbb{R}$, is symmetric, negative, and increasing in both arguments. Moreover, if $x < x' < y' < y$ while $x + y = x' + y'$, $g_4(x,y) < g_4(x',y')$.*

*Proof.* Lemma 2 implies that the numerator of $g_4$ is positive if and only if $x < y$, hence $g_4(x,y) < 0$ for all $x, y \in \mathbb{R}$. Symmetry, i.e. $g_4(x,y) = g_4(y,x)$, is obvious, and for $g_4$ being increasing, it means we only need to show that $\frac{\partial g_4}{\partial x} > 0$ for a fixed $y$. Differentiating (A-6) w.r.t. $x$, we have

$$\frac{\partial g_4}{\partial x} = -\frac{1}{(x-y)^2}\left[\frac{1-e^{-x\tau}}{x\tau} - \frac{1-e^{-x\tau} - x\tau e^{-x\tau}}{x^2\tau}(y-x) - \frac{1-e^{-y\tau}}{y\tau}\right].$$

Lemma 2 also implies that the term inside the bracket is negative, and hence $g_4$ is increasing in $x$ and $y$. Moreover, for an arbitrary constant $y$ we have $\lim_{x\to\infty} g_4(x,y) = 0$, so if $g_4$ is increasing in $x$, it must be that for all $x, y \in \mathbb{R}$, $g_4(x,y) < 0$.

Notice that the last claim of the Lemma is equivalent to showing that fixing $z \equiv \frac{x+y}{2}$, $g_4(x,y) = g_4(x, 2z-x)$ is increasing in $x$ whenever $x < z$. Differentiating with respect to $x$ we obtain

$$\begin{aligned}
\frac{dg_4(x,2z-x)}{dx} &= \frac{d}{dx}\frac{g_1(x) - g_1(2z-x)}{x-(2z-x)} = \frac{[g_1'(x) + g_1'(2z-x)](x-z) - [g_1(x) - g_1(2z-x)]}{2(x-z)^2} \\
&= \frac{2}{y-x}\left[\frac{g_1(y) - g_1(x)}{y-x} - \frac{g_1'(x) + g_1'(y)}{2}\right],
\end{aligned}$$

where in the last equality we substituted $y$ back. Therefore, $g_4(x, 2z-x)$ is increasing in $x$ if and only if

$$\frac{g_1(y) - g_1(x)}{y-x} - \frac{g_1'(x) + g_1'(y)}{2} > 0 \tag{A-7}$$

for all $x < y$.

Since $g_1$ is twice differentiable everywhere (see Lemma 2), we can write

$$\begin{aligned}
g_1(y) - g_1(x) &= \int_x^y g_1'(t)\,dt = \int_x^y \left[g_1'(x) + \int_x^t g_1''(w)\,dw\right] dt \qquad \text{(A-8)} \\
&= g_1'(x)(y-x) + \int_x^y \int_x^t g_1''(w)\,dwdt \\
&= g_1'(x)(y-x) + \int_x^y \int_w^y g_1''(w)\,dtdw \\
&= g_1'(x)(y-x) + \int_x^y g_1''(w)(y-w)\,dw,
\end{aligned}$$

where the fourth equality is an application of Fubini's theorem. From Lemma 2 we know $g_1''(x) > 0$ for all $x \in \mathbb{R}$; moreover, the third derivative of $g_1$ is simply

$$g_1'''(x) = \frac{dg_1''(x)}{dx} = -\frac{6e^{-x\tau}}{x^4\tau}\left[e^{x\tau} - \left(1 + x\tau + \frac{x^2\tau^2}{2} + \frac{x^3\tau^3}{6}\right)\right] < 0 \tag{A-9}$$

for all $x \in \mathbb{R}$, thus $g_1''(x)$ is a positive decreasing function. Therefore, we can write

$$\int_x^y g_1''(w)(y-w)\,dw > \int_x^y g_1''(x)(y-w)\,dw = g_1''(x)\int_x^y (y-w)\,dw = g_1''(x)\frac{(y-x)^2}{2}. \tag{A-10}$$

Combining (A-8) and (A-10), we obtain

$$\begin{aligned}
\frac{g_1(y)-g_1(x)}{y-x}-\frac{g_1'(x)+g_1'(y)}{2} &= \frac{1}{y-x}\left[g_1'(x)(y-x)+\int\limits_x^y g_1''(w)(y-w)\,dw\right]-\frac{g_1'(x)+g_1'(y)}{2}\\
&> \frac{1}{y-x}\left[g_1'(x)(y-x)+g_1''(x)\frac{(y-x)^2}{2}\right]-\frac{g_1'(x)+g_1'(y)}{2}\\
&= \frac{1}{2}\left[g_1'(x)+g_1''(x)(y-x)-g_1'(y)\right].
\end{aligned}$$

But (A-9) also means that $g_1'$ is a concave function, i.e. it lies below all its tangents. Therefore, $g_1'(x)+g_1''(x)(y-x)-g_1'(y)>0$, which implies (A-7) and concludes the proof of the lemma. □

**Lemma 6.** *Fix $\tau>0$. The function*

$$g_5(x,y)=\frac{1-e^{-x\tau}}{x}+\frac{x(y-x)}{y}\left(-\frac{1-e^{-x\tau}-x\tau e^{-x\tau}}{x^2}\right)-\frac{1-e^{-y\tau}}{y}, \qquad \text{(A-11)}$$

*$x,y\in\mathbb{R}^+$, satisfies $g_5(x,y)=0$ if $x=y$ and $g_5(x,y)>0$ whenever $x\neq y$.*

*Proof.* If $x=y$, the first and last terms of $g_5$ are equivalent and the middle one is zero, hence $g_5(x,y)=0$. Next we differentiate $g_5$ with respect to $y$ while keeping $x$ fixed to obtain

$$\frac{dg_5(x,y)}{dy}=\frac{\left(1-e^{-y\tau}-y\tau e^{-y\tau}\right)-\left(1-e^{-x\tau}-x\tau e^{-x\tau}\right)}{y^2}=\frac{F(y)-F(x)}{y^2},$$

where $F$ is defined in the proof of Lemma 2. As shown there, $F$ is increasing on $\mathbb{R}^+$, so $0<x<y$ implies the numerator is positive and thus $\frac{dg_5(x,y)}{dy}>0$. On the other hand, $0<y<x$ implies the numerator is negative and $\frac{dg_5(x,y)}{dy}<0$. Therefore, $g_5$ is decreasing in $y$ before $x$, reaches zero, then increasing, i.e., is positive for all $y\neq x$. □

### *Appendix C.1.2 Covariance matrix of $(r_t,D_t)^\top$*

In this appendix we derive the variance-covariance matrix of $(r_t,D_t)^\top$ under $\mathbb{P}$ from (1) and (11). Following the standard technique, applying Ito's lemma to $e^{\kappa t}r_t$ and combining it with (1), we obtain

$$d\left(e^{\kappa t}r_t\right)=\left[\kappa e^{\kappa t}r_t+e^{\kappa t}\kappa(\theta-r_t)\right]dt+e^{\kappa t}\sigma dB_t=e^{\kappa t}\kappa\theta dt+e^{\kappa t}\sigma dB_t.$$

Integrating both sides between zero and $t$ and rearranging gives

$$r_t=\theta\left(1-e^{-\kappa t}\right)+r_0e^{-\kappa t}+\sigma e^{-\kappa t}\int\limits_0^t e^{\kappa s}dB_s. \qquad \text{(A-12)}$$

As $r_t$ shows up in the drift of $D_t$ under $P$ whenever $\delta_r\neq 0$, we cannot simply use the same calculation for $D_t$. Instead, let us define

$$\bar{D}_t=D_t+\frac{\delta_r}{\delta_D-\kappa}r_t; \qquad \text{(A-13)}$$

we then have

$$d\bar{D}_t = dD_t + \frac{\delta_r}{\delta_D - \kappa} dr_t = \delta_D \left(\bar{\theta}_D - \bar{D}_t\right) dt + \bar{\sigma}_D dB_t,$$

where in the last step we use (15) and introduce the notation

$$\bar{\theta}_D = \theta_D + \frac{\delta_r}{\delta_D - \kappa}\theta \text{ and } \bar{\sigma}_D = \eta_y \sigma_y^{\bar{\tau}} + \frac{\delta_r}{\delta_D - \kappa}\sigma = \delta_D \frac{\eta_y \sigma_y^{\bar{\tau}}}{\delta_D - \kappa}.$$

That is, $\bar{D}_t$ is a Vasicek process with a speed of mean reversion $\delta_D$. Applying the same steps as for $r_t$, we also obtain

$$\bar{D}_t = \bar{\theta}_D \left(1 - e^{-\delta_D t}\right) + \bar{D}_0 e^{-\delta_D t} + \bar{\sigma}_D e^{-\delta_D t} \int_0^t e^{\delta_D s} dB_s, \tag{A-14}$$

where, importantly, the Brownian increments $dB_s$ are the same as in (A-12).

From (A-12) and (A-14) we can compute the conditional means and variances of $r$ and $D$ and the conditional covariance between them $t$ time ahead. First, as the expectation of the increments of the Brownian motion is zero, we have

$$E_0 [r_t] = \theta \left(1 - e^{-\kappa t}\right) + r_0 e^{-\kappa t} \text{ and } E_0 \left[\bar{D}_t\right] = \bar{\theta}_D \left(1 - e^{-\delta_D t}\right) + \bar{D}_0 e^{-\delta_D t}; \tag{A-15}$$

this, together with (A-14), also yields

$$\begin{aligned} E_0 [D_t] &= E_0 \left[\bar{D}_t\right] - \frac{\delta_r}{\delta_D - \kappa} E_0 [r_t] \qquad\qquad (A\text{-}16) \\ &= \bar{\theta}_D \left(1 - e^{-\delta_D t}\right) + D_0 e^{-\delta_D t} - \frac{\delta_r}{\delta_D - \kappa} \left[\theta \left(1 - e^{-\kappa t}\right) + r_0 \left(e^{-\kappa t} - e^{-\delta_D t}\right)\right]. \end{aligned}$$

Next, from (A-12), we have

$$Var_0 [r_t] = \sigma^2 e^{-2\kappa t} E_0 \left[\left(\int_0^t e^{\kappa s} dB_s\right)^2\right] = \sigma^2 e^{-2\kappa t} \int_0^t e^{2\kappa s} ds = \frac{\sigma^2}{2\kappa} \left(1 - e^{-2\kappa t}\right), \tag{A-17}$$

where we use that $dB_s \sim N(0, ds)$ i.i.d. over time. Similarly, from (A-14) we obtain

$$Var_0 \left[\bar{D}_t\right] = \bar{\sigma}_D^2 e^{-2\delta_D t} E_0 \left[\left(\int_0^t e^{\delta_D s} dB_s\right)^2\right] = \bar{\sigma}_D^2 e^{-2\delta_D t} \int_0^t e^{2\delta_D s} ds = \frac{\bar{\sigma}_D^2}{2\delta_D} \left(1 - e^{-2\delta_D t}\right). \tag{A-18}$$

Finally, for the covariance, we have

$$\begin{aligned} Cov_0 \left[r_t, \bar{D}_t\right] &= \sigma \bar{\sigma}_D e^{-(\kappa + \delta_D)t} E_0 \left[\int_0^t e^{\kappa s} dB_s \int_0^t e^{\delta_D s} dB_s\right] \qquad (A\text{-}19) \\ &= \sigma \bar{\sigma}_D e^{-(\kappa + \delta_D)t} \int_0^t e^{(\kappa + \delta_D)s} ds = \frac{\sigma \bar{\sigma}_D}{\kappa + \delta_D} \left(1 - e^{-(\kappa + \delta_D)t}\right). \end{aligned}$$

From (A-13), (A-17), and (A-19) we then have

$$\begin{aligned} Cov_0\left[r_t, D_t\right] &= Cov_0\left[r_t, \bar{D}_t\right] - \frac{\delta_r}{\delta_D - \kappa} Var_0\left[r_t\right] \\ &= \frac{\sigma \bar{\sigma}_D}{\kappa + \delta_D}\left(1 - e^{-(\kappa+\delta_D)t}\right) - \frac{\delta_r}{\delta_D - \kappa}\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right), \end{aligned} \tag{A-20}$$

and from (A-13), (A-17), (A-18), and (A-20) we get

$$Var_0\left[D_t\right] = Var_0\left[\bar{D}_t\right] - 2\frac{\delta_r}{\delta_D - \kappa} Cov_0\left[r_t, D_t\right] - \left(\frac{\delta_r}{\delta_D - \kappa}\right)^2 Var_0\left[r_t\right] \tag{A-21}$$

$$= \frac{\bar{\sigma}_D^2}{2\delta_D}\left(1 - e^{-2\delta_D t}\right) - 2\frac{\delta_r}{\delta_D - \kappa}\frac{\sigma \bar{\sigma}_D}{\kappa + \delta_D}\left(1 - e^{-(\kappa+\delta_D)t}\right) + \left(\frac{\delta_r}{\delta_D - \kappa}\right)^2 \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right).$$

A notable special case that we use to determine the coefficients of the predictive regressions is the unconditional variance-covariance matrix of $(r_t, D_t)^\top$. Taking the limit $t \to \infty$ in (A-17), (A-20) and (A-21) yields

$$V = \begin{pmatrix} \frac{\sigma^2}{2\kappa} & \frac{\delta_r \sigma^2}{2\kappa(\kappa+\delta_D)} \\ \frac{\delta_r \sigma^2}{2\kappa(\kappa+\delta_D)} & \frac{\delta_r^2 \sigma^2}{2\kappa^2(\kappa+\delta_D)} \end{pmatrix}, \tag{A-22}$$

implying that in general when $\delta_D \neq 0$ the two factors are not collinear.

### *Appendix C.2 Proofs and derivations*

*Proof of Lemma 1.* For notational simplicity let us write bond prices in the form

$$\frac{d\Lambda_t^\tau}{\Lambda_t^\tau} = \mu_t^\tau dt - \sigma_t^\tau dB_t. \tag{A-23}$$

Substituting (A-23) into intermediaries' budget constraint, (2), we get

$$dW_t = \left[r_t W_t + \int_0^T x_t^\tau \Lambda_t^\tau \left(\mu_t^\tau - r_t\right) d\tau\right] dt - \left[\int_0^T x_t^\tau \Lambda_t^\tau \sigma_t^\tau d\tau\right] dB_t,$$

therefore (3) simplifies to

$$\max_{\{x_t^\tau\}_{\tau \in (0,T]}} \int_0^T x_t^\tau \Lambda_t^\tau \left(\mu_t^\tau - r_t\right) d\tau - \frac{\alpha}{2}\left[\int_0^T x_t^\tau \Lambda_t^\tau \sigma_t^\tau d\tau\right]^2. \tag{A-24}$$

Because markets are complete, by no-arbitrage, there exists a unique market price of interest rate risk across all bonds that satisfies

$$\lambda_t = \frac{E_t\left(\frac{d\Lambda_t^\tau}{\Lambda_t^\tau}\right)/dt - r_t}{\frac{1}{\Lambda_t^\tau}\frac{d\Lambda_t^\tau}{dr_t}\sigma} = \frac{\mu_t^\tau - r_t}{-\sigma_t^\tau}, \tag{A-25}$$

and introducing

$$x_t = \frac{d\left(\int_0^T x_t^\tau \Lambda_t^\tau d\tau\right)}{dr_t} = \int_0^T x_t^\tau \frac{d\Lambda_t^\tau}{dr_t} d\tau = -\frac{1}{\sigma}\int_0^T x_t^\tau \Lambda_t^\tau \sigma_t^\tau d\tau, \tag{A-26}$$

for the total exposure of interest rate risk borne by intermediaries, their maximization problem (A-24) reduces to

$$\max_{x_t} \lambda_t x_t - \frac{\alpha\sigma}{2} x_t^2. \tag{A-27}$$

The first order condition of (A-27) together with the market clearing condition (4) determine the equilibrium market price of risk and provides (8). □

*Proof of Theorem 1.* We conjecture that equilibrium yields in the model defined by (12) and (13) are in the form (14), i.e., bond prices are

$$\Lambda_t^\tau = e^{-[\tau A(\tau) + \tau B(\tau) r_t + \tau C(\tau) D_t]}. \tag{A-28}$$

Applying Ito's Lemma to (A-28), substituting in (12) and (13), and imposing the condition that the bond price drift under $\mathbb{Q}$ must be $r_t \Lambda_t^\tau dt$, we obtain an equation affine in the factors $r_t$ and $D_t$. Collecting the $r_t$, $D_t$, and constant terms, respectively, we get a set of ODEs:

$$\begin{aligned}
1 &= \tau \mathcal{B}'(\tau) + \mathcal{B}(\tau) + \kappa\tau\mathcal{B}(\tau) + \delta_r \tau \mathcal{C}(\tau), && \text{(A-29)}\\
0 &= \tau \mathcal{C}'(\tau) + \mathcal{C}(\tau) + \delta_D^{\mathbb{Q}} \tau \mathcal{C}(\tau) - \alpha\sigma\sigma_y^{\bar{\tau}} \tau \mathcal{B}(\tau), \text{ and} && \text{(A-30)}\\
0 &= \tau \mathcal{A}'(\tau) + \mathcal{A}(\tau) - \kappa\theta\tau\mathcal{B}(\tau) - \delta_0 \tau \mathcal{C}(\tau) + \frac{1}{2}\sigma^2 \tau^2 \mathcal{B}^2(\tau) + \frac{1}{2}\eta_y^2 \left(\sigma_y^{\bar{\tau}}\right)^2 \tau^2 \mathcal{C}(\tau)^2, && \text{(A-31)}
\end{aligned}$$

with terminal conditions $\mathcal{A}(0) = \mathcal{C}(0) = 0$ and $\mathcal{B}(0) = 1$. Combining (A-29) and (A-30), we write the following second order ODE for $\mathcal{C}$:

$$0 = \tau \mathcal{C}''(\tau) + 2\mathcal{C}'(\tau) + \left(\kappa + \delta_D^{\mathbb{Q}}\right)\left(\tau \mathcal{C}'(\tau) + \mathcal{C}(\tau)\right) + \left(\kappa \delta_D^{\mathbb{Q}} + \alpha\sigma\sigma_y^{\bar{\tau}} \delta_r\right) \tau \mathcal{C}(\tau) - \alpha\sigma\sigma_y^{\bar{\tau}}. \tag{A-32}$$

Solving (A-32) for $\mathcal{C}$, from there deriving $\mathcal{B}$ and $\mathcal{A}$, and applying the terminal conditions, yields the following solution:

$$\begin{aligned}
\mathcal{C}(\tau) &= -\frac{\alpha\sigma\sigma_y^{\bar{\tau}}}{(\kappa+\varepsilon) - \left(\delta_D^{\mathbb{Q}} - \varepsilon\right)} \left[\frac{1 - e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} - \frac{1 - e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)\tau}}{\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)\tau}\right], && \text{(A-33)}\\
\mathcal{B}(\tau) &= \frac{1 - e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} - \frac{\varepsilon}{(\kappa+\varepsilon) - \left(\delta_D^{\mathbb{Q}} - \varepsilon\right)} \left[\frac{1 - e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} - \frac{1 - e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)\tau}}{\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)\tau}\right] && \text{(A-34)}
\end{aligned}$$

and

$$\mathcal{A}(\tau) = \frac{1}{(\kappa+\varepsilon)}\left[\kappa\theta\frac{(\kappa+\varepsilon)-\delta_D^{\mathbb{Q}}}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)} - \delta_0\frac{\alpha\sigma\sigma_y^{\bar{\tau}}}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)}\right]\omega\left((\kappa+\varepsilon)\tau\right) \tag{A-35}$$

$$+\frac{1}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)}\left[\kappa\theta\frac{\varepsilon}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)} + \delta_0\frac{\alpha\sigma\sigma_y^{\bar{\tau}}}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)}\right]\omega\left(\left(\delta_D^{Q}-\varepsilon\right)\tau\right)$$

$$+\frac{1}{2}\sigma^2\frac{\left[(\kappa+\varepsilon)-\delta_D^{\mathbb{Q}}\right]^2+\left[\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2\right]^2}{(\kappa+\varepsilon)^2\left[(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\right]^2}\left[\frac{1}{2}\omega\left(2(\kappa+\varepsilon)\tau\right)-\omega\left((\kappa+\varepsilon)\tau\right)\right]$$

$$+\frac{1}{2}\sigma^2\frac{\varepsilon^2+\left[\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2\right]^2}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)^2\left[(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\right]^2}\left[\frac{1}{2}\omega\left(2\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau\right)-\omega\left(\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau\right)\right]$$

$$+\sigma^2\frac{\left[(\kappa+\varepsilon)-\delta_D^{\mathbb{Q}}\right]\varepsilon-\left[\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2\right]^2}{(\kappa+\varepsilon)\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\left[(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\right]^2}\left[\omega\left(\left(\kappa+\delta_D^{\mathbb{Q}}\right)\tau\right)-\omega\left((\kappa+\varepsilon)\tau\right)-\omega\left(\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau\right)\right],$$

where the function $\omega(.)$ is defined as $\omega(x) = 1-\frac{1-e^{-x}}{x}$ for all $x \neq 0$ and $\omega(0) = 0$, and where

$$\varepsilon = \frac{\delta_D^{\mathbb{Q}}-\kappa-\sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2-4\alpha\sigma\sigma_y^{\bar{\tau}}\delta_r}}{2} \tag{A-36}$$

as long as it exists, i.e., the determinant is positive.

Next we pin down the endogenous parameters of the model. First, from (1), (11), and (14) the volatility of the reference yield has to solve

$$\sigma_y^{\bar{\tau}} = \mathcal{B}(\bar{\tau})\,\sigma + \mathcal{C}(\bar{\tau})\,\eta_y\sigma_y^{\bar{\tau}}. \tag{A-37}$$

Moreover, again from (14), we have

$$dy_t^{\bar{\tau}} = \mathcal{B}(\bar{\tau})\,dr_t + \mathcal{C}(\bar{\tau})\,dD_t. \tag{A-38}$$

Plugging (A-38) into (11) and using (1), we get

$$dD_t = \kappa_D(\theta_D - D_t)\,dt + \mathcal{B}(\bar{\tau})\,\eta_y\left[\kappa(\theta - r_t)\,dt + \sigma dB_t\right] + \mathcal{C}(\bar{\tau})\,\eta_y dD_t,$$

and collecting all $dD_t$ terms on the LHS yields

$$\left[1-\eta_y\mathcal{C}(\bar{\tau})\right]dD_t = \left[\kappa_D(\theta_D - D_t) + \kappa\eta_y\mathcal{B}(\bar{\tau})(\theta - r_t)\right]dt + \mathcal{B}(\bar{\tau})\,\sigma dB_t. \tag{A-39}$$

Matching the $r_t$, $D_t$, and constant terms in the drift of $dD_t$ from (A-39) with those in (11), we obtain (15).

We make use of the following result:

**Lemma 7.** *As long as $\varepsilon$ exists, we have (i) $\kappa+\varepsilon < \delta_D^{\mathbb{Q}}-\varepsilon$ always; and (ii) $\varepsilon$ has the same sign as $\delta_D^{\mathbb{Q}}-\kappa$. Finally, (iii) $\kappa+\varepsilon$ and $\delta_D^{\mathbb{Q}}-\varepsilon$ are always "between" $\kappa$ and $\delta_D^{\mathbb{Q}}$. That is, if $\kappa < \delta_D^{\mathbb{Q}}$, we have*

$$\kappa < \kappa+\varepsilon < \frac{\kappa+\delta_D^{\mathbb{Q}}}{2} < \delta_D^{\mathbb{Q}}-\varepsilon < \delta_D^{\mathbb{Q}}; \tag{A-40}$$

*if $\kappa > \delta_D^{\mathbb{Q}}$, we have*

$$\delta_D^{\mathbb{Q}} < \kappa+\varepsilon < \frac{\kappa+\delta_D^{\mathbb{Q}}}{2} < \delta_D^{\mathbb{Q}}-\varepsilon < \kappa. \tag{A-41}$$

*Proof.* First, notice that (A-36) and (15) together imply $\varepsilon$ can be rewritten as

$$\varepsilon = \frac{\left(\delta_D^{\mathbb{Q}}-\kappa\right) - \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}}{2}. \tag{A-42}$$

From (A-42), we have

$$\kappa+\varepsilon = \frac{\delta_D^{\mathbb{Q}}+\kappa - \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}}{2}$$

and

$$\delta_D^{\mathbb{Q}}-\varepsilon = \frac{\delta_D^{\mathbb{Q}}+\kappa + \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}}{2},$$

and since the square-root is non-negative, we always have

$$\kappa+\varepsilon < \frac{\kappa+\delta_D^{\mathbb{Q}}}{2} < \delta_D^{\mathbb{Q}}-\varepsilon.$$

Second, revisiting (A-42), if $\kappa > \delta_D^{\mathbb{Q}}$, both components of the RHS are negative and thus $\varepsilon < 0$. On the other hand, since $4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2 > 0$, we have

$$\left|\delta_D^{\mathbb{Q}}-\kappa\right| > \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}.$$

Therefore, if $\delta_D^{\mathbb{Q}} > \kappa$, the first component of $\varepsilon$ is positive and greater than the second, and thus $\varepsilon > 0$.

Third, we can write

$$\kappa+\varepsilon-\delta_D^{\mathbb{Q}} = \frac{-\left(\delta_D^{\mathbb{Q}}-\kappa\right) - \sqrt{\left(\delta_D^{\mathbb{Q}}-\kappa\right)^2 - 4\kappa\alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2}}{2},$$

and with similar reasoning as above, $\kappa > \delta_D^{\mathbb{Q}}$ implies $\kappa-\delta_D^{\mathbb{Q}}+\varepsilon > 0$, i.e. $\delta_D^{\mathbb{Q}} < \kappa+\varepsilon$ and $\delta_D^{\mathbb{Q}}-\varepsilon < \kappa$. Combining the three results gives inequalities (A-40) and (A-41). $\square$

To complete the proof of the Theorem, we need to provide sufficient conditions such that the set of equations given by (15) has a solution. First, we show that all meaningful $\sigma_y^{\bar{\tau}}$

solutions of (A-37) are non-negative. Notice that with the help of (A-6), (A-34) and (A-33) can be written as

$$\mathcal{C}(\tau) = -\alpha\sigma\sigma_y^{\bar{\tau}} g_4\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}}-\varepsilon\right) \text{ and } \tag{A-43}$$

$$\mathcal{B}(\tau) = -\left[\frac{e^{-(\kappa+\varepsilon)\tau}-e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)} + \delta_D^{\mathbb{Q}}\, g_4\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}}-\varepsilon\right)\right]. \tag{A-44}$$

Lemma 5 and (A-43) together imply that $\mathcal{C}(\tau)$ and $\sigma_y^{\bar{\tau}}$ have the same sign; therefore, the second term of the RHS of (A-37) is always non-negative. Regarding (A-44), as the function $x \mapsto e^{-x}$ is decreasing, the first term inside the bracket is negative. On the other hand, according to Lemma 5, the second term has the opposite sign as $\delta_D^{\mathbb{Q}}$. But $\delta_D^{\mathbb{Q}}$ must be positive, otherwise the duration process under $\mathbb{Q}$, (13), would explode. Hence, both terms inside the bracket are negative, i.e., $\mathcal{B}(\tau) \geq 0$. Going back to (A-37), we have shown that both components of the RHS are non-negative, and thus in all meaningful solutions $\sigma_y^{\bar{\tau}} \geq 0$. Notice that this also implies $\mathcal{C}(\tau) \geq 0$ for all $\tau \geq 0$, and from (15) it also means $0 < 1 - \eta_y\mathcal{C}(\bar{\tau}) \leq 1$.

Second, a sufficient condition for the existence of a solution to (A-37) is that its LHS is smaller than the RHS for $\sigma_y^{\bar{\tau}} = 0$ but greater than the RHS when $\sigma_y^{\bar{\tau}}$ is large enough. It is easy to see that $\sigma_y^{\bar{\tau}} = 0$ leads to $\mathcal{C}(\tau) = 0$ for all $\tau \geq 0$, and yields

$$\mathcal{B}(\tau) = \frac{1-e^{-\kappa\tau}}{\kappa\tau}.$$

Therefore, the RHS of (A-37) is zero while the LHS equals

$$\mathcal{B}(\bar{\tau})\sigma + \mathcal{C}(\bar{\tau})\eta_y\sigma_y^{\bar{\tau}} = \frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\sigma > 0.$$

For inequality in the other direction, notice that Lemmas 5 and 7 together imply

$$g_4\left(\kappa, \delta_D^{\mathbb{Q}}\right) < g_4\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}}-\varepsilon\right) < 0, \tag{A-45}$$

regardless of the order of $\kappa$ and $\delta_D^{\mathbb{Q}}$. Further, Lemma 5 states that $g_4$ is increasing in both arguments, therefore

$$g_4\left(\kappa, \kappa_D^{\mathbb{Q}}\right) < g_4\left(\kappa, \delta_D^{\mathbb{Q}}\right) < 0, \tag{A-46}$$

where $\kappa_D^{\mathbb{Q}} \equiv \kappa_D - \alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2 \leq \delta_D^{\mathbb{Q}}$ always, because $\kappa_D \leq \delta_D$ holds due to (15) and $\mathcal{C}(\tau) \geq 0$. Combining (A-43), (A-45), and (A-46), we obtain

$$0 < \mathcal{C}(\tau) < -\alpha\sigma\sigma_y^{\bar{\tau}} g_4\left(\kappa, \kappa_D^{\mathbb{Q}}\right). \tag{A-47}$$

We also approximate $\mathcal{B}(\tau)$ from above with the help of Lemma 5. First, if we assume $\kappa < \delta_D^{\mathbb{Q}}$, which also implies $\varepsilon > 0$ and $\kappa < \delta_D^{\mathbb{Q}} - \varepsilon$ according to Lemma 7, since $g_4$ is negative and increasing in both arguments, we have

$$g_4(\kappa+\varepsilon, \kappa) < g_4\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}}-\varepsilon\right) < 0.$$

Rearranging and using (A-34), we obtain

$$\mathcal{B}(\tau) = \frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\,\tau} - \varepsilon\, g_4\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}} - \varepsilon\right) < \frac{1-e^{-\kappa\tau}}{\kappa\tau}. \tag{A-48}$$

If, on the other hand, $\delta_D^{\mathbb{Q}} < \kappa$, we rewrite (A-34) as

$$\mathcal{B}(\tau) = \frac{1-e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau} + \left(\kappa+\varepsilon-\delta_D^{\mathbb{Q}}\right)\, g_4\left(\kappa+\varepsilon, \delta_D^{\mathbb{Q}} - \varepsilon\right).$$

Notice that Lemma 7 in this case yields $\kappa+\varepsilon-\delta_D^{\mathbb{Q}} > 0$, and since $g_4$ is negative, we get

$$\mathcal{B}(\tau) < \frac{1-e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau}}{\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)\tau} < \frac{1-e^{-\delta_D^{\mathbb{Q}}\tau}}{\delta_D^{\mathbb{Q}}\tau} \le \frac{1-e^{-\kappa_D^{\mathbb{Q}}\tau}}{\kappa_D^{\mathbb{Q}}\tau}, \tag{A-49}$$

where in the last two steps we use $\delta_D^{\mathbb{Q}}-\varepsilon > \delta_D^{\mathbb{Q}} \ge \kappa_D^{\mathbb{Q}}$. Combining (A-48) and (A-49), we obtain that under any circumstances we have

$$\mathcal{B}(\tau) < \max\left\{\frac{1-e^{-\kappa\tau}}{\kappa\tau}, \frac{1-e^{-\kappa_D^{\mathbb{Q}}\tau}}{\kappa_D^{\mathbb{Q}}\tau}\right\}, \tag{A-50}$$

and, together with (A-47),

$$\mathcal{B}(\tau)\,\sigma + \mathcal{C}(\tau)\,\eta_y\sigma_y^{\bar{\tau}} < \left[\max\left\{\frac{1-e^{-\kappa\tau}}{\kappa\tau}, \frac{1-e^{-\kappa_D^{\mathbb{Q}}\tau}}{\kappa_D^{\mathbb{Q}}\tau}\right\} - \alpha\eta_y\left(\sigma_y^{\bar{\tau}}\right)^2 g_4\left(\kappa, \kappa_D^{\mathbb{Q}}\right)\right]\sigma. \tag{A-51}$$

We want to give a sufficient condition for the LHS of (A-37) to be larger than the RHS when $\sigma_y^{\bar{\tau}}$ is large enough to make $\kappa_D^{\mathbb{Q}} = 0$, that is, $\left(\sigma_y^{\bar{\tau}}\right)^2 = \frac{\kappa_D}{\alpha\eta_y}$. For this it is sufficient if we make $\sigma_y^{\bar{\tau}}$ larger than the RHS of (A-51), which, after some algebra, is equivalent to

$$\sigma_y^{\bar{\tau}} > \left[1 - \frac{\kappa_D}{\kappa}\left(\frac{1-e^{-\kappa\tau}}{\kappa\tau} - 1\right)\right]\sigma,$$

because $\kappa_D^{\mathbb{Q}} = 0$ makes the RHS of (A-49) equal to 1. Taking squares of both sides and using $\left(\sigma_y^{\bar{\tau}}\right)^2 = \frac{\kappa_D}{\alpha\eta_y}$ again, after some algebra we obtain

$$\alpha < \frac{\kappa_D}{\eta_y\left(\frac{\kappa+\kappa_D}{\kappa} - \frac{\kappa_D}{\kappa}\frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\right)^2\sigma^2}. \tag{A-52}$$

Defining $\bar{\alpha}$ as the RHS of (A-52), which is certainly positive, (A-37) has at least one solution whenever $0 \le \alpha < \bar{\alpha}$. $\square$

*Proof of Propositions 1 and 3.* The excess return over horizon $(t, t+h)$ on a maturity-$\tau$ bond is

$$rx_{t,t+h,\tau} = \log\Lambda_{t+h,\tau-h} - \log\Lambda_{t,\tau} + \log\Lambda_{t,h} = -(\tau-h)\,y_{t+h,\tau-h} + \tau y_{t,\tau} - h y_{t,h}. \tag{A-53}$$

From (14) we have

$$
\begin{aligned}
E_t\left[y_{t+h,\tau-h}\right] &= \mathcal{A}\left(\tau-h\right)+\mathcal{C}\left(\tau-h\right)E_t\left[D_{t+h}\right]+\mathcal{B}\left(\tau-h\right)E_t\left[r_{t+h}\right] \\
&= \ldots + e^{-\delta_D h}\mathcal{C}\left(\tau-h\right)D_t+\left[e^{-\kappa h}\mathcal{B}\left(\tau-h\right)-\frac{\delta_r}{\delta_D-\kappa}\left(e^{-\kappa h}-e^{-\delta_D h}\right)\mathcal{C}\left(\tau-h\right)\right]r_t,
\end{aligned}
$$

where in the second equality we use (A-15) and (A-16) for the time period $(t, t+h)$ and ignore the additive constant terms.

We take the expectation of the sides in (A-53) as of time $t$ and plug in the above result. Omitting additive constants again, we obtain that the forecastable part of excess returns is

$$
\begin{aligned}
E_t\left[rx_{t,t+h,\tau}\right] &= \ldots + \left[\tau\mathcal{C}\left(\tau\right)-h\mathcal{C}\left(h\right)-e^{-\delta_D h}\left(\tau-h\right)\mathcal{C}\left(\tau-h\right)\right]D_t \\
&+\left[\tau\mathcal{B}\left(\tau\right)-h\mathcal{B}\left(h\right)-e^{-\kappa h}\left(\tau-h\right)\mathcal{B}\left(\tau-h\right)-\delta_r\frac{e^{-\kappa h}-e^{-\delta_D h}}{\kappa-\delta_D}\left(\tau-h\right)\mathcal{C}\left(\tau-h\right)\right]r_t.
\end{aligned}
\tag{A-54}
$$

When running a multivariate regression of $rx_{t,t+h,\tau}$ on $D_t$ and $r_t$ in the form

$$
rx_{t,t+h,\tau}=\alpha_{D,r}+\left(\beta_1^{\tau,h},\beta_2^{\tau,h}\right)\begin{pmatrix} D_t \\ r_t \end{pmatrix}+\epsilon_{t+h},
$$

the vector of theoretical coefficients becomes $\left(\beta_1^{\tau},\beta_2^{\tau}\right)^{\top}=V^{-1}Cov\left[rx_{t+h,\tau},\left(r_t,D_t\right)^{\top}\right]$, where $V$ is given by (A-22). Using (A-54), after some algebra we get

$$
\begin{aligned}
\beta_1^{\tau} &= \tau\mathcal{C}\left(\tau\right)-h\mathcal{C}\left(h\right)-e^{-\delta_D h}\left(\tau-h\right)\mathcal{C}\left(\tau-h\right) \text{ and} \\
\beta_2^{\tau} &= \tau\mathcal{B}\left(\tau\right)-h\mathcal{B}\left(h\right)-e^{-\kappa h}\left(\tau-h\right)\mathcal{B}\left(\tau-h\right)-\delta_r\frac{e^{-\kappa h}-e^{-\delta_D h}}{\kappa-\delta_D}\left(\tau-h\right)\mathcal{C}\left(\tau-h\right),
\end{aligned}
\tag{A-55}
$$

i.e. the loadings in (A-54). On the other hand, when running a univariate regression of $rx_{t+h,\tau}$ on $D_t$ in the form $rx_{t,t+h,\tau}=\alpha_D+\beta^{\tau}D_t+\epsilon_{t+h}$, the slope coefficient is

$$
\beta^{\tau}=\frac{Cov\left[rx_{t,t+h,\tau},D_t\right]}{Var\left[D_t\right]}=\beta_1^{\tau}+\frac{Cov\left[r_t,D_t\right]}{Var\left[D_t\right]}\beta_2^{\tau}=\beta_1^{\tau}+\frac{\sigma}{\eta_y\sigma_y^{\bar{\tau}}}\beta_2^{\tau},
\tag{A-56}
$$

where the last equality is due to (15) and (A-22).

We start by looking at $\beta_1^{\tau}$. First, it is easy to see from (A-33) that $\lim_{\tau\to h}\beta_1^{\tau}=0$. Second, substituting (A-33) into (A-55), differentiating with respect to $\tau$, and rearranging, we obtain

$$
\frac{d\beta_1^{\tau}}{d\tau}=-\alpha\sigma\sigma_y^{\bar{\tau}}\frac{\left[e^{-(\kappa+\varepsilon)h}-e^{-\delta_D h}\right]e^{-(\kappa+\varepsilon)(\tau-h)}-\left[e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)h}-e^{-\delta_D h}\right]e^{-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)(\tau-h)}}{(\kappa+\varepsilon)-\left(\delta_D^{\mathbb{Q}}-\varepsilon\right)}.
\tag{A-57}
$$

We focus on the parameter set that satisfies $\kappa<\delta_D$, that is, the speed of mean-reversion of the short rate is smaller than the speed of mean-reversion of the duration, which is true for the calibrated real-world parameters (see Table 12; $\kappa<\kappa_D$ and (15) together imply $\kappa<\delta_D$). We have the following result:

**Lemma 8.** *$\kappa<\delta_D$ implies $\delta_D^{\mathbb{Q}}-\varepsilon<\delta_D$.*

*Proof.* Suppose instead $\delta_D \leq \delta_D^{\mathbb{Q}} - \varepsilon$; together with $\kappa < \delta_D$ we get $\kappa < \delta_D^{\mathbb{Q}} - \varepsilon$. Revisiting Lemma 7, it must be that (A-40) holds, i.e. $\varepsilon > 0$. But then $\delta_D^{\mathbb{Q}} - \varepsilon < \delta_D^{\mathbb{Q}} < \delta_D$; contradiction. $\square$

Combining Lemmas 7 and 8, we obtain that $\kappa + \varepsilon < \delta_D^{\mathbb{Q}} - \varepsilon < \delta_D$. As $0 < h \leq \tau$, we have both $0 < e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)(\tau - h)} < e^{-(\kappa+\varepsilon)(\tau-h)}$ and $0 < e^{-(\kappa+\varepsilon)h} < e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)h} < e^{-\delta_D h}$. In turn, the latter implies $0 < e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)h} - e^{-\delta_D h} < e^{-(\kappa+\varepsilon)h} - e^{-\delta_D h}$. Therefore, the two terms in the numerator of the LHS are positive, but both (positive) components of the first expression are larger than the corresponding component from the second. Hence, the numerator and the total RHS of (A-57) are both positive: $\beta_1^\tau$ is increasing in $\tau$. Since we also have $\beta_1^h = 0$, we conclude that $\beta_1^\tau$ is positive and increasing across maturities.

Next we look at $\beta_2^\tau$. First, it is easy to see from (A-34) and (A-33) that $\lim_{\tau \to h} \beta_2^\tau = 0$. Second, substituting (A-33) and (A-34) into (A-55), differentiating with respect to $\tau$, and rearranging, we obtain

$$\frac{d\beta_2^\tau}{d\tau} = \kappa\alpha\eta_y \left(\sigma_y^{\bar{\tau}}\right)^2 \frac{\left[g_2(\delta_D) - g_2(\kappa+\varepsilon)\right] e^{-(\kappa+\varepsilon)(\tau-h)} - \left[g_2(\delta_D) - g_2\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)\right] e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)(\tau-h)}}{(\kappa+\varepsilon) - \left(\delta_D^{\mathbb{Q}} - \varepsilon\right)}, \tag{A-58}$$

where $g_2(.)$ is defined in (A-3). As $\kappa + \varepsilon < \delta_D^{\mathbb{Q}} - \varepsilon < \delta_D$, Lemma 3 implies $g_2(\kappa+\varepsilon) < g_2\left(\delta_D^{\mathbb{Q}} - \varepsilon\right) < g_2(\delta_D)$, i.e. $0 < g_2(\delta_D) - g_2\left(\delta_D^{\mathbb{Q}} - \varepsilon\right) < g_2(\delta_D) - g_2(\kappa+\varepsilon)$. On the other hand $\kappa + \varepsilon < \delta_D^{\mathbb{Q}} - \varepsilon$ yields $0 < e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)h} < e^{-(\kappa+\varepsilon)h}$. Hence, the numerator is positive, and thus the total RHS of (A-58) is negative: $\beta_2^\tau$ is decreasing in $\tau$. Since we also have $\beta_2^h = 0$, we conclude that $\beta_2^\tau$ is negative and decreasing (i.e., becoming more negative) across maturities.

Total univariate slope is given by (A-56). First, it is easy to confirm that $\lim_{\tau \to h} \beta^\tau = 0$. Second, using (A-57) and (A-58) and rearranging, we obtain

$$\begin{aligned} \frac{d\beta^\tau}{d\tau} &= \frac{d\beta_1^\tau}{d\tau} + \frac{\sigma}{\eta_y \sigma_y^{\bar{\tau}}} \frac{d\beta_2^\tau}{d\tau} \qquad \text{(A-59)} \\ &= -\alpha\sigma\sigma_y^{\bar{\tau}} \frac{\left[g_3(\kappa+\varepsilon) - g_3(\delta_D)\right] e^{-(\kappa+\varepsilon)(\tau-h)} - \left[g_3\left(\delta_D^{\mathbb{Q}} - \varepsilon\right) - g_3(\delta_D)\right] e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)(\tau-h)}}{(\kappa+\varepsilon) - \left(\delta_D^{\mathbb{Q}} - \varepsilon\right)}, \end{aligned}$$

where $g_3(.)$ is defined in (A-4). Lemma 4 implies that for small $h > 0$, the function $g_3$ is decreasing, hence $g_3(\delta_D) < g_3\left(\delta_D^{\mathbb{Q}} - \varepsilon\right) < g_3(\kappa+\varepsilon)$, i.e. $0 < g_3\left(\delta_D^{\mathbb{Q}} - \varepsilon\right) - g_3(\delta_D) < g_3(\kappa+\varepsilon) - g_3(\delta_D)$. On the other hand $\kappa + \varepsilon < \delta_D^{\mathbb{Q}} - \varepsilon$ implies $0 < e^{-\left(\delta_D^{\mathbb{Q}} - \varepsilon\right)h} < e^{-(\kappa+\varepsilon)h}$. Hence, the numerator and the total RHS of (A-59) are positive: $\beta^\tau$ is increasing in $\tau$. Since we also have $\beta^h = 0$, we conclude that $\beta^\tau$ is positive and increasing across maturities.

Finally, from (14), the effect of duration on yields is given by $\mathcal{C}(\tau)$. From the Proof of Theorem 1, $\mathcal{C}(\tau) \geq 0$. Moreover, from (A-33) it is easy to show that

$$\lim_{\tau \to 0} \mathcal{C}(\tau) = \lim_{\tau \to \infty} \mathcal{C}(\tau) = 0,$$

with $\mathcal{C}(\tau)$ increasing for small but decreasing for large $\tau$ values, which implies that the effect is either increasing across maturities if $T$ is small, or first increasing then decreasing if $T$ is sufficiently large. This completes the proof. $\square$

*Proof of Proposition 2.* From (14), (A-33), and (A-34), bond yield volatility is given by

$$\begin{aligned}\sigma_y^\tau &= \mathcal{B}(\tau)\sigma + \mathcal{C}(\tau)\eta_y\sigma_y^{\bar\tau} \\ &= \frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau}\sigma - \frac{\varepsilon+\alpha\eta_y\left(\sigma_y^{\bar\tau}\right)^2}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)}\left[\frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau} - \frac{1-e^{-\left(\delta_D^Q-\varepsilon\right)\tau}}{\left(\delta_D^Q-\varepsilon\right)\tau}\right]\sigma.\end{aligned} \tag{A-60}$$

Due to the complexity the feedback mechanism introduces into the endogenous parameters, we cannot compute the exact effect of convexity $-\eta_y$ on yield volatilities in closed form. Instead, we derive its effect by considering (A-60) around $\alpha = 0$. From (A-60) we write $\sigma_y^\tau \approx h_0(\tau) + \alpha\eta_y h_1(\tau)$, where

$$h_0(\tau) \equiv \left(\mathcal{B}(\tau)\sigma + \mathcal{C}(\tau)\eta_y\sigma_y^{\bar\tau}\right)|_{\alpha=0} \text{ and } h_1(\tau) \equiv \frac{1}{\eta_y}\frac{d\left(\mathcal{B}(\tau)\sigma + \mathcal{C}(\tau)\eta_y\sigma_y^{\bar\tau}\right)}{d\alpha}|_{\alpha=0}.$$

We start with $h_0$. It is straightforward from (A-36) and (15) that taking the limit $\alpha \to 0$ yields

$$\lim_{\alpha\to0}\varepsilon = 0 \text{ and } \lim_{\alpha\to0}\delta_D^Q = \lim_{\alpha\to0}\delta_D = \kappa_D, \tag{A-61}$$

hence

$$h_0(\tau) = \frac{1-e^{-\kappa\tau}}{\kappa\tau}\sigma. \tag{A-62}$$

and as a special case, the volatility of the reference-maturity yield is

$$\lim_{\alpha\to0}\sigma_y^{\bar\tau} = h_0(\bar\tau) = \frac{1-e^{-\kappa\bar\tau}}{\kappa\bar\tau}\sigma. \tag{A-63}$$

Second, differentiating (A-60) with respect to $\alpha$, we get

$$\begin{aligned}\frac{1}{\sigma}\frac{d\sigma_y^\tau}{d\alpha} &= -\frac{d(\kappa+\varepsilon)}{d\alpha}\frac{1-e^{-(\kappa+\varepsilon)\tau}-(\kappa+\varepsilon)\tau e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)^2\tau} \\ &\quad -\left(\varepsilon+\alpha\eta_y\left(\sigma_y^{\bar\tau}\right)^2\right)\frac{d}{d\alpha}\left(\frac{\frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau}-\frac{1-e^{-\left(\delta_D^Q-\varepsilon\right)\tau}}{\left(\delta_D^Q-\varepsilon\right)\tau}}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)}\right) \\ &\quad -\left(\frac{d\varepsilon}{d\alpha}+\eta_y\left(\sigma_y^{\bar\tau}\right)^2+2\alpha\eta_y\sigma_y^{\bar\tau}\frac{d\sigma_y^{\bar\tau}}{d\alpha}\right)\frac{\frac{1-e^{-(\kappa+\varepsilon)\tau}}{(\kappa+\varepsilon)\tau}-\frac{1-e^{-\left(\delta_D^Q-\varepsilon\right)\tau}}{\left(\delta_D^Q-\varepsilon\right)\tau}}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)}.\end{aligned} \tag{A-64}$$

As

$$\frac{d(\kappa+\varepsilon)}{d\alpha} = \frac{d\varepsilon}{d\alpha} = \frac{\varepsilon}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)}\frac{d\delta_D^Q}{d\alpha} - \frac{\kappa\eta_y\left(\sigma_y^{\bar\tau}\right)^2}{(\kappa+\varepsilon)-\left(\delta_D^Q-\varepsilon\right)},$$

we get

$$\lim_{\alpha\to0}\frac{d(\kappa+\varepsilon)}{d\alpha} = \lim_{\alpha\to0}\frac{d\varepsilon}{d\alpha} = \frac{\kappa\eta_y}{\kappa_D-\kappa}\lim_{\alpha\to0}\left(\sigma_y^{\bar\tau}\right)^2 = \frac{\kappa\eta_y\sigma^2}{\kappa_D-\kappa}\left(\frac{1-e^{-\kappa\bar\tau}}{\kappa\bar\tau}\right)^2, \tag{A-65}$$

where in the last step we used (A-63). Hence, after some algebra, (A-64) yields

$$\begin{aligned}\frac{1}{\sigma}\lim_{\alpha\to 0}\frac{d\sigma_y^{\tau}}{d\alpha} &= \frac{\kappa_D\eta_y\sigma^2}{(\kappa_D-\kappa)^2}\left(\frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\right)^2\left[\frac{1-e^{-\kappa\tau}}{\kappa\tau}-\frac{\kappa_D-\kappa}{\kappa_D}\frac{1-e^{-\kappa\tau}-\kappa\tau e^{-\kappa\tau}}{\kappa\tau}-\frac{1-e^{-\kappa_D\tau}}{\kappa_D\tau}\right]\\ &= \frac{\kappa_D\eta_y\sigma^2}{(\kappa_D-\kappa)^2}\left(\frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\sigma\right)^2 g_5(\kappa,\kappa_D),\end{aligned}$$

and thus

$$h_1(\tau)=\frac{\kappa_D\sigma^3}{(\kappa_D-\kappa)^2}\left(\frac{1-e^{-\kappa\bar{\tau}}}{\kappa\bar{\tau}}\right)^2 g_5(\kappa,\kappa_D),$$

where $g_5$ is defined in (A-11). But $g_5\geq 0$ always according to Lemma 6, hence $h_1(\tau)\geq 0$, which implies that bond yield volatilities are increasing in negative convexity: $\frac{d\sigma_y^{\tau}}{d\eta_y}>0$.

Third, we trivially verify that

$$\lim_{\tau\to 0}\sigma_y^{\tau}=\sigma \text{ and } \lim_{\tau\to\infty}\sigma_y^{\tau}=0,$$

independent of $\eta_y$. Therefore, the effect of negative convexity on yield volatilities tends to zero at very short and very long maturities, and hence it must be hump-shaped.

Regarding bond return volatilities, given by $\sigma_y^{\tau}\tau$, we have $\lim_{\tau\to 0}\sigma_y^{\tau}\tau=0$ and

$$\lim_{\tau\to\infty}\sigma_y^{\tau}\tau=\frac{\delta_D}{(\kappa+\varepsilon)\left(\delta_D^Q-\varepsilon\right)}\sigma=\frac{\sigma}{\kappa},$$

again independent of $\eta_y$, where the last equality is due to (A-36). Hence, the effect of negative convexity on bond return volatilities tends to zero at very short and very long maturities. However, as $\eta_y$ increases $\sigma_y^{\tau}$, it also increases $\sigma_y^{\tau}\tau$, and thus the effect must be hump-shaped. □

**Corollary 1** (Background calculations for Section 4.4)**.** *The theoretical $R^2$s of univariate regressions of the duration factor $D_t$ on the short rate factor $r_t$, the long-term yield $y_t^{\bar{\tau}}$, and the slope $y_t^{\bar{\tau}}-r_t$ are given by*

$$R^2_{D,r} = 1-\frac{\delta_D}{\kappa+\delta_D}, \tag{A-66}$$

$$R^2_{D,y} = 1-\frac{\delta_D}{\kappa\left(1+\frac{\mathcal{C}(\bar{\tau})}{\mathcal{B}(\bar{\tau})}\frac{\delta_r}{\kappa}\right)^2+\delta_D}, \text{ and} \tag{A-67}$$

$$R^2_{D,y-r} = 1-\frac{\delta_D}{\kappa\left(1+\frac{\mathcal{C}(\bar{\tau})}{\mathcal{B}(\bar{\tau})-1}\frac{\delta_r}{\kappa}\right)^2+\delta_D}. \tag{A-68}$$

*Proof.* When running a linear regression in the form $Y_t=\alpha+\gamma X_t+\epsilon_t$, the theoretical slope coefficient is simply $\gamma=Cov\left[X_t,Y_t\right]/Var[X_t]$, whereas the $R^2$ of the regression is

$$R^2=\frac{\gamma^2 Var[X_t]}{Var[Y_t]}=\frac{Cov^2\left[X_t,Y_t\right]}{Var[X_t]Var[Y_t]}. \tag{A-69}$$

Applying this to $Y_t = D_t$ and $X_t = r_t$ and using (A-22) yields (A-66). Applying (A-69) to $Y_t = D_t$ and $X_t = y_t^{\bar{\tau}}$ and combining it with (14), we obtain

$$R^2_{D,y} = \frac{Cov^2\left[y_t^{\bar{\tau}}, D_t\right]}{Var[y_t^{\bar{\tau}}]Var[D_t]} = \frac{\left(\mathcal{B}\left(\bar{\tau}\right)Cov\left[r_t, D_t\right] + \mathcal{C}\left(\bar{\tau}\right)Var\left[D_t\right]\right)^2}{\left(\mathcal{B}^2\left(\bar{\tau}\right)Var[r_t] + 2\mathcal{B}\left(\bar{\tau}\right)\mathcal{C}\left(\bar{\tau}\right)Cov[r_t, D_t] + \mathcal{C}^2\left(\bar{\tau}\right)Var[D_t]\right)Var[D_t]}.$$

Equation (A-22) and rearranging gives (A-67). Finally, applying (A-69) to $Y_t = D_t$ and $X_t = y_t^{\bar{\tau}} - r_t$ and combining it with (14), we obtain

$$R^2_{D,y-r} = \frac{\left(\left(\mathcal{B}\left(\bar{\tau}\right) - 1\right)Cov\left[r_t, D_t\right] + \mathcal{C}\left(\bar{\tau}\right)Var\left[D_t\right]\right)^2}{\left(\left(\mathcal{B}\left(\bar{\tau}\right) - 1\right)^2 Var[r_t] + 2\left(\mathcal{B}\left(\bar{\tau}\right) - 1\right)\mathcal{C}\left(\bar{\tau}\right)Cov[r_t, D_t] + \mathcal{C}^2\left(\bar{\tau}\right)Var[D_t]\right)Var[D_t]}.$$

Using (A-22) and rearranging gives (A-68). □

### *Appendix C.3 Time-varying convexity*

Here we present a tractable way to relax the assumption of constant MBS convexity and capture the non-linearities inherent to the prepayment option. This version of the model allows for an additional degree of freedom and provides a better statistical description of MBS duration and convexity series. However, the qualitative implications of the model are identical to the ones outlined in Section 3. More precisely, we allow the sensitivity of outstanding MBS to the short rate to be quadratic:

$$\frac{dMBS_t}{dr_t} = z_t + \phi z_t^2, \quad \text{and} \tag{A-70}$$

$$dz_t = -\kappa_z^{\mathbb{Q}} z_t dt + \sigma_z dB_t^{\mathbb{Q}}. \tag{A-71}$$

In the data, when interest rates and MBS duration decrease, the negative convexity of MBS increases. In other words MBS duration has negative skewness. The skewness of the monthly series of MBS duration in our sample is equal to $-1.32$ compared to the 10 year yield which displays only a moderate skewness of $-0.06$. The parameter $\phi$ can be calibrated to match this feature of the data. From an economic point of view negative skewness corresponds to the asymmetry in MBS duration response to changes in interest rates: it reacts more to falling than to rising interest rates.

In the model described by (1), (9) and (A-70)-(A-71) yields are given by

$$y_t^{\tau} = \mathcal{A}_z(\tau) + \mathcal{B}_z(\tau)r_t + \mathcal{C}_z(\tau)z_t + \mathcal{D}_z(\tau)z_t^2,$$

The key to quadratic closed form solution is that while quadratic terms appear under $\mathbb{Q}$ in the dynamics of $r_t$, $z_t$ is still affine under $\mathbb{Q}$ (and therefore not affine under $\mathbb{P}$); see also Cheng and Scaillet (2007). Bond prices are given by

$$\Lambda_t^{\tau} = e^{-\left[A_z(\tau) + B_z(\tau)r_t + C_z(\tau)z_t + D_z(\tau)z_t^2\right]},$$

where $A_z\left(\tau\right) \equiv \mathcal{A}_z(\tau)\tau$, $B_z\left(\tau\right) \equiv \mathcal{B}_z(\tau)\tau$, $C_z\left(\tau\right) \equiv \mathcal{C}_z(\tau)\tau$, and $D_z\left(\tau\right) \equiv \mathcal{D}_z(\tau)\tau$. No-arbitrage pricing of bonds results in the following system of ODEs (where we remove the time-dependence and subscript to simplify the notation):

$$0 = A' - \kappa\theta B + \frac{1}{2}\sigma^2 B^2 + \frac{1}{2}\sigma_z^2\left(C^2 - 2D\right) + \sigma\sigma_z BC,$$

and

$$1 = B' + \kappa B,$$

and

$$0 = C' + \left(\kappa_z^{\mathbb{Q}} + 2\sigma_z^2 D\right) C - \alpha\sigma^2 B + 2\sigma\sigma_z BD,$$

and

$$0 = D' + 2\kappa_z^{\mathbb{Q}} D + 2\sigma_z^2 D^2 - \alpha\sigma^2\phi B,$$

together with the boundary conditions $A(0) = B(0) = C(0) = D(0) = 0$. The solution to the system above can be written in terms of $J$- and $Y$-type Bessel functions. To simplify, we can also solve for $A_z(\tau)$, $B_z(\tau)$, $C_z(\tau)$, and $D_z(\tau)$ recursively using a discrete time approximation of the dynamics of the state variables.

By Itô's lemma the second order dollar sensitivity of outstanding MBS to short rate shocks $\left(\frac{d^2 MBS_t}{dr_t^2} \equiv -\gamma\right)$ is equal to:

$$\sigma_z + 2\phi\sigma_z z_t,$$

implying time-varying convexity. The instantaneous volatility of maturity-$\tau$ yield is given by

$$\mathcal{B}_z(\tau)\sigma + \mathcal{C}_z(\tau)\sigma_z + 2\mathcal{D}_z(\tau)\sigma z_t.$$

The code that calculates $A_z(\tau)$, $B_z(\tau)$, $C_z(\tau)$, and $D_z(\tau)$ and allows to verify Propositions 1 and 2 in the context of stochastic convexity is available upon request.