Examining Macroeconomic Models through the Lens of Asset Pricing*

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Abstract

We develop new methods for representing the asset-pricing implications of stochastic general equilibrium models. We provide asset-pricing counterparts to impulse response functions and the resulting *dynamic value decompositions (DVDs)*. These methods quantify the exposures of macroeconomic cash flows to shocks over alternative investment horizons and the corresponding prices or investors' compensations. We extend the continuous-time methods developed in Hansen and Scheinkman (2012) and Borovička et al. (2011) by constructing discrete-time, state-dependent, shock-exposure and shock-price elasticities as functions of the investment horizon. Our methods are applicable to economic models that are nonlinear, including models with stochastic volatility.

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1 Introduction

It is standard practice to represent implications of dynamic macroeconomic models by showing how featured time series respond to shocks. Alternative current period shocks influence the future trajectory of macroeconomic processes such as consumption, investment or output, and these impacts are measured by impulse response functions. From an asset pricing perspective, these functions reflect the *exposures* of the underlying macroeconomic processes to shocks. These exposures depend on how much time has elapsed between the time the shock is realized and time of its impact on the macroeconomic time series under investigation. Changing this gap of time gives a trajectory of exposure elasticities that we measure. In this manner we build *shock-exposure* elasticities that are very similar to and in some cases coincide with impulse response functions.

In a fully specified dynamic stochastic general equilibrium model, exposures to macroeconomic shocks are priced because investors must be compensated for bearing this risk. To capture this compensation, we produce pricing counterparts to impulse response functions by representing and computing *shock-price* elasticities implied by the structural model. These prices are the risk compensations associated with the shock exposures. The shock-exposure and shock-price elasticities provide us with dynamic value decompositions (DVDs) to be used in analyzing alternative structural models that have valuation implications. Quantity dynamics reflect the impact of *current* shocks on future distributions of a macroeconomic process, while pricing dynamics reflect the current period compensation for the exposure to *future* shocks.

In our framework the shock-exposure and shock-price elasticities have a common underlying mathematical structure. We build processes that grow or decay stochastically in a geometric fashion. They capture the compounding of the discount and/or growth rates over time. We construct the shock elasticities that measure the intertemporal responses to changing exposures of these processes to alternative shocks. We interpret the objects of interest as 'elasticities' because they reflect the sensitivity of the logarithm of expected returns or expected cash flows to a change in the exposure to a shock normalized to have a unit standard deviation. The shock elasticities are state-dependent and reflect the nonlinearities of the dynamic model. We provide an abstract construction of the elasticities and ways to compute them in practice, including tractable frameworks suitable for applications in dynamic, stochastic general equilibrium (DSGE) modeling.

While these elasticities have not been explored in the quantitative literature in macroeconomics, they have antecedents in the asset pricing literature. The intertemporal structure of risk premia has been featured in the term structure of interest rates, but this literature purposefully abstracts from the pricing of stochastic growth components in the macroeconomy. Recently Lettau and Wachter (2007) and Hansen et al. (2008) have explored the term structure of risk premia explicitly in the context of equity claims that grow over time. Risk premia reflect contributions from exposures and prices of those exposures. Here we build on an analytical framework developed in Alvarez and Jermann (2005), Hansen and Scheinkman (2009), Hansen and Scheinkman (2012) and Borovička et al. (2011) to distinguish exposure elasticities and price elasticities.

The shock elasticities are also conceptually close to nonlinear versions of impulse response functions, introduced in Gallant et al. (1993), Koop et al. (1996) or Gourieroux and Jasiak (2005). In a loglinear framework, the shock elasticities exactly correspond to impulse response functions familiar from VAR analysis applied to the logarithms of stochastic growth or discount factor processes. In nonlinear models, our elasticities trace out changes in conditional expectations of future quantities in response to a *marginal* change in shock exposures. We design our approach to give a direct link to familiar characterizations of risk prices extended to multiple payoff horizons. We provide a way to operationalize the continuous-time formulations in Hansen and Scheinkman (2012), Borovička et al. (2011) and Hansen (2012) in a discrete-time setting.

In Section 2, we develop the concept of shock elasticities in a general framework. The shock elasticities arise naturally in decompositions of risk premia into the contribution of shocks at different horizon. In Section 3, we show that similar decompositions can be employed in deconstructing entropy measures of Backus et al. (2011) used to analyze the dynamics of the stochastic discount factor. An important goal of this paper is a tractable implementation of DVDs. We therefore devote Sections 4 and 5 to the discussion of methods that solve for approximate dynamics in a broad class of DSGE models. We pay particular attention to the approximation of recursive preferences of Kreps and Porteus (1978) and Epstein and Zin (1989) since these preferences play a prominent role in the asset pricing literature. We show that a second-order perturbation approximation of the DSGE models derived using the series expansion methods can be nested within an exponential-quadratic framework in which the shock elasticities are available in quasi-analytical form. We introduce this framework in Section 6 and discuss details of the solution in the Appendix. We also provide Matlab codes for the computation of the shock elasticities in models solved by Dynare.

Finally, in Section 7, we illustrate the developed tools in measuring shock exposures and model-implied prices of exposure to those shocks in a model with physical and intangible capital constructed by Ai et al. (2012). A reader immediately interested in the applicability of the introduced methods can read this section directly after, or in parallel to, Section 2.

2 Analytical framework

In this section we describe some basic tools for valuation accounting, by which we provide measures of shock exposures and shock prices for alternative investment horizons. In our framework the shock-exposure and shock-price elasticities have a common underlying mathematical structure. Let M be a process that grows or decays stochastically in a geometric fashion. It captures the compounding discount and/or growth rates over time in a stochastic fashion and is constructed from an underlying Markov process X. Let W be a sequence of independent and identically distributed standard normal random vectors. The common ingredient in our analysis is the ratio:

$$\varepsilon_m(x,t) = \alpha_h(x) \cdot \frac{E\left[M_t W_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]}. \tag{1}$$

where x is the current Markov state and α_h selects the linear combination of the shock vector W_1 of interest. The state dependence in α_h allows for analysis of stochastic volatility. We interpret this entity as a "shock elasticity" used to quantify the date t impact on values of exposure to the shock $\alpha_h(x)W_1$ at date one.

We add more structure to this formulation, by considering dynamic systems of the form

$$X_{t+1} = \psi(X_t, W_{t+1}) \tag{2}$$

where W is a sequence of independent shocks distributed as a multivariate standard normal. In much of what follows we will focus on stationary solutions for this system. By imposing appropriate balanced growth restrictions, we suppose that the logarithms of many macroeconomic processes that interest us grow or decay over time and can be represented as:

$$Y_t = Y_0 + \sum_{s=0}^{t-1} \kappa(X_s, W_{s+1}) \tag{3}$$

where Y_0 is an initial condition, which we will set conveniently to zero in much of our discussion. A typical example of the increment to this process is

$$\kappa(X_s, W_{s+1}) = \beta(X_s) + \alpha(X_s) \cdot W_{s+1}$$

where the function β allows for nonlinearity in the conditional mean and the function α introduces stochastic volatility. We call such a process Y an *additive functional* since it accumulates additively over time, and can be built from the underlying Markov process X provided that W_{t+1} can be inferred from X_{t+1} and X_t. By a suitable construction of the state vector, this restriction can always be met. The state vector X thus determines the dynamics of the increments in Y. When X is stationary Y has stationary increments.

While the additive specification of Y is convenient for modeling logarithms of economic processes, to represent values of uncertain cash flows it is necessary to study levels instead of logarithms. We therefore use the exponential of an additive functional, $M = \exp(Y)$, to capture growth or decay in levels. We will refer to M as a *multiplicative functional* represented by κ or sometimes the more restrictive specification (α, β).

In what follows we will consider two types of multiplicative functionals, one that captures macroeconomic growth, denoted by G, and another that captures stochastic discounting, denoted by S. The stochastic nature of discounting is needed to adjust consumption processes or cash flows for risk. Thus S, and sometimes G as well, are computed from the underlying economic model to reflect equilibrium price dynamics. For instance, G might be a consumption process or some other endogenously determined cash flow, or it might be an exogenously specified technology shock process that grows through time. The interplay between S and G will dictate valuation over multi-period investment horizons.

Our aim is to use a structural stochastic equilibrium model with identified macroeconomic shocks to *deconstruct* the asset-pricing implications. Such a model will imply a stochastic discount factor process S and benchmark stochastic growth processes. While for empirical purposes the pricing implications are conveniently captured by the stochastic discount factor process, with DVD methods we use the identified macroeconomics shocks as vehicles for interpreting the resulting pricing implications. These methods measure two things: i) how exposed are future macroeconomic processes to next-period shocks, and ii) what are the implied prices for these shock exposures. Measurements i) are very closely related to familiar impulse response functions. Our use of measurements ii) reflects a more substantive departure from common practice in the macroeconomics literature. We view these latter measurements as the pricing counterparts to impulse response functions.

2.1 One-period asset pricing

It is common practice in the asset pricing literature to represent prices of risk in terms of expected return on an investment per unit of exposure to risk. For instance, the familiar Sharpe ratio measures the difference between the expected return on a risky and a risk-free cash flow scaled by the volatility of the risky cash flow. We are interested in using this approach to assign prices to shock exposures.

As a warm up for subsequent analysis, consider one-period asset pricing for conditionally normal models. Suppose that

$$\log G_1 = \beta_g(X_0) + \alpha_g(X_0) \cdot W_1$$
$$\log S_1 = \beta_s(X_0) + \alpha_s(X_0) \cdot W_1$$

where G_1 is the payoff to which we assign values and S_1 is the one-period stochastic discount factor used to compute these values. The one-period return on this investment is:

$$R_1 = \frac{G_1}{E\left[S_1 G_1 \mid X_0\right]}$$

Applying standard formulas for lognormally distributed random variables, the logarithm of the expected return is:

$$\log E\left[G_1 \mid X_0 = x\right] - \log E\left[S_1 G_1 \mid X_0 = x\right] = \underbrace{-\beta_s(x) - \frac{|\alpha_s(x)|^2}{2}}_{\text{risk-free rate}} \underbrace{- \alpha_s(x) \cdot \alpha_g(x)}_{\text{risk premium}} .$$

Imagine applying this to a family of such payoffs parameterized in part by α_g. The vector α_g defines a vector of exposures to the components of the normally distributed shock W_1. Then $-\alpha_s$ is the vector of shock "prices" representing the compensation for exposure to the shocks. This compensation is expressed in terms of expected returns as is typical in asset pricing.

While this calculation is straightforward, we now explore a related derivation that will extend directly to multiple horizons. We parameterize a family of payoffs using:

$$\log H_1(\mathsf{r}) = \mathsf{r}\alpha_h(X_0) \cdot W_1 - \frac{\mathsf{r}^2}{2}|\alpha_h(X_0)|^2 \tag{4}$$

where r is an auxiliary scalar parameter and impose

$$E[|\alpha_h(X_0)|^2] = 1$$

as a normalization. In what follows we use the vector α_h as an exposure direction to compute a directional derivative as $\mathsf{r} \to 0$. We have built $H_1(\mathsf{r})$ so that it has conditional expectation equal to one, but other constructions are also possible. We allow α_h to depend on the state vector X to provide flexibility in the scaling of the perturbation. The state dependence allows α_h to capture fluctuations in shock exposures induced by stochastic volatility.

Form a parameterized family of payoffs $G_1 H_1(\mathsf{r})$ where by design:

$$\log G_1 + \log H_1(\mathsf{r}) = [\alpha_g(X_0) + \mathsf{r}\alpha_h(X_0)] \cdot W_1 + \beta_g(X_0) - \frac{\mathsf{r}^2}{2}|\alpha_h(X_0)|^2. \tag{5}$$

By changing r we alter the exposure in direction α_h. These payoffs imply a corresponding parameterized family of logarithms of expected returns:

$$\log E[G_1 H_1(\mathsf{r}) \mid X_0 = x] - \log E[S_1 G_1 H_1(\mathsf{r}) \mid X_0 = x].$$

Since we are using the logarithms of the expected returns measure and our exposure direction $\alpha_h(X_0) \cdot W_1$ has a unit standard deviation, by differentiating with respect to r we compute an elasticity:

$$\frac{d}{d\mathsf{r}} \log E[G_1 H_1(\mathsf{r}) \mid X_0 = x]\bigg|_{\mathsf{r}=0} - \frac{d}{d\mathsf{r}} \log E[S_1 G_1 H_1(\mathsf{r}) \mid X_0 = x]\bigg|_{\mathsf{r}=0}.$$

This calculation leads us to define counterparts to quantity and price elasticities from microeconomics:

1. *shock-exposure elasticity*:

$$\varepsilon_g(x,1) = \frac{d}{d\mathsf{r}} \log E[G_1 H_1(\mathsf{r}) \mid X_0 = x]\bigg|_{\mathsf{r}=0} = \alpha_g(x) \cdot \alpha_h(x)$$

2. *shock-price elasticity*:

$$\begin{aligned}\varepsilon_p(x,1) &= \frac{d}{d\mathsf{r}} \log E[G_1 H_1(\mathsf{r}) \mid X_0 = x]\bigg|_{\mathsf{r}=0} - \frac{d}{d\mathsf{r}} \log E[S_1 G_1 H_1(\mathsf{r}) \mid X_0 = x]\bigg|_{\mathsf{r}=0} \\ &= -\alpha_s(x) \cdot \alpha_h(x).\end{aligned}$$

For this conditional log-normal specification, α_g measures the exposure vector, $-\alpha_s$ measures the price vector and α_h captures which combination of shocks is being targeted. The shock price elasticity "conditional covariance" between $-\log S_1$ and $\alpha_h \cdot W_1$. Notice that our *elasticities* measure the sensitivity of the logarithm of the expected return or expected cash flow to a perturbation $\alpha_h \cdot W_1$ to $\log G$ that has a unit standard deviation.

Since exposure to risk requires compensation, notice that a "value elasticity" is the difference between an exposure elasticity and a price elasticity:

$$\frac{d}{d\mathsf{r}} \log E[S_1 G_1 H_1(\mathsf{r}) \mid X_0 = x]\bigg|_{\mathsf{r}=0} = \varepsilon_g(x,1) - \varepsilon_p(x,1).$$

The value of an asset responds to changes in exposure of the associated cash flow to a shock (a quantity effect), and to changes in the compensation resulting from the change in exposure (a price effect). The shock elasticity of the asset value is then obtained by taking into account both effects operating in opposite directions. Specifically, the shock price elasticity enters with a negative sign because exposure to risk requires compensation reflected in a decline in the asset value.

Our formulas for the shock elasticities exploit conditional log-normality of the payoffs to be priced and of the stochastic discount factor. In this formulation we are using the possibility of conditioning variables to fatten tails of distributions as in models with stochastic volatility. This conditioning is captured by the Markov state x in our elasticity formulas. We use one as the second argument for the elasticities to denote that we are pricing a one-period payoff. We extend the analysis to multi-period cash flows in the next subsection. While the one-period price elasticity does not depend on our specification of α_g, the dependence on α_g emerges when we consider longer investment horizons.

2.2 Multiple-period investment horizons

Next we analyze how our analysis extends to longer investment horizons. Consider the parameterized payoff $G_t H_1(\mathsf{r})$ with a date-zero price $E\left[S_t G_t H_1(\mathsf{r}) \mid X_0 = x\right]$. Notice that we are changing the exposure at date one and looking at the consequences on a t-period investment. The logarithm of the expected return is:

$$\log E\left[G_t H_1(\mathsf{r}) \mid X_0 = x\right] - \log E\left[S_t G_t H_1(\mathsf{r}) \mid X_0 = x\right].$$

Following our previous analysis, we construct two elasticities:

1. *shock-exposure elasticity*:

$$\varepsilon_g(x,t) = \frac{d}{d\mathsf{r}} \log E[G_t H_1(\mathsf{r}) \mid X_0 = x]\Big|_{\mathsf{r}=0}$$

2. *shock-price elasticity*:

$$\varepsilon_p(x,t) = \frac{d}{d\mathsf{r}} \log E[G_t H_1(\mathsf{r}) \mid X_0 = x]\Big|_{\mathsf{r}=0} - \frac{d}{d\mathsf{r}} \log E[S_t G_t H_1(\mathsf{r}) \mid X_0 = x]\Big|_{\mathsf{r}=0}.$$

These two elasticities are functions of the investment horizon t, and thus we obtain a term structure of elasticities. The components of these elasticities have a common mathematical form. This is revealed by using a multiplicative functional M to represent either G or the product SG. Taking the derivative with respect to r yields equation (1). This formula provides a target for computation and interpretation. Consider the pricing of a vector of payoffs $G_t W_1$ in comparison to the scalar payoff G_t. The shock-exposure elasticity is constructed from the ratio of expected payoffs $E\left[G_t W_1 \mid X_0 = x\right]$ relative to $E\left[G_t \mid X_0 = x\right]$. The shock-price elasticity includes an adjustment for the values of the payoffs $E\left[S_t G_t W_1 \mid X_0 = x\right]$ relative to $E\left[S_t G_t \mid X_0 = x\right]$. Our interest in elasticities leads us to the use of ratios in these computations.

Notice that

$$\frac{E\left[M_t W_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]} = E\left[\frac{E\left[M_t \mid W_1, X_0\right]}{E\left[M_t \mid X_0\right]} W_1 \middle| X_0 = x \right].$$

Thus a major ingredient in the computation is the covariance between $\frac{E[M_t | W_1, X_0]}{E[M_t | X_0]}$ and the shock vector W_1, which shows how the shock elasticity measures the impact of the shock W_1 on the conditional conditional expectation of M_t.

Prior to our more general discussion, consider the case in which M is lognormal,

$$E\left[\log M_t \mid W_1, X_0\right] - E\left[\log M_t \mid X_0\right] = \phi_t \cdot W_1$$

where ϕ_t is the (state-independent) vector of "impulse responses" or moving-average coefficients of M for horizon t. Then

$$\frac{E\left[M_t \mid W_1, X_0\right]}{E\left[M_t \mid X_0\right]} = \exp\left(\phi_t \cdot W_1 - \frac{1}{2} |\phi_t|^2 \right), \tag{6}$$

and its covariance with W_1 is:

$$\frac{E\left[M_t W_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]} = \phi_t.$$

Thus when M is constructed as a lognormal process and α_h is state-independent, our elasticities coincide with the impulse response functions typically computed in empirical macroeconomics.[1] The shock-exposure elasticities are the responses for $\log G$ and the shock-price elasticities are the impulse response functions for $-\log S$.

Our interest is in calculating elasticities for nonlinear models and in particular for models with stochastic volatility in which α_g and possibly α_h are state-dependent. One possibility is to let α_h be a coordinate vector. More generally, α_h is allowed to be state-dependent and

[1] Our dating is shifted by one period *vis-à-vis* an impulse response function. In macroeconomic modeling what we denote as ϕ_t is the vector of responses of $\log M_{t-1}$ to the components of the shock vector W_0. The responses are indexed by the gap of time $t-1$ between the shock date and the outcome date.

thus may change its magnitude over time, subject to the unconditional scaling constraint $E[|\alpha_h(X_0)|^2] = 1$. A suitable choice of the functional form of $\alpha_h(x)$ is typically driven by the specification of the cash-flow and stochastic discount factor dynamics. In models with stochastic volatility, it is often advantageous to mimic the stochastic-volatility exposure of the cash-flow process.

The construction of shock elasticities is based on the comparison of the conditional expectation of M under the perturbed and unperturbed dynamics, which resembles the analysis of nonlinear impulse response functions in Gallant et al. (1993), Koop et al. (1996) or Gourieroux and Jasiak (2005). We choose a construction that is particularly appealing in the structural macroeconomics and asset pricing literatures where *logarithms* of quantities are frequently modeled as additive but where the conditional expectations of the *levels* of quantities are relevant for the computation of asset values. This is why we explore additive perturbations to $\log M$ but compute conditional expectations of M.[2]

As in the literature on nonlinear impulse response functions, our shock elasticities take into account the full nonlinear dynamics of the model between the time of the shock and the maturity of the cash flow. We differ, however, in the specification of the initial shock impulse. Rather than specifying a discrete impulse, which would require us to take a stand on the magnitude of the shock, we compute the sensitivity to a marginal perturbation, represented by the derivative in the formula for the shock elasticity.

2.3 Alternative representation

To contrast transitory and long-term implications of structural shocks for the exposure and price dynamics, we isolate growth rate and martingale components of multiplicative functionals. Hansen and Scheinkman (2009) justify the following factorization of the multiplicative functional:

$$M_t = \exp(\eta t)\hat{M}_t \frac{e(X_0)}{e(X_t)} \tag{7}$$

where $\hat{M}$ is multiplicative martingale and η is the growth or decay rate. Associated with the martingale is a change of probability measure given by

$$\hat{E}\left[Z \mid X_0\right] = E\left[\hat{M}_t Z \mid X_0\right]$$

for a random variable Z that is a (Borel measureable) function of the Markov process between dates zero and t. This change of measure preserves the Markov structure for X although it

[2]We gain both intuition and tractability because we narrow down our analysis to frameworks with Gaussian shocks although the approach can be extended to allow for other shock distributions. See Borovička et al. (2011) for one such extension.

changes the transition probabilities. To study long-horizon limits, we consider only measure changes that preserve stochastic stability in the sense that

$$\lim_{t \to \infty} \hat{E}\left[f(X_t) \mid X_0 = x\right] \to \int f(x) d\hat{Q}(x)$$

where $\hat{Q}$ is a stationary distribution under the change of measure.[3]

Using factorization (7),

$$\frac{E\left[M_t W_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]} = \frac{\hat{E}\left[\hat{e}(X_t) W_1 \mid X_0 = x\right]}{\hat{E}\left[\hat{e}(X_t) \mid X_0 = x\right]}$$

where $\hat{e} = \frac{1}{e}$. In the large t limit, the right-hand side converges to the conditional mean of W_1 under the altered distribution:

$$\hat{E}\left[W_1 \mid X_0 = x\right]. \tag{8}$$

The dependence of $\hat{e}(X_t)$ on W_1 governs the dependence of the shock elasticities on the investment horizon and eventually decays as $t \to \infty$.

2.4 Multi-period risk elasticities and a decomposition result

To build assets with differential exposures to risk over multiple investment horizons, consider a multi-period parameterization of an underlying cash flow $GH(\mathsf{r})$, constructed as a generalization of the family of payoffs from equation (4):

$$\log H_t(\mathsf{r}) = \sum_{s=0}^{t-1} \left[-\frac{1}{2} \mathsf{r}^2 \left| \alpha_h(X_s) \right|^2 + \mathsf{r} \alpha_h(X_s) \cdot W_{s+1} \right].$$

The perturbed cash flow $GH(\mathsf{r})$ is now more exposed to the shock vector W in the direction α_h at all times between the current period and the maturity date. We capture the sensitivity of the expected return to such a multi-period perturbation using the *risk-price elasticity* $\varrho_p(x,t)$

$$\varrho_p(x,t) = \frac{1}{t} \frac{d}{d\mathsf{r}} \log E\left[G_t H_t(\mathsf{r}) \mid X_0 = x\right] \Big|_{\mathsf{r}=0} - \frac{1}{t} \frac{d}{d\mathsf{r}} \log E\left[S_t G_t H_t(\mathsf{r}) \mid X_0 = x\right] \Big|_{\mathsf{r}=0}. \tag{9}$$

The risk-price elasticity measures the marginal increase in the expected return on a cash flow

[3] Notice that we did not specify the initial distribution for X_0 in our use of $\hat{M}$. The convergence is presumed to hold at least for almost all x under the $\hat{Q}$ distribution.

in response to a marginal increase in exposure of the cash flow functional in the direction α_h in every period. Scaling by t annualizes the elasticity.

The risk-price elasticity again consists of two terms, reflecting the contribution of the exposure of the expected cash flow, and the contribution of the valuation of this cash flow. Both terms have a common mathematical structure. Using a general multiplicative functional M that substitutes either for S or SG, the derivative in (9) can be expressed as

$$\varrho(x,t) = \frac{1}{t}\frac{d}{d\mathsf{r}} \log E\left[M_t H_t(\mathsf{r}) \mid X_0 = x\right]\Big|_{\mathsf{r}=0} = \frac{1}{t}\frac{E\left[M_t D_t \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]}$$

where D is an *additive* functional

$$D_t = \sum_{s=0}^{t-1} \alpha_h(X_s) \cdot W_{s+1}.$$

By interchanging summation and integration in the conditional expectation, and utilizing the martingale decomposition from Section 2.3, we write the risk elasticity as[4]

$$\varrho(x,t) = \frac{1}{t}\sum_{s=0}^{t-1} \frac{E\left[M_t \varepsilon(X_s, t-s) \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]} = \frac{1}{t}\sum_{s=0}^{t-1} \frac{\hat{E}\left[\hat{e}(X_t)\,\varepsilon(X_s, t-s) \mid X_0 = x\right]}{\hat{E}\left[\hat{e}(X_t) \mid X_0 = x\right]}.$$

This formula reveals how a risk elasticity is constructed by averaging across time the contributions of the shock elasticities in different periods. The contributions of future shocks are weighted by the term

$$\frac{\hat{e}(X_t)}{\hat{E}\left[\hat{e}(X_t) \mid X_0 = x\right]} \tag{10}$$

which represents the contribution of the nonlinear dynamics of the model arising from both the stationary component captured by $\hat{e}$, and by the martingale component incorporated in the change of probability measure $\hat{\cdot}$. The shock elasticities are essential inputs into this computation because of the recursive construction of valuation as reflected by the multiplicative functional M.

The resulting elasticity of a payoff maturing in period $t+\tau$ to a shock that occurs in period $\tau+1$ then is

$$\varepsilon(x,t;\tau) = \frac{\hat{E}\left[\hat{e}(X_{t+\tau})\,\varepsilon(X_\tau, t) \mid X_0 = x\right]}{\hat{E}\left[\hat{e}(X_{t+\tau}) \mid X_0 = x\right]}.$$

By construction, $\varepsilon(x,t;0) = \varepsilon(x,t)$.

The impact of $\hat{e}$ in the weighting (10) is transient in two particular senses. First, fix the

[4] While we are being casual about this interchange, Hansen and Scheinkman (2012) provide a rigorous analysis of such formulas.

time of the shock τ and extend the maturity of the cash flow by $t \to \infty$. Then the limiting elasticity generalizes result (8):

$$\varepsilon(x, \infty; \tau) = \hat{E}\left[\varepsilon(X_\tau, \infty) \mid X_0 = x\right] = \hat{E}\left[\alpha_h(X_\tau) \cdot W_{\tau+1} \mid X_0 = x\right]$$

The impact of proximate shocks on cash flows far in the future remains state-dependent but is only determined by the change in probability measure constructed from the contribution of permanent shocks.

Second, fix the distance between the time of the shock and the maturity date, t, but extend the date of the shock by $\tau \to \infty$. The resulting elasticity

$$\varepsilon(x, t; \infty) = \frac{\hat{E}\left[\hat{e}(X_t)\,\varepsilon(X_0, t)\right]}{\hat{E}\left[\hat{e}(X_t)\right]} = \frac{\hat{E}\left[\hat{e}(X_t)\alpha_h(X_0) \cdot W_1\right]}{\hat{E}\left[\hat{e}(X_0)\right]}$$

is independent of the current state, and depends on the transient term $\hat{e}$ only through its dynamics between the date of the shock and the maturity of the cash flow. Transient dynamics preceding the date of the shock become irrelevant.

2.5 Partial shock elasticities

In our application in Section 7, we explore how shock elasticities are altered when we change the shock configuration. We are interested in measuring the approximate impact of introducing new shocks. Among other things, this will allow us to quantify the contribution of different propagation channels of the dynamics (2)–(3) to the shock elasticity. In a dynamical system a given shock may operate through multiple channels as is the case in the example economy we investigate. To feature a specific channel, we introduce a new shock and study the sensitivity of the elasticities. Because of the potential nonlinear nature of the model, we *do not* calculate this sensitivity by zeroing out the existing shocks. Instead we perturb the system by exposing it to new hypothetical shocks.

We motivate and compute the following object:

$$\widetilde{\varepsilon}_m(x, t) = \widetilde{\alpha}_h(x) \cdot \frac{d}{d\mathsf{q}} \left. \frac{E\left[M_t(\mathsf{q})\widetilde{W}_1 \mid X_0 = x\right]}{E\left[M_t(\mathsf{q}) \mid X_0 = x\right]} \right|_{\mathsf{q}=0} . \tag{11}$$

where $\widetilde{W}_1$ is a new shock vector. We use the auxiliary parameter q as a way to parameterize equilibrium outcomes when the economic model includes a marginal perturbation to this new shock vector $\widetilde{W}_1$. The vector $\widetilde{\alpha}_h(x)$ determines which combination of $\widetilde{W}_1$ is the target of the computation. We refer to this entity as a *partial shock elasticity*.

Formally, we consider the perturbed model:

$$X_{t+1}(\mathsf{q}) \quad = \quad \widetilde{\psi}\left(X_t(\mathsf{q}), W_{t+1}, \mathsf{q}\widetilde{W}_{t+1}, \mathsf{q}\right) \quad \text{for } t \geq 0$$

where we assume that the shock vector $\widetilde{W}$ is independent of W and X_0. Changing the real number q alters the stochastic dynamics for the Markov process $X(\mathsf{q})$, and formula (11) reveals that we are interested in the impact of small perturbations as $\mathsf{q} \to 0$. We nest our original construction by imposing that

$$\psi(x, w) = \widetilde{\psi}(x, w, 0, 0).$$

Similarly, we let

$$Y_{t+1}(\mathsf{q}) - Y_t(\mathsf{q}) = \widetilde{\kappa}\left(X_t(\mathsf{q}), W_{t+1}, \mathsf{q}\widetilde{W}_{t+1}, \mathsf{q}\right) \quad \text{for } t \geq 0,$$

where

$$\kappa(x, w) = \widetilde{\kappa}(x, w, 0, 0).$$

We consider the multiplicative functional $M(\mathsf{q}) = \exp[Y(\mathsf{q})]$, which depends implicitly on q. The functions $\widetilde{\psi}$ and $\widetilde{\kappa}$ are assumed to be smooth in what follows in order that we may compute derivatives needed to characterize sensitivity.

We measure the sensitivity to the new shock $\widetilde{W}$ to characterize a specific transmission mechanism within the model. In the example in Section 7, one shock influences the dynamics of the model through two channels, as a shock to the production of final output and a shock to the production of new capital. We utilize the partial shock elasticity to reveal crucial differences in the role of the two channels.

As in our construction of shock elasticities, we specify a parameterized perturbation $\widetilde{H}_1(\mathsf{r})$ analogous to (4):

$$\log \widetilde{H}_1(\mathsf{r}) = \mathsf{r}\widetilde{\alpha}_h(X_0) \cdot \widetilde{W}_1 - \frac{\mathsf{r}^2}{2}|\widetilde{\alpha}_h(X_0)|^2.$$

We restrict $\widetilde{\alpha}_h$ so that

$$E|\widetilde{\alpha}_h(X_t)|^2 = 1$$

analogous to our previous elasticity computation. Since $\widetilde{W}_1$ is independent of X_0 and W, the shock elasticity for $\widetilde{W}_1$ is degenerate:

$$\lim_{\mathsf{q}\to 0} \widetilde{\alpha}_h(x) \cdot \frac{E\left[M_t(\mathsf{q})\widetilde{W}_1 \mid X_0 = x\right]}{E\left[M_t(\mathsf{q}) \mid X_0 = x\right]} = \widetilde{\alpha}_h(x) \cdot \frac{E\left[M_t\widetilde{W}_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]} = 0.$$

where M is $M(\mathsf{q})$ evaluated at $\mathsf{q} = 0$. In what follows we compute a partial elasticity by differentiating with respect to q:

$$\widetilde{\varepsilon}_m(x,t) = \frac{d}{d\mathsf{q}}\, \widetilde{\alpha}_h(x) \cdot \left. \frac{E\left[M_t(\mathsf{q})\widetilde{W}_1 \mid X_0 = x\right]}{E\left[M_t(\mathsf{q}) \mid X_0 = x\right]} \right|_{\mathsf{q}=0}.$$

We use this derivative to quantify the impact of the shock elasticity when we introduce a new shock into the dynamical system. When there are multiple components to $\widetilde{W}_1$, we will be able to conduct relative comparisons of their importance by evaluating the derivative vector:

$$\frac{d}{d\mathsf{q}} \left. \frac{E\left[M_t(\mathsf{q})\widetilde{W}_1 \mid X_0 = x\right]}{E\left[M_t(\mathsf{q}) \mid X_0 = x\right]} \right|_{\mathsf{q}=0}.$$

2.5.1 Construction

Let $X_{1,\cdot}$ and $Y_{1,\cdot}$ denote the "first derivative processes" obtained by differentiating the functions $\widetilde{\psi}$ and $\widetilde{\kappa}$ and evaluated at $\mathsf{q} = 0$. These processes are represented using the recursion

$$\begin{aligned} X_{1,t+1} &= \widetilde{\psi}_x(X_t, W_{t+1}, 0, 0)X_{1,t} + \widetilde{\psi}_{\widetilde{w}}(X_t, W_{t+1}, 0, 0)\widetilde{W}_{t+1} + \widetilde{\psi}_q(X_t, W_{t+1}, 0, 0) \\ Y_{1,t+1} - Y_{1,t} &= \widetilde{\kappa}_x(X_t, W_{t+1}, 0)X_{1,t} + \widetilde{\kappa}_{\widetilde{w}}(X_t, W_{t+1}, 0, 0)\widetilde{W}_{t+1} + \widetilde{\kappa}_q(X_t, W_{t+1}, 0, 0) \end{aligned} \tag{12}$$

To implement these recursions, we include $X_{1,t}$ as an additional state vector but we have initialized it to be zero at date zero. The process X used in this recursion is the one associated with the original ($\mathsf{q} = 0$) dynamics.

By imitating our previous analysis, we compute:

$$\begin{aligned} \widetilde{\varepsilon}_m(x,t) &= \widetilde{\alpha}_h(x) \cdot \frac{E\left[M_t Y_{1,t}\widetilde{W}_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]} \\ &\quad - \widetilde{\alpha}_h(x) \cdot \left(\frac{E\left[M_t Y_{1,t} \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]}\right) \left(\frac{E\left[M_t \widetilde{W}_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]}\right) \end{aligned}$$

where M is evaluated at $\mathsf{q} = 0$. Since $\widetilde{W}_1$ is independent of X_0 and W, the second term on the right-hand side is zero but the first term is not. Thus formula (11) for the partial elasticity is valid.

We compute this expectation in two steps. Since $\widetilde{W}_1$ is independent of X and W and future $\widetilde{W}_t$'s, in the first step we compute expectations $\widetilde{X}_{1,t} = E\left[X_{1,t}(\widetilde{W}_1)' \mid \mathcal{F}_t\right]$ and $\widetilde{Y}_{1,t} =$

$E\left[Y_{1,t}(\widetilde{W}_1)' \mid \mathcal{F}_t\right]$ recursively using

$$\begin{aligned}
\widetilde{X}_{1,t+1} &= \widetilde{\psi}_x(X_t, W_{t+1}, 0, 0)\widetilde{X}_{1,t} \\
\widetilde{Y}_{1,t+1} - \widetilde{Y}_{1,t} &= \widetilde{\kappa}_x(X_t, W_{t+1}, 0, 0)\widetilde{X}_{1,t}
\end{aligned}$$

for $t \geq 1$ and with initial conditions:

$$\begin{aligned}
\widetilde{X}_{1,1} = \widetilde{\psi}_{\widetilde{w}}(x, W_1, 0, 0)E\left[\widetilde{W}_1(\widetilde{W}_1)' \mid \mathcal{F}_1\right] = \widetilde{\psi}_{\widetilde{w}}(x, W_1, 0, 0) \\
\widetilde{Y}_{1,1} = \widetilde{\kappa}_{\widetilde{w}}(x, W_1, 0, 0)E\left[\widetilde{W}_1(\widetilde{W}_1)' \mid \mathcal{F}_1\right] = \widetilde{\kappa}_{\widetilde{w}}(x, W_1, 0, 0).
\end{aligned} \tag{13}$$

For the recursions in (12), notice that

$$\begin{aligned}
\widetilde{\psi}_x(X_t, W_{t+1}, 0, 0) = \psi_x(X_t, W_{t+1}) \\
\widetilde{\kappa}_x(X_t, W_{t+1}, 0, 0) = \kappa_x(X_t, W_{t+1}).
\end{aligned}$$

With this construction, we may view $\widetilde{Y}_{1,t}$ as the approximate vector of "impulse responses" of Y_t to unit "impulses" of the components of $\widetilde{W}_1$. For a nonlinear model, the date t response will be a random variable. In the second step we use $\widetilde{Y}_{1,t}$ to represent the partial elasticity:

$$\widetilde{\varepsilon}_m(x, t) = \widetilde{\alpha}_h(x) \cdot \frac{E\left[M_t\left(\widetilde{Y}_{1,t}\right)' \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]}.$$

2.5.2 An interesting special case

The following special case will be of interest in our application. Suppose that we construct the perturbed model so that

$$\widetilde{\psi}_{\widetilde{w}}(x, w, 0, 0)\Upsilon = \psi_w(x, w), \tag{14}$$

and similarly,

$$\widetilde{\kappa}_{\widetilde{w}}(x, w, 0, 0)\Upsilon = \kappa_w(x, w) \tag{15}$$

for some matrix Υ with the same number of rows as in the shock vector $\widetilde{W}_{t+1}$ and the same number of columns as in the vector W_{t+1}. In this construction, Υ has at least as many rows as columns and $\Upsilon'\Upsilon = I$.

Given a random vector $\alpha_h(x)$ used to model state dependence in the exposure to W_{t+1}, form:

$$\widetilde{\alpha}_h(x) = \Upsilon\alpha_h(x)$$

In light of equalities (14) and (15), and our initialization in (13),

$$\widetilde{\varepsilon}_m(x,t) = \widetilde{\alpha}_h(x) \cdot \frac{E\left[M_t\left(\widetilde{Y}_{1,t}\right)' \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]} \approx \alpha_h(x) \cdot \frac{E\left[M_t W_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]}, \tag{16}$$

where the right-hand side is a shock elasticity and the left-hand side is a partial shock elasticity. The approximation becomes arbitrarily good in a continuous-time limit. See Borovička et al. (2011) for a continuous-time characterization of the right-hand side of this equation. In Appendix B.3, we analyze the discrete-time approximation (16) in more detail and provide an alternative way to characterize this approximation.

In our application in Section 7, $\widetilde{W}$ has twice as many entries as W. We construct the model perturbed by $\widetilde{W}$ in order to explore implications of alternative transmission mechanisms when individual shocks have multiple impacts on the dynamic economic system. When a component of W_{t+1} influences the economic system through two channels, we design the perturbed system in which two distinct components of $\widetilde{W}_{t+1}$ are independent inputs into each of the channels. In this manner the partial elasticities in conjunction with formula (16) allow us to unbundle the impacts of the original set of shocks.

3 Entropy decomposition

Our shock-price elasticities target particular shocks. It is also of interest to have measures of the overall magnitude across shocks. In the construction that follows we build on ideas from Bansal and Lehmann (1997), Alvarez and Jermann (2005), and especially Backus et al. (2011). The relative entropy of a multiplicative functional M for horizon t is given by:

$$\frac{1}{t}\left[\log E\left(M_t | X_0 = x\right) - E\left(\log M_t | X_0 = x\right)\right],$$

which is nonnegative as an implication of Jensen's Inequality. When M_t is log-normal, this notion of entropy yields one-half the conditional variance of $\log M_t$ conditioned on date zero information, and Alvarez and Jermann (2005) propose using this measure as a "generalized notion of variation." Our primary task is to construct a decomposition that provides a more refined quantification of how entropy depends on the investment horizon t. While our approach in this section is similar to the construction of shock elasticities, the analysis of entropy is global in nature and does not require localizing the risk exposure. On the other hand, it necessarily bundles the pricing implications of alternative shocks.

For a multiplicative functional M, form:

$$\frac{E[M_t \mid W_1, X_0]}{E[M_t \mid X_0]} \tag{17}$$

which has conditional expectation one conditioned on X_0. By Jensen's inequality we know that the expected logarithm of this random variable conditioned on X_0 must be less than or equal to zero, which leads us to construct:

$$\zeta_m(x,t) = \log E\left[M_t \mid X_0 = x\right] - E\left[\log E\left(M_t \mid W_1, X_0\right) \mid X_0 = x\right] \geq 0$$

which is a measure of "entropy" of the random variable in (17). It measures the magnitude of new information that arrives between date zero and date one for the process M. This is the building block for a variety of computations. We think of these measures as the entropy counterparts to our shock elasticity measures considered previously. These measures do not feature specific shocks but they also do not require that we localize the exposures.

Consider the case in which M is lognormal. As we showed in (6),

$$\frac{E\left[M_t \mid W_1, X_0\right]}{E\left[M_t \mid X_0\right]} = \exp\left(\phi_t \cdot W_1 - \frac{1}{2}\left|\phi_t\right|^2\right),$$

where ϕ_t is the (state-independent) vector of "impulse responses" or moving-average coefficients of M for horizon t. Then

$$\zeta_m(x,t) = \frac{1}{2}\left|\phi_t\right|^2.$$

which is one-half the variance of the contribution of the random vector W_1 to $\log M_t$.

Returning to our more general analysis, a straightforward calculation justifies:

$$\lim_{t \to \infty} \zeta_m(x,t) = -E\left[\log \hat{M}_1 \mid X_0 = x\right]$$

where $\hat{M}$ is the martingale component of M in factorization (7) of the multiplicative functional.

To see why $\zeta_m(x,t)$ are valuable building blocks, we use the multiplicative Markov structure of M to obtain:

$$\frac{E\left[M_t \mid \mathcal{F}_{j+1}\right]}{E\left[M_t \mid \mathcal{F}_j\right]} = \frac{E\left[\frac{M_t}{M_j} \mid \mathcal{F}_{j+1}\right]}{E\left[\frac{M_t}{M_j} \mid \mathcal{F}_j\right]} = \frac{E\left[\frac{M_t}{M_j} \mid W_{j+1}, X_j\right]}{E\left[\frac{M_t}{M_j} \mid X_j\right]},$$

and thus

$$\log E\left[M_t \mid \mathcal{F}_j\right] - E\left[\log E\left(M_t \mid \mathcal{F}_{j+1}\right) \mid \mathcal{F}_j\right] = \zeta_m(X_j, t-j)$$

for $j = 0, 1, ..., t-1$. Taking expectations as of date zero,

$$E\left[\log E\left(M_t \mid \mathcal{F}_j\right) \mid \mathcal{F}_0\right] - E\left[\log E\left(M_t \mid \mathcal{F}_{j+1}\right) \mid \mathcal{F}_0\right] = E\left[\zeta_m(X_j, t-j) \mid X_0\right].$$

We now have the ingredients for representing entropy over longer investment horizons. Notice that

$$\frac{M_t}{E\left[M_t \mid \mathcal{F}_0\right]} = \prod_{j=1}^{t} \frac{E\left[M_t \mid \mathcal{F}_j\right]}{E\left[M_t \mid \mathcal{F}_{j-1}\right]}.$$

Taking logarithms and expectations conditioned on date zero information, the entropy over investment horizon-t is

$$\frac{1}{t}\left[\log E\left(M_t \mid X_0\right) - E\left(\log M_t \mid X_0\right)\right] = \frac{1}{t}\sum_{j=1}^{t} E\left[\zeta_m(X_{t-j}, j) \mid X_0\right]. \tag{18}$$

The left-hand side is a conditional version of the entropy measure for alternative prospective horizons t. The right-hand side represents the horizon t entropy in terms of averages of the building blocks $\zeta_m(x,t)$.

The structure of the entropy is similar to that of the risk elasticity function $\varrho(x,t)$ from Section 2.4. Both are constructed as averages over the investment horizon of the expected one-period contributions captured by our fundamental building blocks.

Recall the multiplicative martingale decomposition of M constructed in Section 2.3. Hansen (2012) compares this to an additive decomposition of $\log M$:

$$\log M_t = \rho t + \log \tilde{M}_t + g(X_0) - g(X_t)$$

where $\log \tilde{M}$ is an additive martingale. Backus et al. (2011) propose the average entropy over a t period investment horizon as a measure of horizon dependence. The large t limit of equation (18) then is

$$\lim_{t\to\infty} \frac{1}{t}\left[\log E\left(M_t \mid X_0\right) - E\left(\log M_t \mid X_0\right)\right] = \eta - \rho.$$

The asymptotic entropy measure is state-independent and is expressed as the difference of two asymptotic growth rates, one arising from the multiplicative martingale decomposion and the other from the additive martingale decompositions in logarithms.

We now suggest some applications of our entropy decomposition. First, to relate our calculations to the work of Backus et al. (2011), let $M = S$. Backus et al. (2011) study the left-hand side of (18) averaged over the initial state X_0. They view this entropy measure for different investment horizons as an attractive alternative to the volatility of stochastic discount factors featured by Hansen and Jagannathan (1991). To relate these entropy measures to asset pricing models and data, Backus et al. (2011) note that

$$-\frac{1}{t}E\left[\log E\left(S_t \mid X_0\right)\right]$$

is the average yield on a t-period discount bond where we use the stationary distribution for X_0. Following Bansal and Lehmann (1997),

$$-\frac{1}{t}E\left[\log S_t\right] = -E\left[\log S_1\right],$$

is the average one-period return on the maximal growth portfolio under the same distribution. The right-hand side of (18) extends this analysis by featuring the role of conditioning information captured by the state vector X_0 and the entropy-building blocks $\zeta(x,t)$. Notice that we may write

$$\zeta_s(x,t) = -E\left[\log S_1 \mid X_0 = x\right] + \log E\left[S_t \mid X_0 = x\right] - E\left[\log E\left(\frac{S_t}{S_1} \mid X_1\right) \mid X_0 = x\right]. \quad (19)$$

Observe that an input into the formula is

$$\log E\left(\frac{S_t}{S_1} \mid X_1\right) - \log E\left(S_t \mid X_0\right),$$

which is the logarithm of the one-period holding period return on a t period discount bond at date one. The expectation of this logarithm contributes to entropy building block $\zeta(x,t)$. By featuring S only, these calculations by design feature the term structure of interest rates but not the term structure of exposures of stochastic growth factors.

As an alternative application, following Rubinstein (1976), Lettau and Wachter (2007), Hansen et al. (2008), Hansen and Scheinkman (2009), and Hansen (2012) we consider the interaction between stochastic growth and stochastic discounting. For instance, as in Section 2.4 the logarithm of the risk premium for a t-period investment in a cash flow G_t is:

$$\begin{aligned}
\frac{1}{t}\log E\left[G_t \mid X_0 = x\right] - \frac{1}{t}\log E\left[S_t G_t \mid X_0 = x\right] + \frac{1}{t}\log E\left[S_t \mid X_0 = x\right] = \\
= \frac{1}{t}\left(\log E\left[G_t \mid X_0 = x\right] - E\left[\log G_t \mid X_0 = x\right]\right) \\
+ \frac{1}{t}\left(\log E\left[S_t \mid X_0 = x\right] - E\left[\log S_t \mid X_0 = x\right]\right) \\
- \frac{1}{t}\left(\log E\left[S_t G_t \mid X_0 = x\right] - E\left[\log S_t + \log G_t \mid X_0 = x\right]\right).
\end{aligned}$$

The formula relates the t-period risk premium on a stochastically growing cash flow on the left-hand side to the entropy measures for three multiplicative functionals on the right-hand side: G, S and SG.[5] Our decompositions can be applied to all three components to measure how important one-period ahead exposures are to t-period risk premia.

4 Perturbation methods

In the preceding sections, we developed formulas for shock-price and shock-exposure elasticities for a wide class of models driven by a state vector with Markov dynamics (2). While the general analysis is revealing, we now propose a tractable implementation. Our interest lies in providing tools for valuation analysis in structural macroeconomic models, and we now feature a special dynamic structure for which we obtain closed-form solutions for the shock elasticities.

We start by introducing a special class of approximate solutions to dynamic macroeconomic models constructed using perturbation methods. We show how to approximate the equilibrium dynamics, additive and multiplicative functionals, and the resulting shock elasticities. These approximations will share a common exponential-quadratic functional form which we discuss in detail in Section 6.

Consider a parameterized family of the dynamic systems specified in (2):

$$X_{t+1}(\mathsf{q}) = \psi(X_t(\mathsf{q}), \mathsf{q}W_{t+1}, \mathsf{q}) \tag{20}$$

where we let q parameterize the sensitivity of the system to shocks. The dynamics of $X(\mathsf{q})$ for $\mathsf{q} = 1$ coincide with the dynamics for X in the original model as introduced in equation (2). We consider a limit in which $\mathsf{q} = 0$ and first- and second-order approximations around this limit system. Specifically, following Holmes (1995) and Lombardo (2010), we form an approximating system by deducing the dynamic evolution for the pathwise derivatives with respect to q and evaluated at $\mathsf{q} = 0$. To build a link to the parameterization in Section 6, we feature a second-order expansion:

$$X_t \approx X_{0,t} + \mathsf{q}X_{1,t} + \frac{\mathsf{q}^2}{2}X_{2,t}$$

where $X_{m,t}$ is the m-th order, date t component of the stochastic process. We abstract from the dependence on initial conditions by restricting each component process to be stationary. Our approximating process will similarly be stationary.[6]

[5] We thank Ian Martin for suggesting this link to entropy.

[6] As argued by Lombardo (2010), this approach is computationally very similar to the pruning approach

4.1 Approximating the state vector process

While X_t serves as a state vector in the dynamic system (20), the state vector itself depends on the parameter q. Let $\mathcal{F}_t$ be the σ-algebra generated by the infinite history of shocks $\{W_j : j \leq t\}$. For each dynamic system, we presume that the state vector X_t is $\mathcal{F}_t$ measurable and that in forecasting future values of the state vector conditioned on $\mathcal{F}_t$ it suffices to condition on X_t. Although X_t depends on q, the construction of $\mathcal{F}_t$ does not. As we will see, the approximating dynamic system will require a higher-dimensional state vector for a Markov representation, but the construction of this state vector will not depend on the value of q. We now construct the dynamics for each of the component processes. The result will be a recursive system that has the same structure as the triangular system (28).

Define $\bar{x}$ to be the solution to the equation:

$$\bar{x} = \psi(\bar{x}, 0, 0),$$

which gives the fixed point for the deterministic dynamic system. We assume that this fixed point is locally stable. That is $\psi_x(\bar{x}, 0, 0)$ is a matrix with stable eigenvalues, eigenvalues with absolute values that are strictly less than one. Then set

$$X_{0,t} = \bar{x}$$

for all t. This is the zeroth-order contribution to the solution constructed to be time-invariant.

In computing pathwise derivatives, we consider the state vector process viewed as a function of the shock history. Each shock in this history is scaled by the parameter q, which results in a parameterized family of stochastic processes. We compute derivatives with respect to this parameter where the derivatives themselves are stochastic processes. Given the Markov representation of the family of stochastic processes, the derivative processes will also have convenient recursive representations. In what follows we derive these representations.[7]

Using the Markov representation, we compute the derivative of the state vector process with respect to q, which we evaluate at $\mathsf{q} = 0$. This derivative has the recursive representa-

described by Kim et al. (2008) or Andreasen et al. (2010).

[7]Conceptually, this approach is distinct from the approach often taken in solving dynamic stochastic general equilibrium models. The common practice is to a compute a joint expansion in q and state vector x around zero and $\bar{x}$ respectively in approximating the one-period state dynamics. This approach often results in approximating processes that are not globally stable, which is problematic for our calculations. We avoid this problem by computing an expansion of the stochastic process solutions in q alone, which allows us to impose stationarity on the approximating solution. In conjunction with the more common approach, the method of "pruning" has been suggested as an *ad hoc* way to induce stochastic stability, and we suspect that it will give similar answers for many applications. See Lombardo (2010) for further discussion.

tion:

$$X_{1,t+1} = \psi_q + \psi_x X_{1,t} + \psi_w W_{t+1} \tag{21}$$

where ψ_q, ψ_x and ψ_w are the partial derivative matrices:

$$\psi_q \doteq \frac{\partial \psi}{\partial \mathsf{q}}(\bar{x}, 0, 0), \quad \psi_x \doteq \frac{\partial \psi}{\partial x'}(\bar{x}, 0, 0), \quad \psi_w \doteq \frac{\partial \psi}{\partial w'}(\bar{x}, 0, 0).$$

In particular, the term $\psi_w W_{t+1}$ reveals the role of the shock vector in this recursive representation. Recall that we have presumed that $\bar{x}$ has been chosen so that ψ_x has stable eigenvalues. Thus the first derivative evolves as a Gaussian vector autoregression. It can be expressed as an infinite moving average of the history of shocks, which restricts the process to be stationary. The first-order approximation to the original process is:

$$X_t \approx \bar{x} + \mathsf{q} X_{1,t}.$$

In particular, the approximating process on the right-hand side has $\bar{x} + \mathsf{q}(I - \psi_x)^{-1}\psi_q$ as its unconditional mean.

In many applications, the first-derivative process $X_{1,\cdot}$ will have unconditional mean zero, $\psi_q = 0$. This includes a large class of models solved using the familiar log approximation techniques, widely used in macroeconomic modeling. This applies to the example economy we consider in Section 7. In Section 5 we suggest an alternative approach motivated by models in which economic agents have a concern for model misspecification. This approach, when applied to economies with production, results in a $\psi_q \neq 0$.

We compute the pathwise second derivative with respect to q recursively by differentiating the recursion for the first derivative. As a consequence, the second derivative has the recursive representation:

$$\begin{aligned} X_{2,t+1} &= \psi_{qq} + 2\left(\psi_{xq} X_{1,t} + \psi_{wq} W_{t+1}\right) + \\ &\quad + \psi_x X_{2,t} + \psi_{xx}\left(X_{1,t} \otimes X_{1,t}\right) + 2\psi_{xw}\left(X_{1,t} \otimes W_{t+1}\right) + \psi_{ww}\left(W_{t+1} \otimes W_{t+1}\right) \end{aligned} \tag{22}$$

where matrices ψ_{ij} denote the second-order derivatives of ψ evaluated at $(\bar{x}, 0, 0)$ and formed using the construction of the derivative matrices described in Appendix A.2. As noted by Schmitt-Grohé and Uribe (2004), the mixed second-order derivatives ψ_{xq} and ψ_{wq} are often zero using second-order refinements to the familiar log approximation methods.

The second-derivative process $X_{2,\cdot}$ evolves as a stable recursion that feeds back on itself and depends on the first derivative process. We have already argued that the first derivative process $X_{1,t}$ can be constructed as a linear function of the infinite history of the shocks. Since the matrix ψ_x has stable eigenvalues, $X_{2,t}$ can be expressed as a linear-quadratic function of this same shock history. Since there are no feedback effects from $X_{2,t}$ to $X_{1,t+1}$, the joint process $(X_{1,\cdot}, X_{2,\cdot})$ constructed in this manner is necessarily stationary.

With this second-order adjustment, we approximate X_t as

$$X_t \approx \bar{x} + \mathsf{q} X_{1,t} + \frac{\mathsf{q}^2}{2} X_{2,t}.$$

When using this approach we replace X_t with these three components, thus increasing the number of state variables. Since $X_{0,t}$ is invariant to t, we essentially double the number of state variables by using $X_{1,t}$ and $X_{2,t}$ in place of X_t.

Further, the dynamic evolution for $(X_{1,\cdot}, X_{2,\cdot})$ becomes a special case of the the triangular system (28) given in Section 6. When the shock vector W_t is a multivariate standard normal, we can utilize results from Section 6 to produce exact formulas for conditional expectations of exponentials of linear-quadratic functions in $(X_{1,t}, X_{2,t})$. We exploit this construction in the subsequent subsection. For details on the derivation of the approximating formulas see Appendix A.

4.2 Approximating an additive functional and its multiplicative counterpart

Consider the approximation of a parameterized family of additive functionals with increments given by:

$$Y_{t+1}(\mathsf{q}) - Y_t(\mathsf{q}) = \kappa(X_t(\mathsf{q}), \mathsf{q} W_{t+1}, \mathsf{q})$$

and an initial condition $Y_0(\mathsf{q}) = 0$. We use the function κ in conjunction with q to parameterize implicitly a family of additive functionals. We approximate the resulting additive functionals by

$$Y_t \approx Y_{0,t} + \mathsf{q} Y_{1,t} + \frac{\mathsf{q}^2}{2} Y_{2,t} \tag{23}$$

where each additive functional is initialized at zero and has stationary increments.

Following the steps of our approximation of X, the recursive representation of the zeroth-order contribution to Y is

$$Y_{0,t+1} - Y_{0,t} = \kappa(\bar{x}, 0, 0) \doteq \bar{\kappa};$$

the first-order contribution is

$$Y_{1,t+1} - Y_{1,t} = \kappa_q + \kappa_x X_{1,t} + \kappa_w W_{t+1}$$

where κ_x and κ_w are the respective first derivatives of κ evaluated at $(\bar{x}, 0, 0)$; and the second-order contribution is

$$\begin{aligned} Y_{2,t+1} - Y_{2,t} &= \kappa_{qq} + 2\left(\kappa_{xq} X_{1,t} + \kappa_{wq} W_{t+1}\right) + \\ &\quad + \kappa_x X_{2,t} + \kappa_{xx}\left(X_{1,t} \otimes X_{1,t}\right) + 2\kappa_{xw}\left(X_{1,t} \otimes W_{t+1}\right) + \kappa_{ww}\left(W_{t+1} \otimes W_{t+1}\right) \end{aligned}$$

where the κ_{ij}'s are the second derivative matrices constructed as in Appendix A.2. The resulting component additive functionals are special cases of the additive functional given in (29) that we introduced in Section 6.

Consider next the approximation of a multiplicative functional:

$$M_t = \exp\left(Y_t\right).$$

The corresponding components in the second-order expansion of M_t are

$$\begin{aligned} M_{0,t} &= \exp\left(t\bar{\kappa}\right) \\ M_{1,t} &= M_{0,t} Y_{1,t} \\ M_{2,t} &= M_{0,t}\left(Y_{1,t}\right)^2 + M_{0,t} Y_{2,t}. \end{aligned}$$

Since Y has stationary increments constructed from X_t and W_{t+1}, errors in approximating X and κ may accumulate when we extend the horizon t. Thus caution is required for this and other approximations to additive functionals and their multiplicative counterparts. In what follows we will be approximating elasticities computed as conditional expectations of multiplicative functionals that scale the shock vector or functions of the state vector. Previously, we have argued that the nonstationary martingale component of multiplicative functionals can be absorbed conveniently into a change of measure. Thus for our purposes, this problem of approximation of a multiplicative functional is essentially equivalent to the problem of approximating a change in measure. Since our elasticities are measured per unit of time, the potential accumulation of errors is at least partly offset by this scaling. In our applications we will perform some *ad hoc* checks, but such approximation issues warrant further investigation.

4.3 Approximating shock elasticities

We consider two alternative approaches to approximating shock elasticities of the form:

$$\varepsilon(x,t) = \alpha_h(x) \cdot \frac{E\left[M_t W_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]}. \tag{24}$$

Recall that we produced this formula by localizing the risk exposure and computing a (logarithmic) derivative.

4.3.1 Approach 1: Approximation of elasticity functions

Our first approach is a direct extension of the perturbation method just applied. We will show how to construct a second-order approximation to the shock elasticity function of the form

$$\varepsilon(X_0, t) \approx \varepsilon_0(t) + \mathsf{q}\varepsilon_1(t) + \frac{\mathsf{q}^2}{2}\varepsilon_2(X_{1,0}, X_{2,0}, t)$$

where only the second-order component is state-dependent. First, observe that the zeroth-order approximation is

$$\varepsilon_0(t) = 0$$

because the zeroth-order contribution in the numerator of (24) is

$$E\left[\exp(t\bar{\kappa})W_1 | X_0 = x\right] = 0.$$

This result replicates the well-known fact that first-order perturbations of a smooth deterministic system do not lead to any compensation for risk exposure.

The first-order approximation is:

$$\varepsilon_1(t) = \alpha_h(\bar{x}) \cdot E\left[Y_{1,t}W_1 \mid \mathcal{F}_0\right] = \alpha_h(\bar{x}) \cdot \left[\sum_{j=1}^{t-1} \kappa_x \left(\psi_x\right)^{j-1} \psi_w + \kappa_w\right]'$$

which is state-independent. This approximation shows the explicit link between the impulse response function for a log-linear approximation and the shock elasticity function.

The second-order adjustment to the approximation is:

$$\begin{aligned}\varepsilon_2(X_{1,0}, X_{2,0}, t) = \alpha_h(\bar{x}) \cdot \left\{E\left[\left(Y_{1,t}\right)^2 W_1 + Y_{2,t}W_1 \mid \mathcal{F}_0\right] - 2E\left[Y_{1,t}W_1 \mid \mathcal{F}_0\right] E\left[Y_{1,t} \mid \mathcal{F}_0\right]\right\} + \\ + 2\left[\frac{\partial \alpha_h}{\partial x'}(\bar{x})\right] X_{1,0} \cdot E\left[Y_{1,t}W_1 \mid \mathcal{F}_0\right].\end{aligned}$$

This adjustment can be expressed as a function of $X_{1,0}$ and $X_{2,0}$ since $(X_{1,\cdot}, X_{2,\cdot})$ is Markov.

Notice that the second-order approximation can induce state dependence in the shock elasticities. Often it is argued that higher than second-order approximations are required to capture state dependence in risk premia. Since we have already performed a differentiation to construct an elasticity, the second-order approximation of an elasticity implicitly include third-order terms. Relatedly, in approximating elasticities using representation (24), we have normalized the exposure to have a unit standard deviation and this magnitude is held fixed even when q declines to zero. By fixing the exposure we reduce the order of differentiation required for state dependence to be exposed.

To illustrate these calculations, consider a special case in which

$$Y_{t+1} - Y_t = \kappa(X_t, \mathsf{q}W_{t+1}, \mathsf{q}) = \beta(X_t) + \mathsf{q}\alpha(X_t) \cdot W_{t+1}.$$

Then

$$\varepsilon(x, 1) = \alpha_h(x) \cdot \frac{E\left[M_1 W_1 \mid X_0 = x\right]}{E\left[M_1 \mid X_0 = x\right]} = \mathsf{q}\alpha_h(x) \cdot \alpha(x).$$

We may use our previous formulas or perform a direct calculation to show that

$$\begin{aligned}
\varepsilon_1(1) =&\ \alpha_h(\bar{x}) \cdot \alpha(\bar{x}) \\
\varepsilon_2(X_{1,0}, X_{2,0}, 1) =&\ 2(X_{1,0})' \left[\frac{\partial \alpha_h}{\partial x'}(\bar{x})\right]' \alpha(\bar{x}) + 2(X_{1,0})' \left[\frac{\partial \alpha}{\partial x'}(\bar{x})\right]' \alpha_h(\bar{x})
\end{aligned}$$

In comparison, suppose that we compute a risk premium for the one-period cash flow

$$G_1 = \exp\left[\beta_g(X_0) + \mathsf{q}\alpha_g(X_0) \cdot W_1\right]$$

priced using the one-period stochastic discount factor:

$$S_1 = \exp\left[\beta_s(X_0) + \mathsf{q}\alpha_s(X_0) \cdot W_1\right]$$

The one-period risk premium (in logarithms) is:

$$\log E\left[G_1 \mid X_0 = x\right] - \log E\left[S_1 G_1 \mid X_0 = x\right] + \log E\left[S_1 \mid X_0 = x\right] = (\mathsf{q})^2 \alpha_g(x) \cdot \alpha_s(x).$$

The first two terms on the left when taken together give the logarithm of the expected one period return, and the negative of the third term is an adjustment for the risk-free rate. Since we scaled the cash flow exposure by q, the risk premium scales in q^2 and the second-order approximation to this premium will be constant in contrast to our shock elasticities.

4.3.2 Approach 2: Exact elasticities under approximate dynamics

As an alternative approach, we exploit the fact that the second-order approximation is a special case of the convenient functional form that we discussed in Section 6. This allows us to compute elasticities using the quasi-analytical formulas we described in that section. With this second approach, we calculate approximating stochastic growth and discounting functionals and then use these to represent arbitrage-free pricing. This second approach leads us to include some (but not all) third-order terms in q as we now illustrate.

Recall that in the example just considered, we approximated the one-period shock elasticity as

$$\varepsilon(x,1) = \mathsf{q}\alpha_h(x) \cdot \alpha(x).$$

With this second approach, we obtain

$$\varepsilon(x,1) \approx \mathsf{q}\left[\alpha_h(\bar{x}) + \mathsf{q}\frac{\partial \alpha_h}{\partial x'}(\bar{x})X_{1,0}\right] \cdot \left[\alpha(\bar{x}) + \mathsf{q}\frac{\partial \alpha}{\partial x'}(\bar{x})X_{1,0}\right].$$

The q and q^2 terms agree with the outcome of our first approach, but we now include an additional third-order term in q. Both approaches are straightforward to implement and can be compared.

There are applications where it is natural to make the perturbation vector $\alpha_h(x)$ depend on x, for example, when calculating shock elasticities in models with stochastic volatility. However, in line with the literature on impulse response functions, $\alpha_h(x)$ will often be chosen to be a constant vector of zeros with a single one. In this case, both notions of the second-order approximation of a shock elasticity function coincide.

4.4 Approximating partial shock elasticities

In Section 2.5 we defined the partial shock elasticity function as a way to explore alternative transmission mechanisms and the impact of introducing new shocks. We may either compute direct expansions or we may use the second-order expansion in q as a starting point. The formulas in Section 2.5 are directly applicable to these, except that we must compute the initializations:

$$\begin{aligned}\widetilde{X}_{1,1} &= \widetilde{\psi}_{\widetilde{w}}(x, W_1, 0, 0)\\ \widetilde{Y}_{1,1} &= \widetilde{\kappa}_{\widetilde{w}}(x, W_1, 0, 0).\end{aligned}$$

We may approximate these initial conditions by constructing a joint expansion based on scaling W_{t+1} by q and $\mathsf{q}\tilde{W}$ and including first-order terms in q. This allows us to exploit the analytical tractability of the convenient functional form in Section 6.

In Appendix B.3, we show that the first-order expansion in r of the partial elasticity function

$$\widetilde{\varepsilon}(X_0, t) \approx \widetilde{\varepsilon}_0(t) + \mathsf{q}\widetilde{\varepsilon}_1(X_{1,0}, t).$$

corresponds to the second-order expansion of the shock elasticity function for appropriately chosen shock configurations. The differentiation in q that we used to construct the partial elasticity (11) implies that the partial elasticity function is nonzero already in its zeroth-order:

$$\widetilde{\varepsilon}_0(t) \quad = \quad \widetilde{\alpha}(\bar{x}) \cdot \left[\sum_{j=1}^{t-1} \widetilde{\kappa}_x \left(\widetilde{\psi}_x \right)^{j-1} \widetilde{\psi}_{\widetilde{w}} + \widetilde{\kappa}_{\widetilde{w}} \right]'$$

where the derivative matrices are evaluated at the deterministic steady state $(\bar{x}, 0, 0, 0)$.

Observe that $\widetilde{\varepsilon}_0(t)$ is linear in the partial derivatives with respect to $\widetilde{W}$ evaluated at the deterministic steady state, which is also true for the higher-order terms in the expansion of $\widetilde{\varepsilon}(x, t)$. This illustrates why partial elasticities decompose additively in shock configurations, as we documented in the 'interesting special case' in Section 2.5. We utilize this additive decomposition in Section 7 to quantify the contribution of different shock propagation channels to shock elasticities in an example economy.

4.5 Equilibrium conditions

In our discussion for pedagogical simplicity we took as a starting point the Markov representation for the law of motion (20). In economic applications, this law of motion is expressed in terms equilibrium conditions that involve conditional expectations of state and co-state variables. Using the perturbation methods described in Judd (1998), we may compute the necessary derivatives at the deterministic steady state without explicitly computing the function ψ in advance. As in our calculations there is a convenient recursive structure to the derivatives in which higher-order derivatives can be built easily from the lower-order counterparts. The requisite derivatives can be constructed sequentially, order by order.

4.6 Related approaches

There also exist *ad-hoc* approaches which mix orders of approximation for different components of the model or state vector. The aim of these methods is to improve the precision of the approximation along specific dimensions of interest, while retaining tractability in the computation of the derivatives of the function ψ. Justiniano and Primiceri (2008) use a first-order approximations but augment the solution with heteroskedastic innovations. Benigno et al. (2010) study second-order approximations for the endogenous state variables in which exogenous state variables follow a conditionally linear Markov process. Malkhozov and Shamloo (2011) combine a first-order perturbation with heteroskedasticity in the shocks to the exogenous process and corrections for the variance of future shocks. These solution methods are designed to produce nontrivial roles for stochastic volatility in the solution of the model and in the pricing of exposure to risk. The approach of Benigno et al. (2010) or Malkhozov and Shamloo (2011) give alternative ways to construct the functional form used in Section 6.

5 Recursive utility investors

In this section we contrast two preference specifications which share some common features but can lead to different approaches for local approximation. The first preference specification is the recursive utility of Kreps and Porteus (1978). By design, this specification avoids presuming that investors reduce intertemporal, compound consumption lotteries. Instead investors may care about the intertemporal composition of risk. As an alternative, we consider an investor whose preferences are influenced by his concern for robustness, which leads him to evaluate his utility under alternative distributions and checking for sensitivity.

5.1 Recursive preferences and the robust utility interpretation

We follow Epstein and Zin (1989) and others by using a homogeneous aggregator in modeling recursive preferences in the study of asset pricing implications. For simplicity we focus on the special case in which investors' preferences exhibit a unitary elasticity of intertemporal substitution. In this case the continuation value process satisfies the forward recursion:

$$\log V_t = [1 - \exp(-\delta)] \log C_t + \frac{\exp(-\delta)}{1-\gamma} \log E\left[(V_{t+1})^{1-\gamma} | \mathcal{F}_t \right]. \tag{25}$$

where V_t is the date t continuation value associated with the consumption process $\{C_{t+j} : j = 0, 1, ...\}$. The parameter δ is the subjective rate of discount and γ is used for making a risk adjustment in the continuation value. The limiting $\gamma = 1$ version gives the separable logarithmic utility. We focus on the case in which $\gamma > 1$. As we will see, the forward-looking nature of the continuation value process can amplify the role of beliefs and uncertainty about the future in asset valuation.

We suppose that the equilibrium consumption process from an economic model is a multiplicative functional of the type described previously. For numerical convenience, subtract $\log C_t$ from both sides of this equation:

$$\log V_t - \log C_t = \frac{\exp(-\delta)}{1-\gamma} \log E\left[\left(\frac{V_{t+1}}{C_t} \right)^{1-\gamma} | \mathcal{F}_t \right],$$

or

$$\log U_t = \frac{\exp(-\delta)}{1-\gamma} \log E \left(\exp \left[(1-\gamma) \log U_{t+1} + (1-\gamma)(\log C_{t+1} - \log C_t) \right] \mid \mathcal{F}_t \right)$$

where $\log U_t = \log V_t - \log C_t$. The stochastic discount factor process is given by the recursion:

$$\begin{aligned} \frac{S_{t+1}}{S_t} &= \exp(-\delta) \left(\frac{C_t}{C_{t+1}} \right) \frac{(V_{t+1})^{1-\gamma}}{E\left[(V_{t+1})^{1-\gamma} \mid \mathcal{F}_t \right]} \\ &= \exp(-\delta) \left(\frac{C_t}{C_{t+1}} \right) \frac{(U_{t+1})^{1-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma}}{E\left[(U_{t+1})^{1-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \mid \mathcal{F}_t \right]}, \end{aligned} \tag{26}$$

which gives the one-period intertemporal marginal rate of substitution for a recursive utility investor. When $\gamma = 1$ the expression for the stochastic discount factor simplifies and reveals the intertemporal marginal rate of substitution for discounted logarithmic utility. When $\gamma > 1$, there is a potentially important contribution from the forward-looking continuation value process reflected in V_{t+1} or U_{t+1}.

Allowing the parameter γ in the recursive utility specification to be large has become common in the macro-asset pricing literature. For this reason we are led to consider motivations other than risk aversion for large values of this parameter. Anderson et al. (2003) extend the literature on risk-sensitive control by Jacobson (1973), Whittle (1990) and others and provide a "concern for robustness" interpretation of the utility recursion (25). Under this interpretation the decision maker explores alternative specifications of the transition dynamics as part of the decision-making process. This yields a substantially different interpretation of the utility recursion and the parameter γ. An outcome of this robustness assessment is an exponentially-tilted worst case model (subject to penalization) in which the term

$$\frac{\widetilde{S}_{t+1}}{\widetilde{S}_t} \equiv \frac{(V_{t+1})^{1-\gamma}}{E\left[(V_{t+1})^{1-\gamma} \mid \mathcal{F}_t \right]} = \frac{(U_{t+1})^{1-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma}}{E\left[(U_{t+1})^{1-\gamma} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma} \mid \mathcal{F}_t \right]}$$

in the stochastic discount factor ratio (26) induces an alternative specification of the transitional dynamics used to implement robustness. Notice that this term has conditional expectation equal to one, and as a consequence it implies an alternative density for the shock vector W_{t+1} conditioned on date t information.

5.2 Expansion approaches

Since an essential ingredient for the evolution of the logarithm of the stochastic discount factor process is the continuation value process, as a precursor to approximating the stochastic discount factor process we first approximate $\log U$. As previously, we seek an approximation of the form:

$$\log U_t \approx \log U_{0,t} + \mathsf{q} \log U_{1,t} + \frac{\mathsf{q}^2}{2} \log U_{2,t}$$

where the terms on the right-hand side are themselves components of stationary processes. We will construct the approximation of the continuation value as a function of a corresponding approximation of the logarithm of the consumption process $\log C$ given by equation (23). For ease of comparison, we will hold fixed the second-order approximation for consumption as we explore two different approaches. In a production economy the approximation of the consumption process will itself change as we alter the specification of preferences.

The typical approach that is valid for the recursive utility specification dictates to treat both the scaled continuation value process U as well as the consumption process C as functions of the perturbation parameter q:

$$\log U_t(\mathsf{q}) = \frac{\exp(-\delta)}{1-\gamma} \log E\left(\exp\left[(1-\gamma)\left(\log U_{t+1}(\mathsf{q}) + \log C_{t+1}(\mathsf{q}) - \log C_t(\mathsf{q})\right)\right] \mid \mathcal{F}_t\right).$$

The zero-th order expansion implies a constant contribution

$$\log U_{0,t} \equiv \bar{u} = \frac{\exp(-\delta)}{1-\exp(-\delta)} \left(\log C_{0,t+1} - \log C_{0,t}\right) \tag{27}$$

and the higher-order terms can be represented recursively as

$$\begin{aligned}
\log U_{1,t} &= \exp(-\delta) E\left[\log U_{1,t+1} + \log C_{1,t+1} - \log C_{1,t} \mid \mathcal{F}_t\right] \\
\log U_{2,t} &= \exp(-\delta) E\left[\log U_{2,t+1} + \log C_{2,t+1} - \log C_{2,t} \mid \mathcal{F}_t\right] + \\
&\quad + (1-\gamma)\exp(-\delta) E\left[\left(\log U_{1,t+1} + \log C_{1,t+1} - \log C_{1,t}\right)^2 \mid \mathcal{F}_t\right] \\
&\quad - (1-\gamma)\exp(-\delta)\left[E\left(\log U_{1,t+1} + \log C_{1,t+1} - \log C_{1,t} \mid \mathcal{F}_t\right)\right]^2
\end{aligned}$$

and can be solved forward. This approach assures that both $\log U$ and $\log C$ will conform functional forms introduced when constructing expansions of additive functionals in Section 4.2. Observe that only the second-order term $\log U_{2,\cdot}$ in the expansion of the continuation value depends on the risk aversion parameter γ, and only scales the first-order terms.[8]

[8] In related work we derive an alternative approximation for stochastic discount factors motivated by a concern for robustness and calibrations of that concern. This change in perspective alters the expansion in

Under the recursive utility preferences, the terms in the expansion of the stochastic discount factor are linear in continuation values and changes in consumption:

$$
\begin{aligned}
\log S_{0,t+1} - \log S_{0,t} &= -\delta + \log C_{0,t} - \log C_{0,t+1} \\
\log S_{1,t+1} - \log S_{1,t} &= \log C_{1,t} - \log C_{1,t+1} \\
&\quad + (1-\gamma)\left[\log U_{1,t+1} + \log C_{1,t+1} - \log C_{1,t} - \exp(\delta) \log U_{1,t}\right] \\
\log S_{2,t+1} - \log S_{2,t} &= \log C_{2,t} - \log C_{2,t+1} \\
&\quad + (1-\gamma)\left[\log U_{2,t+1} + \log C_{2,t+1} - \log C_{2,t} - (1-\gamma)\exp(\delta) \log U_{2,t}\right]
\end{aligned}
$$

6 Convenient functional form

In Sections 4 and 5, we developed second-order approximations of dynamic macroeconomic models and the resulting shock elasticities. We now introduce a more general exponential-quadratic framework that nests these approximate solutions and that generates quasi-analytical solutions for the shock elasticities. The exact mapping from the second-order approximation to this framework together with detailed computations is provided in Appendix A.

Consider the following triangular state vector system:

$$
\begin{aligned}
X_{1,t+1} &= \Theta_{10} + \Theta_{11} X_{1,t} + \Lambda_{10} W_{t+1} \\
X_{2,t+1} &= \Theta_{20} + \Theta_{21} X_{1,t} + \Theta_{22} X_{2,t} + \Theta_{23}\left(X_{1,t} \otimes X_{1,t}\right) \\
&\quad + \Lambda_{20} W_{t+1} + \Lambda_{21}\left(X_{1,t} \otimes W_{t+1}\right) + \Lambda_{22}\left(W_{t+1} \otimes W_{t+1}\right). \qquad (28)
\end{aligned}
$$

Such a system allows for stochastic volatility, and we restrict the matrices Θ_{11} and Θ_{22} to have stable eigenvalues. A comparison with equations (21) and (22) reveals that the dynamics of X_1 and X_2 capture the laws of motion of the first and second derivative of the state vector X introduced in Section 4. The additive functionals that interest us satisfy

$$
\begin{aligned}
Y_{t+1} - Y_t = {} & \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3\left(X_{1,t} \otimes X_{1,t}\right) \\
& + \Psi_0 W_{t+1} + \Psi_1\left(X_{1,t} \otimes W_{t+1}\right) + \Psi_2\left(W_{t+1} \otimes W_{t+1}\right). \qquad (29)
\end{aligned}
$$

In what follows we use a $1 \times k^2$ vector Ψ to construct a $k \times k$ symmetric matrix $\text{sym}\left[\text{mat}_{k,k}(\Psi)\right]$ such that[9]

$$
w' \left(\text{sym}\left[\text{mat}_{k,k}(\Psi)\right]\right) w = \Psi (w \otimes w).
$$

a substantive way by allowing for the robustness concern to present in lower order terms.

[9] In this formula $\text{mat}_{k,k}(\Psi)$ converts a vector into a $k \times k$ matrix and the sym operator transforms this square matrix into a symmetric matrix by averaging the matrix and its transpose. Appendix A introduces convenient notation for the algebra underlying the calculations in this and subsequent sections.

This representation will be valuable in some of the computations that follow. We use additive functionals to represent stochastic growth via a technology shock process or aggregate consumption, and to represent stochastic discounting used in representing asset values. This setup is rich enough to accommodate stochastic volatility, which has been featured in the asset pricing literature and to a lesser extent in the macroeconomics literature.

A virtue of parameterization (28)–(29) is that it gives quasi-analytical formulas for our dynamic elasticities. The implied model of the stochastic discount factor has been used in a variety of reduced-form asset pricing models. Such calculations are free of any approximation errors to the dynamic system (28)–(29) and, as a consequence, ignore the possibility that approximation errors compound and might become more prominent as we extend the investment or forecast horizon t. On the other hand, we will use an approximation to deduce this dynamical system, and we have research in progress that explores the implications of approximation errors in the computations that interest us.

We illustrate the convenience of this functional form by calculating the logarithms of conditional expectations of multiplicative functionals of the form (29). Consider a function that is linear/quadratic in $x = (x_1', x_2')'$:

$$\log f(x) = \Phi_0 + \Phi_1 x_1 + \Phi_2 x_2 + \Phi_3 (x_1 \otimes x_1)$$

Then conditional expectations are of the form:

$$\begin{aligned} \log E\left[\left(\frac{M_{t+1}}{M_t}\right) f(X_{t+1}) \mid X_t = x\right] &= \log E\left[\exp\left(Y_{t+1} - Y_t\right) f(X_{t+1}) \mid X_t = x\right] \\ &= \Phi_0^* + \Phi_1^* x_1 + \Phi_2^* x_2 + \Phi_3^* (x_1 \otimes x_1) \\ &= \log f^*(x) \end{aligned} \tag{30}$$

where the formulas for $\Phi_i^*,\ i = 0, \ldots, 3$ are given in Appendix A. This calculation maps a function f into another function f^* with the same functional form. Our multi-period calculations exploit this link. For instance, repeating these calculations compounds stochastic growth or discounting. Moreover, we may exploit the recursive Markov construction in (30) initiated with $f(x) = 1$ to obtain:

$$\log E\left[M_t \mid X_0 = x\right] = \Phi_{0,t}^* + \Phi_{1,t}^* x_1 + \Phi_{2,t}^* x_2 + \Phi_{3,t}^* (x_1 \otimes x_1)$$

for appropriate choices of $\Phi_{i,t}^*$.

6.1 Shock elasticities

To compute shock elasticities given in (1) under the convenient functional form, we construct:

$$\frac{E\left[M_t W_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]} = \frac{E\left[M_1 E\left(\frac{M_t}{M_1} \mid X_1\right) W_1 \mid X_0 = x\right]}{E\left[M_1 E\left(\frac{M_t}{M_1} \mid X_1\right) \mid X_0 = x\right]}.$$

Notice that the random variable:

$$L_{1,t} = \frac{M_1 E\left(\frac{M_t}{M_1} \mid X_1\right)}{E\left[M_1 E\left(\frac{M_t}{M_1} \mid X_1\right) \mid X_0 = x\right]}$$

has conditional expectation one. Multiplying this positive random variable by W_1 and taking expectations is equivalent to changing the conditional probability distribution and evaluating the conditional expectation of W_1 under this change of measure. Then under the transformed measure, using a complete-the-squares argument we may show that W_1 remains normally distributed with a covariance matrix:

$$\widetilde{\Sigma}_t = \left[I_k - 2\,\mathrm{sym}\left(\mathrm{mat}_{k,k}\left[\Psi_2 + \Phi^*_{2,t-1}\Lambda_{22} + \Phi^*_{3,t-1}\left(\Lambda_{10} \otimes \Lambda_{10}\right)\right]\right)\right]^{-1}.$$

where I_k is the identity matrix of dimension k.[10] We suppose that this matrix is positive definite. The conditional mean vector for W_1 under the change of measure is:

$$\tilde{E}\left[W_1 | X_0 = x\right] = \widetilde{\Sigma}_t \left[\mu_{t,0} + \mu_{t,1} x_1\right],$$

where $\tilde{E}$ is the expectation under the change of measure and the coefficients $\mu_{t,0}$ and $\mu_{t,1}$ are given in Appendix B.

Thus the shock elasticity is given by:

$$\begin{aligned} \varepsilon(x,t) &= \alpha_h(x) \cdot E\left[L_{1,t} W_1 \mid X_0 = x\right] \\ &= \alpha_h(x)' \widetilde{\Sigma}_t \left[\mu_{t,0} + \mu_{t,1} x_1\right] \end{aligned}$$

The shock elasticity function in this environment depends on the first component, x_1, of the state vector. Recall from (28) that this component has linear dynamics. The coefficient matrices for the evolution of the second component, x_2, nevertheless matter for the shock elasticities even though these elasticities do not depend on this component of the state vector.

[10] This formula uses the result that $(\Lambda_{10} W_1) \otimes (\Lambda_{10} W_1) = (\Lambda_{10} \otimes \Lambda_{10})(W_1 \otimes W_1)$.

6.2 Entropy increments

The convenient functional form (28)–(29) also provides a tractable formula for the entropy components. Observe that

$$\zeta(x,t) = -E\left[\log L_{1,t} | X_0 = x\right].$$

Consistent with our previous calculations, $L_{1,t}$ is the likelihood ratio built from two normal densities for the shock vector: a multivariate normal density for the altered distribution and a multivariate standard normal density. A consequence of this construction is that the negative of the resulting expected log-likelihood satisfies:

$$\zeta(x,t) = \frac{1}{2}\left[\left(\widetilde{E}\left[W_1 | X_0 = x\right]\right)' \left(\widetilde{\Sigma}_t\right)^{-1} \left(\widetilde{E}\left[W_1 | X_0 = x\right]\right) + \log|\widetilde{\Sigma}_t| + \operatorname{trace}\left(\widetilde{\Sigma}_t^{-1}\right) - k\right]$$

Thus the mean distortion $\widetilde{E}\left[W_1 | X_0 = x\right]$ is a critical input into both the shock elasticities and the entropy increments.[11]

7 Application: Intangible risk

We use the model of Ai et al. (2012) to illustrate our methodology by analyzing shock elasticities associated with consumption and capital dynamics in a model with two types of capital. The two capital stocks face different risk exposures, which leads to differences in their valuation. We decompose shock elasticities to understand the mechanism how risk propagates in the model economy.

The model is motivated by an extensive literature that confronts challenges in measuring capital. In this literature, one component of the capital stock, *tangible capital*, is measured while another one, *intangible capital*, is not. In what follows we will refer to the tangible component as physical capital. Intangible capital is introduced to account fully for firm values. For instance, if firms accumulate large quantities of unmeasured productive intangible capital, their market valuation will differ from valuation based on the replacement value of the stock of physical capital. Hall (2000, 2001) uses this argument to understand the secular movement in asset values relative to measures of capital. Similarly, McGrattan and Prescott (2010a,b) argue that accounting properly for the accumulation of intangible capital explains the heterogeneity in measured returns and the observed macroeconomic dynamics including

[11] In a continuous-time limit, the only term that will remain is the counterpart to the quadratic form in the conditional mean distortion for the shock.

the period of the 1990's.[12]

Following Hansen et al. (2005) we consider a related question by exploring risk-based explanations for the heterogeneity in the returns to physical and intangible capital. Hansen et al. (2005) use the return heterogeneity documented by Fama and French (1992, 1996) to motivate studies of the risk exposure differences between returns on tangible and intangible capital. Among other things, Fama and French (1992, 1996) show that firms with high book-to-market (B/M) ratios (value firms) have systematically higher expected returns compared to their low B/M counterparts (growth firms).[13] Ai et al. (2012) build a stylized model to investigate formally the link between the value premium featured by Fama and French and the differential contribution of intangible capital to what are classified as growth or value firms. In the Ai et al. (2012) model growth firms are those with relatively large amounts of intangible capital, are less exposed to aggregate risk, and therefore earn lower expected returns.

7.1 The model

We use the aggregate version of the Ai et al. (2012) model inclusive of adjustment costs. Ai et al. (2012) suggest a more primitive starting point meant to provide microfoundations for the model. We use shock elasticities to characterize the valuation of measured and intangible capital stocks. Parameters and specification of some of the functional forms can be found in Appendix C. While a more explicit use of econometric methods to the estimation of this model is a welcome extension, we find it useful to exposit properties of the model as given in the Ai et al. (2012) paper.

7.1.1 Technology

The economy consists of two sectors. Final output is produced using physical capital K and labor, and allocated to consumption C and investment into physical capital I and intangible capital I^*:

$$C_t + I_t + I_t^* = (K_t)^{\nu} (Z_t)^{1-\nu} .$$

The model abstracts from endogenous labor supply and instead normalizes the labor input to be one. The technology process Z is specified exogenously. To produce new capital,

[12] This literature implicitly confronts the potential fragility in asset values because to the extent tangible capital is used to explain increases in asset values, it must also account for large declines in these values.

[13] For related empirical motivation see the cross-sectional heterogeneity in cash-flow risk exposures of growth and value firms documented by Bansal et al. (2005) and Hansen et al. (2008).

investment I must be combined with the stock of intangible capital K^*

$$K_{t+1} = (1-\lambda) K_t + \left(\frac{Z^*_{t+1}}{Z^*_t}\right) G(I_t, K^*_t).$$

The investment-specific technology process Z^* is also specified exogenously. In the process of capital accumulation $G(I_t, K^*_t)$ units of intangible capital are depleted in the production of one unit of new physical capital. With this adjustment, intangible capital accumulates in accordance with:

$$K^*_{t+1} = (1-\lambda^*)\left[K^*_t - G(I_t, K^*_t)\right] + H(I^*_t, K_t).$$

The functions G and H used to model adjustment costs are both concave.

7.1.2 Exogenous inputs

The technology processes Z and Z^* evolve according to:

$$\begin{aligned}
\log Z_{t+1} - \log Z_t &= \Gamma_0 + \Gamma_1 X_t + \Psi W_{t+1} \\
\log Z^*_{t+1} - \log Z^*_t &= \Gamma^*_0 + \Gamma^*_1 X_t + \Psi^* W_{t+1} \\
X_{t+1} &= \Theta_1 X_t + \Lambda W_{t+1}.
\end{aligned} \tag{31}$$

where X_t and W_{t+1} are both two-dimensional. The first component of the shock vector W is a *direct shock* to the growth rate of technology Z, while the second component represents a *long-run risk shock* to the expected growth rates. The persistence in these expected growth rates is modeled using a first-order, bivariate Markov process X. Correspondingly, Ψ and Ψ^* are two-dimensional row vectors with a zero in their second columns, and Λ is a two-dimensional square matrix with zeros in its first column.

The matrix Θ_1 is a diagonal matrix with common diagonal entries strictly less then one, and Λ has identical entries in the second column. By design, the two components of X remain the same when they have a common initialization. We include both components to the state vector because we will consider perturbations of the original dynamics (31) where the two components will have distinct roles. Observe that the first component of W impacts both Z and Z^*. Moreover, we impose the restrictions

$$\Psi^* = -\frac{1-\nu}{\nu}\Psi, \quad \Gamma_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \Gamma^*_1 = \begin{bmatrix} 0 & -\frac{1-\nu}{\nu} \end{bmatrix}.$$

Under the maintained restrictions,

$$\Gamma_1^* X_t + \Psi^* W_{t+1} = -\left(\frac{1-\nu}{\nu}\right)(\Gamma_1 X_t + \Psi W_{t+1})$$

and shocks thus have offsetting impacts on the technology processes Z and Z^*. A positive shock movement increases the growth rate in the neutral technology process Z but simultaneously decreases the investment-specific process Z^*. Ai et al. (2012) interpret Z^* as a wedge that temporarily mitigates the risk exposure of newly installed capital. In summary, there are two underlying shocks whose impacts we seek to characterize: a direct shock and a long-run risk shock.

To understand better the shock transmission mechanisms in this model, we also consider a less rigid specification by introducing an independent shock vector $\widetilde{W}_{t+1}$ that has four components:

$$\begin{aligned}
\log Z_{t+1}(\mathsf{q}) - \log Z_t(\mathsf{q}) &= \Gamma_0 + \Gamma_1 X_t(\mathsf{q}) + \Psi W_{t+1} + \mathsf{q}\widetilde{\Psi}\widetilde{W}_{t+1} \\
\log Z^*_{t+1}(\mathsf{q}) - \log Z^*_t(\mathsf{q}) &= \Gamma_0^* + \Gamma_1^* X_t(\mathsf{q}) + \Psi^* W_{t+1} + \mathsf{q}\widetilde{\Psi}^*\widetilde{W}_{t+1} \\
X_{t+1}(\mathsf{q}) &= \Theta_1 X_t(\mathsf{q}) + \Lambda W_{t+1} + \mathsf{q}\widetilde{\Lambda}\widetilde{W}_{t+1}
\end{aligned}$$

where

$$\widetilde{\Psi} = \sqrt{2}\begin{bmatrix}\Psi & 0\end{bmatrix}, \quad \widetilde{\Psi}^* = \sqrt{2}\begin{bmatrix}0 & \Psi\end{bmatrix}, \quad \widetilde{\Lambda} = \sqrt{2}\begin{bmatrix}\Lambda_1 & 0 \\ 0 & \Lambda_1\end{bmatrix}$$

and Λ_1 is the first row (or the second row as they are the same) of Λ. We construct $\widetilde{W}_{t+1}$ in order to explore independent shocks that impinge directly on each technology as well as independent shocks that shift the predictable components to these technologies. The first two components of $\widetilde{W}$ only impact the neutral technology process Z while the remaining two components impact the investment-specific technology process Z^*. We compute partial elasticities by exploring small changes in the exposure to $\widetilde{W}_{t+1}$ parameterized by q. By design, the constructed impact matrices for $\widetilde{W}_{t+1}$ satisfy:

$$\widetilde{\Psi}\Upsilon = \Psi, \quad \widetilde{\Psi}^*\Upsilon = \Psi^*, \quad \widetilde{\Lambda}\Upsilon = \Lambda.$$

where

$$\Upsilon = \frac{1}{\sqrt{2}}\begin{bmatrix}I \\ I\end{bmatrix}.$$

Notice that $\Upsilon'\Upsilon = I$. We impose these restrictions to ensure that restrictions (14) and (15) given in Section 2.5 are satisfied.

7.1.3 Preferences

The model is closed by introducing a representative household with recursive preferences of the Epstein and Zin (1989) type:

$$V_t = \left\{ \left[1 - \exp(-\delta)\right] (C_t)^{1-\rho} + \exp(-\delta) E\left[(V_{t+1})^{1-\gamma} \mid \mathcal{F}_t \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}. \tag{32}$$

This specification is more general than the recursion considered in Section 5 by allowing the elasticity of intertemporal substitution ρ^{-1} to be different from one. We obtain equation (25) by taking the limit as $\rho \to 1$. The preference recursion (32) implies a stochastic discount factor which is a generalization of expression (26):

$$\frac{S_{t+1}}{S_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\left(E\left[(V_{t+1})^{1-\gamma} \mid \mathcal{F}_t \right] \right)^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma}.$$

The first-order conditions from a fictitious planner problem then lead to recursive formulas for the (shadow) prices of existing physical and intangible capital Q and Q^*, respectively:

$$\begin{aligned} Q_t &= \nu \left(\frac{Z_t}{K_t} \right)^{1-\nu} + E\left[\frac{S_{t+1}}{S_t} \left[H_K\left(I_t^*, K_t\right) Q_{t+1}^* + (1-\lambda) Q_{t+1} \right] \mid \mathcal{F}_t \right] \\ Q_t^* &= E\left[\frac{S_{t+1}}{S_t} \left(\left(\frac{Z_{t+1}^*}{Z_t^*} \right) G_{K^*}\left(I_t, K_t^*\right) Q_{t+1} + (1-\lambda^*) \left[1 - G_{K^*}\left(I_t, K_t^*\right)\right] Q_{t+1}^* \right) \mid \mathcal{F}_t \right] \end{aligned}$$

This equation system can be solved forward to compute the prices of the two capital stocks.[14] The resulting solution will, at least implicitly, use the multi-period stochastic discount factors to make risk adjustments in future time periods. Dividing both equations by the right-hand side variables gives the pricing formula for one-period returns to physical and intangible capital. The conditional expectation of the one-period stochastic discount factor times the one-period return is equal to one.

7.2 Dynare implementation

Following Ai et al. (2012), we solve the model using a second-order perturbation around the deterministic steady state. We provide online the Dynare code for the model, and the toolbox

[14] Alternative formulas can be obtained by looking at the first-order conditions for investment.

that computes shock elasticities from the solution generated by Dynare.[15] The toolbox is general and can be employed to analyze shock elasticities in conjunction with Dynare using only minor modifications to the model files.[16]

We exploit Dynare to construct the equilibrium dynamics for the increments of additive functionals that are of our interest. With the characterization of the dynamics (28)–(29), we only need to implement the elasticity formulas developed in Section 4.

7.3 Shock price and exposure dynamics

We use elasticities and partial elasticities to obtain a more complete characterization of the equilibrium expected return heterogeneity. We analyze the dynamics of aggregate consumption which determines the characteristics of the stochastic discount factor, and the pricing implications for the two capital stocks.

7.3.1 Consumption price and exposure elasticities

We first consider the shock elasticities for the equilibrium consumption process. To make comparisons to the literature on long-run consumption risk, we use consumption as the growth functional. The resulting elasticities are reported in Figure 1.

The top left panel gives the shock-price elasticities. The flat trajectories are familiar from our earlier analysis of consumption-based models of the type suggested by Bansal and Yaron (2004). See Hansen (2012) and Borovička et al. (2011). As is shown in these two papers, with large specifications of the risk aversion coefficient γ, a forward-looking martingale component associated with the continuation value process dominates the pricing implications. Expected future growth in consumption is an important contributor to this martingale component. The magnitudes of the shock-price elasticities reported in Figure 1 are about double of those reported in our earlier work.

There is a substantive difference in the structure of the Bansal and Yaron (2004) and the Ai et al. (2012) models. Bansal and Yaron (2004) specify directly predictability in the growth rates in consumption whereas Ai et al. (2012) specify the predictability in technology processes that are inputs into production. The two models in fact produce very different implied predictability for consumption, reflected in the shock-exposure elasticities. For in-

[15] See https://files.nyu.edu/jb4457/public/software.html.

[16] Dynare produces a full second-order approximation of the model solution as in Schmitt-Grohé and Uribe (2004). This approximation is globally unstable, and does not fit the convenient triangular structure introduced in Section 6. However, we can apply the perturbation methods from Section 4 to the second-order solution itself. This step effectively doubles the number of state variables, generating separate vectors of variables for the first- and second-order dynamics. This method also corresponds to the algorithms used in Andreasen et al. (2010).

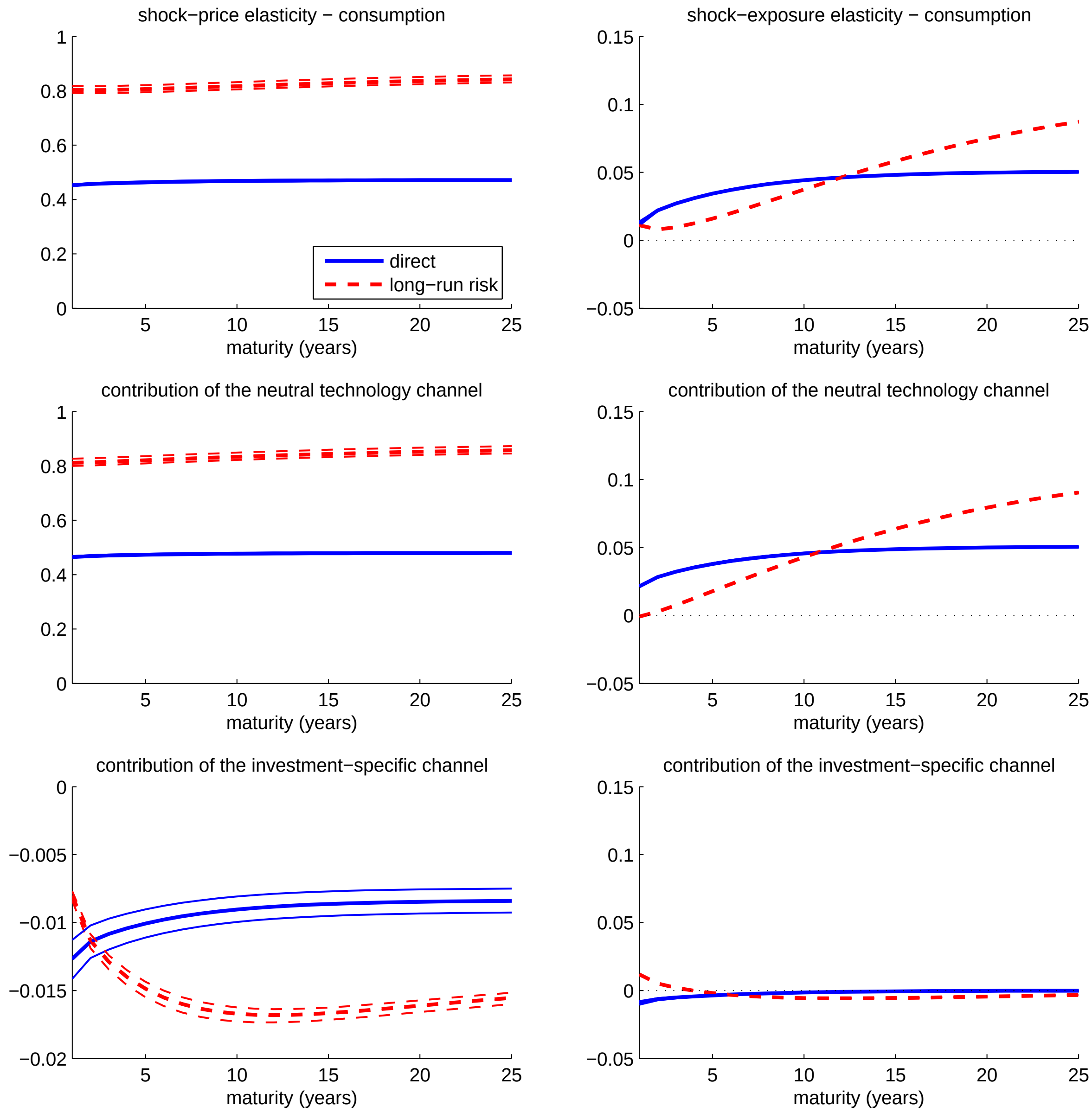

Figure 1: Shock elasticities for consumption. The left panels give the shock-price elasticities and the right panels give the shock-exposure elasticities. The top row shows elasticities for alternative investment horizons in the original model. The second and third rows show the corresponding elasticities using the perturbed specification. The second row features the transmission mechanism for neutral technology shocks, and the third row for investment-specific shocks. To capture the state dependence in the elasticities, we report three quartiles.

stance, the limiting shock-exposure elasticity for the shock to the growth rates in technology reported in the top right panel of Figure 1 is about double that implied by the Bansal and Yaron (2004) model. Given the forward-looking role for continuation values in pricing, the approximate doubling of the long-run responses also doubles the entire trajectory of the shock-price elasticity function.

The direct empirical evidence for the long-run predictability in consumption is weak, however. For instance, see Hansen et al. (2008). This has led one of us to view long-run risk models as models of sentiments (Hansen (2012)) and to explore related models in which investors have skepticism about their model as in Hansen (2007) and Hansen and Sargent (2010). Given the even more prominent role of this forward-looking channel in the Ai et al. (2012) model, it would be valuable either to reconsider the evidence for predictability in growth using other macroeconomic time series or to reduce the degree of the confidence that investors have in the long-run risk model.[17]

Since the long-run risk shocks have a common impact on both technology processes, we use partial elasticities to explore the two channels of influence: i) neutral technology channel and ii) investment-specific channel. As is evident from comparing the panels in rows two and three, the neutral technology channel is much more important for equilibrium consumption as reflected by the larger exposure elasticities. This same channel dominates pricing again with a flat trajectory. The investment-specific channel has only a small and transitory impact on equilibrium consumption dynamics, reflected in elasticities that start small and decay quickly to zero. The partial shock-price elasticities for the investment specific channel are also very small, although they do not decay to zero due to the forward-looking channel of the recursive preference specification.

Another difference between the model used Bansal and Yaron (2004) and that used by Ai et al. (2012) is that Bansal and Yaron introduce stochastic volatility in consumption as an exogenously specified process. There is no counterpart process in the Ai et al. (2012) model, although stochastic volatility could be generated endogenously by the nonlinearity in the equilibrium evolution. Stochastic volatility would be manifested in the state dependence of the shock elasticities. Figure 1 shows that this endogenous source is only noticeable for the partial elasticities associated with the investment channel and these elasticities are small in magnitude.

7.3.2 Elasticities for capital and the associated prices

The Ai et al. (2012) model features differences in valuation of physical and intangible capital. To understand what underlies the differences, we report exposure elasticities for quantities and prices of capital. Figure 2 shows the differential exposures of the two capital stocks, K and K^*, to the underlying shocks, and Figure 3 complements the analysis by depicting the exposures of the corresponding prices of capital, Q and Q^*. The prices are of direct interest,

[17] Hansen et al. (2008) feature corporate earnings but do not report findings for other macroeconomic aggregates.

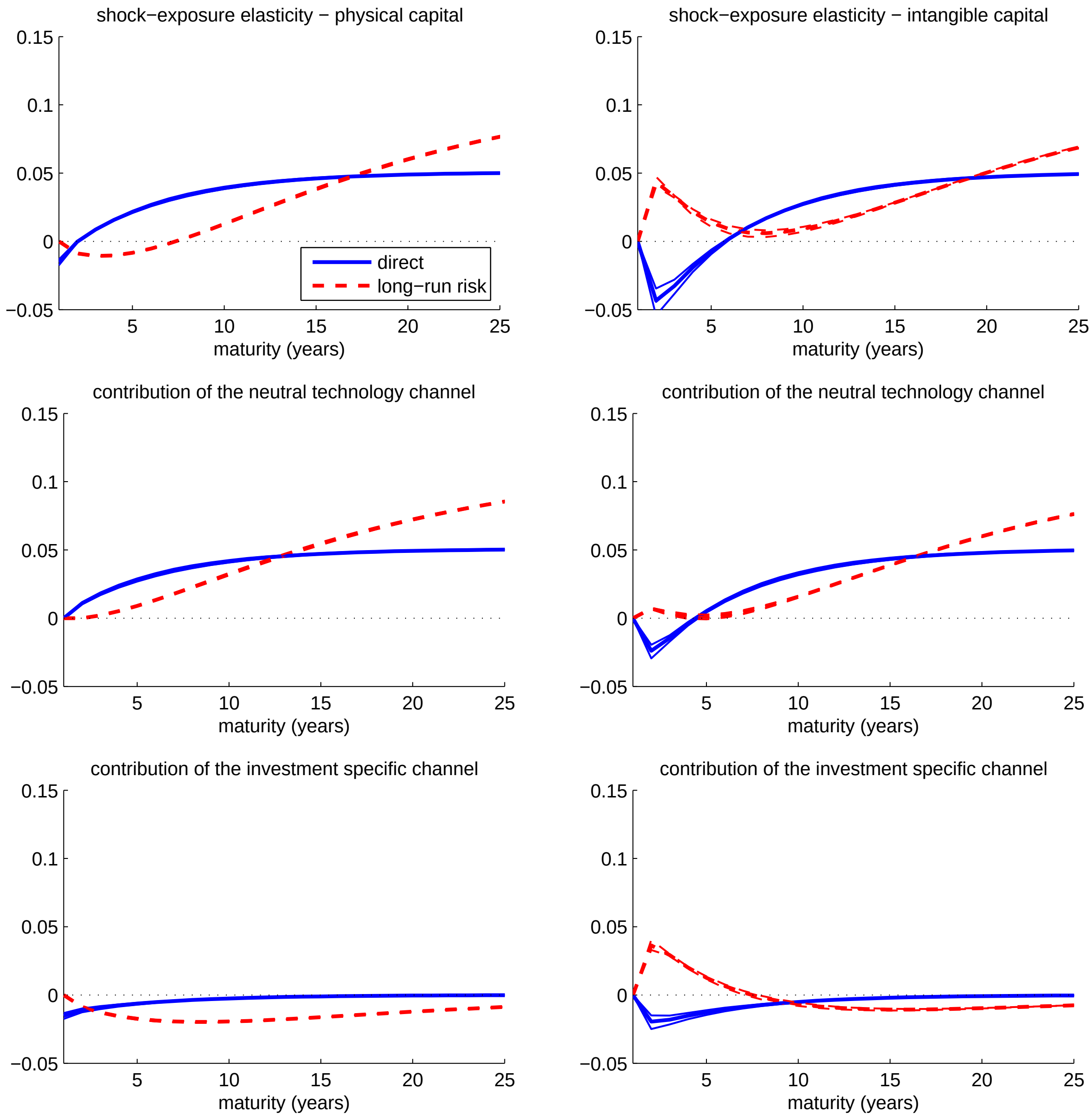

Figure 2: Shock-exposure elasticities for physical and intangible capital. The left panels give the elasticities for physical capital and the right panels give the elasticities for intangible capital. The top row shows elasticities for alternative investment horizons in the original model. The second and third rows show the corresponding partial elasticities using the perturbed specification. The second row features the transmission mechanism for neutral technology shocks, and the third row for investment-specific shocks. To capture the state dependence in the elasticities, we report three quartiles.

but they are also important components to returns to holding capital over time.

The responses of physical capital (top left panel in Figure 2) start small and build up over time, as is typically the case in business cycle models. The long-run responses of intangible

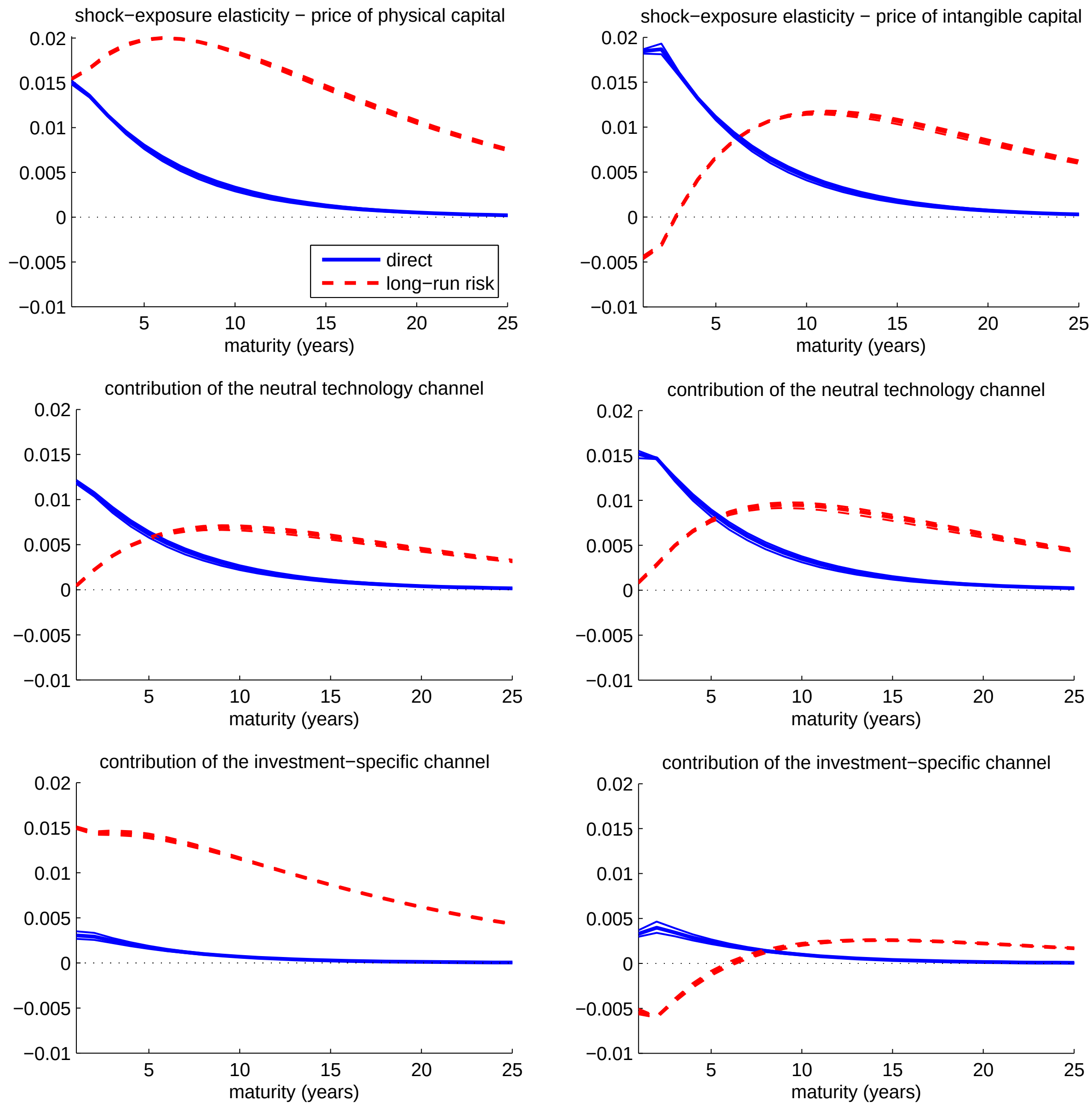

Figure 3: Shock-exposure elasticities for the prices of physical and intangible capital. The left panels give the elasticities for the price of physical capital Q and the right panels give the elasticities for the price of intangible capital Q^*. The top row shows elasticities for alternative investment horizons in the original model. The second and third rows show the corresponding partial elasticities using the perturbed specification. The second row features the transmission mechanism for neutral technology shocks, and the third row for investment-specific shocks. To capture the state dependence in the elasticities, we report three quartiles.

capital (top right panel) necessarily coincide with the positive responses for physical capital but the short-run responses are very different for both shocks. The exposure of intangible capital to the direct shock to the technology processes is initially strongly negative (beginning after a one-period delay), while the exposure elasticity for the long-run risk shock provides a mirror image of the direct shock elasticity in the short run. For the physical capital the short-run exposure elasticities are slightly negative for both shocks but then both eventually become positive and more pronounced.

The partial elasticities in the second and third row of Figure 2 show that the neutral technology shock channel dominates the long-term responses for both capital stocks as might be expected. The investment-specific channel is important for intangible capital for the shorter investment horizons but not for the physical capital stock. In fact, the investment-specific channel inhibits the accumulation of physical capital after a positive shock because new vintages of physical capital are temporarily less productive.

Consider next the exposure elasticities for the prices of the two types of capital reported in Figure 3. Overall these exposure elasticities are much smaller than the corresponding quantity elasticities and are only transitory because prices of capital in this model are stationary. The important differences are in the elasticities to the long-run risk shock. They are initially negative for the price of intangible capital but substantially positive for the physical capital stock. Recall that intangible capital is expected to increase in response to such a shock in contrast to the physical capital stock, but the physical capital stock becomes more valuable. From the partial elasticity plots it is evident that the important differences are accounted for by the investment-specific channel.

Overall, the partial elasticities illuminate the interaction between the quantity and price dynamics for the two types of capital. While the neutral technology shock channel dominates the long-term quantity responses for both capital stocks, the investment-specific channel plays a crucial role in the short-run dynamics after a long-run risk shock. This latter channel drives both the quantity response of intangible capital, and the price response of physical capital.

7.3.3 Exposure elasticities for cumulative returns

The Ai et al. (2012) model generates a large expected return on physical capital, much larger than for intangible capital. To enhance our understanding of the differences in the risk premia associated with the two capital investments, we study the shock-exposure elasticities of their associated excess returns. An n-period return is a cash flow delivered in n periods for a unitary initial investment. Figure 4 plots the shock-exposure elasticities of the cumulative excess returns on physical and intangible capital and their decomposition into partial elasticities.

The elasticities of the cumulative excess returns are flat. The excess return exposures for the physical capital are essentially the same for both shocks, but they are substantially

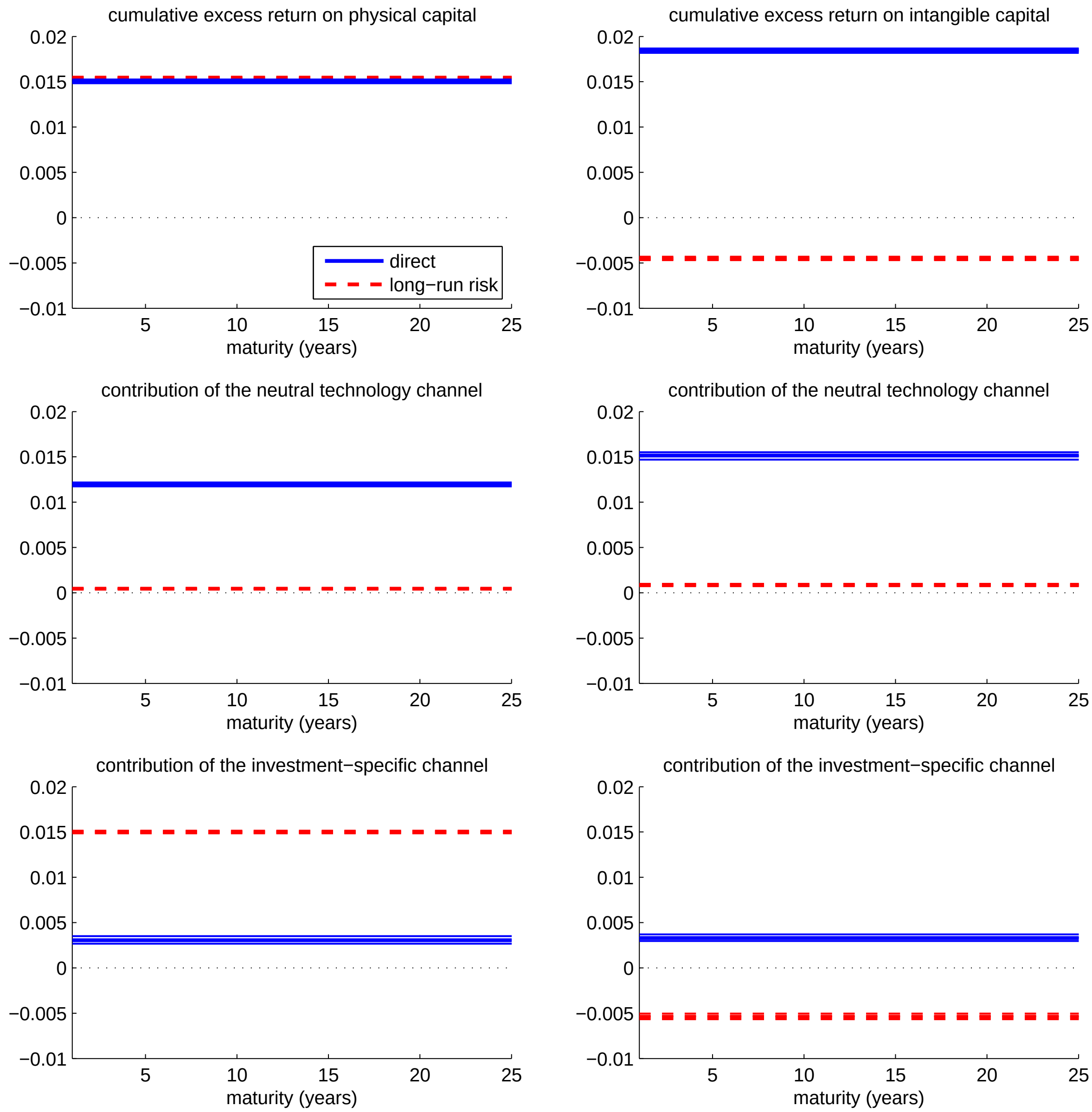

Figure 4: Shock exposure elasticities for cumulative excess returns on physical and intangible capital in the Ai et al. (2012) model. The left column gives the elasticities and partial elasticities for physical capital, the right column for intangible capital. The top row shows elasticities for alternative investment horizons in the original model. The second and third rows show the corresponding partial elasticities using the perturbed specification. The second row features the transmission mechanism for neutral technology shocks, and the third row for investment-specific shocks. To capture the state dependence in the elasticities, we report three quartiles.

different for the excess returns on intangible capital. The exposure elasticity for the long-run risk shock is slightly negative for the intangible capital excess return whereas this exposure elasticity is much bigger in magnitude and positive for the direct shock. Recall that the shock-price elasticities are much larger for the long-run risk shock and hence investors in the physical capital are compensated more than investors in intangible capital. The negative exposure elasticity of intangible capital to the long-run risk shock makes intangible capital a good hedge against such a shock and this is reflected in equilibrium expected returns.

The partial elasticities are particularly revealing for the excess return to the physical capital asset. The primary channel for the large exposure to the direct shock is through the impact of the neutral technology process, while the primary channel for the long-run risk shock is through the impact of the investment-specific technology. Consider the partial elasticities for the long-run risk shock. The impact on the expected returns via the neutral technology process Z is very small. This same impact via the investment-specific technology Z^* is large for the physical capital stock but small and actually negative for the intangible capital stock for the reasons given in our discussion of exposure elasticities for the quantities and prices of capital. This investment-specific channel is the critical one for generating large expected returns for physical capital *vis-à-vis* intangible capital.

In summary, distinguishing price from exposure elasticities and exploring separately channels with two technological inputs reveal key features underlying the differences in risk premia between physical and intangible capital investments. As in the earlier literature, shocks to long-run risk are central to understanding these differences. The partial elasticities for the shock prices are large for the *neutral technology* process. Exposure to the shock to long-run risk in this technology requires compensation . At the same time, excess returns to physical capital have large exposure elasticities to the long-run risk shocks to the *investment-specific technology* process. The large premium for returns to physical capital are generated by the high (in fact perfect) correlation between the two long-run risk shocks.

8 Conclusion and directions for further research

In this paper, we build on our previous work in Hansen and Scheinkman (2012), Borovička et al. (2011), and Hansen (2012) by developing tractable ways to measure the sensitivity of expected cash-flows with macroeconomic components and the associated expected returns to structural shocks. These shock elasticities measure prices and quantities of risk in macro-asset pricing models. They constitute fundamental building blocks for dynamic value decompositions within stochastic equilibrium models. We show that the same approach can be used to deconstruct dynamic entropy measures analyzed in Alvarez and Jermann (2005) and Backus et al. (2011) by taking account of the role of conditioning information for alternative investment horizons.

This paper focuses on tractable implementability in contrast to Hansen and Scheinkman (2012), who provide a more rigorous basis for some of our calculations by taking continuous-time limits. We show that a second-order perturbation approach to model solution along the lines of Holmes (1995) and Lombardo (2010) results in tractable closed-form formulas for the shock elasticities. To support the use of our methodology, we provide a set of Matlab codes[18] that can be integrated with Dynare/Dynare++ and generate the shock elasticities for second-order solutions to dynamic macroeconomic models. It remains to provide more rigor to some of these approximations and to explore other more global approaches to approximation.

This paper also sketches an approach for constructing low-order expansions applicable to economies in which either private agents or policy makers have a concern for robustness. Our emphasis is to show how robustness can have consequences for even first-order approximations to continuation values and for initial terms in expansions for stochastic discount factors and the resulting elasticities. We suspect this same approach will also provide additional insights into the study and design of robust macroeconomic policy rules.

In this paper we used shock elasticities as interpretive diagnostics for comparing the asset valuation implications of alternative macroeconomic models and for understanding better the channels by which exogenous shocks influence equilibrium outcomes. We have not described formally shock identification and statistical uncertainty in our measurements, but we should be able to build on the related macroeconomic literature on identification and inference for impulse response functions. Also methods like the ones we describe here should provide useful complements for the recent empirical work by Binsbergen et al. (2011) and others on the decomposition of cash flow contributions to equity returns for alternative investment horizons.

[18] See https://files.nyu.edu/jb4457/public/software.html.

Appendix

A Conditional expectations of multiplicative functionals

Let $X = (X_1', X_2')'$ be a $2n \times 1$ vector of states, $W \sim N(0, I)$ a $k \times 1$ vector of independent Gaussian shocks, and $\mathcal{F}_t$ the filtration generated by $(X_0, W_1, \ldots, W_t)$. In this appendix, we show that given the law of motion from equation (28)

$$\begin{aligned}
X_{1,t+1} &= \Theta_{10} + \Theta_{11} X_{1,t} + \Lambda_{10} W_{t+1} \\
X_{2,t+1} &= \Theta_{20} + \Theta_{21} X_{1,t} + \Theta_{22} X_{2,t} + \Theta_{23} \left(X_{1,t} \otimes X_{1,t} \right) + \\
&\quad + \Lambda_{20} W_{t+1} + \Lambda_{21} \left(X_{1,t} \otimes W_{t+1} \right) + \Lambda_{22} \left(W_{t+1} \otimes W_{t+1} \right)
\end{aligned} \tag{33}$$

and a multiplicative functional $M_t = \exp(Y_t)$ whose additive increment is given in equation (29):

$$\begin{aligned}
Y_{t+1} - Y_t &= \Gamma_0 + \Gamma_1 X_{1,t} + \Gamma_2 X_{2,t} + \Gamma_3 \left(X_{1,t} \otimes X_{1,t} \right) + \\
&\quad + \Psi_0 W_{t+1} + \Psi_1 \left(X_{1,t} \otimes W_{1,t+1} \right) + \Psi_2 \left(W_{t+1} \otimes W_{t+1} \right),
\end{aligned} \tag{34}$$

we can write the conditional expectation of M as

$$\log E \left[M_t \mid \mathcal{F}_0 \right] = \left(\bar{\Gamma}_0 \right)_t + \left(\bar{\Gamma}_1 \right)_t X_{1,0} + \left(\bar{\Gamma}_2 \right)_t X_{2,0} + \left(\bar{\Gamma}_3 \right)_t \left(X_0 \otimes X_0 \right) \tag{35}$$

where $\left(\bar{\Gamma}_i \right)_t$ are constant coefficients to be determined.

The dynamics given by (33)–(34) embeds the perturbation approximation constructed in Section 4 as a special case. The Θ and Λ matrices needed to map the perturbed model into the above structure are constructed from the first and second derivatives of the function $\psi(x, w, \mathsf{q})$ that captures the law of motion of the model, evaluated at $(\bar{x}, 0, 0)$:

$$\begin{array}{llll}
\Theta_{10} = \psi_q & \Theta_{11} = \psi_x & \Lambda_{10} = \psi_w & \\
\Theta_{20} = \psi_{qq} & \Theta_{21} = 2\psi_{xq} & \Theta_{22} = \psi_x & \Theta_{23} = \psi_{xx} \\
\Lambda_{20} = 2\psi_{wq} & \Lambda_{21} = 2\psi_{xw} & \Lambda_{22} = \psi_{ww} &
\end{array}$$

where the notation for the derivatives is defined in Appendix A.2.

A.1 Definitions

To simplify work with Kronecker products, we define two operators vec and $\text{mat}_{m,n}$. For an $m \times n$ matrix H, $\text{vec}(H)$ produces a column vector of length mn created by stacking the columns of H:

$$h_{(j-1)m+i} = \left[\text{vec(H)} \right]_{(j-1)m+i} = H_{ij}.$$

For a vector (column or row) h of length mn, $\mathrm{mat}_{m,n}(h)$ produces an $m \times n$ matrix H created by 'columnizing' the vector:

$$H_{ij} = [\mathrm{mat}_{m,n}(h)]_{ij} = h_{(j-1)m+i}.$$

We drop the m, n subindex if the dimensions of the resulting matrix are obvious from the context. For a square matrix A, define the sym operator as

$$\mathrm{sym}(A) = \frac{1}{2}\left(A + A'\right).$$

Apart from the standard operations with Kronecker products, notice that the following is true. For a row vector $H_{1\times nk}$ and column vectors $X_{n\times 1}$ and $W_{n\times 1}$

$$H(X \otimes W) = X' [\mathrm{mat}_{k,n}(H)]' W$$

and for a matrix $A_{n\times k}$, we have

$$X'AW = \left(\mathrm{vec}A'\right)'(X \otimes W). \tag{36}$$

Also, for $A_{n\times n}$, $X_{n\times 1}$, $K_{k\times 1}$, we have

$$\begin{aligned}(AX) \otimes K &= (A \otimes K) X \\ K \otimes (AX) &= (K \otimes A) X.\end{aligned}$$

Finally, for column vectors $X_{n\times 1}$ and $W_{k\times 1}$,

$$(AX) \otimes (BW) = (A \otimes B)(X \otimes W)$$

and

$$(BW) \otimes (AX) = [B \otimes A_{\bullet j}]_{j=1}^{n} (X \otimes W)$$

where

$$[B \otimes A_{\bullet j}]_{j=1}^{n} = [B \otimes A_{\bullet 1} \quad B \otimes A_{\bullet 2} \quad \ldots \quad B \otimes A_{\bullet n}].$$

A.2 Concise notation for derivatives

Consider a vector function $f(x, w)$ where x and w are column vectors of length m and n, respectively. The first-derivative matrix f_i where $i = x, w$ is constructed as follows. The k-th row $[f_i]_{k\bullet}$ corresponds to the derivative of the k-th component of f

$$[f_i(x, w)]_{k\bullet} = \frac{\partial f^{(k)}}{\partial i'}(x, w).$$

Similarly, the second-derivative matrix is the matrix of vectorized and stacked Hessians of individual components with k-th row

$$[f_{ij}(x,w)]_{k\bullet} = \left(\mathrm{vec}\frac{\partial^2 f^{(k)}}{\partial j \partial i'}(x,w)\right)'.$$

It follows from formula (36) that, for example,

$$x'\left(\frac{\partial^2 f^{(k)}}{\partial x \partial w'}(x,w)\right)w = \left(\mathrm{vec}\frac{\partial^2 f^{(k)}}{\partial w \partial x'}(x,w)\right)'(x\otimes w) = [f_{xw}(x,w)]_{k\bullet}(x\otimes w).$$

A.3 Conditional expectations

Notice that a complete-the squares argument implies that, for a $1\times k$ vector A, a $1\times k^2$ vector B, and a scalar function $f(w)$,

$$\begin{aligned}
& E\left[\exp\left(B\left(W_{t+1}\otimes W_{t+1}\right)+AW_{t+1}\right)f\left(W_{t+1}\right)\mid \mathcal{F}_t\right] = \\
&\quad = E\left[\exp\left(\frac{1}{2}W_{t+1}'\left(\mathrm{mat}_{k,k}\left(2B\right)\right)W_{t+1}+AW_{t+1}\right)f\left(W_{t+1}\right)\mid \mathcal{F}_t\right] \\
&\quad = \left|I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2B\right)\right]\right|^{-1/2}\exp\left(\frac{1}{2}A\left(I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2B\right)\right]\right)^{-1}A'\right)\tilde{E}\left[f\left(W_{t+1}\right)\mid \mathcal{F}_t\right]
\end{aligned} \tag{37}$$

where $\tilde{\cdot}$ is a measure under which

$$W_{t+1}\sim N\left(\left(I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2B\right)\right]\right)^{-1}A',\left(I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2B\right)\right]\right)^{-1}\right).$$

We start by utilizing formula (37) to compute

$$\begin{aligned}
\bar{Y}\left(X_t\right) &= \log E\left[\exp\left(Y_{t+1}-Y_t\right)\mid \mathcal{F}_t\right] = \\
&= \Gamma_0+\Gamma_1X_{1,t}+\Gamma_2X_{2,t}+\Gamma_3\left(X_{1,t}\otimes X_{1,t}\right)+ \\
&\quad +\log E\left[\exp\left(\left[\Psi_0+X_{1t}'\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right]'\right]W_{t+1}+\frac{1}{2}W_{t+1}'\left[\mathrm{mat}_{k,k}\left(\Psi_2\right)\right]W_{t+1}\right)\mid \mathcal{F}_t\right] \\
&= \Gamma_0+\Gamma_1X_{1,t}+\Gamma_2X_{2,t}+\Gamma_3\left(X_{1,t}\otimes X_{1,t}\right)- \\
&\quad -\frac{1}{2}\log\left|I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right|+\frac{1}{2}\mu'\left(I_k-\mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\mu
\end{aligned}$$

with μ defined as

$$\mu = \Psi_0'+\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right]X_{1,t}.$$

Reorganizing terms, we obtain

$$\bar{Y}\left(X_t\right) = \bar{\Gamma}_0+\bar{\Gamma}_1X_{1,t}+\bar{\Gamma}_2X_{2,t}+\bar{\Gamma}_3\left(X_{1,t}\otimes X_{1,t}\right)$$

where

$$
\begin{aligned}
\bar{\Gamma}_0 &= \Gamma_0 - \frac{1}{2}\log\left|I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right| + \frac{1}{2}\Psi_0\left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\Psi_0' \\
\bar{\Gamma}_1 &= \Gamma_1 + \Psi_0\left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right] \\
\bar{\Gamma}_2 &= \Gamma_2 \\
\bar{\Gamma}_3 &= \Gamma_3 + \frac{1}{2}\mathrm{vec}\left[\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right]'\left(I_k - \mathrm{sym}\left[\mathrm{mat}_{k,k}\left(2\Psi_2\right)\right]\right)^{-1}\left[\mathrm{mat}_{k,n}\left(\Psi_1\right)\right]\right]'.
\end{aligned}
\tag{38}
$$

For the set of parameters $\mathcal{P} = (\Gamma_0, \ldots, \Gamma_3, \Psi_0, \ldots, \Psi_2)$, equations (38) define a mapping

$$
\bar{\mathcal{P}} = \bar{\mathcal{E}}\left(\mathcal{P}\right),
$$

with all $\bar{\Psi}_j = 0$. We now substitute the law of motion for X_1 and X_2 to produce $\bar{Y}\left(X_t\right) = \tilde{Y}\left(X_{t-1}, W_t\right)$. It is just a matter of algebraic operations to determine that

$$
\begin{aligned}
\tilde{Y}\left(X_{t-1}, W_t\right) &= \log E\left[\exp\left(Y_{t+1} - Y_t\right) \mid \mathcal{F}_t\right] = \\
&= \tilde{\Gamma}_0 + \tilde{\Gamma}_1 X_{1,t-1} + \tilde{\Gamma}_2 X_{2,t-1} + \tilde{\Gamma}_3\left(X_{1,t-1} \otimes X_{1,t-1}\right) \\
&\quad + \tilde{\Psi}_0 W_t + \tilde{\Psi}_1\left(X_{1,t-1} \otimes W_t\right) + \tilde{\Psi}_2\left(W_t \otimes W_t\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\Gamma}_0 &= \bar{\Gamma}_0 + \bar{\Gamma}_1\Theta_{10} + \bar{\Gamma}_2\Theta_{20} + \bar{\Gamma}_3\left(\Theta_{10} \otimes \Theta_{10}\right) \\
\tilde{\Gamma}_1 &= \bar{\Gamma}_1\Theta_{11} + \bar{\Gamma}_2\Theta_{21} + \bar{\Gamma}_3\left(\Theta_{10} \otimes \Theta_{11} + \Theta_{11} \otimes \Theta_{10}\right) \\
\tilde{\Gamma}_2 &= \bar{\Gamma}_2\Theta_{22} \\
\tilde{\Gamma}_3 &= \bar{\Gamma}_2\Theta_{23} + \bar{\Gamma}_3\left(\Theta_{11} \otimes \Theta_{11}\right) \\
\tilde{\Psi}_0 &= \bar{\Gamma}_1\Lambda_{10} + \bar{\Gamma}_2\Lambda_{20} + \bar{\Gamma}_3\left(\Theta_{10} \otimes \Lambda_{10} + \Lambda_{10} \otimes \Theta_{10}\right) \\
\tilde{\Psi}_1 &= \bar{\Gamma}_2\Lambda_{21} + \bar{\Gamma}_3\left(\Theta_{11} \otimes \Lambda_{10} + \left[\Lambda_{10} \otimes \left(\Theta_{11}\right)_{\bullet j}\right]_{j=1}^{n}\right) \\
\tilde{\Psi}_2 &= \bar{\Gamma}_2\Lambda_{22} + \bar{\Gamma}_3\left(\Lambda_{10} \otimes \Lambda_{10}\right).
\end{aligned}
\tag{39}
$$

This set of equations defines the mapping

$$
\tilde{\mathcal{P}} = \tilde{\mathcal{E}}\left(\bar{\mathcal{P}}\right).
$$

A.4 Iterative formulas

We can write the conditional expectation in (35) recursively as

$$
\log E\left[M_t \mid \mathcal{F}_0\right] = \log E\left[\exp\left(Y_1 - Y_0\right) E\left[\frac{M_t}{M_1} \mid \mathcal{F}_1\right] \mid \mathcal{F}_0\right].
$$

Given the mappings $\bar{\mathcal{E}}$ and $\tilde{\mathcal{E}}$, we can therefore express the coefficients $\bar{\mathcal{P}}$ in (35) using the recursion

$$\bar{\mathcal{P}}_t = \bar{\mathcal{E}}\left(\mathcal{P} + \tilde{\mathcal{E}}\left(\bar{\mathcal{P}}_{t-1}\right)\right)$$

where the addition is by coefficients and all coefficients in $\bar{\mathcal{P}}_0$ are zero matrices.

B Shock elasticity calculations

In this appendix, we provide details on some of the calculations underlying the derived shock elasticity formulas.

B.1 Shock elasticities under the convenient functional form

To calculate the shock elasticities in Section 6.1, utilize the formulas derived in Appendix A to deduce the one-period change of measure

$$\log L_{1,t} = \log M_1 + \log E\left(\frac{M_t}{M_1} \mid X_1\right) - \log E\left[M_1 E\left(\frac{M_t}{M_1} \mid X_1\right) \mid X_0 = x\right].$$

In particular, following the set of formulas (39), define

$$\begin{aligned}
\mu_{0,t} &= \left[\Psi_1 + \Phi_{1,t-1}^{*}\Lambda_{1,0} + \Phi_{2,t-1}^{*}\Lambda_{20} + \Phi_{3,t-1}^{*}\left(\Theta_{10}\otimes\Lambda_{10} + \Lambda_{10}\otimes\Theta_{10}\right)\right]' \\
\mu_{1,t} &= \operatorname{mat}_{k,n}\left[\Psi_1 + \Phi_{2,t-1}^{*}\Lambda_{21} + \Phi_{3,t-1}^{*}\left(\Theta_{11}\otimes\Lambda_{10} + \left[\Lambda_{10}\otimes\left(\Theta_{11}\right)_{\bullet j}\right]_{j=1}^{n}\right)\right] \\
\mu_{2,t} &= \operatorname{sym}\left[\operatorname{mat}_{k,k}\left(\Psi_2 + \bar{\Gamma}_2\Lambda_{22} + \bar{\Gamma}_3\left(\Lambda_{10}\otimes\Lambda_{10}\right)\right)\right].
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\log L_{1,t} &= \left(\mu_{0,t} + \mu_{1,t}X_{1,0}\right)' W_1 + \left(W_1\right)' \mu_{2,t} W_1 - \\
&\quad -\frac{1}{2}\log E\left[\exp\left(\left(\mu_{0,t} + \mu_{1,t}X_{1,0}\right)' W_1 + \left(W_1\right)' \mu_{2,t} W_1\right) \mid \mathcal{F}_0\right].
\end{aligned}$$

Expression (37) then implies that

$$\begin{aligned}
E\left[L_{1,t}W_1 \mid \mathcal{F}_0\right] &= \widetilde{E}\left[W_1 \mid \mathcal{F}_0\right] = \\
&= \left(I_k - 2\mu_{2,t}\right)^{-1}\left(\mu_{0,t} + \mu_{1t}X_{1,0}\right).
\end{aligned}$$

The variance of W_1 under the $\tilde{\cdot}$ measure satisfies

$$\widetilde{\Sigma}_t = \left(I_k - 2\operatorname{sym}\left[\operatorname{mat}_{k,k}\left(\Psi_2 + \bar{\Gamma}_2\Lambda_{22} + \bar{\Gamma}_3\left(\Lambda_{10}\otimes\Lambda_{10}\right)\right)\right]\right)^{-1}.$$

B.2 Approximation of the shock elasticity function

In Section 4, we constructed the approximation of the shock elasticity function $\varepsilon(x,t)$. The first-order approximation is constructed by differentiating the elasticity function under the perturbed dynamics

$$\varepsilon_1(X_{1,0},t) = \frac{d}{d\mathsf{q}}\, \alpha(X_0(\mathsf{q})) \cdot \left. \frac{E\left[M_t(\mathsf{q})\, W_1 \mid X_0 = x\right]}{E\left[M_t(\mathsf{q}) \mid X_0 = x\right]} \right|_{\mathsf{q}=0} = \alpha(\bar{x}) \cdot E\left[Y_{1,t} W_1 \mid X_0 = x\right].$$

The first-derivative process $Y_{1,t}$ can be expressed in terms of its increments, and we obtain a state-independent function

$$\varepsilon_1(t) = \alpha(\bar{x}) \cdot E\left[\sum_{j=1}^{t-1} \kappa_x (\psi_x)^{j-1} \psi_w + \kappa_w\right]'$$

where $\kappa_x, \psi_x, \kappa_w, \psi_w$ are derivative matrices evaluated at the steady state $(\bar{x}, 0)$.

Continuing with the second derivative, we have

$$\begin{aligned}
\varepsilon_2(X_{1,0}, X_{2,0}, t) &= \frac{d^2}{d\mathsf{q}^2}\, \alpha(X_0(\mathsf{q})) \cdot \left. \frac{E\left[M_t(\mathsf{q})\, W_1 \mid X_0 = x\right]}{E\left[M_t(\mathsf{q}) \mid X_0 = x\right]} \right|_{\mathsf{q}=0} = \\
&= \alpha(\bar{x}) \cdot \left\{ E\left[(Y_{1,t})^2 W_1 + Y_{2,t} W_1 \mid \mathcal{F}_0\right] - 2E\left[Y_{1,t} W_1 \mid \mathcal{F}_0\right] E\left[Y_{1,t} \mid \mathcal{F}_0\right] \right\} + \\
&\quad +2\left[\frac{\partial \alpha}{\partial x'}(\bar{x})\right] X_{1,0} \cdot E\left[Y_{1,t} W_1 \mid \mathcal{F}_0\right].
\end{aligned}$$

However, notice that

$$\begin{aligned}
E\left[(Y_{1,t})^2 W_1 \mid \mathcal{F}_0\right] &= 2\left(\sum_{j=0}^{t-1} \kappa_x (\psi_x)^j X_{1,0}\right) \left(\sum_{j=1}^{t-1} \kappa_x (\psi_x)^{j-1} \psi_w + \kappa_w\right)' \\
E\left[Y_{1,t} W_1 \mid \mathcal{F}_0\right] &= \left(\sum_{j=1}^{t-1} \kappa_x (\psi_x)^{j-1} \psi_w + \kappa_w\right)' \\
E\left[Y_{1,t} \mid \mathcal{F}_0\right] &= \sum_{j=0}^{t-1} \kappa_x (\psi_x)^j X_{1,0}
\end{aligned}$$

and thus

$$E\left[(Y_{1,t})^2 W_1 \mid \mathcal{F}_0\right] - 2E\left[Y_{1,t} W_1 \mid \mathcal{F}_0\right] E\left[Y_{1,t} \mid \mathcal{F}_0\right] = 0.$$

The second-order term in the approximation of the shock elasticity function thus simplifies to

$$\varepsilon_2(X_{1,0}, X_{2,0}, t) = \alpha(\bar{x}) \cdot E\left[Y_{2,t} W_1 \mid \mathcal{F}_0\right] + 2\left[\frac{\partial \alpha}{\partial x'}(\bar{x})\right] X_{1,0} \cdot E\left[Y_{1,t} W_1 \mid \mathcal{F}_0\right]. \tag{40}$$

The expression for the first term on the right-hand side is

$$\begin{aligned} E\left[Y_{2,t}W_1 \mid \mathcal{F}_0\right] &= E\left[\sum_{j=0}^{t-1}\left(Y_{2,j+1}-Y_{2,j}\right)W_1 \mid \mathcal{F}_0\right] = 2\mathrm{mat}_{k,n}\left(\kappa_{xw}\right)X_{1,0} + \\ & +2\sum_{j=1}^{t-1}\left[\psi_w'\left(\psi_x'\right)^{j-1}\mathrm{mat}_{n,n}\left(\kappa_{xx}\right)\left(\psi_x\right)^j + \mathrm{mat}_{k,n}\left[\kappa_x\left(\psi_x\right)^{j-1}\psi_{xw}\right]\right]X_{1,0} \\ & +2\sum_{j=1}^{t-1}\sum_{k=1}^{j-1}\left[\psi_w'\left(\psi_x'\right)^{k-1}\mathrm{mat}_{n,n}\left[\kappa_x\left(\psi_x\right)^{j-k-1}\psi_{xx}\right]\left(\psi_x\right)^k\right]X_{1,0}. \end{aligned}$$

To obtain this result, notice that repeated substitution for $Y_{1,j+1}-Y_{1,j}$ into the above formula yields a variety of terms but only those containing $X_{1,0}\otimes W_1$ have a nonzero conditional expectation when interacted with W_1.

B.3 Partial shock elasticities

In Section 4.4, we constructed the first-order approximation of the partial shock elasticity function, and argued that it is equivalent to the second-order approximation of the shock elasticity function.

Recall that for a shock vector $\widetilde{W}$ that is independent of W,

$$\widetilde{\varepsilon}(x,t) = \widetilde{\alpha}(x)\cdot\frac{E\left[M_tY_{1,t}\widetilde{W}_1 \mid X_0 = x\right]}{E\left[M_t \mid X_0 = x\right]}$$

where

$$\begin{aligned} Y_{1,t} &= \sum_{j=0}^{t-1}\left(Y_{1,j+1}-Y_{1,j}\right) = \widetilde{\kappa}_{\widetilde{w}}\left(X_0,W_1,0,0\right)\widetilde{W}_1 + \\ & +\sum_{j=1}^{t-1}\widetilde{\kappa}_x\left(X_j,W_{j+1},0,0\right)\left(\prod_{k=1}^{j-1}\widetilde{\psi}_x\left(X_k,W_{k+1},0,0\right)\right)\widetilde{\psi}_{\widetilde{w}}\left(X_0,W_1,0,0\right)\widetilde{W}_1 = \\ &= \sum_{j=0}^{t-1}\left(\widetilde{Y}_{1,j+1}-\widetilde{Y}_{1,j}\right)\widetilde{W}_1 \end{aligned}$$

where $\widetilde{Y}_{1,t} = E\left[Y_{1,t}\left(\widetilde{W}_1\right)' \mid \mathcal{F}_t\right]$, with $\mathcal{F}_t$ being the σ-algebra generated by $(X_0, W_1, \ldots, W_t)$. Once $\widetilde{W}_1$ is conditioned out, we proceed with the parameterization of the sensitivity to the shock W given by (20), and follow the approximations from Section 4.

We construct a first-order approximation of the partial shock elasticity function

$$\widetilde{\varepsilon}\left(x,t\right) \approx \widetilde{\varepsilon}_0\left(x,t\right) + \mathsf{q}\widetilde{\varepsilon}_1\left(x,t\right).$$

The zero-th order approximation to the partial shock elasticity function evaluates $\widetilde{Y}_{1,t}$ at the deterministic steady state

$$\widetilde{\varepsilon}_0(x,t) = \widetilde{\alpha}(\bar{x}) \cdot \left[\sum_{j=1}^{t-1} \widetilde{\kappa}_x(\bar{x},0,0,0) \left[\widetilde{\psi}_x(\bar{x},0,0,0) \right]^{j-1} \widetilde{\psi}_{\widetilde{w}}(\bar{x},0,0,0) + \widetilde{\kappa}_{\widetilde{w}}(\bar{x},0,0,0) \right].$$

Notice that derivatives $\widetilde{\kappa}_x$ and $\widetilde{\psi}_x$ evaluated at the deterministic steady state coincide with κ_x and ψ_x. In line with the interesting special case from Section 2.5.2, consider the following positioning of the shock vector $\widetilde{W}$:

$$\begin{aligned} \widetilde{\psi}(x,w,\mathsf{q}\widetilde{w},\mathsf{q}) &\equiv \psi\left(x, w + \mathsf{q}\Upsilon'\widetilde{w}, \mathsf{q}\right) \\ \widetilde{\kappa}(x,w,\mathsf{q}\widetilde{w},\mathsf{q}) &\equiv \kappa\left(x, w + \mathsf{q}\Upsilon'\widetilde{w}, \mathsf{q}\right). \end{aligned} \tag{41}$$

Then the derivatives evaluated at $\mathsf{q} = 0$ satisfy:

$$\begin{aligned} \widetilde{\psi}_{\widetilde{w}}(x,w,0,0) &\equiv \psi_w(x,w,0)\,\Upsilon' \\ \widetilde{\kappa}_{\widetilde{w}}(x,w,0,0) &\equiv \kappa_w(x,w,0)\,\Upsilon', \end{aligned}$$

and post-multiplying by Υ yields expressions (14)–(15). Choosing the exposure direction vector as $\widetilde{\alpha}_h = \Upsilon\alpha_h$, we obtain $\widetilde{\varepsilon}_0(x,t) = \varepsilon_1(x,t)$. By constructing alternative configurations of the shock vector $\widetilde{W}$ in the functions $\widetilde{\psi}$ and $\widetilde{\kappa}$, the partial elasticity function allows us to study a richer class of dynamic responses.

In order to construct the first-order approximation, notice that

$$\begin{aligned} \widetilde{\varepsilon}_1(X_{1,0},t) &= \left. \frac{d}{d\mathsf{q}} \widetilde{\alpha}(X_0) \frac{E\left[M_t \left(\widetilde{Y}_{1,t} \right)' \mid X_0 = x \right]}{E\left[M_t \mid X_0 = x \right]} \right|_{\mathsf{q}=0} = \\ &= \widetilde{\alpha}(\bar{x}) \cdot E\left[\left. \frac{d}{d\mathsf{q}} \left(\widetilde{Y}_{1,t} \right)' \right|_{\mathsf{q}=0} \mid \mathcal{F}_0 \right] + \frac{\partial \widetilde{\alpha}}{\partial x'}(\bar{x})\, X_{1,0} \cdot \widetilde{\varepsilon}_0(x,t). \end{aligned}$$

The second term on the second line corresponds to one half of the second term in expression (40). It remains to express the derivative in the first term. Recall that

$$\begin{aligned} \widetilde{Y}_{1,1}(\mathsf{q}) &= \widetilde{\kappa}_{\widetilde{w}}(X_0(\mathsf{q}), \mathsf{q}W_1, 0, 0) \\ \widetilde{Y}_{1,j+1}(\mathsf{q}) - \widetilde{Y}_{1,j}(\mathsf{q}) &= \widetilde{\kappa}_x(X_j(\mathsf{q}), \mathsf{q}W_{j+1}, 0, 0) \\ &\quad \left(\prod_{k=1}^{j-1} \widetilde{\psi}_x(X_k(\mathsf{q}), \mathsf{q}W_{k+1}, 0, 0) \right) \widetilde{\psi}_{\widetilde{w}}(X_0(\mathsf{q}), \mathsf{q}W_1, 0, 0), \quad j > 0. \end{aligned}$$

We then have

$$E\left[\left. \frac{d}{d\mathsf{q}} \left(\widetilde{Y}_{1,1} \right)' \right|_{\mathsf{q}=0} \mid \mathcal{F}_0 \right] = \mathrm{mat}_{\widetilde{k},n}(\widetilde{\kappa}_{x\widetilde{w}})\, X_{1,0}$$

and, for $j > 0$,

$$E\left[\frac{d}{d\mathsf{q}}\left(\widetilde{Y}_{1,j+1}(\mathsf{q}) - \widetilde{Y}_{1,j}(\mathsf{q})\right)'\Big|_{\mathsf{q}=0} \mid \mathcal{F}_0\right] =$$

$$= \widetilde{\psi}'_{\widetilde{w}}\left(\widetilde{\psi}'_x\right)^{j-1} \mathrm{mat}_{n,n}\left(\widetilde{\kappa}_{xx}\right) E\left[X_{1,j} \mid \mathcal{F}_0\right] + \mathrm{mat}_{\widetilde{k},n}\left[\widetilde{\kappa}_x\left(\widetilde{\psi}_x\right)^{j-1}\widetilde{\psi}_{x\widetilde{w}}\right] X_{1,0} +$$

$$+\sum_{k=1}^{j-1}\widetilde{\psi}'_{\widetilde{w}}\left(\widetilde{\psi}'_x\right)^{k-1}\mathrm{mat}_{n,n}\left[\widetilde{\kappa}_x\left(\widetilde{\psi}_x\right)^{j-k-1}\widetilde{\psi}_{xx}\right] E\left[X_{1,k} \mid \mathcal{F}_0\right].$$

Collecting the terms and substituting for $E\left[X_{1,k} \mid \mathcal{F}_0\right]$, we obtain a result that is analogous to the first term of $\frac{1}{2}\varepsilon_2\left(X_{1,0}, X_{2,0}, t\right)$ in expression (40):

$$E\left[\frac{d}{d\mathsf{q}}\left(\widetilde{Y}_{1,t}\right)'\Big|_{\mathsf{q}=0} \mid \mathcal{F}_0\right] =$$

$$= E\left[\frac{d}{d\mathsf{q}}\sum_{j=0}^{t-1}\left(\widetilde{Y}_{1,j+1} - \widetilde{Y}_{1,j}\right)'\Big|_{\mathsf{q}=0} \mid \mathcal{F}_0\right] = \mathrm{mat}_{\widetilde{k},n}\left(\widetilde{\kappa}_{x\widetilde{w}}\right) X_{1,0} +$$

$$+\sum_{j=1}^{t-1}\left[\widetilde{\psi}'_{\widetilde{w}}\left(\widetilde{\psi}'_x\right)^{j-1}\mathrm{mat}_{n,n}\left(\widetilde{\kappa}_{xx}\right)\left(\widetilde{\psi}_x\right)^{j} + \mathrm{mat}_{\widetilde{k},n}\left[\widetilde{\kappa}_x\left(\widetilde{\psi}_x\right)^{j-1}\widetilde{\psi}_{x\widetilde{w}}\right]\right] X_{1,0} +$$

$$+\sum_{j=1}^{t-1}\sum_{k=1}^{j-1}\widetilde{\psi}'_{\widetilde{w}}\left(\widetilde{\psi}'_x\right)^{k-1}\mathrm{mat}_{n,n}\left[\widetilde{\kappa}_x\left(\widetilde{\psi}_x\right)^{j-k-1}\widetilde{\psi}_{xx}\right]\left(\widetilde{\psi}_x\right)^{k} X_{1,0}.$$

Once again, if we construct $\widetilde{\psi}$ and $\widetilde{\kappa}$ to satisfy (41), then all partial derivatives of $\widetilde{\kappa}$ and $\widetilde{\psi}$ with respect to $\widetilde{W}$ correspond to those of κ and ψ with respect to W multiplied by Υ'. When we choose $\widetilde{\alpha}_h = \Upsilon\alpha_h$, we obtain

$$\widetilde{\varepsilon}_1\left(X_{1,0}, t\right) = \frac{1}{2}\varepsilon_2\left(X_{1,0}, t\right)$$

and thus the approximations coincide.

Moreover, an inspection of the above expressions for $\widetilde{\varepsilon}_0(x,t)$ and $\widetilde{\varepsilon}_1(x_{1,\cdot},t)$ reveals that all terms are linear in a single partial derivative with respect to $\widetilde{W}$. Partial elasticities will thus be additive in shock configurations, and we can naturally additively decompose elasticities by positioning shocks in alternative locations in the functions $\widetilde{\psi}$ and $\widetilde{\kappa}$.

C Parameterization of the Ai et al. (2012) model

For sake of illustration and comparability, we use the same parameters as used by Ai et al. (2012) in their extended model with adjustment costs, $H(I^*, K)$, in the accumulation of intangible capital. The production technology for turning intangible capital into new vintages of physical capital is

Preferences		
Time preference	β	0.971
Risk aversion	γ	10
Intertemporal elasticity of substitution	ρ^{-1}	2
Technology		
Capital share	ν	0.3
Depreciation rate of physical capital	λ	0.11
Depreciation rate of intangible capital	λ^*	0.11
Weight on physical investment	φ	0.88
Elasticity of substitution in $G(I, K^*)$	η	2.5
Elasticity of substitution in $H(I^*, K)$	ξ	5
Scaling parameters $H(I^*, K)$	a_1	0.6645
	a_2	-0.0324
Exogenous shocks		
Mean growth rate	Γ_0	0.02
	Γ_0^*	0
Volatility of the direct shock	Ψ	[0.0508 0]
Autocorrelation of the long-run risk process	$(\Theta_1)_{1,1}$	0.925
Volatility of the long-run risk shock	Λ_1	[0 0.008636]

Table 1: Parameterization of the Ai et al. (2012) model. All parameters correspond to a calibration at the annual frequency.

specified by the CES aggregator

$$G\left(I, K^*\right) = \left(\varphi I^{1-1/\eta} + (1-\varphi)\left(K^*\right)^{1-1/\eta}\right)^{\frac{1}{1-1/\eta}}$$

and the adjustment cost function for the production of new intangible capital is chosen to be

$$H\left(I^*, K\right) = \left[\frac{a_1}{1-1/\xi}\left(\frac{I^*}{K}\right)^{1-1/\xi} + a_2\right] K$$

where a_1 and a_2 are chosen so as to assure that $H\left(\bar{I}^*, \bar{K}\right) = H_{I^*}\left(\bar{I}^*, \bar{K}\right) = 1$ for steady state values $\bar{I}^*$ and $\bar{K}$. The parameter values are summarized in Table 1.

References

Ai, Hengjie, Mariano Massimiliano Croce, and Kai Li. 2012. Toward a Quantitative General Equilibrium Asset Pricing Model with Intangible Capital. *Review of Financial Studies* published online.

Alvarez, Fernando and Urban J. Jermann. 2005. Using Asset Prices to Measure the Persistence of the Marginal Utility of Wealth. *Econometrica* 73 (6):1977–2016.

Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent. 2003. A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection. *Journal of the European Economic Association* 1:69–123.

Andreasen, Martin M., Jesús Fernández-Villaverde, and Juan F. Rubio-Ramírez. 2010. The Pruned State Space System for Non-Linear DSGE Models: Asset Pricing Applications to GMM and SMM. Unpublished manuscript.

Backus, David K., Mikhail Chernov, and Stanley E. Zin. 2011. Sources of Entropy in Representative Agent Models. NBER Working paper W17219.

Bansal, Ravi and Bruce N. Lehmann. 1997. Growth-Optimal Portfolio Restrictions on Asset Pricing Models. *Macroeconomic Dynamics* 1:333–354.

Bansal, Ravi and Amir Yaron. 2004. Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles. *The Journal of Finance* 59 (4):1481–1509.

Bansal, Ravi, Robert F. Dittmar, and Christian T. Lundblad. 2005. Consumption, Dividends, and the Cross Section of Equity Returns. *Journal of Finance* 60 (4):1639–1672.

Benigno, Gianluca, Pierpaolo Benigno, and Salvatore Nisticò. 2010. Second-Order Approximation of Dynamic Models with Time-Varying Risk. NBER Working paper W16633.

Binsbergen, Jules H. van, Michael W. Brandt, and Ralph S. J. Koijen. 2011. On the Timing and Pricing of Dividends. Forthcoming in American Economic Review.

Borovička, Jaroslav, Lars Peter Hansen, Mark Hendricks, and José A. Scheinkman. 2011. Risk-Price Dynamics. *Journal of Financial Econometrics* 9 (1):3–65.

Epstein, Larry G. and Stanley E. Zin. 1989. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica* 57 (4):937–969.

Fama, Eugene F. and Kenneth R. French. 1992. The Cross-Section of Expected Stock Returns. *The Journal of Finance* 47 (2):427–465.

———. 1996. Multifactor Explanations of Asset Pricing Anomalies. *The Journal of Finance* 51 (1):55–84.

Gallant, A. Ronald, Peter E. Rossi, and George Tauchen. 1993. Nonlinear Dynamic Structures. *Econometrica* 61 (4):871–907.

Gourieroux, C. and J. Jasiak. 2005. Nonlinear Innovations and Impulse Responses with Application to VaR Sensitivity. *Annales d'Economie et de Statistique* 78:1–33.

Hall, Robert E. 2000. E-Capital: The Link between the Stock Market and the Labor Market in the 1990s. *Brookings Papers on Economic Activity* 2000 (2):73–102.

———. 2001. The Stock Market and Capital Accumulation. *American Economic Review* 91 (5):1185–1202.

Hansen, Lars Peter. 2007. Beliefs, Doubts and Learning: Valuing Macroeconomic Risk. *American Economic Review* 97:1–30.

———. 2012. Dynamic Valuation Decomposition within Stochastic Economies. *Econometrica* 80:911–967. Fisher-Schultz Lecture at the European Meetings of the Econometric Society.

Hansen, Lars Peter and Ravi Jagannathan. 1991. Implications of Security Market Data for Models of Dynamic Economies. *Journal of Political Economy* 99 (2):225–262.

Hansen, Lars Peter and Thomas Sargent. 2010. Fragile beliefs and the price of uncertainty. *Quantitative Economics* 1 (1):129–162.

Hansen, Lars Peter and José A. Scheinkman. 2009. Long-Term Risk: An Operator Approach. *Econometrica* 77 (1):177–234.

———. 2012. Pricing Growth-Rate Risk. *Finance and Stochastics* 16:1–15.

Hansen, Lars Peter, John C. Heaton, and Nan Li. 2005. Intangible Risk. In *Measuring Capital in the New Economy*, NBER Chapters, 111–152. National Bureau of Economic Research, Inc.

———. 2008. Consumption Strikes Back?: Measuring Long-Run Risk. *Journal of Political Economy* 116:260–302.

Holmes, Mark H. 1995. *Introduction to Perturbation Methods*. Springer.

Jacobson, David H. 1973. Optimal Stochastic Linear Systems with Exponential Performance Criteria and Their Relation to Deterministic Differential Games. *IEEE Transactions for Automatic Control* AC-18:1124–131.

Judd, Kenneth L. 1998. *Numerical Methods in Economics*. The MIT Press, Cambridge, MA.

Justiniano, Alejandro and Giorgio E. Primiceri. 2008. The Time-Varying Volatility of Macroeconomic Fluctuations. *The American Economic Review* 98 (3):604–641.

Kim, Jinill, Sunghyun Kim, Ernst Schaumburg, and Christopher A. Sims. 2008. Calculating and Using Second-Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models. *Journal of Economic Dynamics and Control* 32 (11):3397–3414.

Koop, Gary, M. Hashem Pesaran, and Simon M. Potter. 1996. Impulse Response Analysis in Nonlinear Multivariate Models. *Journal of Econometrics* 74 (1):119–147.

Kreps, David M. and Evan L. Porteus. 1978. Temporal Resolution of Uncertainty and Dynamic Choice Theory. *Econometrica* 46 (1):185–200.

Lettau, Martin and Jessica Wachter. 2007. Why is Long-Horizon Equity Less Risky? A Duration-Based Explanation of the Value Premium. *Journal of Finance* 62:55–92.

Lombardo, Giovanni. 2010. On Approximating DSGE Models by Series Expansions. ECB Working paper No. 1264.

Malkhozov, Aytek and Maral Shamloo. 2011. Asset Prices in Affine Real Business Cycle Models.

McGrattan, Ellen R. and Edward C. Prescott. 2010a. Technology Capital and the US Current Account. *American Economic Review* 100 (4):1493–1522.

———. 2010b. Unmeasured Investment and the Puzzling US Boom in the 1990s. *American Economic Journal: Macroeconomics* 2 (4):88–123.

Rubinstein, Mark. 1976. The Valuation of Uncertain Income Streams and the Pricing of Options. *The Bell Journal of Economics* 7:407–425.

Schmitt-Grohé, Stephanie and Martín Uribe. 2004. Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. *Journal of Economic Dynamics and Control* 28 (4):755–775.

Whittle, Peter. 1990. *Risk Sensitive and Optimal Control*. West Suffix, England: John Wiley and Sons.