Willingness to Wait Under Risk and Ambiguity*

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Abstract

Timing of irreversible decisions depends on decision makers' willingness to wait. This paper studies the distinctive effects of risk and ambiguity on this willingness. We analyze a simple optimal stopping problem in which a decision maker observes an uncertain environment and chooses the timing of an irreversible action. We replicate the model in a laboratory experiment that elicits subjects' willingness to wait. Higher risk increases willingness to wait confirming the model's predictions. Higher ambiguity also increases willingness to wait. Because the model predicts that ambiguity decreases willingness to wait if decision makers are ambiguity averse, this finding is inconsistent ambiguity aversion in timing decisions.

Keywords: Waiting, optimal stopping problem, risk, ambiguity, real options, experiment.

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1 Introduction

Optimal stopping problems are pervasive in economics and finance. Examples are abundant and include a firm's capital investment, job search, industry entry and exit, mergers and acquisitions, and default. In all these situations, agents have a possibility to choose the timing of an irreversible (or partially irreversible) action. Among the factors that characterize agents' waiting and stopping decisions, uncertainty and learning play a prominent role. Uncertainty, by changing the probability of extreme events, affects the attractiveness of waiting. Learning is a natural consequence of the opportunity to wait and to observe the economic environment. This paper studies, both theoretically and experimentally, the effect of uncertainty and learning on willingness to wait.

The literature on timing decisions has mainly interpreted uncertainty as risk. However, it is well known that uncertainty can occur in two different forms, with known probabilities, or risk, and with unknown probabilities, or ambiguity. The distinction between risk and ambiguity not only has a behavioral significance, as documented by an extensive experimental literature, but can also provide a more appropriate description of reality. In many economic problems, information may be so imprecise that decision makers cannot attach probabilities to uncertain events. When decision makers choose between acting immediately and waiting, this fact leads to two separate questions.

(1) Does higher risk increase or decrease willingness to wait?

(2) Does higher ambiguity increase or decrease willingness to wait?

The first question has been largely investigated, in particular in the context of investment models, while recent works by Nishimura and Ozaki (2004, 2007) and Miao and Wang (2010) attempted to answer the second question. However, the extant literature has neglected the role of learning. In our view, learning is a realistic feature of timing problems, in particular under ambiguity. Ambiguity most clearly arises at some relatively infrequent events of regime change (be it economic, political, market or technological) when future outcomes cannot be any more attached to known probabilities. Waiting after such events allows decision makers to observe the new environment and learn about the nature of uncertainty.[1] The other deficiency of the literature is that the empirical evidence for even the most basic predictions of the theory is scant.[2] Particularly in our context, it is difficult for an econometrician to distinguish between risk and ambiguity using field data as this distiction depends on agents' information sets which are not readily observable. Laboratory experiments that control for the level of risk and ambiguty provide an opportunity to circumvent this problem. Our work contributes to the literature in two respects. First, we develop a model with learning to study timing decisions under risk and ambiguity. Second, we test the model in a laboratory experiment and provide empirical evidence for the distinct roles of risk and ambiguity on willingness to wait.

The structure of the model is as follows. A decision maker has an opportunity to invest in a project by paying a fixed cost. The value of the project grows deterministically over time but, at each instant, the option to invest can disappear at an exogenously-specified expiration rate. If the decision maker invests before expiry, he obtains a payoff equal to the current value of the project minus the investment cost and he obtains nothing otherwise. Thus, there is a value in delaying the investment but waiting involves an opportunity cost because the future payoff is uncertain. There are two possible states of the world. In the good state, the expiration rate is low, λ_L, whereas in the bad state, it is high, λ_H. The true value of the expiration rate is unknown at the initial date but the decision maker can learn about the state of the world. As time progresses and the investment opportunity does not expire, the decision maker can infer that the state of the world is more likely to be good, and he updates his belief accordingly.

By featuring irreversibility, uncertainty, and timing flexibility, our model

[1] The importance of combining ambiguity and learning is also stressed in Epstein and Schneider (2008), Leippold, Trojani, and Vanini (2008), Campanale (2011), and Ju and Miao (2012).

[2] Notable exceptions are Guiso and Parigi (1996), Moel and Tufano (2002), Bloom et al. (2007), Kellog (2010), and the experimental works of Oprea et al. (2009), Anderson et al. (2010) and List and Haigh (2010).

includes all the essential elements of real options theory. The original application of real options theory is capital investment, but it has been employed to describe a variety of economic and non-economic problems, like the timing of mergers and acquisitions (Lambrecht, 2004), innovation investments in competitive markets (Weeds, 2002; Huisman and Kort, 2004), debt default (Leland, 1994), labor market fluctuations (Bentolila and Bertola, 1991), and political decisions (Polborn, 2006; Keppo, Smith, and Davydov, 2009). Furthermore, our learning structure, in which the decision maker learns about alternative states of the world, relates our model to bandit problems, a class of problems which found a widespread application in economics. Bandit problems have been used, for example, to study capital investment (Décamps and Mariotti, 2004), R&D financing (Robert and Weitzman, 1981), monopoly pricing (Keller and Rady, 1999), and incentive schemes (Manso, 2011).

We distinguish between a risky and an ambiguous scenario. In a scenario that features risk, the decision maker knows the exact probabilities of the high and low expiration rates. A mean-preserving increase in risk in this environment is given by an increased spread between the high and low rates. In the problem under ambiguity, the decision maker has imprecise information on the relative probability of the expiration rate being low or high. He knows only that this probability lies within a certain interval. A symmetric increase of this interval increases ambiguity.

We show that higher risk increases willingness to wait, a result consistent with the standard real options theory. This happens because, when the spread between high and low expiration rates is larger, the fact that the option does not expire during a given time interval is a more informative signal. Then, the decision maker is more willing to wait to make a more informed decision. We also show that ambiguity decreases willingness to wait of an ambiguity-averse decision maker. Intuitively, the decision maker dislikes the uncertainty associated with the waiting region and is willing to exercise the investment option sooner when ambiguity increases.

With the support of Figure 1, it is useful to interpret our model in comparison with the standard Ellsberg (1961) experiment. In a typical Ellsberg-style setting, subjects decide how much to pay to participate in two lotteries, one risky and one ambiguous, represented by a draw from an urn with balls of different colors, G and B in the figure (see, for example, Fox and Tversky (1995) and Halevy (2007)). In the risky lottery, the composition of the urn, that is, the probability of extracting a B ball, is known and risk is increased by a mean-preserving spread of payoffs from the two types of balls. In the ambiguous lottery, the urn composition is (at least partially) unknown; therefore, the probability of extracting a B ball does not have a unique value, but is defined by a range. Ambiguity is increased by widening the range of probability. In this setting, decision makers disclose their preferences by revealing their *willingness to pay* to participate in the lottery, and a lower willingness to pay for the ambiguous lottery reveals ambiguity aversion.

Our setup is designed to resemble Ellsberg's two-urn environment. In the model, the true expiration rate is determined at the initial date by a draw from a distribution that is known in the risky case but unknown in the ambiguous case. Risk is measured by the spread between λ_H and λ_L, while ambiguity is measured by the probability interval for the good and bad state. Instead of revealing their *willingness to pay* as in the standard Ellsberg setup, decision makers disclose their preferences by revealing their *willingness to wait*. As explained above, a higher risk implies a higher willingness to wait, while higher ambiguity is reduces willingness to wait of an ambiguity-averse decision maker.

To test the predictions, we replicate the model in a laboratory experiment. We first run three treatments, *Benchmark, Risk,* and *Ambiguity*. Relative to *Benchmark*, the *Risk* and *Ambiguity* treatments have increased risk and ambiguity, respectively. The purpose is to test whether individuals respond to risk by delaying investment and to ambiguity by accelerating investment, as predicted by the theoretical analysis.

The experimental data support the predictions about risk. In the *Risk* treatment, investment is delayed compared with *Benchmark*. We further find that investment in *Ambiguity* is also delayed compared with *Benchmark*. The results are robust to a careful treatment of censoring in the experimental data, inter-round learning, and alternative treatments with different levels of ambi-

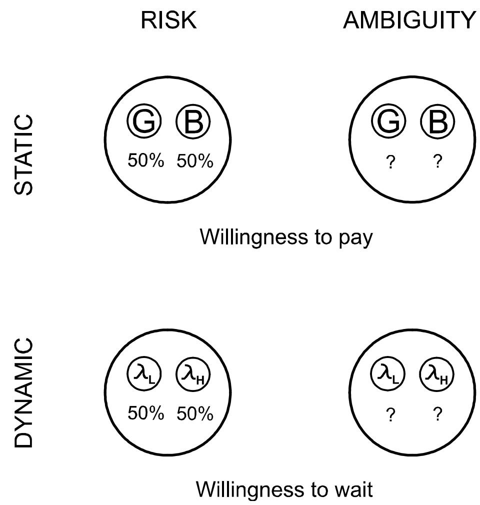

Figure 1: Risk and ambiguity in a static and a dynamic setup.

guity. According to the model, the observed behavior under ambiguity is inconsistent with ambiguity aversion. A plausible interpretation of our findings is that the assumption of ambiguity aversion, which largely derives from evidence obtained in static lotteries, fails to describe behavior in more complex problems. Furthermore, it is also possible that individuals respond differently to new information under ambiguous and unambiguous priors. We propose such a hypothesis as an alternative explanation of our results in Section 6. Overall our stance is that experimental tests of more complex problems can shed further light on theoretical modelling.

The remainder of the paper is organized as follows. The next section discusses how the paper relates to the existing literature. Section 3 develops a model of investment under risk and ambiguity. Section 4 describes the experimental design and testable hypotheses. Section 5 presents the empirical analysis. Section 6 provides a further discussion of the main results and Section 7 concludes.

2 Related Literature

Some recent studies have tested optimal stopping problems in laboratory experiments. Oprea, Friedman, Anderson (2009) take the standard real options model based on a geometric Brownian motion and study whether subjects can learn the optimal investment rule. They find supportive evidence that, by experience, individual behavior converges towards optimality. Anderson, Friedman, and Oprea (2010) study a preemption investment game and find that, for the most part, the predictions of the theory are confirmed. List and Haigh (2010) focus on another facet of the theory of investment under uncertainty, the bad news principle, and conclude that experimental data support it. In contrast to this paper, all these experiments focus exclusively on investment under risk and do not explicitly test the comparative statics prediction with respect to risk.

Theoretical studies that introduce ambiguity into models of investment under uncertainty are presented by Nishimura and Ozaki (2007), Riedel (2009), and Miao and Wang (2011). Nishimura and Ozaki (2007) rely on the assumption of ambiguity aversion and show that increased ambiguity delays investment in projects that generate an infinite flow of ambiguous cash flows. In contrast, our model considers the case in which investment yields a certain (unambiguous) payoff and is, therefore, closer to the job-search model of Nishimura and Ozaki (2004). Consistent with our results, Nishimura and Ozaki (2004) show that an increase in ambiguity decreases the reservation wage and induces the ambiguity-averse worker to end the job search earlier. Miao and Wang (2011) further clarify that the sign of the effect of ambiguity depends on whether the uncertainty is resolved at the time of the investment. When ambiguity affects only the waiting region and the payoff is certain, ambiguity accelerates investment, if the decision maker is ambiguity averse. In contrast, if the final payoff is also ambiguous, investment is delayed. Because we consider the case in which the payoff from investment is certain, our model is consistent with the predictions made by Miao and Wang (2011).[3]

[3] Riedel (2009) provides a more general treatment of stopping-time problems under am-

An important difference between our model and the investment models of Nishimura and Ozaki (2007) and Miao and Wang (2011) is that they assume that ambiguity is not reduced by the observational data. This paper takes a different perspective and considers an environment in which information on the nature of uncertainty is progressively revealed to the decision maker. In this respect, our approach is closer to the work of Cagetti, Hansen, Sargent, and Williams (2002), Epstein and Schneider (2007), Leippold, Trojani, and Vanini (2008), Campanale (2011), and Ju and Miao (2012). These papers show that the combined action of learning and ambiguity, under the assumption of ambiguity aversion, can match a wider set of empirical regularities than ambiguity or learning alone. We already explained in the introduction why we believe that learning combined with ambiguity is a more accurate description of reality. Furthermore, the models without learning like those Nishimura and Ozaki (2007) and Miao and Wang (2011) can be more problematic to be implemented in the laboratory. These models exclude learning but feature independently and indistinguishably distributed ambiguity. In our setting, for example, this would require that at each instant the investment opportunity is assigned a new expiration rate from each time different, but equally ambiguous, distributions. This may be difficult to comprehend especially in continuous time, and the experimenter needs to make sure that subjects are indeed not attempting to learn the underlying distribution. We avoid this concern by studying a setting with only a single, initial source of ambiguity.

3 The Model

3.1 A Simple Stopping Problem

We first present a simple optimal stopping problem that will serve as a building block for our analysis. Time is continuous and labeled by $t \in [0, \infty)$. A risk-neutral decision maker (DM) discounts the future at rate r and has an opportunity (option) to invest in a project of value V_t by paying a fixed cost

biguity applicable also to investment settings.

equal to C.[4] The value of the project grows deterministically over time; therefore, the DM has an incentive to wait. However, the opportunity to invest can expire and disappear at a random time denoted by T. This means that if the DM invests at time τ before the opportunity expires, he obtains a payoff $V_\tau - C$, but he receives nothing otherwise. The payoff can be written as $(V_\tau - C)1_{\tau<T}$. The DM must decide when to invest.

The value of the project V_t evolves according to

$$V_t = V_0 e^{\mu t}, \tag{1}$$

where $\mu > 0$ is the growth rate. At each moment, the investment opportunity may vanish with a strictly positive probability. The expiration of the investment option is modeled as a Poisson shock with a mean arrival rate $\lambda > 0$. This means that, over a period of time Δt, the DM loses the opportunity to invest with probability $\lambda \Delta t$. We assume that $\lambda + r > \mu$ to guarantee that the option to invest will be optimally exercised in finite time.

At time $t \geq 0$ the DM's problem is

$$\max_{\tau \geq t} E_t \left[(V_\tau - C) e^{-r(\tau - t)} 1_{\tau<T} \right]. \tag{2}$$

In general, τ is a stopping time adapted to the filtration of the model but no new unanticipated information arrives before expiry so the relevant filtration is just the calendar time. We can then use that T is exponentially distributed with rate λ, so that $E_t [1_{\tau<T}] = Pr [T > \tau | T > t] = 1 - F_{exp}(\tau - t; \lambda) = e^{-\lambda(\tau - t)}$, to rewrite the problem as

$$\max_{\tau \geq t} \left[(V_t e^{\mu(\tau - t)} - C) e^{-(r+\lambda)(\tau - t)} \right].$$

The first-order condition for optimal τ^* is

$$(\mu - r - \lambda) V_t e^{\mu(\tau^* - t)} + (r + \lambda) C = 0.$$

[4] The assumption of risk neutrality does not affect the qualitative predictions of the model, which are the object of our experimental analysis. In Appendix B, we show that our results hold also when the decision maker is risk averse.

This yields that the DM invests as soon as V_t reaches a threshold $V_K^* = V_{\tau^*}$ given by

$$V_K^* = \frac{\beta_K}{\beta_K - 1} C,$$

where $\beta_K = (r + \lambda)/\mu > 1$. The subscripts K stands for "known," to indicate that the DM has a perfect knowledge of the expiration rate λ. β_K is a ratio of the per-period cost of waiting, $r + \lambda$ (discounting plus the probability of expiry), and the per-period benefit of waiting, μ (the growth rate).

The ratio $\beta_K / (\beta_K - 1) > 1$ provides the proportion by which the value of the project should increase above the cost to induce the DM to invest. A higher λ decreases the investment trigger. Intuitively, if the probability of losing the investment opportunity is higher, the DM will exercise the investment option sooner. In contrast, when μ is higher, he will postpone the investment to exploit the greater growth potential.

3.2 Unknown Expiration Rate

To introduce varying levels of risk and ambiguity, we now consider a slightly modified setting. As in the previous section, the DM has the opportunity to invest in a project of value V_t that grows according to (1). At each moment, the option to invest expires with a strictly positive probability and the expiration rate is determined by the intensity of a Poisson process. However, the expiration rate is unknown. The DM knows that two states of the world are possible. In the bad state, the expiration rate, λ_H, is high. In the good state the expiration rate, λ_L, is low ($\lambda_H > \lambda_L$ holds). The DM knows the values of the two λs but does not know the realization of the state. To ensure that the investment problem always has a finite solution and achieves a maximum, we assume that

$$r + \lambda_L > \mu. \tag{3}$$

Suppose that the DM has a subjective belief about the relative probability of the two states. Specifically, he believes that the intensity of the expiration rate is λ_L with probability $p \in (0, 1)$. At the moment, we do not specify how this belief is formed at the initial time $t = 0$. However, we do specify how it evolves over time. If at $t = 0$ the DM finds it optimal not to invest immediately, he waits for larger values of the project. By waiting, the DM observes the investment payoff to rise according to (1) and the (non)occurrence of expiry. If the DM waits and the option to invest does not expire, he updates his belief in a Bayesian fashion. According to Bayes' rule, after an interval Δt, the DM's posterior belief is

$$p_t + \Delta p_t = \frac{p_t (1 - \lambda_L \Delta t)}{p_t (1 - \lambda_L \Delta t) + (1 - p_t)(1 - \lambda_H \Delta t)}.$$

Taking the limit $\Delta t \to 0$ and rearranging yields an instantaneous change in belief:

$$dp_t = p_t (1 - p_t)(\lambda_H - \lambda_L)\, dt. \tag{4}$$

Equation (4) can be interpreted as the speed at which the DM learns about the true state of the world. Two properties of dp are worth noting. First, the speed of learning is proportional to the difference $\Delta\lambda = \lambda_H - \lambda_L$. The explanation is that, when the difference between λ_H and λ_L is large, it is very informative that during a given time interval the option to invest does not vanish. The DM then becomes rapidly confident that the true expiration rate is low, and p increases quickly. Second, learning is fastest if the realization of λ is still the most uncertain (when p is 0.5) and the slowest if one of λs is nearly certain (when p_t is close to 0 or 1).[5]

Equation (4) implies that p can be written as an explicit function of time t:

$$p_t = \frac{p_0 e^{\lambda_H t}}{(1 - p_0) e^{\lambda_L t} + p_0 e^{\lambda_H t}}, \tag{5}$$

where p_0 is the belief at time $t = 0$. The problem as formulated above has two state variables, the value of the project V and the belief p. Finally, because both V and p are functions of time only, one can rewrite p as a function of V,

[5] These two characteristics of learning dynamics are useful for interpretation of our results. We note that they are identical in other learning problems with binary states and so are not specific to the Poisson signals in our model (see, e.g., Moscarini and Smith (2001) and Keppo, Moscarini and Smith (2008) for models with Gaussian signals).

denoted $p(V)$:

$$p_t = p\left(V_t\right) = \frac{\pi_0 \left(V_t/V_0\right)^{\theta}}{1+\pi_0 \left(V_t/V_0\right)^{\theta}}. \tag{6}$$

where $\pi_0 = p_0/(1-p_0)$ and $\theta = \left(\lambda_H - \lambda_L\right)/\mu$.

In the remainder of the paper, we distinguish two different scenarios. First, we discuss a risk scenario, in which the DM knows the relative probability of the two states of the world at the initial time. Second, we consider an ambiguity scenario, in which the probability is unknown.

3.3 Risk

In the risk scenario, the DM has a single initial prior, denoted by p_0, which describes the probability of the expiration rate being λ_L at time 0. The analysis here is consistent with the assumption that the DM knows the true probability with which the state of the world is selected, or that he can form a single subjective prior that represents his beliefs. In either case, the standard expected utility model can be applied. We measure risk as the spread between λ_H and λ_L. Risk is increased at time 0 by widening the difference $\Delta\lambda = \lambda_H - \lambda_L$ while keeping the expected expiration rate, $p_0\lambda_L + (1-p_0)\,\lambda_H$, constant. The goal of this section is to study the effect of risk on the timing of investment.

The decision problem is analogous to the one described in Section 3.1. At time $t \geq 0$, the problem is again given by (2). The distribution of the expiry time T is now such that

$$\begin{aligned} E_t\left[1_{\tau<T}\right] = Pr\left[T>\tau | T>t\right] = p_t(1-F_{exp}(\tau-t;\lambda_L)) \\ +(1-p_t)(1-F_{exp}(\tau-t;\lambda_H)) = p_t e^{-\lambda_L(\tau-t)} + (1-p_t)e^{-\lambda_H(\tau-t)} \end{aligned}$$

Thus problem (2) can be rewritten as

$$\max_{\tau\geq t}\left[(V_t e^{\mu(\tau-t)} - C)e^{-r(\tau-t)}\left(p_t e^{-\lambda_L(\tau-t)} + (1-p_t)e^{-\lambda_H(\tau-t)}\right)\right]. \tag{7}$$

The first-order condition for optimal τ^* is

$$p_{\tau^*}\left[(\mu - r - \lambda_L)V_{\tau^*} + (r + \lambda_L)C\right] + (1 - p_{\tau^*})\left[(\mu - r - \lambda_H)V_{\tau^*} + (r + \lambda_H)C\right] = 0. \tag{8}$$

Rearranging this yields that the DM invests as soon as V_t reaches a threshold $V_R^* = V_{\tau^*}$ given by

$$V_R^* = \frac{\beta_R(V_R^*)}{\beta_R(V_R^*) - 1}C, \tag{9}$$

where $\beta_R(V) = \left[r + p(V)\lambda_L + (1 - p(V))\,\lambda_H\right]/\mu$. β_R is a cost-benefit ratio similar to β_K but the cost of waiting in the numerator now involves the weighted probability of expiry, $p_t\lambda_L + (1 - p_t)\,\lambda_H$.

Next, we study qualitative effects of a mean-preserving increase in risk.

Proposition 1 *The investment threshold V_R^* is increasing in risk.*

Proposition 1 states that an increase in risk increases willingness to wait. The explanation is related to new information flows. With greater risk, the spread between the positive and negative realizations (λ_H and λ_L) is larger and learning proceeds more quickly, so it is more worthwhile to wait to make a more informed decision.[6]

A delay of investment in response to a higher risk shown above depends on the options effect (risk makes the option to wait more valuable) and it is not dependent on the DM's risk attitude. In Appendix B, we solve the model for a risk-averse DM with CRRA utility and show that, although risk aversion affects the option exercise strategy, the result of Proposition 1 still holds. Specifically, risk aversion accelerates the exercise of the option. However, for a given degree of risk aversion, a higher risk increases the upside potential of the option and delays investment.

[6] Many real options investment models do not feature learning, but the intuition that waiting for new information is more valuable under higher risk is similar to ours. In a popular class of such models, risk is measured by the volatility coefficient of a diffusion process (typically a geometric Brownian motion describing the value of investment). Both upside potential and downside risk increase with higher volatility. Optionality allows the decision maker to benefit from high realizations and limit the impact of low realizations. Therefore, the value of waiting to get new information about realizations increases with risk.

3.4 Ambiguity

In the ambiguity scenario, the DM has only imprecise knowledge of the initial probability of the two states of the world. For this reason, the DM cannot form a single prior and can only identify a set of plausible beliefs. Let $\mathcal{P}_t$ be a closed compact set of plausible beliefs. At the initial time $t = 0$, the initial probability set is defined by $\mathcal{P}_0 = [\overline{p} - \varepsilon, \overline{p} + \varepsilon]$. Here, $\overline{p}$ simply denotes the middle point of the set of plausible initial probabilities while $\varepsilon \in (0, \min[\overline{p}, 1 - \overline{p}]]$ is a measure of ambiguity.

As in the risky case described in Section 3.3, the DM learns about the true state of the world. As time progresses and the investment option does not vanish, the DM becomes more confident that the true expiration rate is λ_L. However, under ambiguity, contrary to the risky case, the learning process involves not a single prior but the entire set of plausible beliefs $\mathcal{P}_t$. To capture the learning process, we assume that the DM updates $\mathcal{P}_t$ prior-by-prior, which means that each belief in $\mathcal{P}_t$ evolves according to the dynamics described in (4) and, at any time t before expiration, satisfies (5).[7]

Because the beliefs remain in the same order, the posteriors originating from $\overline{p} - \varepsilon$ and $\overline{p} + \varepsilon$ represent the worst and best case beliefs, and define the boundaries of $\mathcal{P}_t$. We use p_t^- to denote the posterior belief under the worst case and p_t^+ to denote the posterior belief under the best case scenario (the expressions of p_t^- and p_t^+ are analogous to the expression for p_t in (5) with using $\overline{p} - \varepsilon$ and $\overline{p} + \varepsilon$ in place of p_0). The plausible set of beliefs, which defines the range of ambiguity, is given by $\mathcal{P}_t = \left[p_t^-, p_t^+\right]$.

Theoretical models that embed ambiguity in applied dynamic settings typically assume that economic agents are ambiguity averse (e.g., Epstein and Wang, 1994;, Nishimura and Ozaki 2004, 2007; Miao and Wang, 2010). Following this literature, our goal is to study how ambiguity and ambiguity aversion affects the timing decision. To do so, we adopt the maxmin model of Gilboa and Schmeidler (1989), in which a DM maximizes his utility in the worst

[7] Prior-by-prior Bayesian updating is a common rule to update ambiguous beliefs. It was proposed, among others, by Wasserman and Kadane (1990) and Jaffray (1994), Pires (2002), and Epstein and Schneider (2003).

case scenario over the plausible set of priors. Although the maxmin model describes an extreme form of ambiguity aversion, it allows us to capture the effect of ambiguity in a simple fashion preserving, at the same time, analytical tractability. Furthermore, as we show below, our conclusions about the effect of ambiguity are intuitive and line with the previous literature. Milder forms of ambiguity aversion, as the one proposed by the smooth ambiguity model (Klibanoff, Marinacci, and Mukerji, 2005) and the $\alpha-$maxmin model (Ghirardato, Maccheroni, and Marinacci, 2004), are unlikely to substantially alter our qualitative conclusions.[8]

Under maxmin preferences, the DM's problem is

$$\max_{\tau \geq t} \min_{p \in \mathcal{P}_t} \left[(V_t e^{\mu(\tau-t)} - C) e^{-r(\tau-t)} \left(p_t e^{-\lambda_L(\tau-t)} + (1-p_t) e^{-\lambda_H(\tau-t)} \right) \right]. \tag{10}$$

Since the worst case scenario corresponds the the situation in which $p_t = p_t^-$, the problem can be rewritten as

$$\max_{\tau \geq t} \left[(V_t e^{\mu(\tau-t)} - C) e^{-r(\tau-t)} \left(p_t^- e^{-\lambda_L(\tau-t)} + (1-p_t^-) e^{-\lambda_H(\tau-t)} \right) \right]. \tag{11}$$

The first-order condition for optimal τ^* is

$$p_{\tau^*}^- \left[(\mu - r - \lambda_L) V_{\tau^*} + (r + \lambda_L) C \right] + (1 - p_{\tau^*}^-) \left[(\mu - r - \lambda_H) V_{\tau^*} + (r + \lambda_H) C \right] = 0. \tag{12}$$

Rearranging condition (12), one obtains that the DM invests as soon as V_t reaches a threshold $V_A^* = V_{\tau^*}$ given by

$$V_A^* = \frac{\beta_A(V_A^*)}{\beta_A(V_A^*) - 1} C, \tag{13}$$

where

$$\beta_A(V) = \left[r + p^-(V)\lambda_L + \left(1 - p^-(V)\right) \lambda_H \right] / \mu,$$

[8] In fact, we also solve the model under $\alpha-$maxmin preferences. The qualitative conclusions about the effect of ambiguity are unchanged.

and

$$p^{-}(V) = \pi_0^{-}(V/V_0)^{\theta} / \left[1 + \pi_0^{-}(V/V_0)^{\theta}\right].$$

In the special case of maximum initial ambiguity, that is, if $\mathcal{P}_0 = [0,1]$, the expression for $\beta_A(V)$ simplifies to $\beta_A = (r + \lambda_H)/\mu$.

The next proposition establishes the effect of ambiguity on the optimal threshold.

Proposition 2 *The investment threshold V_A^* is decreasing in ambiguity.*

The intuition of Proposition 2 is straightforward. Willingness to wait depends the expiration rate, which is uncertain. An ambiguity averse DM is pessimistic about the state of the world. Therefore, when ambiguity increases, he prefers to avoid the uncertainty associated with waiting and by investing sooner. This result is intuitive and is line with the findings of the existing stopping problems under ambiguity (Nishimura and Ozaki, 2004; Miao and Wang, 2010). When uncertainty affects the continuation region but not the final payoff, the waiting time decreases with ambiguity.

4 Experimental Design and Hypotheses

To test the model's predictions on willingness to wait, we replicate its setting in a laboratory. Although time is continuous in the theoretical model, in a computerized laboratory implementation time must proceed in discrete steps. We approximate continuous time by setting the time interval equal to 0.1 seconds. In each interval, the project value grows and expires according to the chosen growth and expiration rates. To convey clear information to the subjects, we communicate the growth and expiration rates per second.

We conduct three main treatments, *Benchmark*, *Risk*, and *Ambiguity*. Some basic parameters are identical across all the treatments. The investment cost C is set to equal 10 euros, the initial value V_0 is set to equal 9.8 euros, and the growth rate is equal to $\mu = 0.0036$ every 0.1 seconds (corresponding to

	High expiration rate	Low expiration rate	Probability of high expiration rate
Benchmark	10%	5%	50%
Risk	11%	4%	50%
Ambiguity	10%	5%	no information

Table 1: Parameterization for the Benchmark, Risk, and Ambiguity treatments.

a growth rate of 3% per second[9]). In the *Benchmark* treatment the initial probability that the expiration rate is low is known and equal to 0.5. The high expiration rate is set to equal $\lambda_H = 0.0105$, which means 10% per second, while the low expiration rate is $\lambda_L = 0.0051$, or 5% per second. These rates imply that the initial expected expiration rate is equal to 7.5% per second. In the *Risk* treatment we test the effects of an increase in risk. We leave the initial probability $\overline{p}$ and the growth rate μ unaffected. We then set $\lambda_H = 0.0116$, i.e., 11% per second, and $\lambda_L = 0.0041$, i.e., 4% per second. Hence, risk, the spread between the high and low expiration rates, increases by 40% from 5% to 7%, while the initial expected expiration rate is still equal to 7.5% per second. In the *Ambiguity* treatment, we conduct a first test on the effect of ambiguity. We set λ_H and λ_L as in *Benchmark* but we provide no information about the level of the initial probability that the expiration rate is low (in the notation of our model, this means that ambiguity equals $\varepsilon = 0.5$, centered around $\overline{p} = 0.5$).[10] A summary of the parameterization for the three treatments is found in Table 1.

The choice of parameter values is driven by several considerations. We set the relative probability of the two states of the world equal to 50% in the risky treatments Benchmark and Risk, whereas we provide no information in the ambiguous treatment Ambiguity. This choice is made to conform to a standard version of the Ellsberg experiment, in which good and bad outcomes in the risky urn have equal probabilities, but the subjects are told nothing

[9] If an event occurs with probability x every 0.1 seconds, it occurs with probability $y = 1 - (1 - x)^{10}$ every second.

[10] In practice the probability that the true expiration was low in each period was 50% as in *Benchmark*. But this information was not communicated to the subjects.

about the distribution in the ambiguous urn. Although the task is relatively simple (the subjects must wait and simply click a button when they decide to invest), the time interval and expiration rates should set an environment that is simultaneously challenging and "comfortable". Expiration rates cannot be set too high because subjects should have sufficient time to wait. To avoid trivializing the task of distinguishing between states of the world, the spread between high and low expiration rates should be appreciable but not too large. Furthermore, under the chosen parameterization, the three treatments should provide sufficiently distinguishable theoretical predictions of the investment trigger. For our parameter values, the predicted investment trigger for a risk neutral decision maker is 19.32 in Benchmark, and 30.53 in Risk (calculated using Equation (9) with $\mu = 0.03$, $r = 0$, $p_0 = 0.5$ and $\lambda_L = 0.05$, $\lambda_H = 0.1$ under *Benchmark* or $\lambda_L = 0.04$, $\lambda_H = 0.11$ under *Risk*). For a risk averse decision maker with a risk aversion coefficient γ equal to, e.g., 0.4, the distance between the predicted triggers is smaller but still sufficiently large (13.94 in *Benchmark* and 15.68 in *Risk*, calculated using (15) in Appendix B). In *Ambiguity*, the predicted investment trigger is 14.29 for a risk neutral decision maker and 12.19 for a risk averse decision maker with $\gamma = 0.4$ (calculated using Equations (13) and (16), respectively). The distance with the predicted triggers in Benchmark is appreciable.

The theoretical analysis suggest the following experimental hypotheses:

Hypothesis 1 *Willingness to wait is higher in Risk than in Benchmark.*

Hypothesis 2 *Willingness to wait is lower in Ambiguity than in Benchmark.*

Hypothesis 1 is derived from Proposition 1. Intuitively, a higher risk should increase DMs ability to distinguish the good (low expiration) from the bad (high expiration) projects and induce them to delay investment. Hypothesis 2 comes from the observation that, as shown in Proposition 2, the presence of ambiguity should lead to early investment if DMs are ambiguity averse. Intuitively, without knowing the relative probabilities of the two states of the world, DMs may be pessimistic about the initial draw of the expiration rate and they will try to avoid the threat of losing the investment opportunity by investing earlier.

4.1 Procedures

Subjects played the same investment game for 30 rounds. At the beginning of each round the computer screen displays the parameter values. In the treatments *Benchmark* and *Risk*, the screen displays the values of λ_H, λ_L and the probability that the expiration rate is high. In the *Ambiguity* treatment, the screen displays only the values for λ_H and λ_L, without any information about the probability of the two states of the world. By clicking the "OK" button located at the bottom-right of the screen, the subject begins the experiment and a new screen with the values for V_t and the cost C appears. The subjects observe the project value V_t growing according to (1) and decide when to exercise the investment option by clicking the "INVEST" button. Upon investing, they obtain a payoff $V_t - C$. The experiment was programmed and conducted with the z-Tree software (Fischbacher, 2007). The full instructions and screenshots of computer displays are provided in Online Appendix.

The experiment was conducted at CentERLab at Tilburg University and the experimental subjects were students of Tilburg University (bachelor and master students in economics, business, social sciences, and law) recruited using on-line recruitment software. Participation was voluntary and no subject participated in more that one treatment. In total, 76 subjects participated in the experiment. Groups of 18 subjects participated in the treatments *Benchmark* and *Risk* and 19 subjects participated in the treatment *Ambiguity* (21 subjects participated in a fourth treatment discussed in Section 5.2). We aimed at having 20 subjects per treatment (similar to, e.g., Oprea et al., 2009) and differences are due to some no-show-ups. Since the subjects played the investment game for 30 rounds resulting in a total of 1650 observations in the three treatments. Each treatment was organized in two separate sessions. All subjects were paid 5 euros as a show-up fee.

At the beginning of each treatment, instructions were read aloud. After reading the instructions, the subject played practice rounds. In the practice rounds, as in the game described in Section 3.1, subjects know the true expiration rate. In the treatments *Benchmark* and *Ambiguity*, the subjects played 10 rounds with an expected expiration rate of 10% and 10 rounds with an expected expiration rate of 5%. In the treatment *Risk*, the subjects played 10 rounds with an expected expiration rate of 11% and 10 rounds with an expected expiration rate of 4%. Each subject played 20 practice rounds. In these rounds, the subjects are not compensated and thus have incentives to wait for the natural expiration of the investment opportunity to understand what the specified parametrization for the expiration rates means in practice. By raising their hands, the subjects could call the experimenter and ask for more practice rounds (there were 2 subjects that asked for more practice rounds in *Benchmark*, 0 in *Risk*, 1 in *Ambiguity*).

The subjects were seated at isolated computer terminals and played a game independently from the others. Draws of the initial expiration rate at each round and the exogenous expiration time were different for each subject and randomly chosen by the computer program according to the pre-specified parameters. Earnings were paid at the end of the experimental sessions. To avoid wealth effects, the subjects were paid for only one of the 30 rounds. The payment round was chosen at random at the end of the experiment. The average earning was 9.10 euros, including the show-up fee. A typical round took less than 30 seconds and the sessions lasted approximately fifty minutes, including the reading of the instructions and payment.

5 Results

5.1 Waiting and Investment Timing

As mentioned in Section 4, we set the initial project value below the cost of investment ($C = 10$ and $V_0 = 9.8$). In some cases, the investment opportunity expired when $V_t < C$. There were three of these "early" expirations in *Bench-*

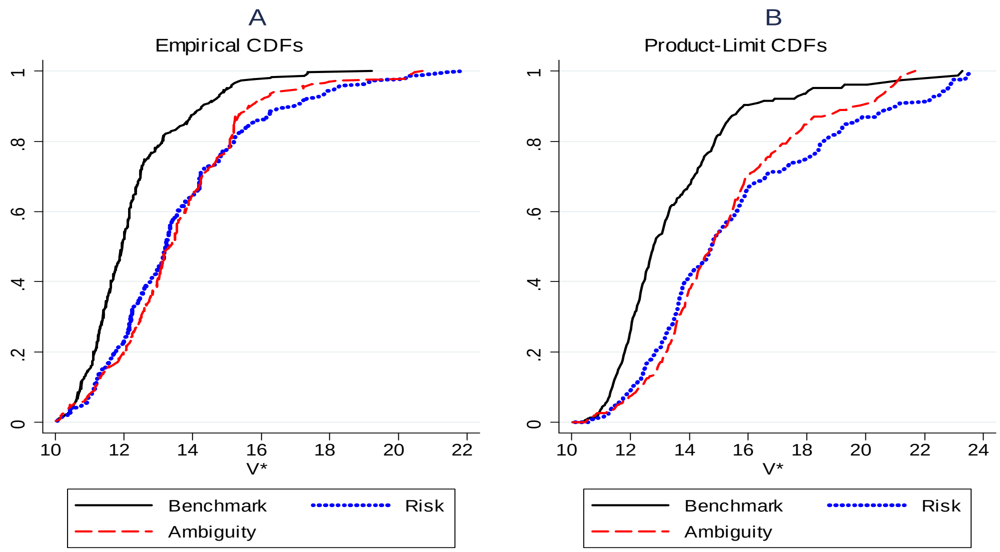

Figure 2: Empirical CDFs of the observed investment trigger (Panel A) and product-limit estimates of CDFs (Panel B) for the *Benchmark*, *Risk*, and *Ambiguity* treatments.

mark, one in *Ambiguity*, and none in *Risk*. Because it is clearly sub-optimal to invest when $V_t < C$ (and the computer program forbids it), early expirations convey no information about the subjects' willingness to wait. For this reason, these early expirations are dropped from our dataset.

The remaining data are right censored. A number of investment decisions are not observed because the option expired before the subjects invested. In the *Benchmark* treatment there were 244 cases out of 537 (45% of the total) in which the option expired before the subjects decided to invest. Similar expirations accounted for 278 of 540 cases (51% of the total) in *Risk* and 307 of 569 cases (54% of the total) in *Ambiguity*.

As a preliminary step, we drop the censored observations, i.e., the cases in which the option expired before subjects decided to invest, and we observe at the investment pattern for the subsample of observed investment decisions. Figure 2.A plots the cumulative distribution functions (CDFs) of the empirically observed exercise trigger. For each value of the project, the figure presents the proportion of subjects who exercised the investment option. A shift of the curve to the right means that, for a given value of V_t, fewer people exercised the option and implies a higher willingness to wait. A visual inspection of the figure immediately reveals a clear pattern. The CDF for the *Risk* treatment lies to the right of that of *Benchmark*, which is consistent with the prediction of Hypothesis 1. Additionally, the figure shows that the CDF for the *Ambiguity* treatment is shifted to the right of the one of *Benchmark*. Following the theory behind Hypothesis 2, this result is sharply in contrast with ambiguity aversion.

The analysis conducted on the restricted sample of empirically observed investment decisions may potentially present a misleading picture. Censored observations are also informative because the subjects have voluntarily decided to wait for the project value to grow (at least) until the moment at which the expiration occurred. This means that the restricted sample of uncensored observations suffers from a downward bias. To address the problem of censored data we use the product-limit estimator (Kaplan and Meier, 1958), which provides a non-parametric method to estimate the CDFs while accounting for random censoring.[11]

By including the right-censored observations, we first assume that observations are i.i.d. and estimate the CDFs using pooled data across the subjects.

[11] The Product-Limit estimator relies on the assumption of independent censoring, which requires that, *within the population under consideration*, the survival time, that is the time at which the event of interest occurs (investment in our model), and censoring time are two statistically independent random variables. This means that the censoring of an observation should not provide any information regarding the potential investment strategy of a subject beyond the censoring time. In our setting, the independent censoring assumption would not be satisfied if we were to estimate the probability distribution of the investment strategy on "aggregate", that is by mixing together observations from different treatments. Since the censoring scheme in Risk is different from the one in Benchmark and Ambiguity, and it determines the investment strategy, we would have a correlation of censoring time and investment time that would bias the results. However, this is not a concern when we employ the Product-Limit estimator to estimate the survival distribution for a group of individuals which are subject to the same censoring scheme. Within each subpopulation the censoring is random (non-informative) as determined by an exogenous Poisson event and the independent censoring assumption is satisfied. See Oprea et al. (2009) for another application of this estimator in a similar context.

Then, we formally compare the pooled product-limit estimates for the CDFs using the log-rank test. Product-limit estimates of the CDFs for the three treatments are reported in Figure 2.B. The figure confirms the intuition suggested by the restricted sample of uncensored observations. Both risk and ambiguity delay investment. We use the log-rank test to verify the null hypothesis of equality between CDFs. A pair wise test rejects the null hypothesis of equality between the *Risk* and *Benchmark* treatments, and the *Ambiguity* and *Benchmark* treatments at a 1% level of significance ($p = 0.001$).

The analysis on the pooled data hinges upon the i.i.d. assumption, which implies that investment decisions are uncorrelated across subjects. If this is not the case, and different subjects behave in a different way, the i.i.d. assumption is violated and standard errors are typically underestimated. To account for within-subject dependence, we construct a product-limit estimate for the average of the investment trigger for each individual. Figure 3 presents histograms of the by-subject means for each of the three treatments. The figure reveals that, on average, subjects in the *Risk* and *Ambiguity* treatments invest later than in *Benchmark*. Table 2 reports the means and standard errors of the investment trigger for the pooled and by-subject data. The means in *Risk* and *Ambiguity* are larger than the estimated mean in *Benchmark*. We apply a pair wise Mann–Whitney test to compare the sample means for the by-subject estimates. The null hypothesis of equality between *Benchmark* and *Risk* and between *Benchmark* and *Ambiguity* is rejected at a 1% level of significance ($p = 0.004$ and $p = 0.007$ respectively). Thus, the results of the analysis with pooled data are confirmed.

The effect of risk is consistent with the theory and supports the logic behind the mechanism inducing DMs to wait to learn. When the spread between the high and low expiration rates widens, subjects become more rapidly confident that the true expiration rate is low and they consistently delay investment. The delay of investment in *Ambiguity* compared with *Benchmark* strongly rejects ambiguity aversion. From a quantitative perspective it is worth noting that on average the subjects exercised the option rather close to the model's predictions. Using the CRRA utility model (Appendix B) with a realistic

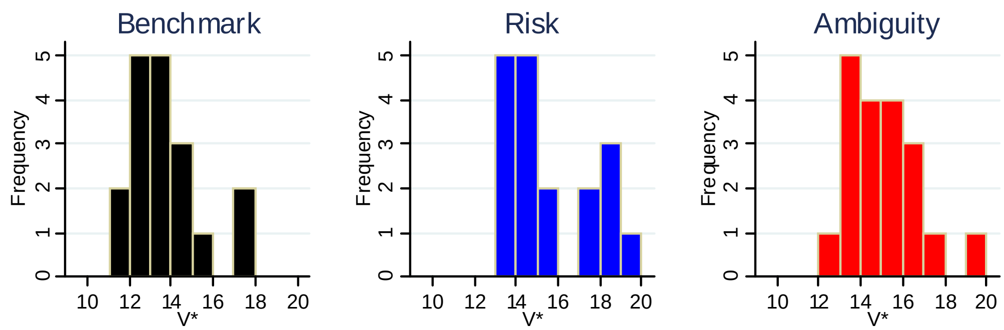

Figure 3: Product-limit by-subject mean estimate of the investment trigger for the *Benchmark*, *Risk* and *Ambiguity* treatments.

	Benchmark	Risk	Ambiguity
	Mean ± Standard Error		
Pooled Empirical	12.89 ± 0.10	14.63 ± 0.17	14.56 ± 0.14
By-subject Empirical	13.31 ± 0.37	14.89 ± 0.47	14.77 ± 0.38
Pooled PL	13.59 ± 0.15	15.68 ± 0.20	15.29 ± 0.15
By-subject PL	13.61 ± 0.42	15.48 ± 0.53	15.14 ± 0.39

Table 2: Empirical means and product-limit estimates of the means of the investment triggers under the Benchmark, Risk, and Ambiguity treatments. The Pooled PL row shows the estimate assuming i.i.d. observations. The By-subject PL row shows the product-limit estimated mean across individual subjects.

estimate of the coefficient of constant relative risk aversion of $\gamma = 0.4$ aligns well theoretical predictions to the sample means (this level of relative risk aversion is in the range of commonly estimated values between 0.3 and 0.5, see, e.g., Holt and Laury (2002) and references therein). Risk-averse theoretical predictions for the thresholds in the *Benchmark* and *Risk* treatments are then 13.94 and 15.68, respectively, and are very close to the by-subject PL means.

As discussed next, there is no evidence in our data of an appreciable change in the subjects' investment strategies over time. In the experimental implementation, for the purpose of data collection, the investment game was repeated for 30 rounds. The repetition of the investment game raises the possibility that subjects may adopt different strategies in different rounds, for example, due to inter-round learning or experimentation. In our experiment, this concern is mitigated by the initial practice period which should help the subjects elaborate an optimal strategy before the actual experiment begins. Furthermore, because subjects are paid for only one of the 30 rounds, possible wealth effects that could alter their behavior over time are reduced. Online Appendix D contains a detailed analysis of the data divided in early and late rounds. It shows that there are no significant differences between the rounds. Moreover, the main results reported above for the full sample are also reconfirmed in the subsamples of early and late rounds.

5.2 The Effect of Ambiguity: Robustness

The experimental data show that subjects are willing to wait longer with increased ambiguity which rejects Hypothesis 2. The purpose of this section is to provide further robustness to these findings. To do so, we conduct an additional treatment, *Mild Ambiguity*, in which we depart from the common practice of providing no information about the probability distribution in the ambiguous scenario. We set the parameter values for the growth and expiration rates like those in *Benchmark* and *Ambiguity* but we tell the experimental subjects that the probability that the expiration rate is high lies somewhere in between 20% and 80% (this means that ambiguity equals $\varepsilon = 0.3$, centered

	High expiration rate	Low expiration rate	Probability of high expiration rate
Benchmark	10%	5%	50%
Mild Ambiguity	10%	5%	between 20% and 80%
Ambiguity	10%	5%	no information

Table 3: Parameterization for the Benchmark, Mild Ambiguity, and Ambiguity treatments.

around $\overline{p} = 0.5$).[12] The relevant information about the treatments of interest in this section is summarized in Table 3.

A group of 21 subjects participated in the *Mild Ambiguity* treatment (in three separate sessions). The subjects played 20 practice rounds as in *Benchmark,* and the investment game was repeated for 30 rounds, resulting in 630 observations. There were no cases in which the option to invest expired when $V_t < C$. In 256 cases (40% of the total) the option expired before the subjects could invest. The analysis follows the same steps as above. First, we consider observations to be i.i.d. and estimate the CDFs, including the censored observations, using the product-limit estimator. The estimated CDF curves are presented in Figure 4.B. The figure shows that the CDF in *Mild Ambiguity* lies slightly to the right of the one in *Benchmark.* A pairwise log-rank test rejects the null hypothesis of equality between the two treatments (marginally significant with $p = 0.061$). We account for within-subject dependence by constructing a product-limit estimate for the average investment trigger of each subject. Figure 5 shows histograms of the by-subject means for each of the three treatments, and Table 4 reports the means and standard errors of the pooled and by-subject data. The mean in *Mild Ambiguity* is larger than the mean in *Benchmark* but the difference is less evident. A pairwise Mann–Whitney test cannot reject the null hypothesis of equality between means ($p = 0.355$). Recalling that an ambiguity averse should accelerate investment in response to ambiguity, the findings taken together confirm the inference that there is no evidence of ambiguity aversion.

[12]In practice, the true probability was 50% as in *Benchmark* and *Ambiguity.*

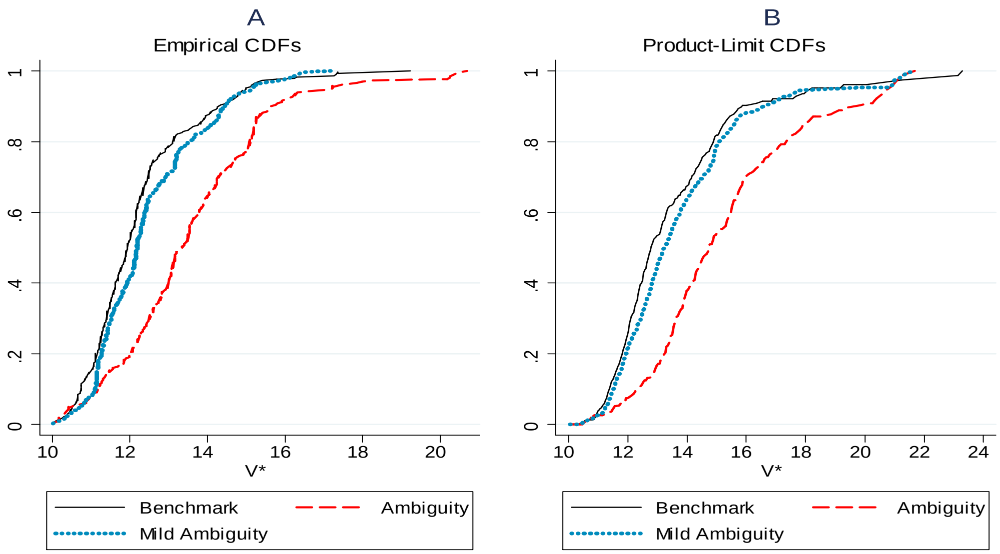

Figure 4: Empirical CDFs of the observed investment trigger (Panel A) and product-limit estimates of CDFs (Panel B) for the *Benchmark*, *Mild Ambiguity* and *Ambiguity* treatments.

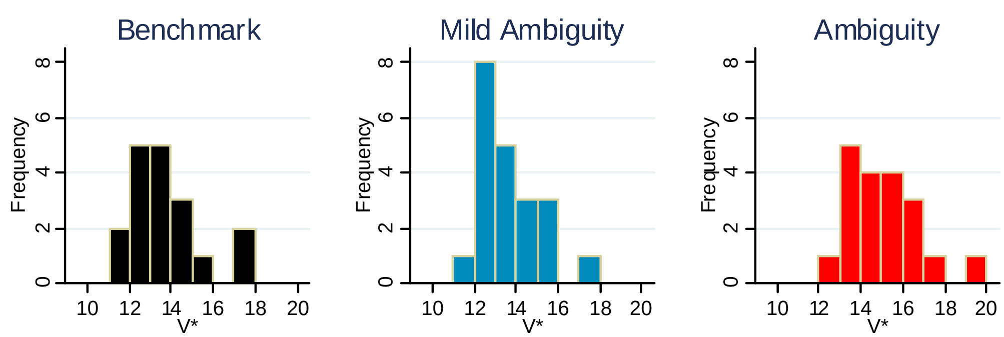

Figure 5: Product-limit by-subject mean estimate of the investment trigger for the *Benchmark*, *Mild Ambiguity* and *Ambiguity* treatments.

	Benchmark	Mild Ambiguity	Ambiguity
	Mean ± Standard Error		
Pooled empirical	12.89 ± 0.10	13.27 ± 0.10	14.56 ± 0.14
By-subject empirical	13.31 ± 0.37	13.75 ± 0.29	14.77 ± 0.38
Pooled PL	13.59 ± 0.15	13.83 ± 0.12	15.29 ± 0.15
By-subject PL	13.61 ± 0.42	13.83 ± 0.29	15.14 ± 0.39

Table 4: Empirical means and product-limit estimates of the means of the investment triggers under the Benchmark, Mild Ambiguity, and Ambiguity treatments. The Pooled PL row shows the estimate assuming i.i.d. observations. The By-subject PL row shows the product-limit estimated mean across individual subjects.

6 Discussion

The experimental data reveal that willingness to wait increases as ambiguity increases, which indicates a clear rejection of the aversion to ambiguity often assumed in economic and finance models. This finding deserves further attention and here we consider possible interpretations of the empirical evidence.

Although ambiguity aversion is a common assumption, prior experimental research indicates that it is not a universal feature and other ambiguity attitudes may prevail in different situations. For example, ambiguity attitudes in willingness-to-pay problems depend on experimental setups. Fox and Tversky (1995) find that ambiguity aversion in willingness-to-pay decisions arises only in comparative contexts, in which subjects are offered both ambiguous and unambiguous lotteries. In non-comparative settings, subjects see only one option and ambiguity aversion disappears. Possible explanations are that in comparative settings, when offered also unambiguous choices, subjects feel less competent with ambiguous choices or are afraid of deception in ambiguous lotteries. In a recent study of willingness to pay in a non-comparative setup, Charness, Karni, and Levin (2012) find that only a small minority of subjects turned out to be ambiguity averse, and actually more subjects displayed ambiguity-seeking behaviors. Our between-subjects experimental design is non-comparative and should be rather interpreted along these analogous willingness-to-pay experiments. Furthermore, past literature has shown that ambiguity attitudes depend on the economic problem at hand. Heath and Tversky (1991) show that people prefer bets on ambiguous events over pure (unambiguous) chance in situations where they feel knowledgeable or competent.

Camerer and Weber (1992) and Wakker (2010) also show that ambiguity seeking is more prevalent than ambiguity aversion for unlikely events and for payoffs expressed in losses. Chen et al. (2007) find that, in first price auctions, bids are lower with the presence of ambiguity, a result consistent with ambiguity seeking attitudes. Ivanov (2011) studies ambiguity attitudes in strategic setups and finds that the majority of subjects are not averse to ambiguity and more are ambiguity seeking than ambiguity averse. Timing decisions in the presence of ambiguity appear to be a separate category and is likely that ambiguity aversion may not emerge as the prevailing behavioral feature.

Our ambiguity model is developed on two main assumptions. The first is the extreme aversion to ambiguity based on the maxmin model of Gilboa and Schmeidler (1989), while the second is the Bayesian updating. Both assumptions are standard in the literature and play an important role in the optimal investment strategy. Apart from the maxmin specification, there are alternative ways to model ambiguity aversion but we have no reason to expect that our qualitative predictions would change by using other transformations on the set of priors. After all, the mechanism that we describe is intuitive. Since investment yields a sure payoff, individuals that dislike ambiguity prefer to resolve uncertainty sooner by investing earlier. Results of other theoretical models point in the same direction, and therefore it is unlikely that the maxmin specification drives this prediction. On the contrary, the assumptions regarding the learning mechanism may be more consequential.

We assume that individuals adopt the same learning mechanism, i.e. Bayes' rule, both in the risky and ambiguous scenarios. Without an empirical guidance on how individuals actually learn, this seems a natural choice. However, it does not need to be a fully accurate description of reality. A learning mechanism is nothing else than a rule which specifies the relative weight attached to new and old information to form posterior beliefs. Departing from a Bayesian framework, which identifies the "optimal" weights, we may think that the way new and old information are actually combined by the individuals may also depend on nature of the uncertainty. One can conjecture, for example, that in ambiguous scenarios individuals attach relatively less weight to prior (vague) beliefs and rely more on new observational data. In other words, due to vague priors, people could overreact (relative to the unambiguous case) to new information. In our setting, the relevant new information, i.e., the non-expiry of the investment option, is good news as it unambiguously signals that the true expiration rate is likely to be low. If, as hypothesized, individuals overreact to new information due to the presence of ambiguity, this mechanism may generate an increased willingness to wait under increased ambiguity even if individuals are ambiguity averse.

7 Conclusions

Risk and ambiguity play important roles in numerous economic situations. In dynamic decision problems that allow for wait-and-see behavior, the acquisition of new information possibly interacts with these two different manifestations of uncertainty. Our paper provides a testable model where risk and ambiguity coexist with learning and an experimental test of the effects of uncertainty on the waiting time to make an irreversible decision. The theoretical predictions are line with the previous literature and are confirmed for what concerns the effect of risk. The empirical results for the effect of ambiguity are less straightforward to interpret but unambiguously reveal an important fact. When ambiguity is embedded in a dynamic framework, individuals' behavior is difficult to reconcile with the aversion towards ambiguity typically found in simpler static settings. We believe that our results point out the importance to understand how the presence of ambiguity influences the acquisition and processing of new information. A careful investigation of how individuals learn under ambiguity appears to be a promising avenue of future experimental research.

A Proofs

A.1 Proof of Proposition 1

We verify the effects of an increase in risk on the optimal investment trigger. An increase in risk is represented by a mean-preserving spread between λ_H and λ_L at the initial date, that is a rise in $\Delta\lambda = \lambda_H - \lambda_L$ that leaves $p_0\lambda_L$ $+ (1 - p_0)\,\lambda_H$ unaffected.

Define function h such that $h = C\beta_R\,(V_R^*)\,/\,(\beta_R\,(V_R^*) - 1)$. At the investment trigger V_R^* it holds that $V_R^* - h = 0$. The effect of an increase in risk on the investment trigger is found as $dV_R^*/d\Delta\lambda = \frac{\partial h/\partial \Delta\lambda}{1-\partial h/\partial V_R^*}$, where the derivatives with respect to $\Delta\lambda$ are for $p_0\lambda_L + (1 - p_0)\,\lambda_H$ held constant.

By the second-order condition for a maximum, it easily follows that $1 - \partial h/\partial V_R^* \geq 0$. Thus the sign $dV_R^*/d\Delta\lambda$ depends on the numerator, $\partial h/\partial\Delta\lambda$.

Now, consider the sign of $\partial h/\partial\Delta\lambda$. It is useful to rewrite the expression for $\beta_R\,(V_R^*)$ as

$$\beta_R\,(V_R^*) = \beta_R\,(V_0) - \frac{\Delta\lambda}{\mu}\,(p\,(V_R^*) - p_0)\,,$$

where $\beta_R\,(V_0) = [r + p_0\lambda_L + (1 - p_0)\,\lambda_H]\,/\mu$. Differentiating with respect to $\Delta\lambda$ yields

$$\frac{\partial\beta_R\,(V_R^*)}{\partial\Delta\lambda} = \frac{\partial\beta_R\,(V_0)}{\partial\Delta\lambda} - \frac{p\,(V_R^*)}{\mu} - \frac{\Delta\lambda}{\mu}\frac{\partial p\,(V_R^*)}{\partial\Delta\lambda}.$$

Because we are considering a mean preserving spread between λ_H and λ_L at date 0 that holds $p_0\lambda_L + (1 - p_0)\,\lambda_H$ constant, it follows that $\partial\beta_R\,(V_0)\,/\partial\Delta\lambda = 0$. Combined with the observation that

$$\frac{\partial p\,(V_R^*)}{\partial\Delta\lambda} = \frac{\pi_0\,(V_R^*/V_0)^{\theta}\,\ln(V_R^*/V_0)}{\mu\left(1 + \pi_0\,(V_R^*/V_0)^{\theta}\right)^2} > 0$$

and $p\,(V_R^*) > 0$, this implies that the sign of $\partial\beta_R\,(V_R^*)\,/\partial\Delta\lambda$ is unambiguously negative. Because $\partial h/\partial\beta_R\,(V_R^*) < 0$, it holds that $\partial h/\partial\Delta\lambda = (\partial h/\partial\beta_R\,(V_R^*))$ $(\partial\beta_R\,(V_R^*)\,/\partial\Delta\lambda) > 0$. The claim in the proposition follows. ■

A.2 Proof of Proposition 2

Consider function g such that $g = C\beta_A(V_A^*)/(\beta_A(V_A^*) - 1)$. At the investment trigger V_A^*, it holds that $V_A^* - g = 0$. The marginal effect of ambiguity on the investment trigger is given by $dV_A^*/d\varepsilon = \frac{\partial g/\partial \varepsilon}{1-\partial g/\partial V_A^*}$. A proof that $\partial g/\partial V_A^* \leq 1$ follows from the second-order condition for a maximum. Thus the sign of $dV_A^*/d\varepsilon$ depends on the numerator $\partial g/\partial \varepsilon$,

$$\frac{\partial g}{\partial \varepsilon} = \frac{\partial g}{\partial \beta_A(V_A^*)} \frac{\partial \beta_A(V_A^*)}{\partial p_A(V_A^*)} \frac{\partial p_A(V_A^*)}{\partial \varepsilon}.$$

It is immediate to show that $\partial g/\partial \beta_A(V_A^*) < 0$, $\partial \beta_A(V_A^*)/\partial p_A^-(V_A^*) < 0$, and $\partial p_A(V_A^*)/\partial \varepsilon < 0$. Therefore, the total effect is such that $dV_A^*/d\varepsilon < 0$. ■

B Risk Aversion

In the main text, we analyzed the investment problem under the assumption of risk neutrality. Here we generalize our arguments allowing the DM to be risk averse. We first solve the stopping problem with known expiration rate of Section 3.1. The reasoning is then easily extended to find the solutions to the risky and ambiguous cases with the uncertain expiration rate of Sections 3.3 and 3.4. We show that the model's fundamental predictions about the effects of risk and ambiguity on the investment timing are not affected by risk aversion under our specification.

Let us consider the setting with a known expiration rate described in Section 2.1 and assume that DM's preferences are described by a constant relative risk aversion utility function (CRRA): $U = w^{1-\gamma}/1-\gamma$, where γ ($\gamma > 0$ and $\gamma \neq 1$[13]) is the coefficient of relative risk aversion, and w is DM's "wealth" or the payoff $V - I$ after the investment is undertaken. The problem for the decision maker is

$$\max_{\tau \geq t} E_t \left[\frac{(V_\tau - C)^{1-\gamma}}{1-\gamma} e^{-r(\tau - t)} 1_{\tau < T} \right] = \max_{\tau > t} \left[\frac{(V_\tau - C)^{1-\gamma}}{1-\gamma} e^{(r+\lambda)(\tau - t)} \right].$$

[13] For $\gamma = 1$, the CRRA utility is $U = \ln(w)$.

The first-order condition can be expressed as

$$\mu V_{\tau^*} - (r+\lambda)\frac{(V_{\tau^*} - C)}{1-\gamma} = 0.$$

The DM invests as soon as V_t reaches a threshold $\widetilde{V}_K^* = V_{\tau^*}$ given by

$$\widetilde{V}_K^* = \frac{\eta_K}{\eta_K - 1} I, \tag{14}$$

where $\eta_K = (r+\lambda)/[\mu(1-\gamma)]$. From (14) and η_K, it is immediately clear that risk aversion leads to an earlier exercise of the investment option, i.e., $\partial\widetilde{V}_K^*/\partial\gamma < 0$. The explanation is that the option value to wait with investment is decreasing in γ, while the final payoff is independent of risk aversion. This implies that a more risk averse DM is less willing to withstand the uncertainty associated with the continuation region and will exercise the investment option sooner.[14]

Now, consider the risk scenario described in Section 3.3 generalized to risk-averse DMs. Following the same steps as above, it is easy to show that the solution for the investment trigger is

$$\widetilde{V}_R^* = \frac{\eta_R(\widetilde{V}_R^*)}{\eta_R(\widetilde{V}_R^*) - 1} C, \tag{15}$$

where $\eta_R(V) = [r + p(V)\lambda_L + (1-p(V))\lambda_H]/[(1-\gamma)\mu]$.

Analogous steps lead to the following expression for the investment trigger in the ambiguity scenario described in Section 3.4 under risk aversion:

$$\widetilde{V}_A^* = \frac{\eta_A(\widetilde{V}_A^*)}{\eta_A(\widetilde{V}_A^*) - 1} C, \tag{16}$$

where $\eta_A(V) = [r + p^-(V)\lambda_L + (1-p^-(V))\lambda_H]/[\mu(1-\gamma)]$.

Note that $\eta_R(V) = \beta_R(V)/(1-\gamma)$ and $\eta_A(V) = \beta_A(V)/(1-\gamma)$. Thus the

[14] This is the same result found in Miao and Wang (2007) when the final payoff is lump sum. They also show that, when the final payoff is given by a flow of an uncertain income, the result is reversed.

model with risk neutrality derived in the main text can be simply generalized to constant relative risk aversion by using "risk-aversion-corrected" drift, $\mu(1-\gamma)$, in place of physical drift, μ. It is then easy to check that expressions (15) and (16) imply that the degree of risk aversion γ does not affect comparative static conclusions about the effects of risk and ambiguity on the timing of investment.

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