Dynamic Equilibrium with Two Stocks, Heterogeneous Investors, and Portfolio Constraints*

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Abstract

We study dynamic general equilibrium in a Lucas economy with two trees, one consumption good, two CRRA investors with heterogeneous risk aversions, and portfolio constraints. We focus on margin and leverage constraints, which restrict access to credit markets. We find positive relationship between the amount of leverage in the economy and magnitudes of conditional stock return correlations and volatilities. Tighter constraints give rise to rich saddle-type patterns in correlations and volatilities, make them less countercyclical, increase risk premia proportionally to assets' margins, and increase price-dividend ratios of low-margin assets more than those of high-margin assets. The paper offers a new methodology for solving models with constraints, and derives closed-form solutions for the unconstrained case and the case of leverage constraints.

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Portfolio constraints are pervasive features of the real world which render financial markets incomplete and prevent efficient risk sharing among investors. Consequently, portfolio constraints are likely to have first order economic effects when they are added to frictionless variations of Lucas (1978) economy, which have been widely used by financial economists to explain dynamic and cross-sectional characteristics of asset prices. Despite wide interest in the effect of constraints on financial markets, especially in the aftermath of the recent financial crisis, long-standing questions on how they influence investor asset allocation, asset prices, and their moments, remain relatively unexplored. In this paper we answer these questions in a parsimonious setup, which accounts for two additional salient features of financial markets, investor heterogeneity in preferences and the multiplicity of risky assets. Our model provides a tractable laboratory for evaluating interactions between constraints and investor heterogeneity without restrictive assumptions of investor myopia and one-stock economy, commonly employed in the literature.

We consider a dynamic general equilibrium Lucas (1978) economy with one consumption good, two Lucas trees, and two groups of investors, which have heterogeneous constant relative risk aversion preferences (CRRA) over consumption and may face portfolio constants. Specifically, we focus on margin constraints, under which an investor can borrow only up to a certain limit using stocks as collaterals, and leverage constraints, under which an investor cannot borrow. These constraints allow us to study how the access to credit markets and differences in stock margins influence stock prices, their conditional correlations, volatilities, and other processes.

First, we demonstrate that heterogeneity in preferences generates significant increase in conditional stock return correlations and volatilities in the unconstrained benchmark, relative to homogeneous-investor economies in the previous literature [e.g., Cochrane, Longstaff and Santa-Clara (2008); Martin (2012)]. Furthermore, the correlations and volatilities are countercyclical, stocks are more correlated and more volatile than dividends, consistent with empirical evidence [e.g., Shiller (1989); Karolyi and Stulz (1996); Ang and Bekaert (2002); Ribeiro and Veronesi (2002)]. Our intuition emphasizes the role of credit markets. In particular, the less risk averse investor levers up and scales portfolio weights up or down, depending on the availability of credit, which increases correlations and volatilities. The effect of leverage is stronger when both borrowers and lenders have large shares in the aggregate consumption.

Second, we find that tighter constraints curb bilateral trades, deleverage the economy, and hence lead to lower conditional correlations and volatilities, consistent with the above intuition on the role of leverage. Therefore, the model predicts a positive relationship between the amount of leverage and sizes of correlations and volatilities. Low correlations also imply that constraints make dividend shocks less contagious, restricting their ability to spread throughout the economy via the common discount factor. Moreover, constraints bind intermittently, make correlations and volatilities spike around times when they start to bind, and reduce their countercyclicality. They also generate rich saddle-type patterns in correlations and market portfolio return volatilities. That is, in an economy where two uncorrelated dividend processes have the same means and volatilities of dividend growth rates, conditional correlations are significantly lower in states where both Lucas trees have equal sizes. Intuitively, trees with equal sizes are equally attractive, and hence if investors cannot borrow (i.e., face leverage constraints) they invest 50% of wealth in each stock. As a result, investors' asset holdings become homogeneous, and correlations decrease towards those in homogeneous-investor economies. Hence, relative sizes of firms or industries, interpreted as trees, can predict correlations and volatilities when credit is tight.

Third, we argue that constrained investors self-select to trade in riskier segments of asset markets. In particular, despite low correlations and better diversification opportunities, the constrained investor substitutes leverage with riskier assets. This substitution effect generates bilateral trades, and hence all equilibrium processes remain stochastic even in the case of leverage constraints. The latter result is in contrast to results in one-tree economy with leverage constraints [e.g., Kogan, Makarov, and Uppal (2007)] where investors do not trade, stock return volatility and the market price of risk are constant, and the interest rate is time-deterministic.

Next, we explore the effect of margin heterogeneity on asset prices. Intuitively, a low-margin asset is a better collateral, that helps relax borrowing limits, and hence its collateral benefits should be priced. To evaluate the pure effect of margin heterogeneity, we look at the case where both trees have the same mean growth rates and volatilities, but different margins. Therefore, when the trees have equal sizes they are symmetric, and hence the differences in their valuations are solely due to the differences in margins. We show that imposing constraints increases price-dividend ratios of low-margin assets more than those of high-margin assets. These results are consistent with the above intuition and the evidence that collateral benefits are reflected in the valuations of assets, such as real estate [e.g., Lustig and Van Nieuwerburgh (2005)] and government bonds [e.g., Brunnermeier (2009)].

We also provide a new consumption CAPM with constraints. In contrast to other works, our expression for the risk premia is in terms of observable processes. Similarly to existing CAPMs [e.g., Cuoco (1997), Gârleanu and Pedersen (2011)], it predicts that low-margin assets should have lower risk premia. Moreover, we extend classical Black's (1972) static mean-variance CAPM with the leverage constraint to our dynamic setting and characterize the deviation from Breeden's C-CAPM in closed form. The model predicts that this deviation increases with the difference in investors' risk aversions and the consumption share of the less risk averse investor. Moreover, similarly to other works [e.g., Basak and Cuoco (1998); Gârleanu and Pedersen (2011)] we find that constraints increase market prices of risk to compensate the more risk averse investor for holding a riskier portfolio, and decrease interest rates due to lower demand for credit.

The decrease in volatilities under constraints is consistent with some empirical evidence. In particular, Hardouvelis and Theodossiou (2002) show that volatilities decrease in normal and bull periods in response to increase in initial margins by the Federal Reserve, while Hardouvelis and Perestiani (1992) document similar effects of margins on daily volatilities using data from Tokyo Stock Exchange. However, some studies [e.g., Schwert (1989)] do not find any effect of margins on volatilities. Accounting for leverage-constrained investors in future tests may strengthen the results for two reasons. First, the constrained investor in our model corresponds to an "average" constrained real-world investor, which subsumes both margin- and leverage-constrained investors. Second, the leverage constraint is important since many investors, e.g. retail investors, some mutual and pension funds, cannot or are not allowed to lever up too much [e.g., Frazzini and Pedersen (2011)].

In light of the discussion above, the leverage constraint is important in its own right. In our model it additionally helps isolate and quantify the effects of credit markets by suppressing borrowing completely. We solve the model with the leverage constraint in closed form, and hence provide the first tractable setting in the literature that allows to explore some of the effects of constraints analytically in a setting with two trees. Using these results we show that constraints have first-order effects on equilibrium. The unconstrained equilibrium is also derived in closed form, which facilitates the comparison with the constrained case. In particular, price-dividend ratios are given by easily computable integrals, and generalize results in Cochrane, Longstaff, and Santa-Clara (2008) and Martin (2012) to the case of heterogeneous preferences.

The paper also offers a new tractable approach for evaluating asset pricing implications of portfolio constraints and investor heterogeneity. This approach preserves the tractability of complete markets by solving constrained optimization in an equivalent fictitious unconstrained economy. Our paper is the first to use fictitious economies for the analysis of general equilibrium without imposing restrictive assumptions of logarithmic investors and one-stock economy, commonly used in the literature. We obtain expressions for market prices of risk and interest rates that preserve the tractability of their complete-market analogues, but additionally incorporate new terms with shadow costs of constraints. The shadow costs are obtained in terms of investors' price-dividend and wealth-consumption ratios satisfying a system of differential equations, which we solve numerically using finite-difference methods.

The previous literature provides tractable characterizations of equilibria primarily in models with logarithmic constrained investors and one-tree economies. We note that logarithmic investors do not have hedging demands, and hence behave myopically. Therefore, their wealth-consumption ratios are constant, and hence play only a limited role in transmitting the effects of constraints into equilibrium processes. For example, in one-stock economies populated only by logarithmic investors, stock prices are unaffected by constraints [e.g., Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000, 2006)]. The importance of incorporating risk aversions into general equilibrium analysis has been underscored even in unconstrained two-trees settings [e.g., Cochrane, Longstaff, and Santa-Clara (2008)].

There is growing literature on equilibrium effects of constraints. In particular, Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000, 2006), Kogan, Makarov, and Uppal (2007), Gallmeyer and Hollifield (2008), Prieto (2010), He and Krishnamurthy (2011) study equilibrium with various one-tree economies with logarithmic constrained investors in one-tree economies. Pavlova and Rigobon (2008), and Schornick (2009) study models with constrained logarithmic investors and two trees, and demonstrate how constraints amplify shocks in an international finance setting. In contrast to their models, our model does not rely on heterogeneous home biases and logarithmic preferences, and finds that constraints can prevent contagions in a one-country setting. Hugonnier (2012) explores asset pricing bubbles in one-tree and two-trees economies with restricted participation.

Chabakauri (2009, 2010) provides the first tractable approach for solving models with restricted participation, short-sale, borrowing, and margin constraints for general CRRA preferences, and finds that margins increase market prices of risk, decrease interest rates and stock return volatilities, and make volatilities less countercyclical. In a similar setting, Rytchkov (2011) incorporates time variations in margins and finds similar results. Gârleanu and Pedersen (2011) study the effects of margins and deviations from the law of one price in an economy with logarithmic constrained investors. Gromb and Vayanos (2002, 2009, 2010), Coen-Pirani (2005), Fostel and Geanakoplos (2008), Brunnermeier and Pedersen (2009), study the impact of margins in economies with different preferences. Frazzini and Pedersen (2011) derive static mean-variance CAPM with margin constraints. Our model is an extension of theirs to the case of a dynamic Lucas economy. Kiyotaki and Moore (1997) consider a production economy with collateral constraints. Brumm et al (2012) consider a model with endogenous collateral constraints in discrete time and show that constraints may increase asset return volatilities. Buss and Dumas (2012), Dumas and Lyasoff (2012) develop new methods for solving incomplete-market models.

Ribeiro and Veronesi (2002), Menzly, Santos, and Veronesi (2004), Cochrane, Longstaff, and Santa-Clara (2008), Buraschi, Trojani, and Vedolin (2010), Chen and Joslin (2011) and Martin (2012) study unconstrained economies with multiple assets. Ehling and Heyerdahl-Larsen (2012) find large countercyclical correlations in a model with heterogeneous CRRA investors and habits. Bhamra and Uppal (2010), and Longstaff and Wang (2012) study equilibrium in one-tree unconstrained economies in closed form, and among other results also demonstrate that the heterogeneity in risk aversions amplifies volatilities, as in our unconstrained model.

The remainder of the paper is organized as follows. Section 1 discusses the economic setup and defines the equilibrium. In Section 2, we provide the characterization of equilibrium processes, discuss their properties, and describe the solution approach. In Section 3 we provide the analysis of equilibrium and the intuition for our numerical results. Section 4 concludes, Appendix A provides the proofs, and Appendix B provides further details of the numerical method.

1. Economic Setup

We consider an infinite horizon Lucas (1978) economy with two stocks and one consumption good, which is generated by two Lucas trees. The economy is populated by two CRRA investors, $i = A$ and $i = B$, with risk aversions γ_A and γ_B, where $\gamma_A \geq \gamma_B$. The uncertainty is generated by a two-dimensional Brownian motion $w = (w_1, w_2)^\top$. The Lucas trees produce streams of dividends D_{jt} that follow geometric Brownian motions (GBMs):

$$dD_{jt} = D_{jt}[\mu_{D_j} dt + \sigma_{D_j} dw_{jt}], \qquad j = 1, 2, \tag{1}$$

where Brownian motions w_1 and w_2 are uncorrelated, and μ_{D_j} and σ_{D_j} are constants. The aggregate dividend $D = D_1 + D_2$ then, by Itô's Lemma, follows a process:

$$dD_t = D_t[\mu_{Dt} dt + \sigma_{Dt}^\top dw_t], \tag{2}$$

where $\mu_D = x\mu_{D_1} + (1-x)\mu_{D_2}$, $\sigma_D = (x\sigma_{D_1}, (1-x)\sigma_{D_2})^\top$, and $x = D_1/D$ is the share of the first tree in the aggregate dividend.

The investors continuously trade in three securities: a riskless bond in zero net supply with instantaneous interest rate r, and two stocks, each in net supply of one unit, which are claims to the output generated by Lucas trees (1). We look for Markovian equilibria in which bond prices, B, and stock prices, $S = (S_1, S_2)^\top$, follow dynamics:

$$dB_t = B_t r_t dt, \tag{3}$$

$$dS_{jt} + D_{jt} dt = S_{jt}[\mu_{jt} dt + \sigma_{jt}^\top dw_t], \qquad j = 1, 2, \tag{4}$$

where $\sigma_j = (\sigma_{j1}, \sigma_{j2})^\top$, and we let $\mu = (\mu_1, \mu_2)^\top$ and $\Sigma = (\sigma_1, \sigma_2)^\top$ denote the vector of mean returns and the volatility matrix of stock returns, respectively. By $\theta_i = (\theta_{i1}, \theta_{i2})^\top$ and α_i we denote the fractions of wealth that investor i allocates to stocks 1 and 2, and bonds, respectively.

1.1. Investors' Optimization and Portfolio Constraints

The investors maximize expected discounted utility of consumption with time discount $\rho > 0$:

$$\max_{c_{it}, \theta_i \in \Theta_i} \mathbb{E}\Big[\int_0^\infty e^{-\rho t} \frac{c_{it}^{1-\gamma_i}}{1-\gamma_i} dt\Big], \quad i = A, B, \tag{5}$$

subject to a self-financing budget constraint

$$dW_{it} = \Big[W_{it}\Big(r_t + \theta_{it}^\top(\mu_t - r_t)\Big) - c_{it}\Big]dt + W_{it}\theta_{it}^\top \Sigma_t dw_t, \qquad i = A, B, \tag{6}$$

and portfolio constraints $\theta_i \in \Theta_i$. For $\gamma_i = 1$ the utility in (5) is replaced with logarithmic utility $\ln(c_{it})$. At $t = 0$ A is endowed with $1 - n_0$ units of stock and b_0 units of bond, while B with n_0 units of stock and $-b_0$ units of bond. Investor A is unconstrained, while investor B faces a margin constraint [e.g., Brunnermeier and Pedersen (2009), Gromb and Vayanos (2009), Gârleanu and Pedersen (2011); among others], and hence sets Θ_i are given by:

$$\Theta_A = \mathbb{R}^2, \quad \Theta_B = \{\theta = (\theta_1, \theta_2)^\top : m^\top \theta \le 1\}, \tag{7}$$

where $0 \leq m_1 \leq 1$, and $0 \leq m_2 \leq 1$, and we let $m = (m_1, m_2)^\top$ denote the vector of margins.[1] We remark in Section 2.2 that our solution method can also handle the case of time-varying margins, short sale constraints, and other portfolio constraints. However, to concentrate on the tightness of access to credit, not confounded by other factors, we do not consider time-varying margins and short-sale constraints.

Similarly to Gârleanu and Pedersen (2011), we note that margin constraint $m^\top \theta_B \leq 1$ can be rewritten as $\theta_{B1} + \theta_{B2} \leq 1 + \theta_{B1}(1 - m_1) + \theta_{B2}(1 - m_2)$, where $\theta_{B1}(1 - m_1) + \theta_{B2}(1 - m_2)$ is the fraction of wealth that can be borrowed using stocks as collateral. Consequently, margin m_j is interpreted as the proportion of asset j's value against which the investor cannot borrow. The *leverage constraint* and the unconstrained case are special cases of constraint (7) when $m = (1, 1)^\top$ and $m = (0, 0)^\top$, respectively.

1.2. Equilibrium

Definition 1. *An equilibrium is a set of processes* $\{r_t, \mu_t, \Sigma_t\}$ *and of consumption and investment policies* $\{c^*_{it}, \alpha^*_{it}, \theta^*_{it}\}_{i \in \{A,B\}}$ *that maximize expected utility* (5) *for each investor, given processes* $\{r_t, \mu_t, \Sigma_t\}$*, and consumption and financial markets clear, i.e.,*

$$\begin{aligned} c^*_{At} + c^*_{Bt} &= D_t, \\ \theta^*_{At} W_{At} + \theta^*_{Bt} W_{Bt} &= S_t, \\ \alpha^*_{At} W_{At} + \alpha^*_{Bt} W_{Bt} &= 0. \end{aligned} \tag{8}$$

Instead of stock mean-returns μ we derive and report the *market prices of risk* $\kappa = \Sigma^{-1}(\mu - r\underline{1})$, where $\underline{1} = (1, 1)^\top$, from which mean-returns μ can be easily recovered. Furthermore, we also solve for investors' *wealth-consumption ratios* and *stock price-dividend ratios*, given by $\Phi_i = W_i/c^*_i$ and $\Psi_j = S_j/D_j$, respectively. We study the equilibria, which are Markovian in two state variables: the first tree's and investor B's consumption shares, $x = D_1/D$ and $y = c^*_B/D$, respectively, which are conjectured to follow Markovian processes

$$dx_t = x_t[\mu_{xt} + \sigma_{xt}^\top dw_t], \quad dy_t = -y_t[\mu_{yt} + \sigma_{yt}^\top dw_t], \tag{9}$$

where $\mu_x = (1 - x)(\mu_{D_1} - \mu_{D_2}) - \sigma^2_{D_1} x(1 - x) + \sigma^2_{D_2}(1 - x)^2$, $\sigma_x = ((1 - x)\sigma_{D_1}, -(1 - x)\sigma_{D_2})^\top$, while μ_y and $\sigma_y = (\sigma_{y1}, \sigma_{y2})^\top$ are determined in equilibrium. The choice of state variables is further discussed in Section 2.2.

[1] We note that margin constraint $m^\top \theta \leq 1$ does not bind for the more risk averse investor A in equilibrium. Consequently, assuming that investor A is unconstrained is without loss of generality. We do not consider economies with restricted participation (i.e., $m_j > 1$) where the analysis is complicated by singularities in equilibrium processes, since the market prices of risk explode when the economy is dominated by the constrained investor [e.g. Basak and Cuoco (1998); Chabakauri (2010)], and leave this case for future research.

2. Characterization of Equilibrium and Consumption CAPM

In this Section we provide the characterization of equilibrium and derive tractable generalizations of capital asset pricing model to the case of margin and leverage constraints. First, in Section 2.1 we obtain optimal consumptions of investors in a partial equilibrium setting by employing the duality approach of Cvitanić and Karatzas (1992). Next, in Section 2.2, we obtain equilibrium processes for market prices of risk, interest rates, and stock return volatilities in terms of shadow costs of constraints. Finally, we derive partial differential equations for price-dividend and wealth-consumption ratios, and discuss economic intuition for equilibrium processes.

2.1. Optimal Consumptions in Partial Equilibrium

As demonstrated in Cvitanić and Karatzas (1992), the utility maximization (5) subject to budget constraint (6) and portfolio constraint (7) can be solved in a *fictitious* unconstrained economy with bond and stock prices following dynamics with adjustments:

$$dB_t = B_t(r_t - \nu^*_t)dt, \tag{10}$$

$$dS_{jt} + D_{jt}dt = S_{jt}[(\mu_{jt} + m_j\nu^*_t - \nu^*_t)dt + \sigma^\top_{jt}dw_t], \quad j = 1, 2, \tag{11}$$

where *adjustment* ν^*_t is the shadow cost of constraint, which can be found from Kuhn-Tucker conditions of optimality [e.g., Cvitanić and Karatzas (1992); Karatzas and Shreve (1998)]. Intuitively, binding constraint reduces constrained investor's share of wealth allocated to stocks, relative to the unconstrained case. The reduction in stock holdings can be mimicked in an unconstrained economy with higher interest rates and lower risk premia than in the original economy. This intuition suggests that $\nu^* \le 0$.

The state price densities in the unconstrained complete-market real and fictitious economies, ξ and ξ_{ν^*t}, evolve as follows [e.g. Duffie (2001)]:

$$d\xi_t = -\xi_t[r_tdt + \kappa_tdw_t], \quad d\xi_{\nu^*t} = -\xi_{\nu^*t}[(r_t - \nu^*_t)dt + (\kappa_t + \nu^*_t\Sigma_t^{-1}m)^\top dw_t], \tag{12}$$

where $\kappa = \Sigma^{-1}(\mu - r\underline{1})$ denotes the vector of market prices of risk. Next, we obtain optimal consumptions of investors from the first order conditions that equate investors' marginal utilities and state price densities [e.g., Huang and Pagés (1992); Cuoco (1997)]:

$$c^*_{At} = \Big(\psi_Ae^{\rho t}\xi_t\Big)^{-\frac{1}{\gamma_A}}, \quad c^*_{Bt} = \Big(\psi_Be^{\rho t}\xi_{\nu^*t}\Big)^{-\frac{1}{\gamma_B}}, \tag{13}$$

where ψ_i denote Lagrange multipliers for static budget constraints in the martingale approach. The equilibrium processes can be derived by substituting consumptions (13) into consumption clearing condition in (8), applying Itô's Lemma to both sides, and matching dt and dw terms.

Remark 1 (Fictitious Economy and Complementary Slackness Condition). The construction of the fictitious economy can be conveniently illustrated via dynamic programming.

Suppose that stock and bond prices are driven by a Markovian state variable $z_t = (z_{1t}, z_{2t})^\top$ which follows process $dz_t = \mu_{zt}dt + \Sigma_{zt}dw_t$, where μ_z and Σ_z denote the vector of drifts and the volatility matrix, respectively. Let J_{Bt} denote investor B's time-t value function, which we conjecture to depend on wealth W, vector z, and time t. Let ℓ_t denote time-t Lagrange multiplier for constraint $m^\top\theta_B \leq 1$, and ν_t^* be the rescaled Lagrange multiplier, given by $\nu_t^* = \ell_t/(W_t\partial J_{Bt}/\partial W_t)$. Then, the value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$0 = \max_{c_{Bt},\theta_{Bt}} \Big\{ e^{-\rho t}\frac{c_{Bt}^{1-\gamma_B}}{1-\gamma_B}dt + \mathbb{E}_t[dJ_{Bt}] + \nu_t^*(m^\top\theta_{Bt} - 1)W_t\frac{\partial J_{Bt}}{\partial W_t}dt \Big\}, \tag{14}$$

which can be further expanded as follows:

$$\begin{aligned} 0 = \max_{c_{Bt},\,\theta_{Bt}} \Big\{ & e^{-\rho t}\frac{c_{Bt}^{1-\gamma_B}}{1-\gamma_B} + \frac{\partial J_{Bt}}{\partial t} + \Big[W_t\Big(r_t - \nu_t^* + (\mu_t - r_t + \nu_t^* m)^\top\theta_{Bt}\Big) - c_{Bt}\Big]\frac{\partial J_{Bt}}{\partial W_t} \\ & + z_t\mu_{zt}^\top\frac{\partial J_{Bt}}{\partial z_t^\top} + \frac{1}{2}\Big[W_t^2\theta_{Bt}^\top\Sigma_t\Sigma_t^\top\theta_{Bt}\frac{\partial^2 J_{Bt}}{\partial W_t^2} + 2W_t\theta_{Bt}^\top\Sigma_t\Sigma_{zt}^\top\frac{\partial^2 J_{Bt}}{\partial W_t\partial z_t^\top} + \mathrm{Tr}\Big(\Sigma_{zt}\Sigma_{zt}^\top\frac{\partial^2 J_{Bt}}{\partial z_t^\top\partial z_t}\Big)\Big]\Big\}, \end{aligned} \tag{15}$$

subject to transversality condition $\mathbb{E}_t[J_{BT}] \to 0$, as $T \to \infty$. We observe, that equation (15) corresponds to an HJB equation in the unconstrained fictitious economy with bond and stock prices following adjusted processes (10)–(11). Furthermore, Kuhn-Tucker optimality conditions imply that $\nu_t^* \leq 0$, and complementary slackness condition $\nu_t^*(m^\top\theta_{Bt}^* - 1) = 0$ is satisfied.

2.2. Characterization of General Equilibrium

The equilibrium is characterized in three steps. First, we assume the existence of a Markovian equilibrium, and from consumption clearing condition recover market prices of risk $\kappa = \Sigma^{-1}(\mu - r\underline{1})$, interest rates r, mean growth μ_y and volatility σ_y of consumption share y in terms of adjustment ν^*, and volatility matrix Σ. Second, we obtain adjustment ν^* in terms of price-dividend and investors' wealth-consumption ratios from Kuhn-Tucker conditions. Finally, we derive PDEs for price-dividend and wealth-consumption ratios, and solve them numerically. We note that since constrained optimization is solved in the fictitious complete-market economy, investors' value functions, portfolio weights, and PDEs for wealth-consumption ratios Φ_i are special cases of those in complete-market portfolio choice literature [e.g., Liu (2007)]. Proposition 1 summarizes the results.

Proposition 1. *In a Markovian equilibrium with state variables $x = D_1/D$ and $y = c_B^*/D$ the market price of risk $\kappa = \Sigma^{-1}(\mu - r\underline{1})$, interest rate r, volatility and drift of consumption share y, μ_y and σ_y, stock return volatilities $\sigma_j = (\sigma_{j1}, \sigma_{j2})^\top$, and portfolio weights θ_i^* are given by:*

$$\kappa_t = \Gamma_t\sigma_{Dt} - \frac{\Gamma_t y_t}{\gamma_B}\nu_t^*\Sigma_t^{-1}m, \tag{16}$$

$$r_t = \rho + \Gamma_t\mu_{Dt} - \frac{\Gamma_t\Pi_t}{2}\sigma_{Dt}^\top\sigma_{Dt} + \frac{\Gamma_t y_t\nu_t^*}{\gamma_B} + (\nu_t^*\Sigma_t^{-1}m)^\top(g_1(y_t)\sigma_{Dt} + g_2(y_t)\nu_t^*\Sigma_t^{-1}m), \tag{17}$$

$$\sigma_{yt} = \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\Big((\gamma_B-\gamma_A)\sigma_{Dt} - \nu_t^*\Sigma_t^{-1}m\Big), \tag{18}$$

$$\mu_{yt} = \mu_{Dt} - \sigma_{Dt}^\top\sigma_{yt} - \frac{1+\gamma_B}{2}(\sigma_{Dt}-\sigma_{yt})^\top(\sigma_{Dt}-\sigma_{yt}) - \frac{r_t-\nu_t^*-\rho}{\gamma_B}, \tag{19}$$

$$\sigma_{jt} = e_j\sigma_{D_j} + \sigma_{xt}\frac{\partial\Psi_{jt}}{\partial x_t}\frac{x_t}{\Psi_{jt}} - \sigma_{yt}\frac{\partial\Psi_{jt}}{\partial y_t}\frac{y_t}{\Psi_{jt}}, \quad j=1,2, \tag{20}$$

$$\theta_{it}^* = (\Sigma_t^\top)^{-1}\Big(\frac{\kappa_t + 1_{\{i=B\}}\nu_t^*\Sigma_t^{-1}m}{\gamma_i} + \sigma_{xt}\frac{\partial\Phi_{it}}{\partial x_t}\frac{x_t}{\Phi_{it}} - \sigma_{yt}\frac{\partial\Phi_{it}}{\partial y_t}\frac{y_t}{\Phi_{it}}\Big), \quad i=A,B, \tag{21}$$

where Γ_t and Π_t are the representative agent's[2] *risk aversion and prudence parameters given by:*

$$\Gamma = \frac{1}{y/\gamma_B + (1-y)/\gamma_A}, \qquad \Pi = \Gamma^2\Big(\frac{1+\gamma_A}{\gamma_A^2}(1-y) + \frac{1+\gamma_B}{\gamma_B^2}y\Big), \tag{22}$$

$g_1(y)$ and $g_2(y)$ are functions given by equations (A1) *in the Appendix, $e_1=(1,0)^\top$, $e_2=(0,1)^\top$, and $1_{\{i=B\}}$ is an indicator function. Adjustment ν_t^* satisfies Kuhn-Tucker conditions*

$$\nu_t^*(m^\top\theta_{Bt}^*(x_t,y_t;\nu_t^*)-1)=0, \quad \nu_t^*\le 0, \quad m^\top\theta_{Bt}^*(x_t,y_t;\nu_t^*)\le 1, \tag{23}$$

and is given by equation (A3) *in Appendix A as a function of price-dividend ratios Ψ_1, Ψ_2, wealth-consumption ratio Φ_B, and their derivatives.*

Investors' value functions are given by $J_{it} = e^{-\rho t}W_{it}^{1-\gamma_i}\Phi_{it}^{\gamma_i}/(1-\gamma_i)$, and price-dividend ratios $\Psi_j(x,y)$ and wealth-consumption ratio $\Phi_B(x,y)$ satisfy PDEs:

$$\begin{aligned}
&\mathcal{D}\Psi_j + x(\mu_x - (\kappa-\sigma_{D_j}e_j)^\top\sigma_x)\frac{\partial\Psi_j}{\partial x} - y(\mu_y - (\kappa-\sigma_{D_j}e_j)^\top\sigma_y)\frac{\partial\Psi_j}{\partial y} \\
&\quad + (\mu_{D_j} - r - \sigma_{D_j}e_j^\top\kappa)\Psi_j + 1 = 0, \qquad j=1,2, &(24)\\
&\mathcal{D}\Phi_B + x\Big(\mu_x + \frac{1-\gamma_B}{\gamma_B}(\kappa+\nu^*\Sigma^{-1}m)^\top\sigma_x\Big)\frac{\partial\Phi_B}{\partial x} - y\Big(\mu_y + \frac{1-\gamma_B}{\gamma_B}(\kappa+\nu^*\Sigma^{-1}m)^\top\sigma_y\Big)\frac{\partial\Phi_B}{\partial y} \\
&\quad + \Big(\frac{1-\gamma_B}{2{\gamma_B}^2}(\kappa+\nu^*\Sigma^{-1}m)^\top(\kappa+\nu^*\Sigma^{-1}m) + \frac{1-\gamma_B}{\gamma_B}(r-\nu^*) - \frac{\rho}{\gamma_B}\Big)\Phi_B + 1 = 0, &(25)
\end{aligned}$$

where $\mathcal{D}$ is a zero-drift Dynkin's operator,[3] *and the boundary conditions are given in Appendix B. The wealth-consumption ratio $\Phi_A(x,y)$ is given by $\Phi_A = (x\Psi_1 + (1-x)\Psi_2 - y\Phi_B)/(1-y)$.*

We derive all processes in Proposition 1 assuming the existence of a Markovian equilibrium with state variables $x = D_1/D$ and $y = c_B^*/D$. If there exist solutions of PDEs (24)–(25)

[2]Similarly to Basak and Cuoco (1998) and Basak (2000, 2005) it can be demonstrated that the equilibrium in this economy is equivalent to the equilibrium in an economy with a *representative investor* with utility

$$u(c;\lambda_t) = \max_{c_A+c_B=c} \frac{c_A^{1-\gamma_A}}{1-\gamma_A} + \lambda_t\frac{c_B^{1-\gamma_B}}{1-\gamma_B},$$

where $\lambda_t = \xi_{\nu^*t}/\xi_t$. The expressions for the relative risk aversion Γ and prudence Π of the representative investor, given by (22), are special cases of those in Basak (2000, 2005), derived for general utility functions.

[3]Zero-drift Dynkin operator $\mathcal{D}$ is defined as follows:

$$\mathcal{D}F = \frac{1}{2}\Big(x^2\sigma_x^\top\sigma_x\frac{\partial^2F}{\partial x^2} + y^2\sigma_y^\top\sigma_y\frac{\partial^2F}{\partial y^2} - 2xy\sigma_x^\top\sigma_y\frac{\partial^2F}{\partial x\partial y}\Big).$$

and equilibrium processes are bounded it can be verified that the economy is in equilibrium [e.g., Remark A.1 in Appendix A] and there are no asset pricing bubbles [e.g., Lemma A.3 in Appendix A]. We note that PDEs are quasilinear, since adjustment ν^* depends on Φ_i, Ψ_j, and their derivatives. General existence results for quasilinear PDEs are not available in the literature. Therefore, the best that we can do is to derive the equilibrium numerically and verify that all processes are bounded and all equilibrium conditions are satisfied.[4]

We further motivate the choice of state variables as follows. First, in the proof of Proposition 1 we demonstrate that expressions (16)–(19) hold in any Markovian equilibrium with state variable vector z following an Itô's process, and hence x, y, and ν^* can be chosen as new state variables. Furthermore, since adjustment ν^* is determined from Kuhn-Tucker conditions, we conjecture that it is not an independent state variable, and derive it as a function of x and y. Second, x and y emerge as state variables in equilibria that admit closed form solutions, e.g. a model with CRRA preferences and leverage constraints in Proposition 2 below. Related literature also uses consumption share y as a state variable in models with logarithmic constrained investors [e.g., Gallmeyer and Hollifield (2008), Prieto (2010), Gârleanu and Pedersen (2011), among others].

Remark 2 (More General Constraints). We also remark that our solution method remains valid for other types of constraints, and time-varying margins. Certain types of time-varying margins are even more tractable than constant margins. For example, it has been pointed out in Prieto (2010) that equilibrium processes become simpler when margins are proportional to volatility. In particular, if margins are given by $m_t = \Sigma_t \tilde{m}$, where $\tilde{m}$ is a vector of constants, we observe that inverse volatility matrix Σ_t^{-1} disappears from expressions (16)–(21) for equilibrium processes, which significantly simplifies the analysis. However, we choose constant margins to switch off extra channels of margin heterogeneity and focus on the tightness of access to credit.

We complete the characterization of equilibrium by providing closed-form equilibrium processes in the unconstrained economy [i.e., $m = (0,0)^\top$] and the economy with the leverage constraint [i.e., $m = (1,1)^\top$], and then discuss the economic intuition. The closed-form solutions are useful for cross-checking numerical methods discussed below, and provide sharper intuition for interactions between constraints and investor heterogeneity. Proposition 2 reports the results.

Proposition 2.

(i) *In the unconstrained economy κ, r, μ_y, σ_y, σ_j are given in closed form by expressions (16)–(20) in which $\nu^* = 0$, and price-dividend ratios $\Psi_j(x,y)$ are given by $\Psi_1(x,y) =$*

[4] Quasilinear differential equations also arise in other models with heterogeneous investors that face portfolio constraints, or are unconstrained but have recursive preferences [e.g., Gârleanu and Pedersen (2011); Gârleanu and Panageas (2010)]. However, we are not aware of existence results demonstrated in the literature.

$\Psi(x, y; \{\mu_{D_1}, \sigma_{D_1}\}, \{\mu_{D_2}, \sigma_{D_2}\})$ *and* $\Psi_2(x, y) = \Psi(1-x, y; \{\mu_{D_2}, \sigma_{D_2}\}, \{\mu_{D_1}, \sigma_{D_1}\})$*, where*

$$\Psi(x, y; \{\mu_{D_1}, \sigma_{D_1}\}, \{\mu_{D_2}, \sigma_{D_2}\}) =$$

$$\int_0^1 \int_0^1 \Big(\frac{s}{x}\frac{y}{z}\Big)^{\gamma_B} \frac{\gamma_B(1-z) + \gamma_A z}{s(1-s)z(1-z)} \frac{e^{q^\top \Sigma^{-1} u(s,z;x,y)} K_0\big(p\sqrt{u(s,z;x,y)^\top \Sigma_u^{-1} u(s,z;x,y)}\big)}{\pi \gamma_A \sqrt{\det(\Sigma_u)}} ds\, dz, \tag{26}$$

$$u(s, z; x, y) = \Big(\ln\Big(\frac{1-z}{1-y}\Big) - \frac{\gamma_B}{\gamma_A}\ln\Big(\frac{z}{y}\Big) + \frac{\gamma_B - \gamma_A}{\gamma_A}\ln\Big(\frac{s}{x}\Big), \ln\Big(\frac{1-s}{1-x}\frac{x}{s}\Big)\Big)^\top, \tag{27}$$

$K_0(z) = \int_0^\infty e^{-z\cosh(s)} ds$ *is a McDonald's function,*[5] *while parameters* $p \in \mathbb{R}$, $q \in \mathbb{R}^2$, *and* $\Sigma_u \in \mathbb{R}^{2\times 2}$ *are given by equations* (A38)–(A39) *in the Appendix as functions of dividend mean growths and volatilities,* μ_{D_1}, μ_{D_2}, σ_{D_1}, *and* σ_{D_2}.

(ii) *In the economy with the leverage constraint* (*i.e.,* $m = (1,1)^\top$) κ, r, μ_y *and* σ_y *are given in closed form by expressions* (16)–(19)*, where* ν^* *and* $\nu^*\Sigma^{-1}m$ *are given by:*

$$\nu_t^* = \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2}, \qquad \nu_t^*\Sigma_t^{-1}m = \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2}\Big(\frac{1}{\sigma_{D_1}}, \frac{1}{\sigma_{D_2}}\Big)^\top. \tag{28}$$

Price-dividend ratios $\Psi_j(x, y)$ *are given by equation* (A50) *in Appendix A in closed form.*

The tractability of the leverage constraint is due to the fact that equilibrium restricts both investors to invest all their wealth in stocks, and hence $\theta_{i1} + \theta_{i2} = 1$, where $i = A, B$. While the leverage constraint is less general than margin constraint (7), Black (1972) and Heaton and Lucas (1996) advocate its economic importance based on evidence that many investors face severe borrowing restrictions. Moreover, by suppressing leverage, this constraint allows to evaluate the importance of access to credit markets analytically. For example, assuming $\sigma_{D_1} = \sigma_{D_2}$ we obtain that $\nu^*\Sigma^{-1}m = 0.5(\gamma_A - \gamma_B)\sigma_{D_1}\underline{1}$. Consequently, the magnitude of $\nu^*\Sigma^{-1}m$ is comparable to that of other terms in expressions (16)–(18) for equilibrium processes, and hence access to credit has first order effect on equilibrium, which we explore further in Section 3.

Substituting expressions for ν^* and $\nu^*\Sigma^{-1}m$ from equations (28) into equations (16)–(19) we observe that equilibrium processes κ, r, σ_y, μ_y, and σ_j remain time-varying and stochastic. This result stands in contrast to one-tree models with leverage constraint where market price of risk κ and volatility σ are constant, volatility σ_y is zero, and hence consumption share y and interest rate r are time-deterministic [e.g., Kogan, Makarov, and Uppal (2007)].

Next, we provide the economic intuition for equilibrium processes in Propositions 1 and 2. We investigate the impact of constraints holding state variables x and y fixed, similarly to the literature [e.g., Basak and Cuoco (1998); Gârleanu and Pedersen (2011); among others]. This means that the analysis is from the standpoint of an observer in the economy who can see x and y and all equilibrium processes, and is trying to infer whether the economy is better described by a model with or without portfolio constraints. For example, if market prices of risk κ in the

[5] This function is implemented in Matlab and is given by integral 8.432.1 in Gradshteyn and Ryzhik (2007).

economy are high, using our analysis the observer may reject the unconstrained model in favor of a model with constraints, which generates higher κ for fixed x and y.[6]

We start with stock risk premia, and derive a tractable consumption CAPM with portfolio constraints in terms of empirically observable processes. By multiplying the market price of risk (16) by Σ we obtain the following C-CAPM:[7]

$$\mu_t - r_t\underline{1} = \Gamma_t \Sigma_t \sigma_{Dt} - \frac{\Gamma_t y_t}{\gamma_B} \nu_t^* m. \tag{29}$$

We note that multiplier $(\Gamma y/\gamma_B)\nu^*$ is the same for all stock risk premia, and hence it can be estimated from the cross-section of stocks. In particular, we obtain ν^* in terms of the risk premium of the market portfolio, $\mu_{Mt} - r_t$, and substituting ν^* back into the expression for risk premia we obtain new C-CAPM. For the case of the leverage constraint ν^* is available in closed form in Proposition 2 and gives another consumption CAPM, which conveniently illustrates the interaction between constraints and investor heterogeneity. Corollary 1 reports the results.

Corollary 1 (Consumption CAPM).

(i) *In the economy with a margin constraint stock risk premia are given by:*[8]

$$\mu_t - r_t\underline{1} = \left(I - \frac{m\theta_{Mt}^\top}{\theta_{Mt}^\top m} \right) \beta_{ct} + \frac{m}{\theta_{Mt}^\top m}(\mu_{Mt} - r_t), \tag{30}$$

where I is an identity matrix, $\theta_M = S/(S_1 + S_2)$ is the vector of market portfolio weights, μ_M is the market portfolio's mean return, and $\beta_c = (\beta_{1c}, \beta_{2c})^\top$ is the vector of consumption betas $\beta_{jc} = \Gamma_t \operatorname{cov}_t(dS_{jt}/S_{jt}, dD_t)/dt$, $j = 1, 2$.

(ii) *In the economy with the leverage constraint stock risk premia are given by:*

$$\mu_t - r_t\underline{1} = \beta_{ct} - \frac{\Gamma_t y_t}{\gamma_B} \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2} \underline{1}. \tag{31}$$

The second terms in equations (29) and (31) capture the direct impact of margin constraints on risk premia. Since $\nu^* \leq 0$, constraints tend to increase risk premia, and the increase is proportional to stock's margin m_i. Intuitively, since the constrained investor B holds less wealth in stocks than in the unconstrained economy, market prices of risk increase to compensate investor A for holding more stocks to clear the market. Furthermore, the low-margin asset with $m_1 < m_2$

[6] We also note that our economy is non-stationary in the sense that only the less risk averse investor B survives in the long-run, where the constraint does not bind. Tighter constraints slow down convergence to the unconstrained one-investor economy. Consequently, the economy with tighter constraints spends more time in constrained region with high market prices of risk and lower interest rates.

[7] Cuoco (1997) and Gârleanu and Pedersen (2011) derive similar CAPMs, but do not express adjustment ν^* in terms of observable processes.

[8] We note, that C-CAPM (30) holds for more general preferences, multiple assets, and time-varying margins.

has smaller risk premium since it is a better collateral that helps relax the portfolio constraint. Equation (31) extends Black's (1972) static mean-variance CAPM with leverage constraint to the dynamic economy, and characterizes the deviation from Breeden's (1979) C-CAPM in closed form. C-CAPM (31) predicts that this deviation increases with the difference in risk aversions $\gamma_A - \gamma_B$ and consumption share y of the constrained investor. The second term in (31) is non-positive (since $\gamma_B \leq \gamma_A$), and hence the constraint tends to increase risk premia.

The effect of constraints on interest rate r is, in general, ambiguous, since r is a non-monotone quadratic function of ν^*. Intuitively, on one hand, the interest rate should decrease since the constrained investor borrows less than in the unconstrained economy. On the other hand, it should increase, since stock risk premia go up, and the unconstrained investor invests more in stocks, and hence less willing to lend. In our calibrations the former effect dominates since the unconstrained investor is very risk averse, and hence the latter effect is weaker.

The effect of the leverage constraint on the interest rate can be explored analytically, since the adjustments ν^* and $\nu^*\Sigma^{-1}m$ are available in closed form. After tedious algebra it can be shown that the leverage constraint unambiguously decreases the interest rate when $\gamma_B \geq 1$. However, the interest rate may be higher than in the unconstrained economy when $\gamma_B < 1$ and the risk averse unconstrained investor A holds large share of consumption, i.e., y is close to zero.

2.3. Computation of Equilibrium

As discussed in Section 2.2, PDEs (24)–(25) are quasilinear, and we solve them numerically. The boundary conditions are derived by passing to limits $x \to 0$, $x \to 1$, $y \to 0$, and $y \to 1$, as discussed in Appendix B. We compute the equilibrium for parameters γ_i, ρ, μ_{D_j}, and σ_{D_j} such that boundary conditions (B4)–(B6) in Appendix B are positive and finite. Otherwise, the equilibria in limiting one-tree/one-investor economies do not exist. Here, we briefly discuss the solution methods. Appendix B provides further details.

First, we use the fixed point iteration method that has been widely used in the economic literature in various settings [e.g., Gomes and Michaelides (2008); Guvenen (2009); Chien, Cole, and Lustig (2011)]. This method at step k evaluates all coefficients of PDEs (24)–(25) using the solution from the previous step $k-1$. Therefore, step k solution satisfies a linear PDE, and is found by solving a system of linear finite-difference equations. We start with some conjectured solutions for PDEs, e.g. solutions for the unconstrained model, and iterate until convergence.

While the fixed point iterations method is the fastest method, the proof of convergence is not available in the literature. Therefore, we demonstrate that the equilibrium can be computed using traditional explicit and implicit-explicit methods employed in the mathematical and economic literatures for solving nonlinear PDEs [e.g., Lapidus and Pinder (1999); Ma and Yong (1999); Cochrane and Saa-Requejo (2000)]. These methods are extensions of Euler's algorithm for ODEs to the case of PDEs. Their theoretical advantage is that they solve PDEs directly, without converting the solution method into a fixed point problem, which comes at a cost of low speed.

In particular, we consider an economy with large finite horizon T, and solve equations backwards in time. The explicit method evaluates all coefficients and derivatives with respect to x and y using time-$(t + \Delta t)$ solutions. Time-t solutions are then expressed as explicit functions of time-$(t + \Delta t)$ solutions and this recursion is iterated backwards without solving any equations. The implicit-explicit method evaluates the coefficients of PDEs using time-$(t+\Delta t)$ solutions, but the derivatives with respect to x and y are discretized using time-t solutions. Consequently, time-t solutions are obtained by solving a system of linear equations. The literature shows that this method is typically more stable than the explicit one [e.g., Lapidus and Pinder (1999)]. However, in our setting the explicit method performs equally well when $\Delta t \leq 0.01$. On a PC with Intel Core i7 CPU fixed point iterations take 2 seconds, explicit and implicit-explicit methods take 233 and 6,055 seconds, respectively, when $T = 350$, $\Delta t = 0.01$, and the number of mesh points for state variables x and y is 100×100.

3. Analysis of Equilibrium

We solve the model numerically and study the impact of constraints on stock return correlations, volatilities, and price-dividend ratios. We also explore the asset pricing implications of the heterogeneity in margins. Our focus is on economic effects that can be analyzed only in a model with two risky assets. Consequently, we do not discuss the effects of constraints on market prices of risk κ and interest rates r, since the intuition for them is provided in Section 2.2, and quantitative results are similar to those in one-tree economies [e.g., Chabakauri (2010)].

In our baseline calibration we set $\mu_{D_1} = \mu_{D_2} = 1.8\%$ and $\sigma_{D_1} = \sigma_{D_2} = 3.6\%$, $\gamma_A = 8$, $\gamma_B = 2$, and $\rho = 0.01$. The values for the Lucas trees parameters μ_{D_j} and σ_{D_j} are within the ranges considered in the literature [e.g., Basak and Cuoco (1998); Campbell (2003); Dumas and Lyasoff (2012); among others]. In the baseline analysis we assume $\sigma_{D_1} = \sigma_{D_2}$ to capture the pure effects of margin heterogeneity. Then, in Section 3.2 we incorporate heterogeneity in volatilities, and study how it interacts with margin heterogeneity. In the analysis of equilibrium processes, following the literature, we call an Itô's process X_t *procyclical* if conditional $\mathrm{corr}_t(dX_t, dD_t)$ is positive, and *countercyclical* if this correlation is negative [e.g., Chan and Kogan (2002); Gârleanu and Panageas (2010); Longstaff and Wang (2012)].

3.1. Stock Return Correlations

Four panels of Figure 1 show conditional stock return correlations $\mathrm{corr}_t(dS_{1t}/S_{1t}, dS_{2t}/S_{2t})$ as functions of the first tree's share x and investor B's consumption share y of aggregate consumption when investors are unconstrained [Panel (a)], and investor B faces constraints with margins $m = (0.7, 0.7)^\top$ [Panel (b)], $m = (0.7, 0.9)^\top$ [Panel (c)], and $m = (1, 1)^\top$ [Panel (d)]. Figure 1 illustrates two important findings. First, constraints decrease correlations and give rise to rich saddle-type patterns. Second, investor heterogeneity significantly amplifies correlations relative

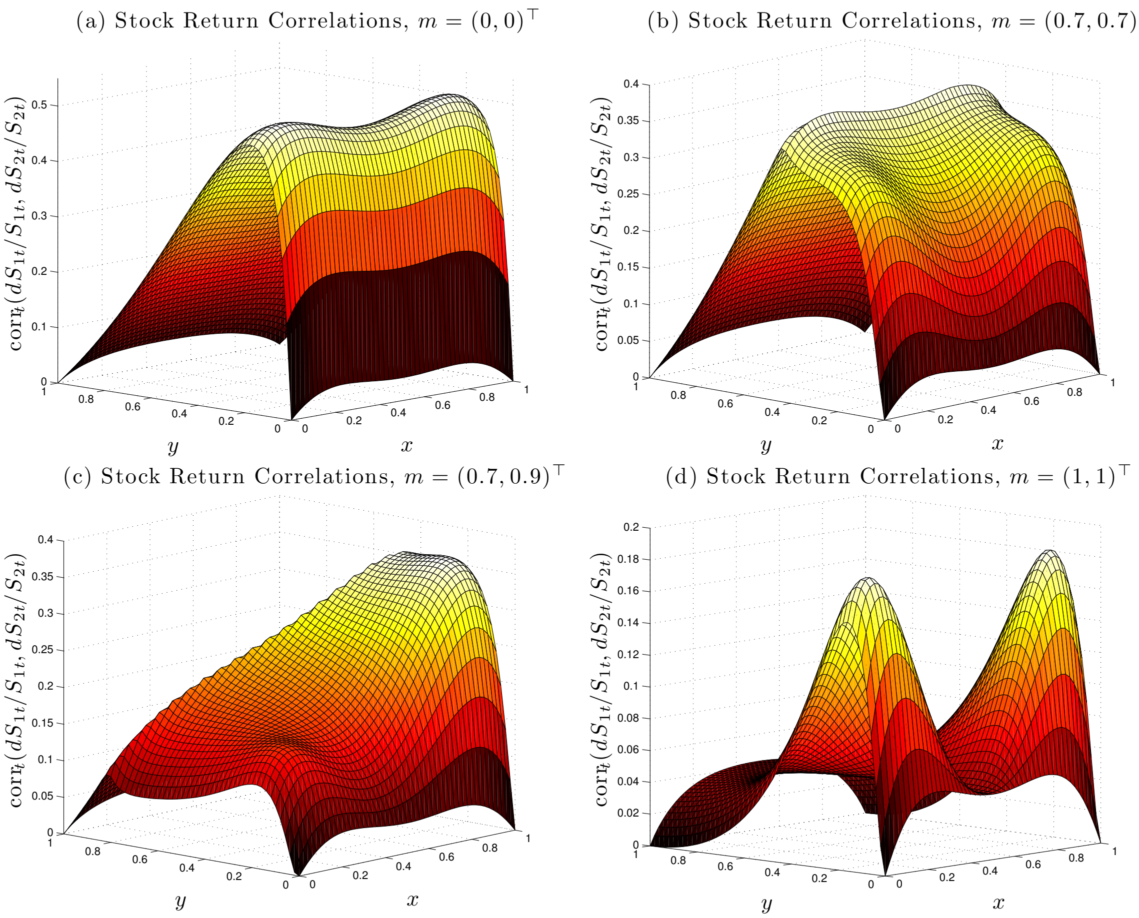

Figure 1: Conditional Stock Return Correlations.

Figure 1 presents the equilibrium stock return correlations when investor B is unconstrained [Panel (a)], and faces constraint $m^\top \theta_B \leq 1$ with $m = (0.7, 0.7)^\top$ [Panel (b)], $m = (0.7, 0.9)^\top$ [Panel (c)], and $m = (1, 1)^\top$ [Panel (d)]. Consumption share $y = c_B^*/D$ is countercyclical, and $x = D_1/D$. The parameters are $\mu_{D_1} = \mu_{D_2} = 1.8\%$, $\sigma_{D_1} = \sigma_{D_2} = 3.6\%$, $\rho = 0.01$, $\gamma_A = 8$, and $\gamma_B = 2$.

to one-investor economies, which correspond to boundaries $y = 0$ or $y = 1$, even in the presence of constraints. Next, we discuss the panels separately.

Panel (a) demonstrates that, consistent with empirical evidence, stock return correlation exceeds the correlation between dividends [e.g., Shiller (1989)], since the latter is zero in the model and the former is positive. The heterogeneity in preferences also generates up to a sixfold increase in correlations relative to homogeneous-investor economies considered in the literature [e.g., Cochrane, Longstaff, and Santa-Clara (2008); Martin (2012)]. Moreover, the correlation is very steep around $y = 0$, when the economy is dominated by the more risk averse investor A. Consequently, in general, the quantitative results of homogeneous investors models are sensitive to introducing a small amount of heterogeneity, measured by consumption share y.

The correlations on Panel (a) are countercyclical, consistent with the empirical evidence [e.g., Karolyi and Stulz (1996); Ang and Bekaert (2002); Ribeiro and Veronesi (2002)]. To demonstrate the countercyclicality we note that consumption share y of investor B is procyclical in the sense that $\mathrm{corr}_t(dy_t, dD_t) = 1$.[9] Intuitively, investor B is less risk averse and invests in stocks more than investor A. As a result, investor B is more exposed to aggregate consumption shocks, and hence positive (negative) shocks to aggregate dividend D_t increase (decrease) investor B's consumption share y. Therefore, the correlation is a decreasing function of a procyclical process y over a large interval, and hence is countercyclical over that interval. We note that consumption share y is a convenient indicator of the state of the economy. Consequently, we label states with high (low) y as good (bad) states of the economy.

The intuition for correlations highlights the economic role of credit markets and risk sharing, the scope for which is provided by the heterogeneity in preferences. In particular, the less risk averse investor B finds optimal to lever up by borrowing from the more risk averse investor A. The amount available for borrowing fluctuates due to the time-variation of dividends and consumption share $1 - y$ of lenders in the economy. Therefore, investor B adjusts portfolio weights of both stocks up or down depending on the availability of liquidity for borrowing, which translates into higher stock return correlations.

The increase in consumption share y of investor B has two opposite effects on correlations, which give rise to a hump-shaped pattern on Panel (a). On one hand, this increase makes the leverage effect more conspicuous by increasing the impact of the levered investor B on the economy, which pushes the correlations up. On the other hand, it reduces consumption share $1 - y$ of the lender A, and hence the availability of liquidity for borrowing, which pushes the correlations down and generates the hump-shape.

Panels (b) and (c) demonstrate a new prediction of the model that margin constraints reduce stock return correlations, and make them spike when the constraint starts to bind. Tighter constraints prevent investor B from levering up, and hence the correlations decrease in accordance with the intuition on the role of credit markets, presented above. This result contributes to the debate on whether constraints dampen or amplify shocks. In particular, Pavlova and Rigobon (2008) show that constraints increase correlations and spread contagions in an international finance setting with three countries. Our results indicate that in a one-country setting constraints decrease correlations, and hence make dividend shocks less contagious.

The margin constraint binds intermittently in the economy, and only in relatively bad times (i.e., y is low), when the aggregate consumption is low. Correlations on Panels (b) and (c) remain countercyclical over intervals where they are decreasing functions of y, which coincide with regions where the constraint does not bind. The comparison with Panel (a) shows that the countercyclicality region shrinks.

Imposing the leverage constraint helps isolate and quantify the effect of credit markets on

[9] The procyclicality of consumption share y can be formally demonstrated by substituting σ_{yt} with $\nu^* = 0$ from (18) into the correlation $\mathrm{corr}_t(dy_t, dD_t) = -\sigma_{yt}^\top \sigma_{Dt}/(|\sigma_{yt}||\sigma_{Dt}|)$.

correlations. The comparison of Panels (a) and (d) reveals that leverage accounts for a significant fraction of the magnitude of correlations. Consequently, larger size of the credit sector is associated with larger correlations. This prediction is consistent with Longstaff and Wang (2012), who demonstrate both theoretically and empirically that the size of the credit sector can help explain first and second moments of asset prices. The correlation on Panel (b) is residual correlation, which is driven by pure common discount factor effect disentangled from the leverage effect. This correlation is highest when share x is close to 0 or 1, and lowest when both trees have equal size. Therefore, another prediction is that correlations under constraints crucially depend on the relative size of industries or firms, interpreted as trees in the model.

Two humps on Panel (d) are due to the combination of two effects that reinforce each other. The first one is the common discount factor effect, which generates small humps even in the unconstrained case on Panel (a). When x is close to 0 or 1, the state price density ξ is mainly driven by one tree. Consequently, irrespective of whether the dividend innovations dD_1 and dD_2 move in the same or opposite directions, there will be extra comovement in stocks. When x is close to 0.5, ξ is less volatile since it is equally affected by both trees, and innovations dD_1 and dD_2 may partially offset each other, which decreases the discount-driven comovement.

The second effect is specific to constraints. When $x = 0.5$ both trees and stocks are "symmetric" in the sense that they have the same expected growth rates μ_{D_j} and volatilities σ_{D_j} in the calibration. Therefore, both stocks are equally attractive to investors, and hence they invest equal fractions of wealth in stocks, i.e., $\theta_{i1} = \theta_{i2}$. In the unconstrained case their asset holdings are still heterogeneous, since investor B can lever up. However, with the leverage constraint, at $x = 0.5$ the shares of wealth invested in stocks have to be the same, $\theta_{11} = \theta_{12} = 0.5$ and $\theta_{21} = \theta_{22} = 0.5$. Therefore, asset holdings become homogeneous, and correlations decrease towards those in homogeneous-investor economies. More formally, substituting $\nu^* \Sigma^{-1} m$ from (28) into the expression for σ_y in (18), for the case $\sigma_{D_1} = \sigma_{D_2}$ we obtain:

$$\sigma_{yt} = \frac{\Gamma_t (1 - y_t)(\gamma_B - \gamma_A)\sigma_{D_1}}{\gamma_A \gamma_B}(x - 0.5, 0.5 - x)^\top. \tag{32}$$

Consistent with the intuition above, consumption share volatility σ_y in equation (32) vanishes when $x = 0.5$. Hence, the third term in volatilities σ_j, given by equation (20), which captures the impact of heterogeneity on volatilities, also vanishes, leading to the reduction in correlations.

When x is around 0 or 1, the investors start trading again, and hence portfolio holdings become heterogeneous, and correlations increase. These bilateral trades are due to constrained investor's desire to substitute leverage with riskier assets in the portfolio. In other words, investor B self-selects to trade in the riskier segment of the asset market. Equation (21) for portfolio weights θ_B^* shows that since adjustment ν^* is negative, investor B may optimally short safer asset j when market price of risk κ_j is low, in the same way as in a static mean-variance CAPM with borrowing restrictions [e.g., Black (1972); Frazzini and Pedersen (2011)]. In the model the shorting occurs only when one tree is very small, i.e., x close to 0 or 1, and investor B's consumption share y

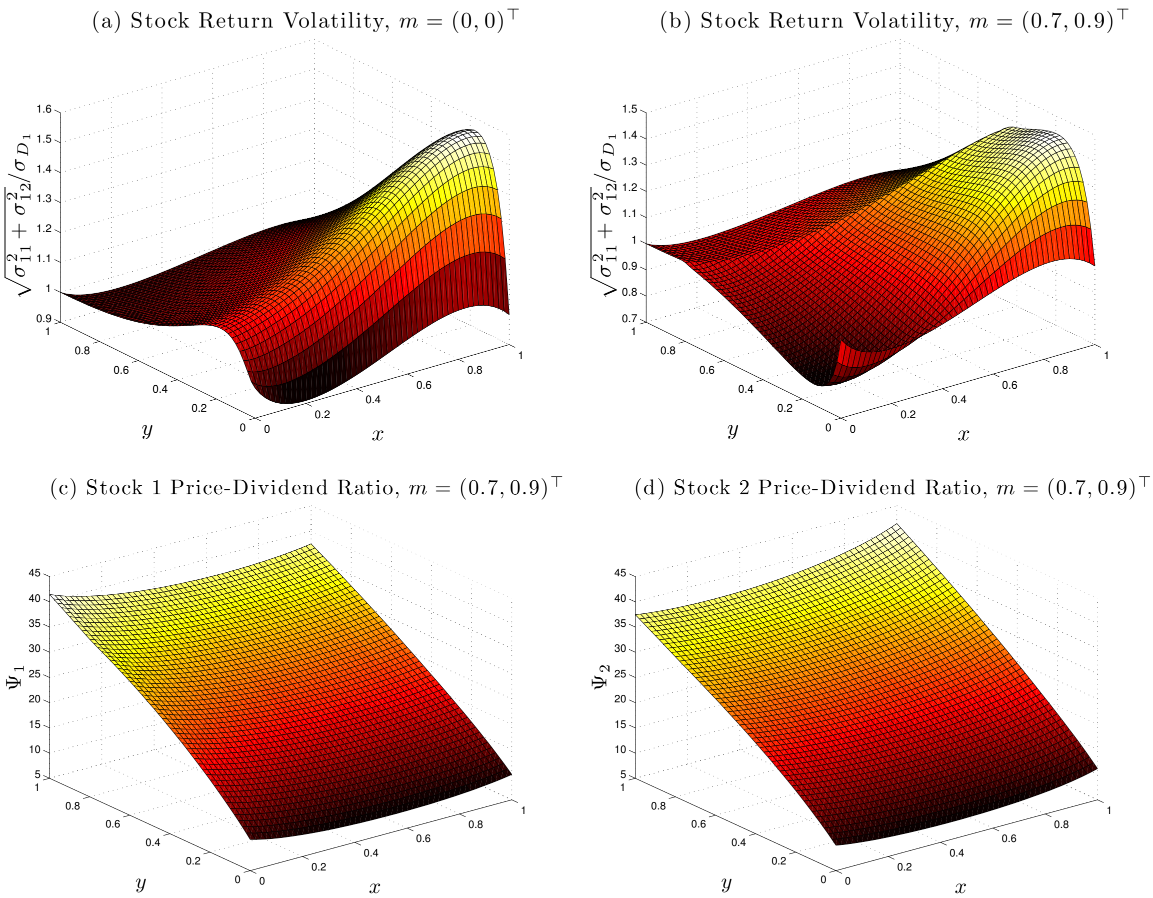

Figure 2: Conditional Return Volatility and Price-Dividend Ratios.

Panels (a) and (b) show the ratio of stock 1 return and dividend volatilities $\sqrt{\sigma_{11}^2 + \sigma_{12}^2}/\sigma_{D_1}$ when $m = (0,0)^\top$ and $m = (0.7, 0.9)^\top$, respectively. Panels (c) and (d) show price-dividend ratios of stocks 1 and 2 when $m = (0.7, 0.9)^\top$. The parameters are: $\mu_{D_1} = \mu_{D_2} = 1.8\%$, $\sigma_{D_1} = \sigma_{D_2} = 3.6\%$, $\rho = 0.01$, $\gamma_A = 8$, and $\gamma_B = 2$.

is small. Lest to complicate the model we do not impose short-sale constraints, but note that under these constraints investor B may optimally hold only one stock in some states.

3.2. Stock Return Volatilities and Price-Dividend Ratios

Panels (a) and (b) of Figure 2 show the ratio of conditional stock 1 return volatility $\sqrt{\sigma_{11t}^2 + \sigma_{12t}^2}$ and dividend volatility σ_{D_1} in the unconstrained economy [Panel (a)] and the economy with margins $m = (0.7, 0.9)^\top$ [Panel (b)]. From this Figure we observe that tighter constraints reduce stock return volatilities. As discussed in the Introduction, this result is consistent with some empirical evidence on the effect of tighter initial margins on stock volatilities [e.g., Hardouvelis and Perestiani (1992); Hardouvelis and Theodossiou (2002)], although there is no consensus in the literature [e.g., Schwert (1989)]. We note that our constrained investor is an approximation of an "average" constrained real-world investor, which subsumes both margin- and leverage-constrained investors. Consequently, a more direct future test of our predictions should look at tightening or relaxing the access to credit markets in general. The effects generated by the combined group of all credit-constrained investors may have larger impact on the volatility.

The ratios σ_M/σ_D of conditional market portfolio return and dividend volatilities have similar shapes to those of correlations on Figure 1, and hence are not reported for brevity. Consistent with the empirical evidence, volatility σ_{Mt} in the unconstrained and constrained economies is high (low) exactly in those periods where stock return correlations are high (low), σ_M exceeds the volatility of the aggregate dividend σ_D, and is countercyclical in good times (i.e., y is large) when the constraint does not bind [e.g., Shiller (1981); Schwert (1989); Karolyi and Stulz (1996); Ang and Bekaert (2002)]. In contrast to the volatility of the market portfolio returns σ_M, the volatility of stock returns on Figure 2 can be either below or above the dividend volatility σ_{D_1}, depending on the size of the tree. We note that volatilities are much lower than in the data. However, the difficulty of matching volatilities has long been recognized in the literature, and is a feature shared by many asset pricing models, as argued in Heaton and Lucas (1996).

Panels (c) and (d) of Figure 2 show price-dividend ratios for stocks 1 and 2 with asymmetric margins $m = (0.7, 0.9)^\top$. The effects of portfolio constraints on asset prices depend on how they influence state price densities, via interest rates r and market prices of risk κ. These effects can be illustrated in a partial equilibrium economy with constant r and κ. In such an economy from the PDEs for price-dividend ratios (24) we observe that price-dividend ratios are given by a generalized Gordon's formula $\Psi_j = 1/(r + \sigma_{D_j}\kappa_j - \mu_{D_j})$. The constraints decrease r and increase κ, as explained in Section 2.2, and hence the latter formula suggests that the impact on Ψ_j is, in general, ambiguous.[10] In our calibrations we find that $\Psi_j^{constr}(x, y) \geq \Psi_j^{unc}(x, y)$, where Ψ_j^{constr} and Ψ_j^{unc} are price-dividend ratios in constrained and unconstrained economies, respectively. The increase in Ψ_j ranges from 0% to 3.5% under the leverage constraint, and from 0% to 2% under the margin constraint with $m = (0.7, 0.7)^\top$.

Low-margin assets are better collaterals and help relax borrowing constraints, which should be reflected in their valuations. To measure this effect we consider two trees with equal sizes, i.e. $x = 0.5$, and equal mean growth rates and volatilities. The trees are symmetric at $x = 0.5$, and hence differences in their valuations are solely due to margin heterogeneity. Panel (a) of Figure 3 shows the *collateral premium* at $x = 0.5$, i.e. price difference between low-margin and high margin assets, expressed as percentage of high-margin asset's price, $100 \times (\Psi_1(0.5, y) - \Psi_2(0.5, y))/\Psi_2(0.5, y)$, for different asset margins. Consistent with our intuition the premium is positive. It ranges from 0% to 3.3% when we set $\sigma_{D_1} = \sigma_{D_2} = 5.6\%$, and disappears when the constraint does not bind. We note that in the baseline calibration with $\sigma_{D_1} = \sigma_{D_2} = 3.6\%$ the premium is smaller, and ranges from 0% to 1.3%. Consequently, we increased baseline dividend volatilities just by 2% to make effects more conspicuous. On Panel (b) we make the first tree less volatile, and set

[10]Gordon's formula $\Psi_j = 1/(r + \sigma_{D_j}\kappa_j - \mu_{D_j})$ is a remarkably accurate approximation price-dividend ratio Ψ_j when consumption share y is close to 0 or 1.

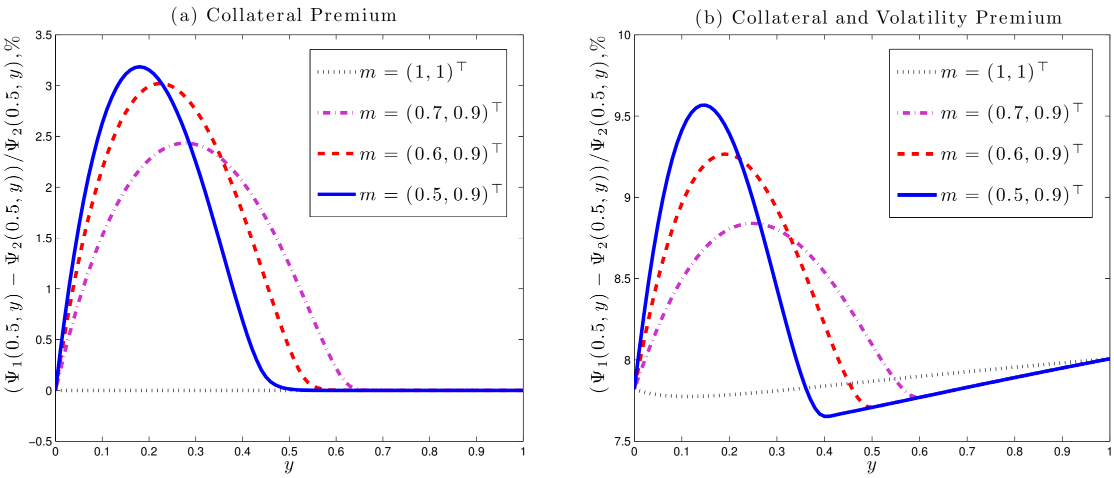

Figure 3: Collateral Premium under Margin Heterogeneity.

Figure 3 shows the price difference as percentage of second asset's price when both trees have equal size, measured as $100 \times (\Psi_1(0.5, y) - \Psi_2(0.5, y))/\Psi_2(0.5, y)$, when $\sigma_{D_1} = \sigma_{D_2} = 5.6\%$ [Panel (a)] and $\sigma_{D_1} = 2.8\%$ and $\sigma_{D_2} = 5.6\%$ [Panel (b)], respectively. We set $\mu_{D_1} = \mu_{D_2} = 1.8\%$, $\rho = 0.01$, $\gamma_A = 8$, and $\gamma_B = 2$.

$\sigma_{D_1} = 2.8\%$ and $\sigma_{D_2} = 5.6\%$. Panel (b) shows that the safer asset is by 8% more expensive than the riskier one even with homogeneous margins, $m = (1, 1)^\top$, and the difference in margins gives an additional premium.

To explain the results, from equation (16) for market price of risk κ we observe that asymmetric margins with $m_2 > m_1$ increase κ_2 more than κ_1. Consequently, Gordon's formula for Ψ_j implies that the value of the low margin asset increases more than the value of the high margin asset. This result is consistent with the empirical evidence that collateral benefits are reflected in prices of assets, such as real estate [e.g., Lustig and Van Nieuwerburgh (2005)], and government bonds [Brunnermeier (2009)]. The results are also consistent with findings in Gârleanu and Pedersen (2011). In particular, they show that the difference in margins gives rise to mispricing between derivative and underlying securities with identical cash flows, and low-margin securities have higher prices. Our paper is the first to evaluate the effects of margin heterogeneity in full general equilibrium model with two trees. Moreover, our approach allows to quantify the effect of margin heterogeneity even when the assets have different cross-sectional characteristics, as on Panel (b) of Figure 3.

4. Conclusion

We study the effects of margin and leverage portfolio constraints in a Lucas (1978) economy with two heterogeneous CRRA investors and two Lucas trees. First, we demonstrate that in the unconstrained benchmark the heterogeneity in preferences generates large countercyclical conditional stock return correlations and the volatilities of the market portfolio returns, which significantly exceed those in models with homogeneous investors.

We show that conditional correlations and volatilities are positively related to the amount of leverage in the economy. In particular, tightening access to credit decreases correlations and volatilities, makes them less countercyclical, and gives rise to rich nonlinear patterns in correlations. Moreover, imposing asymmetric margins increases risk premia proportionally to asset margins, and increases the prices of low-margin assets more than those of high-margin assets. In the case of the leverage constraint we extend Black's (1972) static mean-variance CAPM and quantify the deviation of our C-CAPM from Breeden's one in closed form. The paper also provides a tractable methodology for solving models with portfolio constraints when investors have general CRRA preferences. In the unconstrained economy and the economy with leverage constraints we provide closed form solutions for all equilibrium processes.

Appendix A: Proofs

Proof of Proposition 1. From the duality approach of Cvitanić and Karatzas (1992) it follows that for the margin constraint $m^\top \theta \leq 1$ the constrained investor's optimization can be solved in an unconstrained fictitious economy with asset prices following dynamics (10)–(11) with adjustment parameter ν^*. The construction of the fictitious economy is also demonstrated in Remark 1.

Suppose, there exists a Markovian equilibrium in which all processes are functions of some vector state variable z_t following an Itô's process. Then, the state price densities in complete-market unconstrained and fictitious economies follow dynamics (12), where drifts and volatilities are some functions of z. From the first order conditions, the optimal consumptions are given by equations (13). Substituting consumptions (13) into consumption clearing in (8), applying Itô's Lemma to both sides of it, matching dt and dw terms, and then dividing both sides of the resulting equations by D_t we obtain:

$$\frac{r_t-\rho}{\Gamma_t}-\frac{y_t\nu_t^*}{\gamma_B}+\frac{1}{2}\Big(\frac{1+\gamma_A}{\gamma_A^2}(1-y_t)\kappa_t^\top\kappa_t+\frac{1+\gamma_B}{\gamma_B^2}y_t\Big(\kappa_t+\nu_t^*\Sigma_t^{-1}m\Big)^\top\Big(\kappa_t+\nu_t^*\Sigma_t^{-1}m\Big)\Big)=\mu_{Dt},$$
$$\frac{1-y_t}{\gamma_A}\kappa_t+\frac{y_t}{\gamma_B}\Big(\kappa_t+\nu_t^*\Sigma_t^{-1}m\Big)=\sigma_{Dt}.$$

Solving these equations we obtain κ and r in (16)–(17), where $g_1(y)$ and $g_2(y)$ are given by:

$$g_1(y_t)=\frac{\Gamma_t^3 y_t(1-y_t)(\gamma_B-\gamma_A)}{\gamma_A^2\gamma_B^2},\quad g_2(y_t)=-\frac{\Gamma_t^3 y_t(1-y_t)}{2\gamma_A^2\gamma_B^2}\Big(1+\frac{\gamma_A\gamma_B}{\Gamma_t}\Big). \tag{A1}$$

The expressions (18)–(19) for the volatility σ_y and drift μ_y of consumption share y are obtained by applying Itô's Lemma to $y_t=c_B^*/D_t$ and matching dt and dw terms.

The equilibrium processes (16)–(19) endogenously emerge as functions of shares x and y, and adjustment ν^*. From SDEs (12) we observe that state price densities are then also driven by these variables. Consequently, x, y, and ν^* can be chosen as new state variables. Moreover, adjustment ν^* is not an independent variable, and can be obtained in terms of x and y from Kuhn-Tucker conditions (see Lemma A.1 below). This observation justifies looking for an equilibrium with two state variables x and y, although it is not a rigorous proof that ν^* is a function of x and y.[11]

The expressions (20) for volatilities σ_j are obtained by applying Itô's Lemma to both sides of $S_j=\Psi_j D_j$ and matching dw terms. To obtain the PDEs for the price-dividend ratios $\Psi_j(x,y)$ we apply Itô's Lemma to both sides of equation $\Psi_j=S_j/D_j$, and by matching dt terms we obtain:

$$\begin{aligned}&\frac{1}{2}\Big(x^2\sigma_x^\top\sigma_x\frac{\partial^2\Psi_j}{\partial x^2}-2xy\sigma_y^\top\sigma_x\frac{\partial^2\Psi_j}{\partial y\partial x}+y^2\sigma_y^\top\sigma_y\frac{\partial^2\Psi_j}{\partial x^2}\Big)+x\mu_x\frac{\partial\Psi_j}{\partial x}-y\mu_y\frac{\partial\Psi_j}{\partial y}=\\&\quad(-\mu_{D_j}+\sigma_{D_j}^2+\mu_j-\sigma_{D_j}e_j^\top\sigma_j)\Psi_j-1,\quad j=1,2,\end{aligned} \tag{A2}$$

[11] We note that the choice of state variables may not be unique. For example, one can use the ratio of marginal utilities $\lambda_t=(c_A^*)^{-\gamma_A}/(c_B^*)^{-\gamma_B}$ and aggregate consumption D_t as state variables instead of consumption share y.

where $e_1 = (1, 0)^\top$, $e_2 = (0, 1)^\top$. From the definition of the market price of risk $\kappa = \Sigma^{-1}(\mu - r\underline{1})$, we observe that $\mu_j = r + e_j^\top \Sigma\kappa = r + \sigma_j^\top \kappa$. Next, we substitute $\mu_j = r + \sigma_j^\top \kappa$ into the right-hand side of (A2), and then we replace σ_j by its expression (20) in terms of derivatives of Ψ_j. After some algebra we then obtain PDEs (24).

The PDE for wealth-consumption ratio Φ_B is obtained from HJB equation (15) in the fictitious unconstrained economy, where state variables are $z = (x, y)^\top$. Since the optimization is solved in the fictitious complete market economy, the value functions J_i and portfolio weights θ_i^* are special cases of those in complete-market portfolio choice literature [e.g., Liu (2007)], and have the same functional forms. Substituting the value function $J_B = e^{-\rho t} W_B^{1-\gamma_B} \Phi_B^{\gamma_B}/(1 - \gamma_B)$ into the HJB equation after some algebra, similarly to Liu (2007), we obtain PDE (25).

To derive Φ_A, by summing up bond and stock market clearing conditions in (8) we first obtain: $W_A^* + W_B^* = S_1 + S_2$. The latter equality after some algebra can be rewritten as $(1-y)\Phi_A + y\Phi_B = x\Psi_1 + (1 - x)\Psi_2$, from which we obtain Φ_A. Lemma A.1 derives the adjustment ν^*. $\square$

Remark A.1. If the solutions of PDEs (24)–(25) exist, one can verify that the economy is in equilibrium under certain conditions. Suppose, the ratios Φ_i and Ψ_j and all equilibrium processes are bounded, which can be verified after the equilibrium is computed. Then, Lemmas A.2 and A.3 below show that c_i^* and θ_i^* are optimal, and there are no bubbles. Next, one can verify that all market clearing conditions are satisfied. Market clearing in consumption holds by construction of r and κ, because these processes are chosen to make market clearing hold. Moreover, since wealths are self-financing, and there are no bubbles, W_i^* and S_j are given by $W_{it}^* = \mathbb{E}_t\Big[\int_t^{+\infty} \xi_\tau c_{i\tau}^* d\tau\Big]/\xi_t$, $S_{jt} = \mathbb{E}_t\Big[\int_t^{+\infty} \xi_\tau D_{j\tau} d\tau\Big]/\xi_t$. From the consumption clearing and the above equations we obtain that $W_A^* + W_B^* = S_1 + S_2$. Applying Itô's lemma to both sides and matching terms we obtain market clearing conditions (8) for the stock market.

Lemma A.1. *In a Markovian equilibrium the adjustment ν^* and vector $\Sigma^{-1}m$ are given by:*

$$\nu_t^* = \begin{cases} \dfrac{1 - b_{2t}^\top v_t}{b_{1t} v_t^\top v_t}, & \text{if} \quad m^\top \theta_{Bt}^* = 1, \quad \nu_t^* < 0, \\[2ex] 0, & \text{if} \quad m^\top \theta_{Bt}^* < 1, \quad \nu_t^* = 0, \end{cases} \tag{A3}$$

$$\Sigma_t^{-1} m = \begin{cases} v_t, & \text{if} \quad m^\top \theta_{Bt}^* = 1, \quad \nu_t^* < 0, \\[1ex] \tilde{\Sigma}_t^{-1} m, & \text{if} \quad m^\top \theta_{Bt}^* < 1, \quad \nu_t^* = 0, \end{cases} \tag{A4}$$

where b_1, b_2, b_3, $\tilde{\Sigma}$, v, and $m^\top\theta_B^*$ are given by:

$$b_{1t} = \frac{\Gamma_t(1 - y_t)}{\gamma_A\gamma_B}\Big(1 + \frac{\partial \Phi_{Bt}}{\partial y_t}\frac{y_t}{\Phi_{Bt}}\Big), \quad b_{2t} = \sigma_{Dt} + \sigma_{xt}\frac{\partial \Phi_{Bt}}{\partial x_t}\frac{x_t}{\Phi_{Bt}} - (\gamma_B - \gamma_A)\sigma_{Dt} b_{1t}, \tag{A5}$$

$$b_{3t} = \frac{\Gamma_t(1 - y_t)}{\gamma_A\gamma_B}\Big(\frac{\partial \Psi_{1t}}{\partial y_t}\frac{y_t}{\Psi_{1t}}, \frac{\partial \Psi_{2t}}{\partial y_t}\frac{y_t}{\Psi_{2t}}\Big)^\top, \tag{A6}$$

$$\tilde{\Sigma}_t = \begin{pmatrix} \sigma_{D_1} & 0 \\ & \\ 0 & \sigma_{D_2} \end{pmatrix} + \Big(\frac{\partial \Psi_{1t}}{\partial x_t}\frac{x_t}{\Psi_{1t}}, \frac{\partial \Psi_{2t}}{\partial x_t}\frac{x_t}{\Psi_{2t}}\Big)^\top \sigma_{xt}^\top - (\gamma_B - \gamma_A) b_{3t} \sigma_{Dt}^\top, \tag{A7}$$

$$v_t = \Big(\frac{1}{\sigma_{D_1}}, \frac{1}{\sigma_{D_2}}\Big)^\top + \Big(\tilde{\Sigma}_t - \frac{b_{3t} b_{2t}^\top}{b_{1t}}\Big)^{-1} (m_1 - 1, m_2 - 1)^\top, \tag{A8}$$

$$m^\top \theta_{Bt}^* = b_{2t}^\top (\Sigma_t^{-1} m) + \nu_t^* b_{1t} (\Sigma_t^{-1} m)^\top (\Sigma_t^{-1} m). \tag{A9}$$

Proof of Lemma A.1. The idea of the proof is to demonstrate that ν^* and $\Sigma^{-1}m$ satisfy a system of simultaneous equations that can be solved in closed form in terms of Φ_B, Ψ_j, and their derivatives. Substituting θ_B^* from (21) into constraint $m^\top \theta_{Bt}^* \leq 1$ after rearranging terms we obtain an upper bound on ν^* in terms of κ and σ_y. Then, we substitute κ and σ_y from (16) and (18), and obtain the following inequality:

$$\begin{aligned} \nu_t^* &\leq \Big(1 - (\Sigma_t^{-1} m)^\top \Big(\frac{\kappa_t}{\gamma_B} + \sigma_{xt} \frac{\partial \Phi_{Bt}}{\partial x_t}\frac{x_t}{\Phi_{Bt}} - \sigma_{yt} \frac{\partial \Phi_{Bt}}{\partial y_t}\frac{y_t}{\Phi_{Bt}}\Big)\Big) \frac{1}{(\Sigma_t^{-1} m)^\top (\Sigma_t^{-1} m)/\gamma_B} \\ &= \Big(1 - b_{2t}^\top (\Sigma_t^{-1} m) - \nu_t^* \Big(b_{1t} - \frac{1}{\gamma_B}\Big)(\Sigma_t^{-1} m)^\top (\Sigma_t^{-1} m)\Big) \frac{1}{(\Sigma_t^{-1} m)^\top (\Sigma_t^{-1} m)/\gamma_B}. \end{aligned} \tag{A10}$$

Inequality (A10) is satisfied as an equality when the constraint is binding. Furthermore, Kuhn-Tucker condition implies that $\nu_t^* \leq 0$ [e.g., Karatzas and Shreve (1998); Remark 1 in Section 2.1]. Since either the latter or the former inequalities should be satisfied as an equality, depending on whether the constraint is binding or not, we obtain the first equation for ν^* and $\Sigma^{-1}m$:

$$\nu_t^* = \min\Big\{0; 1 - b_{2t}^\top (\Sigma_t^{-1} m) - \nu_t^* \Big(b_{1t} - \frac{1}{\gamma_B}\Big)(\Sigma_t^{-1} m)^\top (\Sigma_t^{-1} m)\Big\} \frac{1}{(\Sigma_t^{-1} m)^\top (\Sigma_t^{-1} m)/\gamma_B}, \tag{A11}$$

where $b_1 \in \mathbb{R}$ and $b_2 \in \mathbb{R}^2$ are given by expressions (A5).

To obtain the second equation for ν^* and $\Sigma^{-1}m$ we substitute σ_y from (18) into volatilities σ_j given by (20), and then construct the volatility matrix $\Sigma = (\sigma_1, \sigma_2)^\top$. Finally, by multiplying both sides of the expression for Σ by $\Sigma^{-1}m$ we derive the following equation:

$$m = \tilde{\Sigma}_t \Sigma_t^{-1} m + \nu_t^* b_{3t} (\Sigma_t^{-1} m)^\top \Sigma_t^{-1} m, \tag{A12}$$

where vector $b_3 \in \mathbb{R}^2$ and matrix $\tilde{\Sigma} \in \mathbb{R}^{2\times 2}$ are given by expressions (A6) and (A7), respectively.

Next, we find ν^* and $\Sigma^{-1}m$ by solving the system of equations (A11) and (A12). First, from equation (A11) we obtain:

$$\nu_t^* = \begin{cases} \dfrac{1 - b_{2t}^\top (\Sigma_t^{-1} m)}{b_{1t} (\Sigma_t^{-1} m)^\top \Sigma_t^{-1} m}, & \text{if constraint binds,} \\ \\ 0, & \text{if constraint does not bind.} \end{cases} \tag{A13}$$

When the constraint binds, ν^* is given by the first line of (A13). Substituting this expression into equation (A12) we obtain a linear equation for $\Sigma^{-1}m$, which has the following solution:

$$\Sigma_t^{-1}m = Q_t^{-1}(m - b_{3t}/b_{1t}), \tag{A14}$$

where $Q_t = \tilde{\Sigma}_t - b_{3t}b_{2t}^\top/b_{1t} \in \mathbb{R}^{2\times 2}$. To simplify the solution in (A14) we note that vector $v^* = (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top$ solves equation $Qv^* = (1,1)^\top - b_3/b_1$. This result can be verified by multiplying matrix Q and vector v^*, and using an easily verifiable fact that $\sigma_D^\top v^* = 1$ and $\sigma_x^\top v^* = 0$, where $\sigma_D = (x\sigma_{D_1}, (1-x)\sigma_{D_2})^\top$ and $\sigma_x = ((1-x)\sigma_{D_1}, -(1-x)\sigma_{D_2})^\top$ are defined in (2) and (9). Substituting $b_3/b_1 = 1 - Qv^*$ into (A14) we obtain $\Sigma^{-1}m = (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top + Q^{-1}(m_1 - 1, m_2 - 1)^\top$. Similarly, when the constraint does not bind, and hence $\nu^* = 0$, (A12) becomes a linear equation for $\Sigma^{-1}m$, which can be solved in closed form. Then, denoting $v = \Sigma^{-1}m$ for the case when the constraint binds, we obtain expression (A4) for vector $\Sigma^{-1}m$. $\square$

Lemma A.2 (Verification of Optimality). *Let $\Phi_B(x,y) \in C^2((0,1)\times(0,1)) \cap C^1([0,1]\times[0,1])$. Suppose, that $|\Sigma_t^{-1}\theta_{Bt}^*| < C_1$, $|\kappa_t + \nu_t^*\Sigma^{-1}m| < C_1$, $0 < \Phi_B(x,y) < C_1$, where C_1 is a constant, and consider function $J_B(W_t, x_t, y_t, t)$, given by:*

$$J_B(W_t, x_t, y_t, t) = e^{-\rho t}\frac{W_t^{1-\gamma_B}\Phi_B(x_t,y_t)^{\gamma_B}}{1-\gamma_B}. \tag{A15}$$

(i) *Consider strategies c_t and θ_t such that θ_t, $J_B(W_t, x_t, y_t, t)$, and $J_B(W_t, x_t, y_t, t)|\Sigma_t^{-1}\theta_t|$, belong to space $\mathcal{H}^2 = \{F : \int_0^T |F_t|^2 dt < \infty, \mathbb{E}_0\Big[\int_0^T |F_t|^2 dt\Big] < \infty,\ \text{for all } T > 0\}$, and the following conditions are satisfied:*

$$\Big|\mathbb{E}_0\Big[\int_0^{+\infty} e^{-\rho t}\frac{c_t^{1-\gamma_B}}{1-\gamma_B}dt\Big]\Big| < \infty, \quad m^\top\theta_t \le 1, \tag{A16}$$

$$\limsup_{T\to+\infty} \mathbb{E}_t[J_B(W_T, x_T, y_T, T)] \ge 0, \tag{A17}$$

where W_τ is wealth, generated by strategies c_t and θ_t. Then, the following inequality holds:

$$J_B(W_t, x_t, y_t, t) \ge \mathbb{E}_t\Big[\int_t^{+\infty} e^{-\rho\tau}\frac{c_\tau^{1-\gamma_B}}{1-\gamma_B}d\tau\Big]. \tag{A18}$$

(ii) *Consumption and investment policies $c_t^* = W_t/\Phi_B(x_t, y_t)$ and θ_{Bt}^* in (21) are optimal, and*

$$J_B(W_t, x_t, y_t, t) = \mathbb{E}_t\Big[\int_t^{+\infty} e^{-\rho\tau}\frac{(c_{B\tau}^*)^{1-\gamma_B}}{1-\gamma_B}d\tau\Big]. \tag{A19}$$

Proof of Lemma A.2.

(i) First, we consider a stochastic process $U_t = \int_0^t e^{-\rho\tau}c_\tau^{1-\gamma_B}/(1-\gamma_B)d\tau + J_B(W_t, x_t, y_t, t)$, which satisfies a stochastic differential equation (SDE)

$$dU_t = \mu_{Ut}dt + \sigma_{Ut}^\top dw_t. \tag{A20}$$

By applying Itô's Lemma to process U_t and adding and subtracting $\nu_t^*(m^\top\theta_t - 1)W_t\partial J_{Bt}/\partial W$ from the drift, after some algebra, we find that μ_U and σ_U are given by:

$$
\begin{aligned}
\mu_{Ut} &= \Big\{e^{-\rho t}\frac{c_t^{1-\gamma_B}}{1-\gamma_B} + \frac{\mathbb{E}_t[J_{Bt}]}{dt} + \nu_t^*(m^\top\theta_t - 1)W_t\frac{\partial J_{Bt}}{\partial W}\Big\} - \nu_t^*(m^\top\theta_t - 1)W_t\frac{\partial J_{Bt}}{\partial W}, \\
\sigma_{Ut} &= J_{Bt}\Big((1-\gamma_B)\Sigma_t^\top\theta_t + \gamma_B\Sigma_t^\top\theta_{Bt}^* - (\kappa_t + \nu_t^*\Sigma_t^{-1}m)\Big),
\end{aligned}
\tag{A21}
$$

where weight θ_B^* is given by expression (21). The first term of μ_U (in curly brackets) is non-positive because J_{Bt} satisfies HJB equation (14) with the "max" operator, whereas the first term in (A21) is evaluated at a sub-optimal strategy (c,θ). The second term in (A21) is non-positive because $\nu_t^* \le 0$ and $m^\top\theta_t - 1 \le 0$ by assumption (A16), and $\partial J_B/\partial W_t \ge 0$, and hence, $\mu_U \le 0$.

Next, we integrate (A20) from t to T and take expectation $\mathbb{E}_t[\cdot]$ on both sides. Because $J_{Bt} \in \mathcal{H}^2$, and $J_{Bt}|\Sigma^\top\theta_t| \in \mathcal{H}^2$, we obtain that $\mathbb{E}_t[\int_0^T \sigma_{U\tau}dw_\tau] = 0$. Taking into account that $\mu_U \le 0$, we find that $U_t \ge \mathbb{E}_t[U_T]$, which can be expanded as follows:

$$
J_B(W_t, x_t, y_t, t) \ge \mathbb{E}_t\Big[\int_t^T e^{-\rho t}\frac{c_\tau^{1-\gamma_B}}{1-\gamma_B}d\tau\Big] + \mathbb{E}_t\Big[J_B(W_T, x_T, y_T, T)\Big]. \tag{A22}
$$

From condition (A17), it follows that there exists a monotonic subsequence T_n such that $T_n \to \infty$, as $n \to \infty$, and $\mathbb{E}_t[J(W_{T_n}, x_{T_n}, y_{T_n}, T_n)] \ge 0$. Moreover, $\int_t^{T_n} c_\tau^{1-\gamma_B}/(1-\gamma_B)d\tau$ is a monotonic sequence of random variables. Therefore, taking the limit of (A22), we obtain inequality (A18) by the monotone convergence theorem [e.g., Shiryaev (1996)].

(ii) First, we demonstrate that the transversality condition $\mathbb{E}_t[J_B(W_T, x_T, y_T, T)] \to 0$ is satisfied as $T \to 0$, where wealth W_T is under strategies c^* and θ^*. To this end, we apply Itô's Lemma to $J_B(W_t, x_t, y_t, t)$. Then, we add and subtract $e^{-\rho t}(c_{Bt}^*)^{1-\gamma_B}/(1-\gamma_B)$ and $\nu_t^*(m^\top\theta_{Bt}^* - 1)W_t\partial J_{Bt}/\partial W_t$ in the drift term of the process, taking into account the condition $\nu_t^*(m^\top\theta_{Bt}^* - 1) = 0$. Noting that $J_B(W_t, x_t, y_t, t)$ satisfies HJB equation (15), after some algebra, we obtain:

$$
dJ_{Bt} = J_{Bt}[\mu_{Jt}dt + \sigma_{Jt}^\top dtw_t], \tag{A23}
$$

where drift μ_J and volatility σ_J are given by:

$$
\mu_{Jt} = -\frac{1}{\Phi_B(x_t, y_t)}, \quad \sigma_{Jt} = \Sigma_t^\top\theta_{Bt}^* - (\kappa_t + \nu_t^*\Sigma_t^{-1}m). \tag{A24}
$$

By assumptions of Lemma A.3, σ_J satisfies Novikov's condition, and hence, process $d\eta_t = \eta_t\sigma_{Jt}^\top dw_t$ is a martingale. Using the martingality of η_t from (A23) and (A24), we obtain:

$$
\begin{aligned}
|\mathbb{E}_t[J_B(W_T, x_T, y_T, T)]| &= \mathbb{E}_t\Big[|J_{Bt}|\exp\Big(-\int_t^T \frac{1}{\Phi_B(x_\tau, y_\tau)}d\tau\Big)\frac{\eta_T}{\eta_t}\Big] \\
&\le |J_{Bt}|e^{-(T-t)/C_1}\mathbb{E}_t\Big[\frac{\eta_T}{\eta_t}\Big] = |J_{Bt}|e^{-(T-t)/C_1}.
\end{aligned}
\tag{A25}
$$

From inequality (A25), it easily follows that $\mathbb{E}_t[J_B(W_T, x_T, y_T, T)] \to 0$, as $T \to \infty$.

Next, we consider a process $U_t^* = \int_0^t e^{-\rho t}(c_{B\tau}^*)^{1-\gamma_B}/(1-\gamma_B)d\tau + J_B(W_t, x_t, y_t, t)$. Applying Itô's Lemma to process U_t^*, we demonstrate that U_t^* satisfies an SDE

$$dU_t^* = U_t^* \frac{J_{Bt}}{U_t^*}\Big(\Sigma_t^\top \theta_{Bt}^* - (\kappa_t + \nu_t^* \Sigma_t^{-1} m)\Big)^\top dw_t. \tag{A26}$$

Because $0 < J_{Bt}/U_t^* < 1$ and given the assumptions of Lemma A.3, the volatility of process U_t^* satisfies Novikov's condition. Therefore process U_t^* is an exponential martingale. Consequently, integrating (A26) from t to T and taking the expectations on both sides, we obtain:

$$J_B(W_t, x_t, y_t, t) = \mathbb{E}_t\Big[\int_t^T e^{-\rho t}\frac{(c_{B\tau}^*)^{1-\gamma_B}}{1-\gamma_B}d\tau\Big] + \mathbb{E}_t\Big[J_B(W_T, x_T, y_T, T)\Big]. \tag{A27}$$

In the limit $T \to +\infty$, the last term in (A27) vanishes due to inequality (A25), whereas the first term converges to (A19) by the monotone convergence theorem. $\square$

Lemma A.3. *Let price-dividend ratios $\Psi_j(x_t, y_t)$, $j = 1, 2$, be such that $0 < \Psi_j(x_t, y_t) \leq C_1$, where C_1 is a constant. Suppose, volatilities σ_{jt} and market price of risk κ_t are such that $(\sigma_{jt} - \kappa_t)^\top(\sigma_{jt} - \kappa_t) < C_2$. Then, there are no bubbles in the economy, and stock prices S_{jt} are given by*

$$S_{jt} = \frac{1}{\xi_t}\mathbb{E}_t\Big[\int_t^{+\infty} \xi_\tau D_{j\tau}\Big], \quad j = 1, 2. \tag{A28}$$

Proof of Lemma A.3. First, we obtain an upper bound for $\mathbb{E}_t[\xi_\tau S_\tau]$. Applying Itô's Lemma to $\xi_t S_t$, where S_t and ξ_t follow processes (4) and (12), respectively, we obtain:

$$\begin{aligned} d(\xi_t S_{jt}) &= -\xi_t D_{jt} dt + \xi_t S_{jt}(\sigma_{jt} - \kappa_t)^\top dw_t \\ &= \xi_t S_{jt}\Big[-\frac{1}{\Psi_j(x_t, y_t)}dt + (\sigma_{jt} - \kappa_t)^\top dw_t\Big], \end{aligned} \tag{A29}$$

SDE (A29) has the following solution:

$$\xi_\tau S_{j\tau} = \xi_t S_t \exp\left(-\int_t^\tau \frac{1}{\Psi_j(x_s, y_s)}ds\right)\frac{\eta_\tau}{\eta_t},$$

where η_t follows a process $d\eta_t = \eta_t(\sigma_{jt} - \kappa_t)^\top dw_t$. Process η_t is an exponential martingale since $\sigma_{jt} - \kappa_t$ is bounded, and hence satisfies Novikov's condition. Consequently, using the fact that $\Psi_j(y) < C_1$, we obtain the following inequality:

$$\mathbb{E}_t[\xi_\tau S_{j\tau}] \leq \xi_t S_{jt} e^{-(\tau - t)/C_1}\mathbb{E}_t\Big[\frac{\eta_\tau}{\eta_t}\Big] \leq \xi_t S_{jt} e^{-(\tau - t)/C_1}. \tag{A30}$$

Next, we consider a process $U_{jt} = \int_0^t \xi_\tau D_{j\tau} d\tau + \xi_t S_{jt}$, which satisfies an SDE

$$dU_{jt} = \xi_t S_{jt}(\sigma_{jt} - \kappa_t)^\top dw_t. \tag{A31}$$

From inequality (A30) and the assumptions of Lemma A.2 we obtain $\mathbb{E}_t[\int_0^T \Big(\xi_\tau S_{j\tau}|\sigma_{j\tau}-\kappa_\tau|\Big)^2 d\tau] < +\infty$, and hence U_{jt} is a martingale. Integrating process (A31) from t to T and taking expectations on both sides we find that $U_{jt} = \mathbb{E}_t[U_{jT}]$, which can be rewritten as follows:

$$\xi_t S_{jt} = \mathbb{E}_t\Big[\int_t^T \xi_\tau D_{j\tau} d\tau\Big] + \mathbb{E}_t\Big[\xi_T S_{jT}\Big]. \tag{A32}$$

inequality (A30) implies that the last term in (A32) converges to zero as $T \to \infty$, while the first term converges to (A28) by the monotone convergence theorem. □

Proof of Proposition 2.

(i) In the unconstrained economy, both investors have the same state price density, i.e., $\xi_{\nu^*} = \xi$. Hence, from the expressions for optimal consumptions (13), we find that $(c^*_{At})^{-\gamma_A}/(c^*_{Bt})^{-\gamma_B} = \lambda$, where λ is a constant. From the latter equation and consumption clearing $c^*_{At} + c^*_{Bt} = D_t$, we find that consumption share $y = c^*_B/D$ is given by

$$y_t = f\Big(\lambda^{-\frac{1}{\gamma_A}} D_t^{\frac{\gamma_B-\gamma_A}{\gamma_A}}\Big), \tag{A33}$$

where $f(\cdot)$ is an implicit function satisfying equation:

$$Zf(Z)^{\frac{\gamma_B}{\gamma_A}} + f(Z) = 1. \tag{A34}$$

The s.p.d. in terms of consumption share is then given by:

$$\xi_t = \tilde{\lambda} e^{-\rho t}(c^*_{Bt})^{-\gamma_B} = \tilde{\lambda} e^{-\rho t} f\Big(\lambda^{-\frac{1}{\gamma_A}} D_t^{\frac{\gamma_B-\gamma_A}{\gamma_A}}\Big)^{-\gamma_B} D_t^{-\gamma_B}, \tag{A35}$$

where $\tilde{\lambda}$ is a constant. Therefore, the price-dividend ratio for the first stock is given by:

$$\begin{aligned}
\Psi_{1t} &= \frac{1}{D_{1t}(c^*_{Bt})^{-\gamma_B}} \mathbb{E}_t\Big[\int_t^{+\infty} e^{-\rho(\tau-t)} D_{1\tau} D_\tau^{-\gamma_B} f\Big(\lambda^{-\frac{1}{\gamma_A}} D_\tau^{\frac{\gamma_B-\gamma_A}{\gamma_A}}\Big)^{-\gamma_B} d\tau\Big] \\
&= \Big(\frac{y_t}{x_t}\Big)^{\gamma_B} \mathbb{E}_t\Big[\int_t^{+\infty} e^{-\hat{\rho}(\tau-t)} \frac{\eta_\tau}{\eta_t}\Big(1+\frac{D_{2\tau}}{D_{1\tau}}\Big)^{-\gamma_B} f\Big(\lambda^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B-\gamma_A}{\gamma_A}}\Big(1+\frac{D_{2\tau}}{D_{1\tau}}\Big)^{\frac{\gamma_B-\gamma_A}{\gamma_A}}\Big)^{-\gamma_B} d\tau\Big] \\
&= \Big(\frac{y_t}{x_t}\Big)^{\gamma_B} \hat{\mathbb{E}}_t\Big[\int_t^{+\infty} e^{-\hat{\rho}(\tau-t)} \Big(1+\frac{D_{2\tau}}{D_{1\tau}}\Big)^{-\gamma_B} f\Big(\lambda^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B-\gamma_A}{\gamma_A}}\Big(1+\frac{D_{2\tau}}{D_{1\tau}}\Big)^{\frac{\gamma_B-\gamma_A}{\gamma_A}}\Big)^{-\gamma_B} d\tau\Big],
\end{aligned} \tag{A36}$$

where $\hat{\rho} = \rho - (1-\gamma_B)\mu_{D_1} + 0.5(1-\gamma_B)\gamma_B\sigma^2_{D_1}$ and η_t is an exponential GBM martingale following process $d\eta_t = \eta_t(1-\gamma_B)\sigma_{D_1} dw_{1t}$, and $\hat{\mathbb{E}}[\cdot]$ is the expectation under the new probability measure $\hat{\mathbb{P}}$ with Radon-Nikodym derivative η_τ/η_t.

Next, rewriting the processes for D_1 and D_2 in (2) under the new measure, we obtain:

$$\lambda^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B-\gamma_A}{\gamma_A}} = \lambda^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B-\gamma_A}{\gamma_A}} e^{u_1}, \quad \frac{D_{2\tau}}{D_{1\tau}} = \frac{D_{2t}}{D_{1t}} e^{u_2}, \tag{A37}$$

where $u = (u_1, u_2)^\top$ has distribution $N(q(\tau - t), \Sigma_u(\tau - t))$ and q and Σ_u are given by:

$$q = \Big(\frac{\gamma_B - \gamma_A}{\gamma_A}(\mu_{D_1} + 0.5(1 - 2\gamma_B)\sigma^2_{D_1}), \mu_{D_2} - \mu_{D_1} + 0.5(2\gamma_B - 1)\sigma^2_{D_1} - 0.5\sigma^2_{D_2}\Big)^\top,$$

$$\Sigma_u = \begin{pmatrix} \Big(\frac{\gamma_B - \gamma_A}{\gamma_A}\Big)^2 \sigma^2_{D_1} & -\frac{\gamma_B - \gamma_A}{\gamma_A}\sigma^2_{D_1} \\ -\frac{\gamma_B - \gamma_A}{\gamma_A}\sigma^2_{D_1} & \sigma^2_{D_1} + \sigma^2_{D_2} \end{pmatrix}. \tag{A38}$$

Next, we define parameter p as follows:

$$p = \sqrt{2\Big(\rho - (1 - \gamma_B)\mu_{D_1} + 0.5(1 - \gamma_B)\gamma_B\sigma^2_{D_1}\Big) + q^\top \Sigma_u^{-1} q}. \tag{A39}$$

Rewriting the expectation in (A36) as an integral involving probability density function (p.d.f.) of distribution $N(q(\tau - t), \Sigma_u(\tau - t))$, we obtain:

$$\begin{aligned} \Psi_{1t} &= \Big(\frac{y_t}{x_t}\Big)^{\gamma_B} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Big(1 + \frac{D_{2t}}{D_{1t}}e^{u_2}\Big)^{-\gamma_B} f\Big(\lambda^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} e^{u_1}\Big(1 + \frac{D_{2t}}{D_{1t}}e^{u_2}\Big)^{\frac{\gamma_B - \gamma_A}{\gamma_A}}\Big)^{-\gamma_B} \times \\ &\qquad \frac{1}{2\pi\sqrt{\det(\Sigma_u)}}\Big[\int_0^{+\infty} \frac{1}{\tau} e^{-\hat{\rho}\tau - \frac{1}{2\tau}(u - q\tau)^\top \Sigma_u^{-1}(u - q\tau)} d\tau\Big] du_1\, du_2 \\ &= \Big(\frac{y_t}{x_t}\Big)^{\gamma_B} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Big(1 + \frac{D_{2t}}{D_{1t}}e^{u_2}\Big)^{-\gamma_B} f\Big(\lambda^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} e^{u_1}\Big(1 + \frac{D_{2t}}{D_{1t}}e^{u_2}\Big)^{\frac{\gamma_B - \gamma_A}{\gamma_A}}\Big)^{-\gamma_B} \times \\ &\qquad \frac{e^{q^\top \Sigma_u^{-1} u} K_0(p\sqrt{u^\top \Sigma_u^{-1} u})}{\pi\sqrt{\det(\Sigma_u)}} du_1\, du_2, \end{aligned} \tag{A40}$$

where the last equality is computed using integral 3.471.9 in Gradshteyn and Ryzhik (2007), and $K_0(\cdot)$ is a McDonald's function.[12]

To eliminate function $f(\cdot)$ from equation (A40), we perform the following change of variables:

$$z = f\Big(\lambda^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} e^{u_1}\Big(1 + \frac{D_{2t}}{D_{1t}}e^{u_2}\Big)^{\frac{\gamma_B - \gamma_A}{\gamma_A}}\Big), \quad s = \Big(1 + \frac{D_{2t}}{D_{1t}}e^{u_2}\Big)^{-1}. \tag{A41}$$

From equation (A34), we observe that $f^{-1}(z) = (1 - z)z^{-\gamma_B/\gamma_A}$. Furthermore, from the definition of share $x = D_1/D$, we note that $D_2/D_1 = (1 - x)/x$, and from equation for share y in (A33), we obtain $\lambda^{-1/\gamma_A} D_1^{(\gamma_B - \gamma_A)/\gamma_A} = f^{-1}(y)/x^{(\gamma_A - \gamma_B)/\gamma_A} = (1 - y)y^{-\gamma_B/\gamma_A}/x^{(\gamma_A - \gamma_B)/\gamma_A}$. Using these expressions, we solve equations (A41) and obtain u_1 and u_2 as functions of s, z, x, and y given by (27). Finally, computing the partial derivatives of u_1 and u_2 in (27) w.r.t. s and z, we obtain:

$$du_1\, du_2 = \begin{vmatrix} \partial u_1/\partial z & \partial u_1/\partial s \\ \partial u_2/\partial z & \partial u_2/\partial s \end{vmatrix} ds\, dz = \frac{\gamma_B(1 - z) + \gamma_A z}{\gamma_A s(1 - s)z(1 - z)} ds\, dz. \tag{A42}$$

[12] Integrals 3.471.9 and 8.432.1 in Gradshteyn and Ryzhik (2007) imply that

$$\int_0^{+\infty} \frac{1}{\tau} e^{-a\tau - \frac{b}{\tau}} d\tau = 2K_0(2\sqrt{ab}), \quad K_0(z) = \int_0^{+\infty} e^{-z\cosh(s)} ds.$$

Using the expression for $du_1 du_2$ in (A42), after performing the change of variables, we obtain price-dividend ratio $\Psi_1(x, y)$ in (26). The expression for $\Psi_2(x, y)$ can be derived analogously.

(ii) We now prove the second part of Proposition 2. When the leverage constraint is binding, substituting $m_1 = 1$ and $m_2 = 1$ into expression (A4), we find that $\Sigma^{-1}\underline{1} = (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top$. Next, we observe that the following easily verifiable equalities hold:

$$\sigma_{Dt}^\top (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top = 1, \quad \sigma_{xt}^\top (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top = 0, \tag{A43}$$

where $\sigma_{Dt} = \Big(x_t\sigma_{D_1}, (1 - x_t)\sigma_{D_1}\Big)^\top$ and $\sigma_{xt} = \Big((1 - x_t)\sigma_{D_1}, -(1 - x_t)\sigma_{D_2}\Big)$ are defined in (2) and (9). Substituting $\Sigma^{-1}m$ into the expression for ν^* in (A13) and using equalities (A43), we obtain the expression for ν^* in (28). Using equalities (A43) one more time, we find that the leverage constraint binds for all x and y, i.e., $(1, 1)^\top \theta_B^* = 1$. Intuitively, the constraint is identically binding because B always wants to borrow but is prevented by constraints.

Then, we derive closed-form expressions for Ψ_j, $j = 1, 2$. Consider the ratio of marginal utilities $\lambda_t = (c_{At}^*)^{-\gamma_A}/(c_{Bt}^*)^{-\gamma_B}$. From the F.O.C. in (13), we find that $\lambda_t = \tilde{\lambda}\xi_t/\xi_{\nu^* t}$, where $\tilde{\lambda}$ is a constant and ξ_t and $\xi_{\nu^* t}$ follow processes (12). Applying Itô's Lemma to λ_t, we obtain:

$$d\lambda_t = -\lambda_t \Big[\nu_t^* \Big(1 - m^\top (\Sigma_t^\top)^{-1}(\kappa_t + \nu_t^* \Sigma_t^{-1} m)\Big) dt - (\nu_t^* \Sigma_t^{-1} m)^\top dw_t\Big]. \tag{A44}$$

Substituting κ_t from (16), $\Sigma^{-1}m = (1/\sigma_{D_1}, 1/\sigma_{D_2})^\top$, and ν^* from (28) into process (A44), we find that λ_t follows a GBM:

$$d\lambda_t = \lambda_t \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2} \Big[(\gamma_B - 1)dt + \Big(\frac{1}{\sigma_{D_1}}, \frac{1}{\sigma_{D_2}}\Big)^\top dw_t\Big]. \tag{A45}$$

Similarly to the proof of the first part of Proposition 2, the consumption share of constrained investor B is given by $y_t = f(\lambda_t^{-1/\gamma_A} D_t^{(\gamma_B - \gamma_A)/\gamma_A})$, where $f(\cdot)$ solves equation (A34). The dividends should be priced using s.p.d. $\xi_t = (1/\psi_A)e^{-\rho t}((1 - y_t)D_t)^{-\gamma_A}$ from F.O.C. (13) for unconstrained investor A. Then, proceeding similarly as in the unconstrained case, we obtain:

$$\Psi_{1t} = \Big(\frac{1 - y_t}{x_t}\Big)^{\gamma_A} \tilde{\mathbb{E}}_t \Big[\int_t^\infty e^{-\tilde{\rho}(\tau - t)} \Big(1 + \frac{D_{2\tau}}{D_{1\tau}}\Big)^{-\gamma_A} \Big(1 - f\Big(\lambda_\tau^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} \Big(1 + \frac{D_{2\tau}}{D_{1\tau}}\Big)^{\frac{\gamma_B - \gamma_A}{\gamma_A}}\Big)\Big)^{-\gamma_A} d\tau\Big], \tag{A46}$$

where $\tilde{\rho} = \rho - (1 - \gamma_A)\mu_{D_1} + 0.5(1 - \gamma_A)\gamma_A\sigma_{D_1}^2$, and $\tilde{\mathbb{E}}[\cdot]$ is an expectation under the new measure $\tilde{\mathbb{P}}$, such that $\tilde{w}_{1t} = w_{1t} - (1 - \gamma_A)\sigma_{D_1} t$ is a Brownian motion under $\tilde{\mathbb{P}}$. From the fact that λ_t, D_{1t} and D_{2t} follow GBMs (A45) and (2), respectively, we obtain that under measure $\tilde{\mathbb{P}}$

$$\lambda_\tau^{-\frac{1}{\gamma_A}} D_{1\tau}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} = \lambda_t^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B - \gamma_A}{\gamma_A}} e^{d_{11}(\tau - t) + d_{12}\varepsilon_t}, \quad \frac{D_{2\tau}}{D_{1\tau}} = \frac{D_{2t}}{D_{1t}} e^{d_{21}(\tau - t) + d_{22}\varepsilon_t}, \tag{A47}$$

where $\varepsilon_t = \Big(-\sigma_{D_1}(\tilde{w}_{1\tau} - \tilde{w}_{1t}) + \sigma_{D_2}(w_{2\tau} - w_{2t})\Big)/\sqrt{(\sigma_{D_1}^2 + \sigma_{D_2}^2)} \sim N(0, \tau - t)$, and

$$
\begin{aligned}
&d_{11} = \frac{\gamma_B - \gamma_A}{\gamma_A}\Big(\mu_{D_1} + \frac{1-2\gamma_A}{2}\sigma_{D_1}^2\Big) - \frac{(\gamma_B - \gamma_A)^2}{2\gamma_A}\frac{1}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2}, \quad && d_{12} = -\frac{\gamma_B - \gamma_A}{\gamma_A}\frac{\sigma_{D_1}^2}{\sqrt{\sigma_{D_1}^2 + \sigma_{D_2}^2}},\\
&d_{21} = \mu_{D_2} - \mu_{D_1} - 0.5\sigma_{D_2}^2 - \frac{1-2\gamma_A}{2}\sigma_{D_1}^2, && d_{22} = \sqrt{\sigma_{D_1}^2 + \sigma_{D_2}^2}.
\end{aligned}
\tag{A48}
$$

Finally, similarly to the unconstrained case, we rewrite the expectation operator in equation (A46) as an integral w.r.t. ε_t, and change variables ε and $\tau - t$ to the following variables:

$$
\begin{aligned}
z &= f\Big(\lambda_t^{-\frac{1}{\gamma_A}} D_{1t}^{\frac{\gamma_B-\gamma_A}{\gamma_A}} e^{d_{11}(\tau-t)+d_{12}\varepsilon_t}\Big(1 + \frac{D_{2t}}{D_{1t}}e^{d_{21}(\tau-t)+d_{22}\varepsilon_t}\Big)^{\frac{\gamma_B-\gamma_A}{\gamma_A}}\Big),\\
s &= \Big(1 + \frac{D_{2t}}{D_{1t}}e^{d_{21}(\tau-t)+d_{22}\varepsilon_t}\Big)^{-1}.
\end{aligned}
\tag{A49}
$$

After the change of variables in equation (A49), similarly to the unconstrained case, we obtain:

$$
\Psi_{1t} = \int_0^1\int_0^1 \Big(\frac{s}{x}\frac{1-y}{1-z}\Big)^{\gamma_A}\frac{\gamma_B(1-z)+\gamma_A z}{z(1-z)s(1-s)}F(s,z;x,y)\,ds\,dz, \tag{A50}
$$

where

$$
F(s,z;x,y) = \begin{cases} \dfrac{e^{-\tilde{\rho}\tau(s,z;x,y) - 0.5\varepsilon(s,z;x,y)^2/\tau(s,z;x,y)}}{\gamma_A|d_{11}d_{22} - d_{12}d_{21}|\sqrt{2\pi\tau(s,z;x,y)}}, & \tau(s,z;x,y) > 0,\\[2ex] 0, & \tau(s,z;x,y) \le 0, \end{cases} \tag{A51}
$$

$\tilde{\rho} = \rho - (1-\gamma_A)\mu_{D_1} + 0.5(1-\gamma_A)\gamma_A\sigma_{D_1}^2$, $\varepsilon(s,z;x,y)$ and $\tau(s,z;x,y)$ are given by:

$$
\varepsilon(s,z;x,y) = \frac{1}{d_{12}d_{21} - d_{11}d_{22}}\Big[d_{21}\Big(\ln\Big(\frac{1-z}{1-y}\Big) - \frac{\gamma_B}{\gamma_A}\ln\Big(\frac{z}{y}\Big) + \frac{\gamma_B-\gamma_A}{\gamma_A}\ln\Big(\frac{s}{x}\Big)\Big) - d_{11}\ln\Big(\frac{1-s}{1-x}\frac{x}{s}\Big)\Big],
$$

$$
\tau(s,z;x,y) = \frac{1}{d_{11}d_{22} - d_{12}d_{21}}\Big[d_{22}\Big(\ln\Big(\frac{1-z}{1-y}\Big) - \frac{\gamma_B}{\gamma_A}\ln\Big(\frac{z}{y}\Big) + \frac{\gamma_B-\gamma_A}{\gamma_A}\ln\Big(\frac{s}{x}\Big)\Big) - d_{12}\ln\Big(\frac{1-s}{1-x}\frac{x}{s}\Big)\Big],
$$

and d_{ij} are given by expressions (A48). Price-dividend ratio Ψ_{2t} is found analogously. □

Proof of Corollary 1.

(i) The risk premium of the market portfolio is given by $\mu_M - r = \theta_M^\top(\mu - r\underline{1})$. Multiplying both sides of the equation for excess returns (29) by θ_M we obtain $\mu_M - r\underline{1} = \theta_M\beta_c - (\Gamma_t y_t\nu^*/\gamma_B)\theta_M^\top m$. From the latter equation we find multiplier $(\Gamma y/\gamma_B)\nu^*$ in terms of $\mu_M - r$, and after substituting this multiplier back into equation (29) we obtain consumption CAPM (30).

(ii) Consumption CAPM (31) is derived by substituting ν^* given by (28) into equation (29). □

Appendix B: Numerical Methods

In this Appendix we discuss the numerical method for solving PDEs for price-dividend and wealth-consumption ratios (24)–(25). We implement and compare three methods. Along with fixed point iterations we use textbook explicit and implicit-explicit methods. The mathematical literature recommends these methods for solving non-linear PDEs in physics and engineering [e.g., Lapidus and Pinder (1999)]. We demonstrate that all these methods work in our setting, but fixed point iterations perform much better in terms of the speed of calculations.

B.1. Finite Differences Approach

We first consider an economy with finite horizon T, and choose T to be sufficiently large so that the equilibrium processes are horizon-independent, as further discussed below. In this economy the PDEs (24)–(25) for price-dividend Ψ_j and wealth-consumption Φ_B ratios now include time-derivatives $\partial\Psi_i/\partial t$ and $\partial\Phi_B/\partial t$, respectively. Then, we solve the resulting PDEs by employing finite difference methods suggested in the literature.

We consider a uniform mesh $\Omega_N = \{(x_n, y_m) : x_n = n \times h, y_m = m \times h; n, m = 0, 1, \dots, N\}$, where $h = 1/N$, and N is an integer, and index time increments by $t = 0, \Delta t, 2\Delta t, \dots, T$. Then, we approximate solutions $\Psi_j(x, y; t)$ and $\Phi_B(x, y; t)$ at points (x_n, y_m) by discrete elements $\Psi^t_{j,n,m}$ and $\Phi^t_{B,n,m}$, respectively. We set terminal conditions for price-dividend and wealth-consumption ratios so that $\Psi^T_{j,n,m} = \Delta t$ and $\Phi^T_{B,n,m} = \Delta t$, since stock prices and investor B's wealth at final date T are given by $S_{jT} = D_{jT}\Delta t$ and $W_{BT} = c_{BT}\Delta t$, respectively. For brevity, we present equations only for price-dividend ratios, and omit subscript j.

We solve the PDEs using both *explicit* and *implicit-explicit* finite-difference methods suggested in the literature. Partial derivatives with respect to variables x and y are discretized using central differences, which give second-order approximation $O(h^2)$. We start with the explicit method, which is essentially backward Euler's method, which approximates our PDEs as follows:

$$\begin{aligned}
&\frac{\Psi^{t+\Delta t}_{n,m} - \Psi^t_{n,m}}{\Delta t} + a_1(Z_{t+\Delta t})\frac{\Psi^{t+\Delta t}_{n+1,m} - 2\Psi^{t+\Delta t}_{n,m} + \Psi^{t+\Delta t}_{n-1,m}}{h^2} + a_2(Z_{t+\Delta t})\frac{\Psi^{t+\Delta t}_{n,m+1} - 2\Psi^{t+\Delta t}_{n,m} + \Psi^{t+\Delta t}_{n,m+1}}{h^2} \\
&+ a_3(Z_{t+\Delta t})\frac{\Psi^{t+\Delta t}_{n+1,m+1} - \Psi^{t+\Delta t}_{n-1,m+1} - \Psi^{t+\Delta t}_{n+1,m-1} + \Psi^{t+\Delta t}_{n-1,m-1}}{4h^2} + b_{1,\Psi}(Z_{t+\Delta t})\frac{\Psi^{t+\Delta t}_{n+1,m} - \Psi^{t+\Delta t}_{n-1,m}}{2h} \\
&+ b_{2,\Psi}(Z_{t+\Delta t})\frac{\Psi^{t+\Delta t}_{n,m+1} - \Psi^{t+\Delta t}_{n,m-1}}{2h} + c_\Psi(Z_{t+\Delta t})\Psi^{t+\Delta t}_{n,m} + 1 = 0,
\end{aligned} \tag{B1}$$

subject to terminal $\Psi^T_{n,m} = \Delta t$ and boundary conditions (discussed below), where a_1, a_2, a_3, $b_{1,\Psi}$, $b_{2,\Psi}$, c_Ψ are coefficients in front of second-order and first-order derivatives in PDEs (24), and Z is a finite-difference approximation of vector $(x, y, \triangledown\Psi_1, \triangledown\Psi_2, \triangledown\Phi_B, \Psi_1, \Psi_2, \Phi_B)^\top$, where $\triangledown F = (\partial F/\partial x, \partial F/\partial y)^\top$ is a gradient vector.

The explicit method is the easiest to implement, and is widely used both for non-linear and linear PDEs [e.g., Lapidus and Pinder (1999)]. The method expresses time-t solution in terms of time-$(t+\Delta t)$ solution, which is known from the previous step $t+\Delta t$. Consequently, solution $\Psi^t_{n,m}$ can be obtained without solving any equations. The explicit method in our model does converge to a horizon-independent solution when Δt is sufficiently small (e.g., $\Delta t = 0.01$).

In general, the literature recommends to use more robust implicit-explicit methods with better convergence properties [e.g., Lapidus and Pinder (1999)]. Consequently, we also consider the following modification of the method:

$$\begin{aligned}
& d\frac{\Psi^{t+\Delta t}_{n,m} - \Psi^t_{n,m}}{\Delta t} + a_1(Z_{t+\Delta t})\frac{\Psi^t_{n+1,m} - 2\Psi^t_{n,m} + \Psi^t_{n-1,m}}{h^2} + a_2(Z_{t+\Delta t})\frac{\Psi^t_{n,m+1} - 2\Psi^t_{n,m} + \Psi^t_{n,m+1}}{h^2} \\
& + a_3(Z_{t+\Delta t})\frac{\Psi^t_{n+1,m+1} - \Psi^t_{n-1,m+1} - \Psi^t_{n+1,m-1} + \Psi^t_{n-1,m-1}}{4h^2} + b_{1,\Psi}(Z_{t+\Delta t})\frac{\Psi^t_{n+1,m} - \Psi^t_{n-1,m}}{2h} \\
& + b_{2,\Psi}(Z_{t+\Delta t})\frac{\Psi^t_{n,m+1} - \Psi^t_{n,m-1}}{2h} + c_\Psi(Z_{t+\Delta t})\Psi^t_{n,m} + 1 = 0,
\end{aligned} \tag{B2}$$

subject to the same terminal and boundary conditions as in equation (B1), and where the coefficients and vector Z are the same as in equation (B1). For $d = 1$ equation (B2) defines an implicit-explicit method, and for $d = 0$ it defines a fixed-point iteration method. The coefficients in equation (B1) are computed using step-$(t+\Delta t)$ solutions, and hence are known at time t of backward iteration. Consequently, function $\Psi^t_{n,m}$ can be obtained by solving a system of linear equations with a sparse 9-diagonal matrix, which can be efficiently inverted numerically.

We cross-check that all three methods give the same result, and evaluate their performance by computing equilibria for different plausible ranges of exogenous model parameters. We choose the model parameters such that the boundary conditions for ratios $\Psi_j(x,y)$ and $\Phi_B(x,y)$ at corner points $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, given in Section B.2 below, are well-defined (i.e., positive and finite). The latter restriction is equivalent to assuming the existence of equilibria in limiting economies with one large investor (A or B) and/or one large tree (tree 1 or tree 2).

We note that specific model parameters have only minor effect on the speed of calculations. Consequently, we report the performance of methods for our benchmark calibration ($\gamma_A = 8$, $\gamma_B = 2$, $m_1 = 0.7$, $m_2 = 0.7$). All calculations were performed on a PC with Intel Core i7 CPU. The fixed point iteration method has the fastest convergence, which is evaluated by looking at $\varepsilon = \max_{(x_n,y_m)\in\Omega_N} |(\Psi^{k+1}_{n,m} - \Psi^k_{n,m})|$, where k is the number of iteration. For $N = 100$, which corresponds to 100^2 mesh points, the convergence with $\varepsilon = 10^{-6}$ requires 2 sec of CPU time.

The numerical methods literature gives preference to implicit-explicit (such as Crank-Nicholson) or fully implicit methods [e.g., Lapidus and Pinder (1999)] over explicit methods since the former have better convergence properties. However, we observe that in our model explicit method performs remarkably well, and does not have problems with convergence. Moreover, explicit method is much faster than the implicit-explicit one. When we set $N = 100$, $\Delta t = 0.01$ and $T = 350$ the explicit method takes 233 sec of CPU time, while the implicit-explicit method takes 6,055 sec. The convergence to a horizon-independent solution is assessed by looking at

$\max_{(x_n, y_m)\in\Omega_N} |(\Psi^{t+\Delta t}_{n,m} - \Psi^t_{n,m})/\Delta t|$, which is around 0.0001 for $T = 350$, but can be reduced to 10^{-6} when the horizon T is doubled, which approximately doubles the computation time.

We note that the portfolio choice in with constraints can be characterized in terms of forward and backward stochastic differential equations (FBSDE), as in Detemple and Rindisbacher (2005). The FBSDE methods can potentially solve models with non-Markovian dynamics, whereas our method assumes the existence of Markovian equilibria. Our solution approach is consistent with these methods, since as demonstrated in the literature [e.g., El Karoui, Peng, and Quenez (1997); Ma and Yong (1999)] in a Markovian setting solving FBSDEs reduces to solving a quasilinear parabolic PDE, exactly as in our model. Moreover, Ma and Yong (1999) recommend solving such PDEs using implicit-explicit methods, discussed above.

B.2. Dealing with Boundary Conditions

We now discuss boundary conditions, which we obtain by passing to limits $x \to 0$, $x \to 1$, $y \to 0$, and $y \to 1$ in PDEs, which correspond to well-defined limiting economies. While some boundary values for Ψ_j and Φ_B can be obtained in closed form, others solve quasilinear ordinary differential equations (ODEs), and must be found numerically. For example, the limit $y \to 0$ corresponds to a two-trees economy populated by a large investor with risk aversion γ_A, and a small investor with risk aversion γ_B, whose trading does not affect equilibrium processes. The price-dividend ratios in this economy give boundary conditions $\Psi_j(x, 0)$, and can be obtained in closed form [e.g., Martin (2012)]. However, the wealth-consumption ratio of a small investor B, which gives the boundary value $\Phi_B(x, 0)$, still has to be solved numerically.

Passing to the limit $y \to 0$ in equation (25) we observe that the coefficients in front of derivatives $\partial^2\Phi_B/\partial y^2$ and $\partial\Phi_B/\partial y$ become zero for $y = 0$. Assuming that $y^2\partial^2\Phi_B/\partial y^2 \to 0$ and $y\partial\Phi_B/\partial y \to 0$, as $y \to 0$, in the limit we obtain an ODE for $\tilde{\Phi}_B(x) = \Phi_B(x, 0)$:

$$\begin{aligned}
&\frac{x^2\sigma_x^\top\sigma_x}{2}\tilde{\Phi}_B''(x) + x\Big(\mu_x + \frac{1-\gamma_B}{\gamma_B}(\kappa + \nu^*\Sigma^{-1}m)^\top\sigma_x\Big)\tilde{\Phi}_B'(x) \\
&\quad + \Big(\frac{1-\gamma_B}{2\gamma_B^2}(\kappa + \nu^*\Sigma^{-1}m)^\top(\kappa + \nu^*\Sigma^{-1}m) + \frac{1-\gamma_B}{\gamma_B}(r - \nu^*) - \frac{\rho}{\gamma_B}\Big)\tilde{\Phi}_B(x) + 1 = 0,
\end{aligned} \tag{B3}$$

where σ_x, μ_x, κ, r, Σ, and ν^* are equilibrium processes in the economy with one unconstrained large investor, one constrained small investor, and two Lucas trees. These processes can be obtained as limits of equilibrium processes (16)-(20) in the original economy as $y \to 0$.

The boundary conditions for equation (B3) can be obtained by passing to limits $x \to 0$ and $x \to 1$ in equation (B3) and observing that the coefficients in front of $\tilde{\Phi}_B''(x)$ and $\tilde{\Phi}_B'(x)$ become zero. These conditions are given by $\tilde{\Phi}_B(0) = \Phi_B(0,0)$ and $\tilde{\Phi}_B(1) = \Phi_B(1,0)$, respectively, where:

$$
\begin{aligned}
\Phi_B(0,0) &= \frac{\gamma_B}{\rho - \dfrac{1-\gamma_B}{2\gamma_B}\kappa_{A2} - (1-\gamma_B)r_{A2} + \dfrac{(1-\gamma_B)\gamma_B}{2}\dfrac{\min(0; 1 - m_2\gamma_A/\gamma_B)^2}{m_1^2/\sigma_{D_1}^2 + m_2^2/\sigma_{D_2}^2}},\\
\Phi_B(1,0) &= \frac{\gamma_B}{\rho - \dfrac{1-\gamma_B}{2\gamma_B}\kappa_{A1} - (1-\gamma_B)r_{A1} + \dfrac{(1-\gamma_B)\gamma_B}{2}\dfrac{\min(0; 1 - m_1\gamma_A/\gamma_B)^2}{m_1^2/\sigma_{D_1}^2 + m_2^2/\sigma_{D_2}^2}},
\end{aligned}
\tag{B4}
$$

where $\kappa_{ij} = \gamma_i\sigma_{D_j}$ and $r_{ij} = \rho + \gamma_i\mu_{D_j} - \gamma_i(1+\gamma_i)\sigma_{D_j}^2/2$ are the equilibrium market price of risk and interest rate in the economy dominated by investor i and tree j. Then, the equation (B3) can be solved using explicit or implicit-explicit finite difference methods discussed in Section B.1. The boundary conditions at corners $(0,1)$ and $(1,1)$ can be obtained similarly, and are given by:

$$
\Phi_B(0,0) = \frac{\gamma_B}{\rho - \dfrac{1-\gamma_B}{2\gamma_B}\kappa_{B2} - (1-\gamma_B)r_{B2}}, \quad \Phi_B(1,0) = \frac{\gamma_B}{\rho - \dfrac{1-\gamma_B}{2\gamma_B}\kappa_{B1} - (1-\gamma_B)r_{B1}}. \tag{B5}
$$

The boundary conditions for ratios $\Psi_j(x,y)$ at corners $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ are given by:

$$
\begin{aligned}
\Psi_j(0,0) &= \frac{1}{r_{A2} + \gamma_A\sigma_{D_2}^2 1_{\{j=2\}} - \mu_{D_j}}, \quad \Psi_j(1,0) = \frac{1}{r_{A1} + \gamma_A\sigma_{D_1}^2 1_{\{j=1\}} - \mu_{D_j}},\\
\Psi_j(0,1) &= \frac{1}{r_{B2} + \gamma_B\sigma_{D_2}^2 1_{\{j=2\}} - \mu_{D_j}}, \quad \Psi_j(1,1) = \frac{1}{r_{B1} + \gamma_B\sigma_{D_1}^2 1_{\{j=1\}} - \mu_{D_j}}.
\end{aligned}
\tag{B6}
$$

Similarly, all other boundary conditions can be found. Boundary conditions (B6) are simple extensions of Gordon's growth formula for price-dividend ratios in economies with one large tree. Finally, we note that the finite difference equations for the boundary condition ODEs can be added to the finite difference equations of the corresponding PDE, and then all these equations can be solved simultaneously. Since these ODEs are special cases of PDEs, their discretizations can be imbedded into finite-difference equations for PDEs in a straightforward way.

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