Using Recursive Methods for Estimating Commercial Damages: Three Case Studies

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Anderson Economic Group working paper 2015-1; for presentation at National Association of Forensic Economics sessions, ASSA conference, Boston MA, January 2015

This draft: 12/30/14.

Abstract

The "recursive" or "value functional" method of estimating the value of investment opportunities and business ventures has recently become practically available to economists skilled in computational methods. This same method provides a natural way to estimate commercial damages caused by breach of contract and other similar causes.

The kernel of the method is modeling the decision of a business owner seeking to maximize the value of the business across the management options that are available. This is an optimization across numerous possible future paths, broken into a series of two-period decisions that result in a recursive structure of equations. This optimization is qualitatively and mathematically different from the simple estimation of future expected profits along a single expected path, which is the kernel of the long-established income method. This mathematical difference allows for the native incorporation of such concepts as mitigation, idiosyncratic and asymmetrical risks, and real options.

This paper presents a small set of early applications of this method in forensic economics settings. These case studies are based on actual controversies involving breach of contract, including one where an estimate of the lost value of a real option was part of a damages award in a jury trial. We present for each case a comparison between the recursive method and the traditional income method. We also provide theoretical references, practical guidance, cautions, software requirements, and other observations regarding the use of this novel method.

JEL codes: K00, C61, C88

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Introduction[2]

There are many business valuation methods in common use in forensic economics. Indeed, at least nine different approaches from economics, finance, and practical work have been used or cited repeatedly in recent decades.

Economics		
Classical	Labor theory of value	Smith, Ricardo (18th century); Marx
Neoclassical	Marginalist school	Walras (19th century); Marshall, Dreze (20th century)
Modern	Recursive general equilibrium	Bewley, Lucas, Sargent (late 20th century)
Traditional		
Asset	Accounting	Pacioli (16th Century); US IRS (20th century)
Market	Traditional	US IRS (20th century)
Income	Rational investor	Fisher, Williams (early 20th century); Modigliani & Miller, Dean (mid-20th century)
Mathematical		
Modern Portfolio Theory	Mean-variance approach	Markowitz, Sharpe (20th century)
Complete Markets	Pricing assuming "no arbitrage" in complete markets	Arrow, Pratt, Black, Scholes, Ross (20th century)
Option Pricing	Financial option in complete markets	Bachelier (early 20th century); Black, Scholes, Merton; McDonald, Siegel (real options)
Value Functional		
Recursive or value functional	Optimality principle in sequential decision problems	Bellman, Blackwell, Stokey, Lucas, Sargent, Anderson (20th century)

Source: Adapted from Anderson (2013, table 2.1)

However, only a subset of these is in practical use for valuing businesses or calculating commercial damages. Of this subset, the income method is the workhorse for most forensic economics applications involving damages to operating businesses.

The income method, when used to estimate commercial damages, relies on a straightforward principle: the lost earnings of a firm or individual caused by the breach of contract or other cause of damages, discounted for time and risk, is the amount of money equivalent necessary to “make whole” the

[2] This section is based on Anderson (2008) and used with permission.

damaged business or individual. For this reason, the income method, when used to estimate commercial damages, is sometimes called loosely the “lost profits” method, and the income method itself is often called “discounted cash flow” or DCF.[3] As we will see further below, these shorthand terms carry with them embedded assumptions that can be contrary to proper methodology in many practical cases.

However common its usage, there are serious weaknesses in both the income principle, and in the practice of the “lost profits” method of estimating commercial damages. These include:

- The doctrine of mitigation requires the forensic economist to consider how a business could *change* its operations to mitigate its damages. Such a doctrine runs directly counter to a standard practice of many practitioners of the DCF method, which is to use avoided costs calculated as static shares of revenue.
- The legal doctrine known as the “new business rule” creates a chicken-and-egg problem when a fledgling business is damaged: without past profits, US courts have in the past refused to accept forecasts of future profits. Fortunately, this legal doctrine has evolved (into the “new business rule”).[4] However, the common practice of many DCF practitioners is to extrapolate past profits, thus making the method inapplicable to start-up and other businesses with little or no historical experience of profitability.
- The existence of real options in many businesses belies the principle underlying the income method. It is clear that a business holding valuable real options (such as leases to natural resources, intellectual property, or contract rights) has valuable assets, and if the business intends to exploit those assets, it has some value as a business. However, the income method does not correctly price such value even under ideal conditions. This failure is deeply rooted. It has been known for approximately 100 years that the net present value algorithm itself, even under ideal conditions, does not accurately price financial or real options.[5]

[3] References to the income method include Pratt et al., (1996), Damadoran (2002), and Hitchner (2003), all of which mention briefly the issues identified below, but generally treat them as special cases or as causes for adjustments to the standard DCF calculations. Indeed, it is common to use significant adjustments, discounts, premia, and other changes as integral parts of the DCF method. However, the fact that such adjustments are *necessary* and *common*, rather than typically unnecessary and unusual, should indicate that there is a weakness in the method.

[4] See the discussions in Gaughan (2003) and Anderson (2005). The Ninth Circuit court of Appeals opinion in *Alaska Rent-A-Car v. Avis Budget Group* also includes a discussion of the evolution of the “new business” rule in the State of New York (*U.S. Court of Appeals, Ninth Circuit, nos. 10-35137, 10-35615 (2013), 709 F.3d 872 (9th Cir. 2013) [2013 BL 61316]).*

[5] For discussion of the failure of the net present value rule, see various papers collected in Schwartz & Trigeorgis (2001); for a seminal treatment of real options (and an introduction to the dynamic programming approach), see Dixit & Pindyck (1994). Our dating of the learned observation that the net present value method is structurally incorrect when pricing financial options refers to Bachelier (1900). One could argue the observation actually extends to ancient times, but Bachelier's contribution is unambiguous. As noted in a recent mathematical biography:
"Seventy three years before Black and Scholes wrote their famous paper in 1973, Bachelier had derived the price of an option where the share price movement is modelled by a Wiener process and derived the price of what is now called a barrier option (namely the option which depends on whether the share price crosses a barrier)." See Farhot (2002).

The Recursive Method

The American mathematician Richard Bellman first introduced recursive methods in 1957 as a means of solving complicated, multi-stage optimization problems.[6] Bellman's book, *Dynamic Programming*, develops the insight that these optimization problems can be reformulated into a series to two-period optimization decisions. This framework defines value as the sum of value over the two periods modeled, where value in the first period is the immediate reward for the current state and value in the second period represents the discounted future benefits, taking into account any impact from the decision made in the current period. This method has been explored extensively in economics by Nancy Stokey and Robert Lucas (1989), Lars Ljungqvist and Thomas Sargent (2001), and in finance by Avinash Dixit and Robert Pindyck (1994), among others. Patrick Anderson (2004, 2005) described the application of this theory to business valuation, and later (2013) provided a guide for its practical use.

Anderson (2005) describes the recursive method for business valuation as follows:

> *This is, for business valuation, analogous to estimating the value of a firm that originates from two parts:*
>
> 1. *The income expected in the current period; and*
> 2. *The value of the firm at the beginning of the next period, taking into account the prospects for future earnings at that point.*[7]
>
> *Note that many discounted cash flow schedules represent data in a similar way: the first columns show income during the next few periods, during which the income can be explicitly forecasted, and then the last column shows a "terminal value" which is the expected value at that time. However, there is a key difference between the approaches: the dynamic programming approach requires the management to* optimize *the sum of the value arising from current-period income and future-period expected earnings. Thus, in contrast to the income statements common to valuation projections, the dynamic programming method assumes that managers will change expenditures when revenues change. Furthermore, it does not implicitly assume that the growth rate for revenue, or [the] ratio of expenses to revenue (even for "variable" expenses) will remain the same.*

The key elements of a practical recursive model for business valuation are shown in Exhibit 1. The mathematical formulation of such a model is shown in Exhibit 2. Further mathematical discussion, including solution algorithms, are discussed in Appendix A.

[6] Bellman, 1957

[7] Anderson, 2005

Exhibit 1: Elements of a Recursive Model of a Firm

1. *A definition of the firm.* The subject firm must be an organization with three characteristics: separate identity from its workers or owners; profit motivation for the owners; and a set of replicable business practices.[8]
2. *Identification of state variables.* State variables define the state of affairs or conditions under which the business may operate in the future.
3. *Identification of control variables.* Control variables, often called actions, are those variables under the control of the firm's management that affect the value of the firm today or in the future.
4. *A transition equation relating the state and control variables.* The transition equation describes how state variable changes over time, taking into account the chosen values for the control variables and any random elements that may also act on the state variable.
5. *A reward function.* The reward function relates the state and control variables to the reward (earnings, or losses) incurred by the subject firm or its investors.
6. *A value functional equation for the firm.* This functional[9] equation relates the value of the firm at a certain time, given a certain state of affairs, to functions of state and control variables, and possibly random processes.

Exhibit 2: The Value Functional Equation

We use the following variables and functions to construct the value functional equation:

$$f(s,x) \equiv \text{ reward function given state and action}$$
$$g\left(s,x\right) \equiv \text{ transition function}$$
$$t = 0,...,T \text{ time index}$$
$$s_t = s_0,...,s_T \text{ state variables}$$
$$x_i = x_0,...,x_M \text{ action or control variables}$$
$$\beta = \text{ discount factor}$$

The functional equation (or Bellman equation) shown below establishes the value of a firm to its owners, given the state of affairs at a certain time.

$$V(s_t) = \max_{x \in \Gamma} \left\{ f(s,x) + \beta E\left[V(s_{t+1})\right] \right\};$$
$$where:$$
$$s = \text{state};$$
$$x = \text{action}; \ \Gamma = \text{feasible set of actions.}$$

Note that the maximization operator applies to the *sum* of current rewards and expected discounted value. Solution algorithms and further mathematical references are contained in Appendix A.

[8] See Chapter 4 of Anderson (2013) for discussion, and comparison between this definition and one that extends to investment portfolios, non-actively managed companies, tax-filing entities, and other "businesses."

[9] See Appendix A for a discussion of the mathematical idea of a "functional."

Differences from the Income Method

The importance of the *optimization* in the recursive method cannot be overstated. Unlike the traditional methods of business valuation, which often assume away the active role played by management, the recursive method is centered upon the important decisions faced by the business manager.

Even in cases where the potential reward from operating a firm is limited to the earnings (profits) of the company, and where assumptions about investor discount rates, typical expense ratios, and trend growth rates are identical, the Income and Recursive methods differ significantly. In particular:

- The recursive method explicitly models actions that can allow the manager to capitalize on new opportunities or to mitigate the effects of damaging business conditions. This allows for the native inclusion of real options, which are not properly modeled by a DCF model. It also places management in an active role, which is different from the passive management implicitly assumed by trend growth rates that are common in DCF models.
- The identification of multiple states, and the explicit use of costs of various management actions, is sharply different from the naïve--but quite common--assumption in DCF models that a business's operations will continue unchanged with conditions that remain stable and growing for an indefinite stretch of time.
- The explicit allowance for transitions into multiple possible states allows for the inclusion of asymmetric risks, in contrast to the implicit assumption in many DCF models that the distribution of possible outcomes is symmetrical and centered on the projected trend growth.
- Barring the erroneous inclusion of an undefined mathematical operation, DCF models always have a readily-calculated answer.[10] On the other hand, functional equations do not always have an answer, and even when they do, finding the answer can be a burdensome task.[11]

Practical Use in Forensic Economics: Estimating Commercial Damages

The theory of recursive models, as described above, is very recent when compared to traditional methods such as the income method of valuation. Even more recent is the ability to practically use the method. The practical use of the method was delayed due to several factors, including:

- The relative obscurity of the theory.
- The mathematical origins of the theory, which lie largely within Control Theory rather than business, economics, finance, or management science.
- Serious computational difficulties, centering on the well-known instability of a recursive (replicating within itself) algorithm within the computing world.
- A host of practical difficulties, including the difficulty in composing, error-checking, ensuring that a solution exists, solving, and describing the results.

[10] Such erroneous operations that can creep into DCF models include: dividing-by-zero; assuming that trend growth rates are higher than trend discount rates; and (less frequently, but more interestingly) attempting to use imaginary numbers to describe business conditions such as revenue, expense ratios, or sales.

[11] The theoretical conditions under which a value functional equation can be solved were outlined by Stokey & Lucas (1989). The practical conditions under which a business valuation problem can be formulated as a value functional equation and solved were stated by Anderson (2013).

However, within the last year or two, tremendous progress has been made toward making the method practical for a mathematically-trained forensic economist. In particular:

1. The book *Economics of Business Valuation* (Anderson, 2013) was published, laying out important theory and practical uses.
2. Commercial software is now available,[12] albeit on a power user platform, that helps a skilled user in composing, error-checking, ensuring the availability of a solution, and then solving a recursive valuation or damages problem.
3. A small set of published works now describe the use of the method.[13]

Relying on all three of these innovations, we provide in the remainder of this paper three separate examples, largely composed and solved by three different economists, of commercial damages estimation problems that were successfully formulated and solved using this method. At least one of these examples can be matched with the damages estimated, using traditional methods, in an actual US breach of contract case, with a jury award and a published federal court decision.

[12] The Rapid Recursive® toolbox, from Supported Intelligence, LLC (www.supportedintelligence.com), is the first commercial software in the world that allows users to compose, error-check, solve using recursive methods, and report the results of sequential decision problems. This software runs on the industrial-strength computing platform MATLAB®, produced by The MathWorks, Inc. (www.mathworks.com). The Rapid Recursive® toolbox is patent-pending.

The authors of this paper all have affiliations with Supported Intelligence, LLC: Mr. Anderson serves as the Executive Chairman and is the inventor of the Rapid Recursive toolbox, Mr. Johnson is the Vice President and Chief Operating Officer and has made significant contributions to the development of the software, and Dr. McManus is an Accredited Consulting Partner of the company.

[13] See Anderson (2013) and Anderson (2014).

Case I: Breach of Contract in *Alaska Rent-A-Car*

This case arose from a dispute over franchise terms in the rental car industry,[14] and pitted the Avis franchisee in the State of Alaska against the Avis-Budget Group, a corporation that combined the operations of the Avis and Budget franchisors nationwide. The case was the subject of a jury decision in a Federal District Court that focused on the amount of damages Avis-Budget must pay Alaska Rent a Car for breaching a settlement agreement involving the franchisees and the franchisor. The jury heard competing experts estimate damages, and witnessed an extensive *voir dire* cross examination of the franchisee's expert on his methodology and data as part of a *Daubert* challenge. After the judge admitted the expert testimony, the jury was presented with vastly different estimates of damages: $15.875 million plus interest from the franchisee's expert, and zero from the franchisor's expert.

The jury returned a verdict of $16 million.

Avis-Budget Group appealed. The Ninth Circuit accepted the appeal, and the decision to admit the expert testimony, as well as the degree to which the methodology used by the expert met a statutory standard for "certainty," was then reviewed by a panel of the Court's judges. The Court affirmed the decisions of the court on admissibility and concluded that the methodology met the statutory standard for "certainty" in the estimation of damages. In January 2014, the US Supreme Court denied Avis-Budget group's petition for further appeal, thus ending an epic confrontation between the small but successful franchisee in Alaska, and the multi-billion-dollar franchisor operating in multiple countries. Avis-Budget was ordered to pay $19.2 million in damages, interest, and attorneys' fees.

This is an important precedent for forensic economics practitioners, for both understanding the legal bases for expert testimony; the lengthy factual and methodological record that is publicly available; and clearly defined conclusions of law, expert testimony, and jury deliberation.

Using that extensive record, we excerpt from the Court of Appeals decision the key facts in the case in Exhibit 3, and a summary of the critical damages issues in Exhibit 4.

For the purposes of evaluating the usefulness of the recursive method, we approximately replicate the "lost profits" estimate presented by the franchisee's expert in the case using the Income method, and then provide a comparison estimate using the Recursive method. As the purpose of this exercise is to review methodology, we have simplified the calculations in several respects from what was presented in the courtroom.

[14] U.S. District Court, Alaska, no. 3:03-cv-00029-TMB (2009);
U.S. Court of Appeals, Ninth Circuit, nos. 10-35137, 10-35615 (2013), 709 F.3d 872 (9th Cir. 2013) [2013 BL 61316];
Petition denied, U.S. Supreme Court, no. 13-330 (2013)

Exhibit 3: Alaska Rent-a-Car v. Avis Budget Group, Summary of Facts

Alaska Rent–A–Car's predecessor began doing business as an Avis licensee in 1956, three years before Alaska attained statehood. Most other Avis licensees had a defined territory in a locality, not an entire state, within which they had the exclusive right to rent cars on behalf of Avis. Avis reasonably considered Alaska different.

In its 1959 agreement, the Alaska Avis licensee was entitled to operate in the “entire State of Alaska,” about 20% of the entire United States, but a negligible percentage of the nation's roads. The license was renewed in 1965, this time giving Alaska Rent–A–Car exclusive rights in specific locations within Alaska. A 1976 amendment added additional locations to the license agreement, and gave Alaska Rent–A–Car a right of first refusal for control of any license Avis planned to grant anywhere in Alaska. It also gave Alaska Rent–A–Car the right to expand into new territory, such as temporary camps during the construction of the oil pipeline from Prudhoe Bay to Valdez during the 1974–1977 period.
...
Avis bought a company called Agency Rent–A–Car in 1995. Some of Avis's licensees claimed that Avis was breaching their license agreements by operating another rental car company in their territories. To protect itself against these claims, Avis sued thirteen of its licensees, and sought class certification, to obtain a judgment that its purchase of Agency Rent–A–Car and its changed operations did not violate licensee rights. Avis and named defendants settled in 1997, without ever litigating to class certification or judgment. Our case arises out of that settlement, which allows Avis to purchase additional rental car companies, but requires that “the sales, marketing and reservation activities, operations and personnel of and for the Avis System will not be utilized to market, provide, and/or make available car rental services” for any additional rental car company purchased by Avis. The settlement agreement protected Avis licensees from the risk of Avis using its personnel to steer customers and potential customers towards another brand.
...
Avis bought Budget Rent–A–Car out of bankruptcy in 2002. It then restructured its central operations, putting the Avis and Budget marketing teams under unified management, creating a single team to answer calls to both Avis and Budget reservation lines, and combining the Avis and Budget national corporate sales forces. The obvious threat from these actions to Avis's licensees was that Budget would bleed off some of their customers and potential customers.
...
Alaska Rent–A–Car sued Avis claiming that Avis had indeed breached the settlement agreement, causing Alaska business to be switched to Budget Rent–A–Car, its local competitor. The district court granted a partial summary judgment, establishing that Alaska Rent–A–Car was a party to the settlement agreement, and that Avis had breached the agreement by using the same personnel to sell and market both Avis and Budget cars. Damages were left for jury trial. The jury returned a verdict in favor of Alaska Rent–A–Car for $16 million.

United States Court of Appeals, Ninth Circuit, Nos. 10–35137, 10–35615, Decided: March 6, 2013. Excerpts retrieved from: http://caselaw.findlaw.com/us-9th-circuit/1624228.html

Exhibit 4: Causes of Damages in Alaska Rent a Car

Each side put on testimony of an expert witness on damages. Avis objected under Federal Rule of Evidence 702 and Daubert v. Merrell Dow Pharmaceuticals24 and its progeny to allowing Alaska–Rent–A–Car's expert to testify.

...

The task, for both sides, was to figure out how much business and how much profit Alaska Rent–A–Car had lost on account of Avis's breach of the settlement agreement. Avis had breached when it bought Budget Rent–A–Car out of bankruptcy in 2002 and then merged much of the two companies' national sales and marketing staffs into one.

As in any damages case, the calculation had to address a hypothetical world that never existed, one in which other things remained the same but the breach had not occurred. To calculate damages from the breach, as opposed to damages from competition, Alaska Rent–A–Car's expert witness compared Avis's and Budget's experience with Alamo–National's (Alamo) experience after Cerberus bought Alamo out of bankruptcy at around the same time. His theory was that Alamo and Budget both got infusions of capital and management enabling them to compete, but differed in that Cerberus did not rent cars through any other company, and Avis did, through Budget. Thus Alamo could not benefit from merging sales and marketing activities because Cerberus had no other car rental company, but Budget could. His assumption was that Budget would have performed much like Alamo but for the benefit of a unified Avis–Budget sales and marketing effort.

Budget rebounded much faster than Alamo. The witness in effect treated the faster rebound of Budget as attributable to the breach of the settlement agreement. He used Alamo's national rate of rebound as a rough approximation of how Budget, had it not had the benefit of the breach, would have performed in Alaska. He then projected how much market share Budget gained each year due to the breach. He testified that he used Alamo's national rate of rebound as an approximation for how Budget in Alaska would have performed. He reasoned that the rental car market is a national market, and that national rebound rates would not be skewed by idiosyncratic local factors.

...

According to Alaska Rent–A–Car's witness, Alamo's national market share dropped 35% after it went into bankruptcy, slowly recovering after Cerberus bought it. Budget was in bankruptcy a shorter time, and recovered faster after Avis bought it. The witness, saying that he wanted to be conservative in his estimates, assumed that Budget would have lost 32.5% of its market share (slightly less than Alamo) had Avis bought it out of bankruptcy but not breached the settlement agreement.

Because the revitalized Budget would draw customers from other car rental companies too, not just Avis, the witness picked the Juneau airport to approximate how much of the bite would come out of Alaska Rent–A–Car. Juneau had the advantage of simplicity, because he could examine a market before Budget entered and after Budget entered, to approximate how much business it took from Alaska–Rent–A–Car. Over the first three years of its entry into the Juneau market, Budget got an average of 23.3% of the Juneau rental car market. About 48% of that market share gain came from Alaska Rent–A–Car customers, 52% from Hertz and other competitors. So to get a statewide figure, the witness made the assumption that after the breach, Budget got about half its customers from Alaska Rent–A–Car statewide. He calculated Budget's market share after the bankruptcy, assumed that but for the breach Budget's rate of market share recovery would have been similar to Alamo's national rate of recovery, and assumed that about half of its faster recovery came at the expense of Alaska Rent–A–Car. These assumptions and inferences generated lost profits calculations of $4.079 million from 2003 to 2008 due to the breach, and future lost profits, discounted to present value, of $11.708 million.

Avis challenges the expert's assumptions and comparisons....

...

The jury returned a unanimous $16 million verdict for Alaska Rent–A–Car, slightly more than the $15,787,182 in damages that Alaska Rent–A–Car's expert witness calculated. Avis's expert witness offered no total number at all to the jury, just critiques of the other expert's assumptions and calculations, with some numbers differing from his for component parts. Avis thus presented the case to the jury as a $16 million or nothing choice. Avis argued in its close that the burden of proof on damages was on Alaska Rent–A–Car, and that its expert was effectively impeached by theirs, so no damages should be awarded.

United States Court of Appeals, Ninth Circuit, Nos. 10–35137, 10–35615, Decided: March 6, 2013. Excerpts retrieved from: http://caselaw.findlaw.com/us-9th-circuit/1624228.html

The expert in this case estimated damages from the breach of the franchise agreement in three categories:

1. Cannibalization: The resurrection of the Budget brand sustained an active competitor in the Alaska market, thus cannibalizing market share Avis had already won. This was a sharp change from the non-breach marketplace
2. Subsidization: The clear effect of the consolidation of Avis and Budget operations was to bring the benefits of Avis's superior customer service and operations to their former competitor, Budget. This would, over time, effectively subsidize the competitor and allow it to slowly gain market share from Avis.
3. Loss of Growth Option: Alaska Rent-a-Car had effective rights to the entire State of Alaska, and had a history of investing in expansion opportunities in smaller towns that, at least potentially, could grow into profitable locations in the future. The breach effectively eliminated these growth options, as the franchisor could now place a direct competitor in the same location, and effectively subsidize its operation as well.

The expert's analysis, summarized in the excerpts from the Ninth Circuit opinion, used historical analogies with other entrances and exits from the market, as well as ample financial and industry data, to form the basis of a forecast of future revenue in the industry in that state. The largest part of the damages estimate originated from the cannibalization and subsidization causes, as these were large and immediate, and the advantages now given to the competitor would cause an erosion of market share over time.[15]

However, as is the situation in most practical cases involving "real options," there were no market observations or specific historical data with which to directly estimate the value of the lost growth option. Thus, the expert was forced to either use indirect methods to estimate this loss, ignore it entirely, or subsume it within an overall trajectory of lost revenue. It is this area where the availability of a recursive method provides an opportunity to dramatically improve the methodology available to skilled experts. Thus, we concentrate on this aspect of damages in this paper.

Income Method Adapted for Real Options

The method used by the expert in the trial testimony could be characterized as following an Income approach heavily modified for real options. It involved the following steps:

- Identify the costs of maintaining the policy of being ready to enter new markets, and to enter them when possible. This is equivalent to the cost of purchasing an option. Financial and operational information, and management policy, provided a basis for this.
- Identify the likely rewards of a successfully executed growth option. In this case, data on a successful expansion into a small town with an airport was available. This was equivalent to the payoff of an option.

[15] The expert's analysis did allow for other entrants into the market to fill part of the vacuum left by the exit of Budget, and, as noted in the Ninth Circuit opinion, used data from a natural experiment at an Alaska airport to estimate the fraction of the former Budget market share that would have gone to Avis.

- Using the “cost” and “payoff” information listed above, construct a simple earnings and cost projection that, in total, could be expected to approximate the value of the lost growth option given by the preceding cost and payoff characteristics.

The expert used for this projection a cash flow schedule incorporating a single successful exercise of such an option during the forthcoming years, occurring in a reasonably near time period but not immediately.

In preparing this schedule, the expert assumed that the business maintained the policy (and related expense) of pursuing expansion opportunities when they were presented, and that the operating costs resulting from the exercise of the growth option would have, at least after an initial period of adjustment, a similar cost structure as the business as a whole. The expert also used the same discount rate as for the business as a whole, reasoning that this was part of the general business risk and financing burden of the business. These assumptions are presented in Table 1 below.

Table 1: Assumptions for Growth Option Analysis

Assumptions	1 Expansion	1.5	2 Expansions
Additional revenue per period when growth option exercised	$ 520,000	$ 680,000	$ 900,000
Cost and margin ratios, excluding costs to sustain option:			
Gross margin on incremental revenue, short term:		38.9%	
Add'l semi-fixed costs to sustain operations, medium term		15.0%	
Add'l fixed costs, reinvestment, and other costs, long term		30.0%	
			83.9%
Cost to "purchase" or be prepared to exercise option:			
Minimum	$ 65,000		
Maximum	$ 150,000		
Chance of favorable event allowing exercise of option, each period:			
Lambda	30.00%		
Company discount rate:	13.20%		
Trend growth rate of revenue in industry, near term:			
near term:	4.39%		
long-term	2.50%		

Source: Data described in text, Author’s analysis

The results of this method, using assumptions that represent a simplified version of the facts in the Alaska Rent-a-Car case, are shown in Table 2 on page 14. The cash flow table incorporates the cost of the option, the payoff when it is exercised, the operating costs (which, over time, move towards the overall cost structure of the company), and also a terminal value representing the ongoing earnings of the operation that originated from the successful exercise of the growth option. Almost all of these required significant professional judgment.

This method, in the expert's opinion, avoided exaggerating the benefits of the growth option, by including the costs of the option and the operating costs, and by using a reasonable payoff. It also took into account the likelihood of successful exercise, by calculating the benefits of one payoff in a close, but not immediate, time period.

However, this method does not take into account the majority of possible scenarios, nor incorporate many of the related managerial decisions that will be made in future years. It is important to note here that there is no closed-form formula for the calculation of the "price" of such a real option, and indeed there is no traded security that incorporates the risks and opportunities involved in the business giving rise to such an option. Furthermore, as has been noted in other critical examinations of available "real option" methods, naively using a method designed for financial options in ideal markets (such as the Black-Scholes-Merton formula) to estimate the purported price of a real option is fraught with problems.[16]

Finally, this methodology, and all the data and assumptions involved, was available for criticism by an opposing expert at trial and the subject of extensive, aggressive cross-examination by opposing counsel in front of the jury.

Comparison with Recursive Method

For this demonstration, we composed and parameterized a recursive model of the growth option, using where possible the same assumptions as in the heavily-modified income method described above. In particular, the same payoff, operating margin, discount rate, and growth rate assumptions were used directly.

However, the recursive methodology allowed us to natively incorporate the proper dynamic for the uncertainty and management decision underlying the growth option. In particular, we used directly the *lambda* parameter for the probability of a favorable event occurring, and used directly the various managerial options related to the "purchase" of the option. This allowed for the possibility that the company could, if it was fortunate, exercise more than one growth option over time if it maintained the expense of preparing to launch new locations. It also allowed for several years to pass before conditions were favorable enough to launch even one. Both of these are possibilities that the management would have considered, and both are contemplated within the recursive model.

The recursive model here allows for different states to account for the existence of zero, one, or more possibly successfully executed growth options, with payoffs related to the same assumptions underlying the income model. It also allows for the company management to decide to pursue, not pursue, or aggressively pursue the growth option. All of these are consistent with what an actual manager of a good-sized business in this industry would consider as possibilities.

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[16] Among the problems: the Black-Scholes formula assumes perfect markets, borrowing at the risk-free rate, and no counterparty risk, all of which are almost certainly incorrect for real options involving privately-held firms; and the lack of traded securities means that the ubiquitous "no arbitrage" assumption of Mathematical Finance must be questioned.
For discussion of available real option methods and their potential misapplication, see Anderson (2013). For a mathematically rigorous presentation of real option values where data are available and markets and businesses are assumed to fulfill strong (and often heroic) conditions, see Dixit & Pindyck (1994).

The estimate of damages was set up by running the model in two scenarios: a “no breach” scenario that allowed for continued “purchase” and occasional execution of the growth option; and a “breach” scenario that allowed for only one such option to be exercised. The difference in value between the two represents the lost value to the company due to the prevention of its expansion strategy in the future. Note that values are shown for the case where no expansion has occurred and where one expansion has already occurred.

Table 2: Growth Option: Comparison of Methods

Recursive Model: Results

	No-Breach	Breach	Difference (Lost Value)	*Note on mitigation*
Value: No Expansion	$ 2,965,700	$ 2,201,400	**$ 764,300**	*Management policy involves expenses for*
Value: One Expansion	$ 3,883,400	$ 2,975,700	$ 907,700	*option when possible to achieve success*
				in no-breach case, but not in breach case

DCF Model of Proxy Growth Option, with Subjective Adjustments: Results

year	0	1	2	3	4	5	6	6
revenue		$ -	$ -	$ 591,534	$ 617,503	$ 644,611	$ 672,910	TV
gross margin				39%	39%	54%	54%	
gross add'l earnings		$ -	$ -	$ 230,107	$ 240,209	$ 347,445	$ 362,698	$ 98,221
cost of option		$ 65,000	$ 65,000	$ 65,000	$ 65,000	$ 65,000	$ 65,000	-
add'l fixed & reinvestment costs		$ -	$ -	$ -	$ -	$ 193,383	$ 201,873	10.70%
net cash flow		$ (65,000)	$ (65,000)	$ 165,107	$ 175,209	$ 89,062	$ 95,825	$ 917,953.56
discount factor		1.132	1.281	1.451	1.642	1.859	2.104	2.104
Present value		$ (57,420)	$ (50,725)	$ 113,822	$ 106,701	$ 47,914	$ 45,541	$ 436,257
Sum of Present Value	**$ 642,089**							

Source: Author’s Analysis

The results of the recursive model incorporate the optimal management decision, and therefore include both a value and a policy element for each state. In this case, the value difference between the “no breach” and “breach” scenarios is slightly higher than, but still close to, the estimate given by the heavily-modified income method. The difference between the two estimates is close enough to be explained by any number of modest differences in assumptions regarding costs, discounting, and treatment of terminal value. Of course, the income method has been forcibly modified to accommodate the existence of a real option, and therefore the professional judgment used to create it is arguably the dominant causal force.

Case II: Lost Intellectual Property of a California Auto Dealer

This section also describes an actual case, arising from alleged intellectual property theft between two California automobile dealers. This case has not been published, and we have obscured the identities and some facts below to preserve the confidentiality of the involved parties.

The Plaintiff in this case operated Dealership A in California throughout the 1990s, selling their interest in 1999. According to another suit, filed by the new owners of Dealership A, during 2009 and 2010, two employees moved from Dealership A to Dealership B (selling the same brand of automobiles), taking with them a list of Dealership A's customers. Dealership B allegedly used this list to solicit customers. That suit was settled before trial. Shortly after settlement, the owners of Dealership A decided to give up their interest in the franchise. The Plaintiff executed a contract with the OEM to resume ownership of Dealership A immediately after termination of the current owner's franchise. The Plaintiff also decided to relocate Dealership A to a nearby location.

Nearly one month before the transition, OEM sent an email to its dealers in the area detailing the hand-off and stating that Dealership A's new location would be "close to the current location." Plaintiff claims that Dealership B ("Defendant") sent a mailer to the customers of Dealership A obtained from the two salesman involved in the case referenced above. This mailer allegedly informed recipients that Dealership A would be "closing its doors," and that Defendant would be the "closest authorized [brand] dealer." This mailer was sent to over 3,000 recipients. Plaintiff claimed that, as a direct result of this mailer, warranty service/maintenance business at Dealership A declined nearly fifty percent. Plaintiff claimed damages in the form of lost profits.

Central Concept: Customer Lifetime Value

In order to provide a basis value for the calculation of damages in this case, we conducted a Customer Lifetime Value (CLV) analysis. [17] CLV is a metric commonly used in the retail industry as a measure of the net present value of future earnings from existing customers. This metric takes into account marketing costs and other costs associated with providing sales.

It is certainly natural to conceive of a long-lived relationship between customers and dealers. The lifecycle of the customer includes the stages of buying a new car, bringing that car in for warranty service/maintenance, continuing to bring the car in for non-warranty service/maintenance, and eventually purchasing another new car. The expected revenue differs at each of these stages, and the discounted sum represents the CLV to the dealer. Estimating this metric would allow us to establish a basis for damages calculations: at most, damages to the Plaintiff could be the sum of CLVs for all customers affected by the mailer in question.

Traditional CLV analysis relies on a discounted cash flow method that naively assumes consistent customer behavior. In the traditional framework without adjustments, a customer would continue to purchase a new car each year throughout their relationship with the dealer. Even with common

[17] The standard methods for conducting such an analysis are described in Bursk (1966), Gupta et al. (2006), and Singh et al. (2013).

adjustments, the traditional model ignores the idea that customer behavior may change in the future, and even more importantly, that the dealer may be able to influence the likelihood or direction of this change.

We describe below the framework and application of a CLV analysis that relies on the recursive method. This new method relaxes the assumption that customers continue behavioral patterns with certainty, and explicitly accounts for the ability of the dealer to influence changes in customer behavior through carefully selected marketing efforts.

Creating a Recursive Model for Customer Lifetime Value

As discussed in Exhibit 1: on page5, creating a recursive model requires the identification of state and control variables, and the description of transition and reward functions. For the specific application of the lifetime value of a customer to an automobile dealer, we set the state variables to the different stages in the customer lifecycle (see above) and the control variables represent the marketing activity of the dealer. We describe each of these, and the related transition and reward functions, in greater detail below.

State Variables

For this model, the states represent the different stages in the customer lifecycle with the auto dealer. We present these stages, along with a description and the assumed margin earned by the dealer from the sale or service associated with each state.

Table 3: State Set

State	Description	Margin before Marketing Expenses
New Care Sale-Premium	This state captures the period where a customer purchases a vehicle that generates a higher-than-average margin for the dealer.	$1,087.50
New Car Sale	Customers in this state purchase a new car, the sale of which provides an average margin to the dealer.	$725.00
New Car Service-Premium	This state represents the customers who have recently purchased a car and are now bringing that car in for warranty repair and service, where the dealer earns a higher-than-average margin.	$17.00
New Car Service	For customers in this state, the dealer performs warranty maintenance/service, earning average margins.	$12.00
High Service-Premium	Customers in this state bring vehicles to the dealer for service that is not covered under the manufacturer's warranty. As with other "premium" states, this state also includes higher margins for the dealer.	$39.60
High Service	Same as above, but at average margins to the dealer.	$26.40
Zero Service	Customers in this state own a car that needs service (and was either purchased at, or previously serviced by, the subject dealer), but have that service performed by an organization other than the subject dealer. These customers are not in the market for a new car.	$0
Zero Sales	This state captures customers who are in the market for a new car but	$0

have not yet purchased one.

Source: Data provided by Client, Author's assumptions

These eight states represent all possible stages of the relationship between a customer and the auto dealer, and form the basis for our recursive model for customer lifetime value.

Control (Action) Variables

A natural control variable in this situation is the marketing effort undertaken by the auto dealer. This works well because dealers are constantly marketing to customers and potential customers in each of the states described above, and it is not unreasonable to assume that these marketing efforts have non-negligible effects on consumer decisions. Additionally, we can construct a relatively compact set of possible marketing actions, which makes calibrating the model a much easier task. We present this set of actions in Table 4 below.

Table 4 also includes the cost to the dealer of taking each of the identified actions (discussed in greater detail in the Reward Function section on page 18).

Table 4: Action Set

Action	Description	Cost
Ignore	Keep the customer in the database, but do not send any promotional materials.	$1.00
Purge	Remove the customer from the database.	$1.50
Baseline	Continue with all non-discretionary marketing: this may include radio and television ads, billboards, etc…	$5.50
Service Offer	Send the customer a coupon for work done in the service department.	$11.50
Sales Offer	Send the customer a coupon for the purchase of a new car.	$15.50
Instant Offer	Send the customer a fast-expiring coupon.	$7.50
Reminder Only	Send the customer a card or letter reminding them of the positive experience of buying/servicing a vehicle at the dealership.	$9.50

Source: Data provided by Client, Author's assumptions

Transition Function

With the discrete state and action spaces defined above, we can represent the transition function as a three dimensional matrix. If we set *S* to be the number of states, and *A* the number of actions, then the transition matrix has size *SxSxA* or 8x8x7. Each entry in the transition matrix contains an estimate of the likelihood of a customer moving from one state in the current time period to a specific state in the next time period, given the action taken by the dealer. For example, the (8,2,5) entry contains the likelihood of a customer moving from the “Zero Sales” state (state 8) to the “New Car Sales” state (state 2), assuming the dealer sends that customer a sales offer (action 5).

For this model, we used both customer data from Dealership A and professional judgment from experience working with automobile dealers to complete the transition matrix. We present one frame from this matrix in Table 5 on page 18. For readability, only non-zero entries are shown.

Table 5: Transition Matrix for Action "Sales Offer"

		Future State							
		New Care Sale-Premium	**New Car Sale**	**New Car Service-Premium**	**New Car Service**	**High Service-Premium**	**High Service**	**Zero Service**	**Zero Sales**
Current State	**New Care Sale-Premium**			0.95				0.05	
	New Car Sale				0.95				0.05
	New Car Service-Premium	0.07		0.35		0.35			0.23
	New Car Service		0.07		0.35		0.35	0.23	
	High Service-Premium	0.08		0.12			0.58		0.22
	High Service		0.08		0.12	0.58		0.22	
	Zero Service				0.05			0.95	
	Zero Sales			0.05					0.95

Source: Data provided by Client, Author’s analysis

Reward Function

The reward function describes the reward to the dealer for each possible state-action pair. For this model, we set the reward equal to the margin received by the dealer for any sales or services occurring in that state, net of the cost of the dealer’s selected action. Similar to the transition function, we also represent this as a matrix, though one with only two dimensions. In the reward matrix, the (i,j) element is the reward received by the dealer for sending offer j to a customer in state i. We present the full reward matrix in Table 6 below.

Table 6: Reward Matrix

	Action						
State	**Ignore**	**Purge**	**Baseline**	**Service Offer**	**Sales Offer**	**Instant Offer**	**Reminder Only**
New Sales Premium	$1086.50	$1086.00	$1082.00	$1076.00	$1072.00	$1080.00	$1078.00
New Sales	$724.00	$723.50	$719.50	$713.50	$709.50	$717.50	$715.50
New Service Premium	$17.00	$16.50	$12.50	$6.50	$2.50	$10.50	$8.50
New Service	$11.00	$10.50	$6.50	$0.50	-$3.50	$4.50	$2.50
High Service Premium	$38.60	$38.10	$34.10	$28.10	$24.10	$32.10	$30.10
High Service	$25.40	$24.90	$20.90	$14.90	$10.90	$18.90	$16.90
Zero Sales	-$1.00	-$1.50	-$5.50	-$11.50	-$15.50	-$7.50	-$9.50
Zero Service	-$1.00	-$1.50	-$5.50	-$11.50	-$15.50	-$7.50	-$9.50

Source: Data provided by Client, Author’s analysis

Results

After composing the problem as described above, we calculated the lifetime value of a customer in each of the eight states. See Recursive and Discounted Cash Flow Methodology for a description of the methods used to solve this problem. Note that the results include both the calculated value and the value-maximizing policy for each state. We present these results in Table 7 below.

Table 7: Results of Recursive Customer Lifetime Value Calculation

State	Value	Policy
New Sales Premium	$1346.27	Baseline
New Sales	$879.60	Ignore
New Service Premium	$306.40	Instant Offer
New Service	$185.01	Baseline
High Service Premium	$355.28	Sales Offer
High Service	$213.90	Sales Offer
Zero Sales	$56.74	Ignore
Zero Service	$25.61	Baseline

Source: Author's analysis

Analysis: Rapid Recursive® toolbox

The values in this table can be interpreted as follows: the lifetime value of a customer in a given segment is the value reported in the middle column, assuming that the dealer follows the recommended strategy over the duration of the customer's "life" at the dealership. For example, a customer from the "New Sales" state will likely move to the "New Service" state, then "High Service" and so on. The value of this "New Sales" customer is roughly $880, assuming that the dealer follows the recommended policy of "ignore" (shown in the third column). This value takes into account both the immediate reward from selling a new car to this customer ($724 as shown in Table 6) and the future stream of earnings expected from this customer as their new car ages and they return to the dealer for service and possibly additional new car purchases.

For the task of estimating damages, we were concerned with the calculated value, shown in the middle column of the table above. To arrive at a total damages estimate, we determined the number of customers of each type affected by the mailer in question and multiplied by the calculated lifetime value of customers in that segment. Assuming that many of the recipients ignored the mailer completely, and another large portion were unaffected, we estimate damages on the basis of 3 customers in each of the new sales states, 10 in each of the service states, and 30 in each of the "zero" states. This yields a total damages estimate of $19,754. The estimate asserted by the opposing expert, assumed to be based on a traditional DCF-based CLV analysis with an exaggerated linear trend, was nearly 10 times the amount calculated using the recursive model.

Outcome of the Case

The parties in this case reached a settlement agreement after the presentation of the damages estimate based on the recursive method and before trial. While the amount of the settlement payment was not disclosed, we presume it was much closer to the recursive than the traditional estimate.

Case III: Misrepresentation of Product for a Class of Consumers

In 2012, Korean automakers Hyundai Motor and Kia Motors announced, after an audit, that they had overstated fuel economy for 1.2 million vehicles sold in the United States for model years 2011-2013. The U.S. Environmental Protection Agency (EPA) said that the overstatement ranged from one to six miles per gallon, depending on the model and powertrain configuration of the vehicles. The EPA and the U.S. Department of Justice (DOJ) launched an investigation into the overstatement by Hyundai and Kia under the Clean Air Act.

Hyundai and Kia launched web-based reimbursement programs for owners of affected vehicles. On the sites, the automakers offered to pay for 10 years the difference in fuel costs between the overstated fuel economy and the actual fuel economy.

Class-action lawsuits were also filed against the Korean automakers. In late 2013, Hyundai and Kia settled class-action suits by agreeing to pay $400 million in compensation to buyers of the affected vehicles.

Hyundai and Kia announced in November 2014 that they had reached an agreement with the EPA and DOJ that settled and clarified their obligations under the Clean Air Act. Together, Hyundai and Kia forfeited emissions credits estimated to be worth $210 million, paid a civil fine of $100 million, and agreed to construct a U.S.-based emissions lab at an estimated cost of $50 million. At $360 million in fines, forfeitures, and required costs the settlement was the largest penalty ever imposed under the Clean Air Act.

Ford Motor Co., BMW, and Mercedes-Benz are other automakers who have either admitted or been accused of overstating fuel-economy estimates recently.

Estimating the Loss Imposed on Drivers by Overstated Fuel Economy

The reimbursement programs initiated by Hyundai[18] and Kia[19] are clear in identifying the drivers of the affected vehicles as the parties who are owed reimbursement for the overstatement. The programs also identified the loss experienced by each affected driver as the difference in fuel costs between what they would have incurred had the fuel economy been as claimed and what their fuel costs actually were. The automakers offered to reimburse costs for ten years from the date of purchase.

A recent survey of the literature on the value of fuel economy[20] found that there does not appear to be a consensus on the empirical value that consumers derive from a marginal increase (or decrease) in fuel economy. However, the traditional approach to valuing fuel economy need not depend on subjective consumer preferences, but can focus on fuel costs and differences in fuel costs between claimed and actual fuel economy. Simply put, the traditional approach measures actual fuel costs for a year of driving and estimates how much lower costs would have been if the claimed fuel economy had been accurate.

[18] https://hyundaimpginfo.com

[19] https://kiampginfo.com

[20] Greene (2010).

Consider a numerical example. Suppose fuel costs $3.50 per gallon, claimed fuel economy is 30 miles per gallon (MPG), and actual fuel economy is 24 MPG. Driving 15,000 miles in a year (assuming the price of fuel does not change from $3.50 in the course of the year) would cost $2,188 at 24 MPG but only $1,750 at 30 MPG. The traditional analysis would consider this $438 difference in annual cost as a measure of the value of 30 MPG over 24 MPG.

Table 8 extends the numerical example to six years of ownership. The assumption is that future expected fuel costs influence the vehicles that consumers choose to purchase today. The length of vehicle operation to include in the present value of future costs could, in principle, extend for the life if the vehicle. However, it is more common to include a shorter term, supposing that new vehicle buyers do not care about future costs that occur after their ownership. The median length of ownership of vehicles by their first owner ranges from about five to six years.[21] The National Research Council committee on fuel economy technology, whose recommendations have generally been adopted by regulatory agencies, favors a three-year payback period.[22]

Table 8: Traditional Approach to Measuring the Value of Fuel Economy

Common Factors and Assumptions					Total Fuel Cost if MPG = 30			Total Fuel Cost if MPG = 24		
Age of Vehicle	Vehicle Survival Probability	Vehicle Miles Driven	Expected Price of Fuel per Gallon	Discount Factor*	Fuel Cost per Mile	Annual Fuel Cost	Discounted Fuel Cost of Surviving Vehicle**	Fuel Cost per Mile	Annual Fuel Cost	Discounted Fuel Cost of Surviving Vehicle**
0	1.00	15,000	$3.50	1.00	$0.12	$1,750	$1,750	$0.15	$2,188	$2,188
1	0.98	15,000	$3.50	0.97	$0.12	$1,750	$1,665	$0.15	$2,188	$2,081
2	0.97	15,000	$3.50	0.94	$0.12	$1,750	$1,600	$0.15	$2,188	$2,000
3	0.96	15,000	$3.50	0.92	$0.12	$1,750	$1,537	$0.15	$2,188	$1,922
4	0.94	15,000	$3.50	0.89	$0.12	$1,750	$1,462	$0.15	$2,188	$1,827
5	0.92	15,000	$3.50	0.86	$0.12	$1,750	$1,389	$0.15	$2,188	$1,736
					Total Present Value		**$9,403**	**Total Present Value**		**$11,754**
								Difference 24 O/(U) 30		**$2,351**

* Discount factor = (1/(1+r)^age; where r = 3%.

** Discounted Fuel Cost of Surviving Vehicle combines effects of both vehicle survival and discounting.

Table 8 has three panels: Common Factors and Assumptions, Total Fuel Cost if MPG = 30 and Total Fuel Cost if MPG = 24. Survival probabilities primarily reflect the impact of vehicular crashes that result in a total loss.[23] Additional assumptions are that the average vehicle is driven 15,000 miles per year, that the price of fuel is $3.50 per gallon, and that the real interest rate is 3 percent. These assumptions produce $1,750 in annual fuel costs if fuel economy is 30 MPG and $2,188 in annual fuel costs if fuel economy is 25 MPG. Annual fuel costs are then discounted to present value (age 0) and summed across the 6 years. The total present value of fuel costs is $9,403 if MPG = 30 and $11,754 if MPG = 24. The difference of $2,351 is the traditional estimate of the value of 30 MPG over 24 MPG.

[21] Based on University of Michigan Survey of Consumers 1980-2003 data on vehicle ownership.

[22] National Research Council (2002). Another example of the three-year payback period is in Greene (2009).

[23] These estimates were published in Busse et al. (2013).

Valuing Fuel Economy Using a Recursive Model

Vehicle fuel economy (miles per gallon, MPG) is a factor in a driver's decision when choosing a new car. For many drivers, finding out that the advertised and actual fuel economies for their vehicle differed significantly would pose a major problem and likely inflict non-negligible additional costs on their operation of the vehicle. We develop a recursive model to investigate the value to a driver of owning and operating a car with two different values for fuel economy. We explicitly account for the driver's ability to mitigate the effects of a lower true fuel economy by decreasing annual mileage or even selling the vehicle altogether in favor of purchasing a new vehicle with better fuel economy. The model is solved with a Rapid Recursive simulation.

Table 9: Assumptions for the Recursive Model

Input	Value
Annual miles driven	15,000
Advertised fuel economy (MPG)	30
Actual fuel economy (MPG)	24
Price of fuel when new vehicle purchased (per gallon)	\$3.50
Purchase price of the vehicle	\$27,000

Source: Facts of the case, Author’s assumptions

To create the recursive model we define states, actions, transitions, and rewards. In addition, we assume that the states characterize a single vehicle. We assume a calendar year as the relevant period during which states of the world hold. We incorporate vehicle aging in whole years from 0 (new) to 5 (1 to 5 are used vehicles), but we assume that fuel economy does not change with vehicle age. When fuel economy is different between two states for the same vehicle age, we assume that fuel economy is the only characteristic of the vehicle that varies; all else remains exactly the same.

The traditional valuation approach ignores benefits to drivers from owning and driving vehicles, and considers only differences in costs of operation for two otherwise identical vehicles with different fuel economy. The recursive model presented here, in contrast, considers both benefits and costs. In particular, we assume that the driver derives a nonzero benefit from driving the vehicle at their intended mileage, which is at least equal to the cost of owning and operating the vehicle.

States, Actions, Transitions, and Rewards

States

The states in this model capture information about the age of the vehicle, its fuel economy, and the price of fuel. An additional absorbing state captures the case in which the vehicle is no longer in use or no longer held by the original owner. The states are three dimensional, where the first dimension represents the age of the car (six values, 0-5), the second represents the fuel economy (two values, 30 MPG advertised or 24 MPG actual), and the third represents the current gas price (5 possible values: \$2.50, \$3.00, \$3.50, \$4.00, and \$4.50). Thus there are 6 x 2 x 5 or 60 possible states, but for vehicles of age 0 we exclude all states with fuel prices other than \$3.50 (the assumed starting fuel price). Thus the model has 53 states: 52 states defined by vehicle age, fuel economy, and fuel price; plus the absorbing 53rd state.

Actions

At the beginning of each year, the vehicle owner learns what the fuel price per gallon will be that year and what the vehicle's actual fuel economy is. There are three actions that the driver can take in this model.

1. Drive (Meaning driving the planned 15,000 miles during the year.)
2. Drive Less (Decrease annual mileage by 5%)
3. Sell the Vehicle

A driver could choose to sell the vehicle immediately upon learning the fuel price and fuel economy of the just-purchased vehicle at the start of the first year of ownership (age zero of the vehicle). To avoid this unrealistic situation, we assume that the benefits of vehicle ownership and driving lead the driver to choose Drive in the first period, when vehicle age is zero.

Rewards

We assume that nominal yearly (or periodic) discount rate (d) equals 12 percent; and the nominal yearly (or periodic) growth rate (g) equals 2.5 percent. The annual discount factor (β) based on both the growth rate and the discount rate is given by: $\beta = (1 + g)/(1 + d)$. We assume an infinite horizon.

Table 10: Reward Matrix for Recursive Model

Panel A: Reward if MPG = 30

State*	Action		
	Drive	Drive_Less	Sell
30-age-2.5	$913	-$223	$500
30-age-3	$663	-$460	$500
30-age-3.5	$413	-$698	$500
30-age-4	$163	-$935	$500
30-age-4.5	-$87	-$1,173	$500

Panel B: Reward if MPG = 24

State*	Action		
	Drive	Drive_Less	Sell
24-age-2.5	$488	-$632	$500
24-age-3	$175	-$929	$500
24-age-3.5	-$137	-$1,226	$500
24-age-4	-$450	-$1,523	$500
24-age-4.5	-$762	-$1,820	$500

***States exist for six vehicle ages, but have identical reward matrices for every vehicle age. States are identified by fuel economy (30 or 24), age of the vehicle, and the per gallon price of fuel ($2.50, $3.00. $3.50, $4.00, or $4.50).**

Source: Author's analysis

Rewards for each period are determined by the driver's benefit of driving one mile, less the cost of driving one mile, multiplied by the number of miles driven during the decision period (one year). If the driver chooses to sell the car, then he incurs a non-negligible adjustment cost. Rewards in the absorbing state are always zero. Elements of the reward matrix represent the current-period net benefits to the driver, for every combination of state and action. Each row represents a state, and each column represents an action. These rewards are shown in Table 10. Rewards do not vary by the age of the vehicle, so the reward matrix in Table 10 is shown in two panels—one for each of the possible fuel economy values, 24 MPG and 30 MPG.

Rewards of Driving

We assume that the per-mile benefit from driving is greater than the per-mile costs of driving as perceived by the driver before purchasing the vehicle. In other words, the per-mile benefit from driving exceeds the expected per-mile cost prior to buying the vehicle in the first place. We assume that the total benefits from driving have a fixed component, based on tastes and expected capital costs, and a variable component based on per mile operating benefits. We assume that the total costs of driving are comprised only of variable per mile operating costs, including per mile fuel costs. For simplicity, we set per mile operating benefits equal to per mile operating costs.

The per-mile cost of driving (whether Drive or Drive Less is the chosen action) is sum of fuel costs per mile and an estimate of the average of all other variable per-mile costs. Note that the per-mile cost of driving depends on the state of the world—it varies by the price of fuel and whether the vehicle's fuel economy is the advertised (30 MPG) or the actual (24 MPG).

$$per\ mile\ cost\ (in\ \$) = \frac{price\ per\ gallon\ of\ fuel}{vehicle\ miles\ per\ gallon} + 0.0936$$

The per-mile benefit of driving is set at the time the vehicle is purchased. It uses the base price per gallon of fuel ($3.50) and the advertised vehicle miles per gallon (30) as the per-mile variable components. The third component is the average fixed component, which is based on the purchase price of the vehicle ($27,000), the sale price of the vehicle (60% of the purchase price), and the expected vehicle miles to be driven (15,000 miles per year times six years = 90,000 miles).

$$\begin{aligned} &per\ mile\ benefit\ (in\ \$) \\ &\qquad = \frac{base\ price\ per\ gallon\ of\ fuel}{advertised\ vehicle\ miles\ per\ gallon} + 0.0936 \\ &\qquad + \frac{\$2{,}476 + purchase\ price - sale\ price}{6 * mileage} \end{aligned}$$

The per-mile net benefit to the driver is simply the difference between the per-mile benefit and the per-mile cost.

If Drive Less is the chosen action, then we assume the driver can reduce vehicle mileage by 5%, but must incur a $1,000 adjustment cost to do so.

In states of the world in which fuel economy is the actual 24 miles per gallon, the sale price of the vehicle is 57.5% of the purchase price rather than the 60% it is in the states of the world in which fuel economy is the advertised 30 miles per gallon. This lowers the third component of per-mile benefit by lowering the fixed benefit. We assume that, all else equal, a car with lower fuel efficiency will have a lower resale value than one with higher fuel efficiency.

The reward for the action Drive is the total annual benefit from driving less annual depreciation. Where depreciation is one-sixth of the difference between the vehicle purchase price and sale price. The reward for Drive Less is similar, but it is based on fewer miles, a lower per-mile net benefit, and subtracts the $1,000 adjustment cost.

For the “sell” action, we assume that the driver sells the vehicle and automatically purchases a new one, which 'resets' them to their desired vehicle. This should have a positive reward, even though the pure financial reward is likely to be slightly negative. We therefore assume a $500 nominal reward for switching to a new, more fuel-efficient vehicle.

Transitions

Transitions in this model occur along two of the three dimensions: after each period the vehicle ages one year and gas prices fluctuate according to a random process. The fuel economy of the car does not change in this model.

Elements of the transition matrix represent the probability of moving to a future state, given the current state and the action taken by the agent. Each row represents a current state, each column represents a state next period, and the third dimension represents the action taken by the company.

Vehicle survival depends on the same year to year survival rates we used in the traditional valuation approach.

Results

The value of 30 MPG over 24 MPG in this model is $1,367. This is measured by the difference in value at the start of a vehicle with 30 MPG compared to a vehicle with 24 MPG, facing all the same possible future states in the model. This is $984 or 42% below the $2,351 value of fuel economy found in the traditional approach.

A large portion of the difference in the traditional and the recursive valuations results from the ability of the driver in the recursive model to change mileage and even sell the vehicle. Table 11 on page 26 shows the computation of the median holding period in the recursive model. If fuel economy is 30 MPG, then the median holding period is 4.59 years. If fuel economy is 24 MPG, then the median holding period is 2.47 years. The traditional approach locks the driver into a six-year holding period, whether the fuel economy is 24 or 30 MPG.

Table 11: Drivers Sell Vehicle Soon if MPG Was Overstated

Year to Year Survival & Holding Period if MPG = 30

Age of Vehicle	Fraction on the Road at Beginning of Period	Remaining Fraction Drive if Vehicle Survives	Vehicle Survival Probability	Remaining Fraction on the Road (End of Period)	Median Holding Period (Age at Disposal)	Memo: Probability of Choosing Drive
1	100%	100%	98%	98%		All choose Drive
2	98%	100%	97%	97%		All choose Drive
3	97%	100%	96%	96%		All choose Drive
4	96%	99%	94%	93%		1 - P(4.5)
5	93%	22%	92%	20%	4.59	P(2.5) + P(3.0)

Year to Year Survival & Holding Period if MPG = 24

Age of Vehicle	Fraction on the Road at Beginning of Period	Remaining Fraction Drive if Vehicle Survives	Vehicle Survival Probability	Remaining Fraction on the Road (End of Period)	Median Holding Period (Age at Disposal)	Memo: Probability of Choosing Drive
1	100%	78%	98%	77%		1 - P(4.0) - P(4.5)
2	77%	78%	97%	76%		1 - P(4.0) - P(4.5)
3	76%	22%	96%	21%	2.47	P(2.5) + P(3.0)
4	21%	22%	94%	20%		P(2.5) + P(3.0)
5	20%	0%	92%	0%		All choose Sell

Source: Author’s analysis

Conclusion

The workhorse method for estimating commercial damages remains the income approach, even though the fundamental principles of the method are known to be violated by the presence of real options and asymmetric risk, and where the projection of income and selection of a discount rate are frequently more art than science. However, for many valuation and damages problems, there has been no practical alternative method.

The three economists that authored this paper have demonstrated that there is now an alternative, and have demonstrated the usability of the approach in three settings based on actual controversies. Based on these demonstrations, the literature reviewed, and the experience in applying the method, we have the following four conclusions:

1. The Recursive method has ample theoretical justification, and now can be applied in practice. While the first part of this conclusion has been asserted for several years, demonstrating the applications contained in this paper and citing other published works using this method would not have been possible even one year ago.
2. The Recursive method is still more difficult than the traditional DCF method in most cases. The advances in software, the availability of published works applying it to operating businesses, and the small but growing number of examples are all very helpful. However, the theoretical basis remains obscure for many possible practitioners, and the available software is more difficult to use than a spreadsheet.
3. The most likely uses for this method in the immediate future involve those situations where significant real options are involved, where asymmetric risks characterize the subject business, and where the decisions of the subject business manager are critical in affecting the value of the company. This is a very large subset of commercial damages cases, including almost all cases involving start-up firms, un-commercialized intellectual property, portfolios with "black swan" risk, and businesses in industries such as natural resources and entertainment where real options can be extremely valuable. In these cases, the additional burden of applying a newer and more powerful method may justify the time and costs involved.
4. We have demonstrated again that simply projecting forward cash flow and then "adjusting" it on a largely subjective basis is not consistent with the principles of the Income method. In the past, when no other method was available, carefully performing these tasks and supporting them may have been the only available method. However, there are now alternatives, and good practitioners should consider them, including the Recursive method, where the facts demand it.

Appendix A. Recursive and Discounted Cash Flow Methodology

Recursive Methodology

In order to construct the recursive model, we extend the familiar concept of the mathematical function, which takes a set of numbers as its inputs, to the idea of a functional, which accepts a set of functions. The dynamic programming pioneer Stuart Dreyfus (1965) descried a functional as a rule relating *curves* to a single number. We call the fundamental equation in dynamic programming a *functional equation*, or the “Bellman Equation.” Recall from the text that this equation has the following variables:

Exhibit A-1: Value Functional Equation Variables

$$
\begin{aligned}
V(s,t) &= \text{ value at time } t \text{ given state } s \\
f(s,x) &\equiv \text{ reward function given state and action} \\
g\left(s,x\right) &\equiv \text{ transition function} \\
t &= 0,\ldots,T \text{ time index} \\
s_t &= s_0,\ldots,s_T \text{ state variables} \\
x_i &= x_0,\ldots,x_M \text{ action or control variables} \\
\beta &= \text{ discount factor} \\
\varepsilon &= \text{ random error term}
\end{aligned}
$$

Exhibit A-2: Value Functional Equation

We combine the variables from Exhibit i: as follows to construct the value functional equation:

$$V(\mathrm{s}_t) = \max_{x\in\Gamma}\{f(s,x) + \beta E_t[V(\mathrm{s}_{t+1})]\}$$

where:

s = state; x = action;

β = discount factor; Γ = set of feasible actions;

$V(s_t)$ = value at time t given state s;

$f(s,x)$ = reward function given state and action.

Resources for more information on the recursive method include Bellman (1957), Stokey and Lucas (1989), Ljungqvist and Sargent (2000), and Anderson (2013).

Two Common Solution Algorithms

The following sections describing common solution algorithms have been reprinted from the *User's Guide* for the Rapid Recursive® toolbox[24] with the permission of Supported Intelligence, LLC.

Value Function Iteration

Value function iteration (also referred to as successive approximations, over-relaxation, backward induction or pre-Jacobi iteration) is, accordingly many sources including Puterman (2005) and Powell (2007), perhaps the most widely used algorithm for solving sequential decision problems.

The value function iteration algorithm used in the Rapid Recursive® Toolbox takes the following steps:

Exhibit A-3: Value Function Iteration Algorithm

1. Choose a finite integer $N \geq 1$, the maximum number of times this algorithm will iterate.
2. Set the number of iterations, $n = 0$, and choose an initial (arbitrary) $V^n(s')$ and $\varepsilon > 0$.
3. Apply the Bellman operator to V^n by computing:

$$V^{n+1}(s) = \max_{a \in A_s} \left\{ R(s,a) + \beta \sum_{s' \in S} P(s'|s,a) V^n(s') \right\}$$

4. (Stopping criterion) If $n + 1 = N$ or

$$\|V^{n+1} - V^n\| < \frac{\varepsilon(1-\beta)}{2\beta}$$

   proceed to Step 5. Otherwise, increase n by 1 and go back to Step 3.
5. The value function is V^{n+1} and the optimal policy is:

$$a_\varepsilon(s) \in \operatorname*{argmax}_{a \in A_s} \left\{ R(s,a) + \beta \sum_{s' \in S} P(s'|s,a) V^{n+1}(s') \right\}$$

   for each $s \in S$.

The "norm," a measure of distance, in step 4 is defined as:

$$\|V^{n+1} - V^n\| \equiv \sup_{s \in S} |(V^{n+1} - V^n)(s)|$$

It turns out that under certain conditions, including that $0 < \beta < 1$, if N is a large enough finite number, this algorithm will find an ε-optimal policy, a policy whose corresponding value function is within ε of the true value function. Further, this algorithm finds a solution for any initial V^0 in a reasonably large set of functions.

It can be shown that under value function iteration V^0 convergences monotonically to the solution. In other words, if V^0 is greater than that optimal solution, as iteration occurs, each V^{n+1} from Step 3 will

[24] Available online at http://www.supportedintelligence.com/files/rrt/docs/Rapid_Recursive_Users_Guide.pdf

be lower than V^n and this will occur until the optimal solution is found. Similarly, when V^0 is less than the optimal solution, each V^{n+1} from Step 3 will be *higher* than V^n.

Policy Iteration

Policy iteration provides an efficient alternative to value function iteration for infinite time problems.

The policy iteration algorithm used in the Rapid Recursive® Toolbox takes the following steps:

Exhibit A-4: Policy Iteration Algorithm

1. Choose a finite integer $N \geq 1$, the maximum number of times this algorithm will iterate.
2. Set the number of iterations, $n = 0$, and choose an initial (arbitrary) policy a_n.
3. Given a_n express $R(s, a_n)$ and $\sum_{s' \in S} P(s'|s, a_n)$ in matrix notation as R_{a_n} and P_{a_n} respectively. Perform the policy evaluation step to find V^n by solving the following system of equations:

$$(I - \beta P_{a_n})V^n = R_{a_n}$$

4. Perform the policy improvement step by finding a_{n+1}:

$$a_{n+1} \in \underset{a \in A_s}{\operatorname{argmax}} \left\{ R(s, a) + \beta \sum_{s' \in S} P(s'|s, a)V^n(s') \right\}$$

   (setting $a_{n+1} = a_n$ is possible).
5. (Stopping criterion) If $n + 1 = N$ or $a_{n+1} = a_n$ stop. Otherwise, increase n by 1 and go back to Step 3.

Step 3 is called the "policy evaluation" step and involves solving the system of linear equations. There are a number of ways to solve a system of linear equations. The policy iteration algorithm in the Rapid Recursive® Toolbox provides users with two choices: Gaussian elimination with partial pivoting or the Jacobi method.

It can be shown that policy iteration converges when the state and action sets are finite and if N is a large enough finite number. Notwithstanding the fact that numeric applications of policy iteration are only practical when state and action sets are finite, it can also be shown that convergence of the policy iteration algorithm also occurs for a broader set of state and action sets as long as an argmax exists in step 4.

A similar monotone convergence result also holds for policy iteration under complete generality: the value function obtained in Step 3, V^n, is always higher than the value function obtained at the previous iteration V^{n-1} i.e. $V^n \geq V^{n-1}$.

Discounted Cash Flow Methodology

The workhorse method in damages estimation for operating business has long been the income, or discounted cash flow, method. This method involves estimating the cash flow of the business at each time period, discounting these values appropriately, and calculating the sum. This process is described in the equation below.

Exhibit A-5: Discounted Cash Flow Equation, Discrete Time

$$NPV = \sum_{t=0}^{T} \beta^t E_0(C_t)$$

where

$t =$ time index,

$\beta =$ discount factor; $0 < \beta < 1$,

$C_t =$ lost cash flow at time t,

$E_0 =$ expectation taken at $t = 0$.

Solving this equation requires straightforward calculations, often performed in spreadsheet software such as Microsoft Excel. References to the income method include Pratt et al., (1996), Damadoran (2002), and Hitchner (2003).

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Software

Rapid Recursive® Toolbox

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