Asset markets with heterogenoeus information

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Abstract

I define a notion of completitive equilibrium for asset markets where assets are heterogeneous and traders have heterogenoeus information about them. I then apply this notion to a model of distressed sales under asymmetric information and examine whether it can account for fire sales: sharp drops in prices when distressed agents need to sell assets. Standard models of asymmetric information with informed sellers, heterogenous assets and identical uninformed buyers predict the opposite phenomenon, as more distressed sellers on average sell less-adversely-selected pools of assets. With heterogeneity among buyers in their ability to distinguish assets of different qualities, the possibility of fire sales depends on the joint distribution of wealth and ability.

Note

This is a preliminary draft. I have tried to point out the things that I haven't finished doing or I am not so sure about. Comments and suggestions are extremely welcome.

1 Introduction

I study an environment where agents trade assets under asymmetric information. Sellers own a portfolio of assets of heterogeneous quality and there are potential gains from trade in selling them to a group of Buyers. Unlike the classic Akerlof (1970) model, which assumes that Buyers are equally uninformed about asset qualities, I allow for different Buyers to have different information about each of the assets. I study the prices and allocations that arise in a competitive equilibrium of this environment.

Defining an equilibrium when there is asymmetric information is not a straightforward matter. Several plausible notions of equilibrium have been proposed for the standard lemons market (Wilson 1980, Hellwig 1987, Gale 1992, 1996, Dubey and Geanakoplos 2002, Guerrieri et al. 2009), including both game-theoretic and Walrasian approaches. In my setting, this difficulty is compounded since one must find a way to describe how different Buyers are able to act on their information while trading in the same market. I adopt a Walrasian definition of equilibrium, letting agents trade anonymously in markets whose features

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they take as given. There can potentially be many markets for the assets, each defined by a price and a clearing mechanism, and Buyers and Sellers can choose in which markets to trade. Crucially, I allow Buyers to impose an acceptance rule that reflects their information and limits the selection of assets they are willing to receive from a given market. I am working on proving existence and uniqueness of equilibrium, I currently have a proof for my leading example but not in general.

I then apply the equilibrium definition to study the problem of fire sales. Imagine that a fraction μ of Sellers have a lower discount factor than Buyers, perhaps because they are banks who are suffering runs and desperately need current funds, and the way to obtain funds is to sell part of their portfolio of assets. How should asset prices vary with μ? Uhlig (2010) has shown that in a standard lemons model, the price of assets should increase with μ: non-distressed banks only sell bad assets, distressed banks may sell good assets as well, so a higher fraction of distressed banks should mean a better selection of sold assets and higher prices. When Buyers have heterogeneous information, this effect depends on the joint distribution of Buyer's wealth and the quality of their information.

2 Environment

Dates

There are two periods, $t = 1$ and $t = 2$. There is actually nothing about the model that requires the temporal interpretation. It could be "apples" and "oranges" rather than "today" and "tomorrow". The key will be that oranges come in boxes called "assets" not everyone knows how many oranges are contained in each box. For concreteness, I will stick the the temporal interpretation. Consumption at time t is denoted c_t.

Agents and preferences

There is a continuum of Buyers of measure 1, indexed by $\theta \in [0, 1]$. Preferences for Buyers are

$$u(c_1, c_2) = \begin{cases} c_1 + c_2 & \text{if } c_1 \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

i.e. they have linear preferences but cannot consume a negative amount. There is also a continuum of Sellers of measure 1, indexed by $t \in [0, 1]$. Preferences for Sellers are

$$u(c_1, c_2, t) = c_1 + \beta(t) c_2$$

with $\beta(t) \leq 1$. Whenever $\beta(t) < 1$ there are potential gains from intertemporal trade between Buyers and Sellers. Linearity in the preferences of Sellers makes things simple because it means that the decision of what to do with one asset does not depend on what the Seller does with any other asset. The case with curvature is more complicated and I haven't solved it yet. For Buyers, linearity is not so essential and is there mostly for convenience.

Endowments and Assets

Buyer θ has an endowment of $w(\theta)$ goods at $t = 1$.

There is a set of assets indexed by $i \in I \subset \mathbb{R}$. For instance, $I = [0, 1]$. Asset i will produce $q(i)$ goods at $t = 2$. I refer to $q(i)$ as the quality of asset i. Assume w.l.o.g. that $q(i)$ is weakly increasing. Seller t is endowed with $e(i, t)$ assets of type i.

Information

Each Seller knows the index i (and therefore the quality $q(i)$) of each asset he owns. Buyers do not observe i but instead Buyer θ observes a signal $x(i, \theta)$ whenever he analyzes asset i. If $x(i, \theta) \neq x(i', \theta)$ whenever $q(i) \neq q(i')$ then Buyer θ is perfectly informed about asset qualitites. Otherwise, he cannot tell apart some asset qualities from others. The interesting case is when at least some Buyers are not perfectly informed about asset qualities.

3 Equilibrium

Markets

There is no market for trading $t = 1$ goods against $t = 2$ goods. If there was such a market, which can be interpreted as a market for uncollateralized borrowing, then the gains from trade would be exhausted and the resulting allocation would be first-best efficient. Instead, the only way to achieve some sort of intertemporal trade is to trade $t = 1$ goods for assets. These assets will in turn produce $t = 2$ goods.

There are many markets, operating simultaneously, where agents can exchange goods for assets. Each market m is defined by a price $p(m)$ of assets in terms of goods and a clearing procedure or algorithm that determines in what order trades are executed. Let M be the set of markets. Gale (1996) uses a similar construct: rather than letting a single price clear markets, all possible prices coexist and the clearing mechanism determines whether trades get executed or not. There are two main differences with Gale's setup. First, the current setup allows more elaborate clearing algorithms than simply rationing the long end of the market. These algorithms make it possible to describe which trades take place when different Buyers place different types or orders in the same market (more on clearing algorithms below). Second, I allow agents to trade in as many markets as they want rather than limiting them to a single market. This is meant to capture the idea of anonymous markets and is clearly more appropriate in some applications than in others.

Seller's problem

Sellers must choose how much to supply of each asset in each market, i.e. they choose a function

$$S : I \times M \to \mathbb{R}^+$$

where $s(i, m)$ is the supply of asset i in market m. From the point of view of the sellers, each market is characterized by a price $p(m) \in \mathbb{R}^+$ and a probability of actually being able to sell asset i, denoted by

$\eta(i,m)$, where $\eta(\cdot,m): I \to [0,1]$. $\eta(\cdot,m)$ is an endogenous object, which results from the clearing algorithm and from the equilibrium supply and demand in market m. Each Seller simply takes it as given.

Seller t solves the following problem:[1]

$$\max_{c_1,c_2,s(i,m)} u(c_1,c_2,t) \tag{1}$$

$$s.t.$$

$$c_1 = \int\left[\int p(m)\, s(i,m)\, \eta(i,m)\, dm\right] di \tag{2}$$

$$c_2 = \int q(i)\left[e(i,t) - \int s(i,m)\, \eta(i,m)\, dm\right] di \tag{3}$$

$$0 \le s(i,m) \le e(i,t) \quad \forall i,m \tag{4}$$

$$\int s(i,m)\, \eta(i,m)\, dm \le e(i,t) \quad \forall i \tag{5}$$

Constraint (2) computes how many goods the Seller gets at $t=1$ as a result of his sales. For each asset i, he supplies $s(i,m)$ in market m, and succeeds in selling with probability $\eta(i,m)$, in which case he gets $p(m)$. Adding up over all markets and qualities results in (2). Constraint (3) computes how many goods the Seller gets at $t=2$ as a result of the assets which he does not sell. For each quality i his unsold assets are equal to his endowment $e(i,t)$ minus what he sold in all markets, and each yields $q(i)$ goods. Constraint (4) says that he can at most attempt to sell his entire endowment of each quality at any given price. This is important when $\eta(i,m) < 1$. It rules out a strategy of offering, say, 100 units for sale when he only owns 30 because he knows that due to rationing, only 30% of the units are actually sold. Constraint (5) just says that the total sales of any given quality (added across all market) are constrained by the Seller's endowment. Note that this embodies the assumption that you can attempt to sell the same asset in many markets, i.e. I do *not* impose

$$\int s(i,m)\, dm \le e(i,t) \quad \forall i \tag{6}$$

If I imposed (6) instead of (5), then a unit that is offered in one market could no longer be offered in other markets, and this commitment could be used as a signal of quality. Gale (1992, 1996), Guerrieri et al. (2009) and Guerrieri and Shimer (2011) all make assumptions similar to (6).

The choice of $s(i,m)$ for values of i,m such that $\eta(i,m)=0$ has no effect on the utility obtained by the Seller. The interpretation of this is that if he is not going to be able to sell, it doesn't matter whether or not he tries. Formally, this means that program (1) has multiple solutions. I am going to assume that when this is the case, the solution has to be robust to small positive $\eta(i,m)$, meaning that the Seller would attempt to sell in all the markets where if he could he would want to.

Definition 1. A solution to program (1) is *robust* if there exists a sequence of functions $\eta^n(i,m) > 0$ and a sequence of consumption and selling decisions $c_1^n, c_2^n, s^n(i,m)$ such that

[1]Pending: come up with a correct way of describing integrals over the set of markets, using a correct measure.

1. $c_1^n, c_2^n, s^n(i,m)$ solve program

$$\max_{c_1, c_2, s(i,m)} u(c_1, c_2, t)$$
$$s.t.$$
$$c_1 = \int \left[\int p(m) s(i,m) \eta^n(i,m) \, dm \right] di$$
$$c_2 = \int q(i) \left[e(i,t) - \int s(i,m) \eta^n(i,m) \, dm \right] di$$
$$0 \le s(i,m) \le e(i,t) \quad \forall i, m$$
$$\int s(i,m) \eta^n(i,m) \, dm \le e(i,t) \quad \forall i$$

2. $\eta^n(i,m) \to \eta(i,m)$

3. $c_1^n \to c_1$, $c_2^n \to c_2$ and $s^n(i,m) \to s(i,m)$

There is probably a simpler way of expressing this robustness condition, but I stick to this for now.

Buyer's orders

When Buyers place orders in a market, they can specify both the quantity of assets that they demand: $d(m)$ and what subset of assets they are willing to accept. An example of an order will be "I offer to buy d assets as long as the indices i of those assets satisfy $i \in [a,b]$". I formalize the idea that Buyers can be selective with the idea of an acceptance rule:

Definition 2. An *acceptance rule* is a function $\chi : I \to \{0,1\}$ that specifies what assets i to accept.

Buyers cannot just impose any selection rule that they want. They are not necessarily able to tell different assets apart from each other since do not observe i but just the imperfect signal $x(i,\theta)$. Feasible acceptance rules are those that only discriminate between assets that Buyers can actually tell apart.

Definition 3. An acceptance rule χ is *feasible for Buyer θ* if it is measurable with respect to the Buyer θ's information set, i.e. if

$$\chi(i) = \chi(i') \quad \text{whenever} \quad x(i,\theta) = x(i',\theta)$$

In general, since different Buyers observe different signals, the set of feasible acceptance rules will be different for each of them and in equilibrium they will end up imposing different acceptance rules.

I denote the set of possible acceptance rules by Ξ and the acceptance rule for Buyer θ in market m by $\chi(\cdot, m, \theta)$.

Allocation functions

Buyers understand that the mix of assets they will get if they buy from market m depends on the acceptance rule they impose. In order to describe this, I introduce the concept of an allocation function, which maps acceptance rules into measures on the space of assets.

Definition 4. An *allocation function* is a function[2]

$$a : I \times \Xi \to \mathbb{R}^+$$

such that for any $I_0 \subseteq I$, the measure of assets with indices $i \in I_0$ that a Buyer obtains if he demands one unit and imposes acceptance rule χ is given by

$$A(I_0, \chi) = \int_{i \in I_0} a(i, \chi)\, di$$

I denote the allocation function that arises in market m by $a(\cdot, \cdot, m)$. It is an endogenous object, which results from the clearing algorithm and from the equilibrium supply and demand in market m. Each Buyer simply takes it as given. Whenever a market m is such that $A(I, \chi, m) < 1$, this means that Buyers who impose acceptance rule χ in market m are rationed, in the sense that they obtain less than one asset for each asset they demand.

Buyer's problem

Buyer θ solves the following problem:

$$\max_{c_1, c_2, d(m), \chi(i,m)} u(c_1, c_2) \tag{7}$$

$$s.t.$$

$$c_1 = w(\theta) - \int p(m) A(I, \chi, m) d(m)\, dm \tag{8}$$

$$c_2 = \int \left(d(m) \int q(i)\, dA(i, \chi, m) \right) dm \tag{9}$$

$$\chi(\cdot, m) \text{ feasible for type } \theta \tag{10}$$

$$d(m) \geq 0 \tag{11}$$

Constraint (8) says that $t = 1$ consumption is equal to the Buyer's endowment minus what he spends on buying assets. For each market, the measure of assets that he obtains is $A(I, \chi, m)\, d(m)$ and he pays $p(m)$ for each asset. Constraint (9) computes the total amount of $t = 2$ goods that the Buyer will obtain, as a result of the measure over qualities that he gets from each market he buys from. Constraint (10) restricts the Buyer to use acceptance rules that are feasible for him.

Clearing algorithms

Each market is defined by a price $p(m)$ and a clearing algorithm. A clearing algorithm is a rule that determines what trades take place as a function of what trades are proposed by Buyers and Sellers.

To see why different clearing algorithms would lead to different results, consider the example in table 1. There are three types of assets. Assets 1 and 2 will pay zero at $t = 2$ while asset 3 will pay one. There are two Buyers in market m, with types θ_1 and θ_2. Type θ_1 cannot tell apart assets 2 and 3 so he must either

[2] Pending: describing the exact conditions on measurability that make this OK.

i	$q(i)$	$\chi(i,m,\theta_1)$	$\chi(i,m,\theta_2)$	$S(i,m)$
1	0	0	0	1.5
2	0	1	0	1.5
3	1	1	1	1.5

$d(m,\theta_1)=1$. $d(m,\theta_2)=1$

Table 1: Example of supplies and demands in a market

accept both of them or reject both of them; assume he is willing to accept both of them but rejects asset 1. Type θ_2 can distingish the worthless assets 1 and 2 from the good asset 3 so he can impose that he will only accept asset 3. Each of the Buyers demands a single unit. The total supply from all Sellers is 1.5 units of each asset.

One possible clearing algorithm would say: "let θ_1 choose first and take a represenative sample of the assets he is willing to accept; then θ_2 can do the same". This would result in the following allocation function:

$$a(i,\chi,m)=\begin{cases} \text{if } \chi=\chi(\cdot,m,\theta_1) & \text{if } \chi=\chi(\cdot,m,\theta_2) & \\ 0 & 0 & \text{if } i=1 \\ 0.5 & 0 & \text{if } i=2 \\ 0.5 & 1 & \text{if } i=3 \end{cases}$$

Type θ_1 picks randomly from the sample excluding the rejected asset 1. Since there are equal amounts of assets 2 and 3 and the total exceeds his demand, he gets a measure 0.5 of each. After that, type θ_2 gets to pick. He only accepts asset 3 and there is one unit left, which is exactly what he wants. From the point of view of sellers, the probability of being able to sell each of the assets in this market is:

$$\eta(i,m)=\begin{cases} 0 & \text{if } i=1 \\ \frac{1}{3} & \text{if } i=2 \\ 1 & \text{if } i=3 \end{cases}$$

Another possible clearing algorithm would say "let θ_2 choose first and take a represenative sample of the assets he is willing to accept; then θ_1 can do the same". This would result in the following allocation function:

$$a(i,\chi,m)=\begin{cases} \text{if } \chi=\chi(\cdot,m,\theta_1) & \text{if } \chi=\chi(\cdot,m,\theta_2) & \\ 0 & 0 & \text{if } i=1 \\ 0.75 & 0 & \text{if } i=2 \\ 0.25 & 1 & \text{if } i=3 \end{cases}$$

After θ_2 picks one unit of asset 3, there are only 0.5 units left, and there are still 1.5 units of asset 2. A respresentative sample from this remainder will give type θ_1 a total of 0.75 units of asset 2 and 0.25 units of asset 3. The probability of selling each of the assets would be:

$$\eta(i,m)=\begin{cases} 0 & \text{if } i=1 \\ \frac{1}{2} & \text{if } i=2 \\ \frac{5}{6} & \text{if } i=3 \end{cases}$$

I will assume that there are markets for trading assets at every possible price and every possible clearing algorithm. To make this statement precise, I need to describe the set of possible clearing algorithms. Clearing algorithms must respect the anonymity of markets, taking as inputs the patterns of asset supply, asset demand and acceptance rules but not the identities of the traders. Furthermore, I will only allow algorithms such that Buyers (in some order) pick representative samples of the assets they accept, as long as any acceptable assets remain. This, for instance, rules out clearing algorithms that "infer" asset qualities from the pattern of demand and attempt to allocate them efficiently.

Formally, let Σ_χ be a σ-algebra on Ξ (the set of possible acceptance rules). Demand in market m is given by a measure $D : \Sigma_\chi \to \mathbb{R}^+$ that says, for each subset of acceptance rules, what is the total measure of assets demanded by Buyers who impose those acceptance rules. Supply in market m is given by by a function $S : I \to \mathbb{R}^+$. Let $\mathcal{D} \equiv \{D : \Sigma_\chi \to \mathbb{R}^+\}$ be the space of possible demand functions and $\mathcal{S} \equiv \{S : I \to \mathbb{R}^+\}$ be the space of possible supply functions and define the set $\Omega \equiv \{\omega : \Xi \times \mathbb{N} \to \mathbb{R}^+ s.t. \sum_{n=1}^{\infty} \omega(\chi, n) = 1, \forall \chi \in \Xi\}$ to be the set of functions that map acceptance rules into a sequence of nonnegative real numbers that add up to 1. Denote the nth number of such a sequence by $\omega(\chi, n)$.

Definition 5. An *clearing algorithm* is a function $\Gamma : \mathcal{D} \times \mathcal{S} \to \Omega$

A clearing algorithm consists of a (possible infinite) number of rounds. Each possible acceptance rule gets to clear some fraction of its total demand in each round, with fractions summing up to 1. In the example above, the first clearing algorithm produced the sequences:

$$\omega(\chi, n) = \begin{cases} \text{if } \chi = \chi(\cdot, m, \theta_1) & \text{if } \chi = \chi(\cdot, m, \theta_2) & \\ 1 & 0 & \text{if } n = 1 \\ 0 & 1 & \text{if } n = 2 \\ 0 & 0 & \text{if } n > 2 \end{cases} \tag{12}$$

while the second clearing algorithm produced the sequences

$$\omega(\chi, n) = \begin{cases} \text{if } \chi = \chi(\cdot, m, \theta_1) & \text{if } \chi = \chi(\cdot, m, \theta_2) & \\ 0 & 1 & \text{if } n = 1 \\ 1 & 0 & \text{if } n = 2 \\ 0 & 0 & \text{if } n > 2 \end{cases} \tag{13}$$

The allocation functions and selling probabilities that result from a clearing algorithm can be computed as follows. Denote the residual supply after the algorithm has gone through n rounds by $S^n(i)$, with $S^0(i) \equiv S(i)$. The nth-round allocation for acceptance rule χ is

$$a^n(i, \chi) = \begin{cases} \frac{\chi(i) S^{n-1}(i)}{\int \chi(i) S^{n-1}(i) di} & \text{if } \int \chi(i) S^{n-1}(i) di > 0 \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

This states that, as long as there is a positive measure of assets that are acceptable, the acceptance rule is assigned a representative sample of those assets, while if there are no acceptable assets left the demand associated with that acceptance rule is left unsatisfied. If the entire demand associated with acceptance rule

χ were cleared in round n, i.e. if $\omega(\chi, n) = 1$, then equation (14) would describe its allocation. In general, the allocation received by acceptance rule χ is the sum over all rounds:

$$a(i, \chi) = \sum_{n=1}^{\infty} \omega(\chi, n) \, a^n(i, \chi) \tag{15}$$

It remains to compute the evolution of residual supply, which is simply given by:

$$S^n(i) = S^{n-1}(i) - \int a^n(i, \chi) \, \omega(\chi, n) \, dD(\chi) \tag{16}$$

The selling probabilities are found as follows. First, for any asset i, let $\Xi_i \equiv \{\chi \in \Xi : \chi(i) = 1\}$ be the set of acceptance rules that accept asset i. Then

$$\eta(i) = \begin{cases} \frac{\int a(i,\chi) dD(\chi)}{S(i)} & \text{if } S(i) > 0 \\ 1 & \text{if } S(i) = 0 \text{ and } D(\Xi_i) > 0 \\ 0 & \text{if } S(i) = 0 \text{ and } D(\Xi_i) = 0 \end{cases} \tag{17}$$

Equation (17) says that, if an asset is in positive supply, then the probability of selling it can simply be computed as the ratio of the total measure that gets allocated (which could be zero) and the measure of supply. If an asset is in zero supply, then there are two possibilities. If there is a positive measure of demand with acceptance rules that would accept it, then any Seller who supplied asset i would be able to sell it, so $\eta(i) = 1$. Instead, if there is no demand that finds asset i acceptable, then it would remain unsold.

As described so far, a clearing algorithm might produce allocations that are not feasible. For instance, suppose in the example above Buyer θ_2 demanded 2 units instead of 1 and one proposed a clearing algorithm that resulted in ω given by (12). This would attempt to assign 0.5 units of asset 3 to Buyer θ_1 and 2 units to Buyer θ_2, a total of 2.5 units when only 1.5 are actually available. Note that, according to equation (16), this would result in negative residual supply. In order to rule out such an outcome I will only allow clearing algorithms such that never result in negative residual supply.

Definition 6. A clearing algorithm Γ is *feasible* if, for any $D \in \mathcal{D}$ and $S \in \mathcal{S}$, $\lim_{n\to\infty} S^n(i) \geq 0, \forall i$, where $S^n(i)$ is defined by equation (16).

The set of markets M is the set of all possible pairs of a nonnegative price and a feasible clearing algorithm.

Definition of equilibrium

An equilibrium consists of:

1. Consumption and supply decisions $c_1(t), c_2(t), s(i, m, t)$ by Sellers
2. Supply in each market $S(i, m)$
3. Consumption, demand and acceptance rule decisions $c_1(\theta), c_2(\theta), d(m, \theta), \chi(i, m, \theta)$ by Buyers

4. Demand in each market $D(\cdot, m)$

5. An allocation function $a(i, \chi, m)$ for each market

6. A probability of selling function $\eta(i, m)$ for each market

such that

1. $c_1(t), c_2(t), s(i, m, t)$ are a robust solution to program (1) for each Seller t, taking $\eta(i, m)$ as given

2. Supply in market m is
$$S(i, m) = \int s(i, m, t)\, dt$$

3. $c_1(\theta), c_2(\theta), d(m, \theta), \chi(i, m, \theta)$ solve program (7) for each Buyer, taking $a(i, \chi, m)$ as given

4. Demand in market m is
$$D(C, m) = \int_{\{\theta : \chi(\cdot, m, \theta) \in C\}} d(m, \theta)\, d\theta$$
for any $C \subseteq \Xi$

5. The allocation function $a(\cdot, \cdot, m)$, the probability of selling function $\eta(\cdot, m)$, supply $S(\cdot, m)$ and demand $D(\cdot, m)$ satisfy equations (15) and (17)

4 An example

This example illustrates how asymmetric information can result in some of the patterns that are often described as "fire sales". The distribution of discount factors for Sellers is

$$\beta(t) = \begin{cases} 1 & \text{if } t \geq \mu \\ 0 & \text{if } t < \mu \end{cases}$$

so there is a fraction μ of Sellers who are really desperate to consume at $t = 0$ (perhaps banks facing runs or hedge funds facing margin calls) while the rest are just as patient as the Buyers.

The set of assets is $I = [0, 1]$, asset qualities are $q(i) = \mathbb{I}(i > \lambda)$ for some $\lambda \in (0, 1)$ and the signals that Buyers observe are $x(i, \theta) = \mathbb{I}(i > \lambda\theta)$, as illustrated in figure 1. This means that a fraction $1 - \lambda$ of assets (those with indices $i > \lambda$) are high quality assets and will pay a dividend of 1 at $t = 2$ and a fraction λ (those with indices $i \leq \lambda$) are "lemons" and will pay nothing. When an asset is of high quality, every Buyer observes $x(i, \theta) = 1$. When an asset is a lemon, those Buyers of types $\theta \leq \frac{i}{\lambda}$ will observe $x(i, \theta) = 1$, so they cannot distinguish it from a high quality asset; instead, Buyers with $\theta > \frac{i}{\lambda}$ will observe $x(i, \theta) = 0$ and conclude that the asset is a lemon. θ can therefore be thought of as an index of expertise: higher values of θ means that there is a smaller subset of lemons that the Buyer might misidentify as high quality assets. Furthermore, expertise is nested: if type θ can identify asset i as a lemon, then so can all types $\theta' > \theta$.

Each seller is endowed with one unit of each asset, i.e. $e(i, t) = 1$ and Buyers' endowments $w(\theta)$ are unspecified for now.

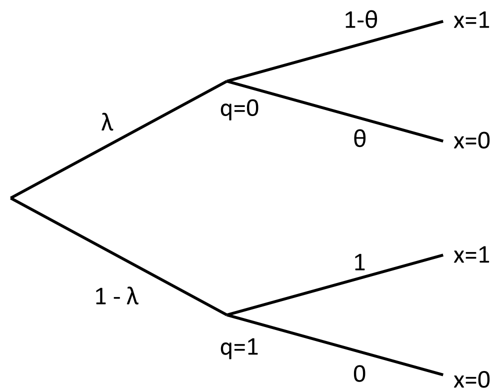

Figure 1: Asset qualities and signals

Equilibrium

In this example, there is an (essentially) unique equilibrium. All trades take place at the same price, $p^* < 1$ in a market m^* whose clearing algorithm (in equilibrium) clears everyone in the first round. Distressed Sellers attempt to sell all their assets; they manage to sell all their high quality assets but only some of their lemons. Non-distressed Sellers only attempt to sell their lemons, and manage to sell them in the same proportions as the distressed Sellers. Buyers only accept assets for which they observe $x(i,\theta) = 1$, which means that lower-θ Buyers accept more lemons mixed in with their high-quality assets. Buyers of types θ below some cutoff prefer to stay out of the market and do not trade at all.

Formally, the equilibrium can be described by a cutoff Buyer type θ^* and price p^* given by the solution to:

$$\int_{\theta^*}^{1} \frac{1}{\lambda(1-\theta)+\mu(1-\lambda)} \frac{w(\theta)}{p^*} d\theta = 1 \tag{18}$$

$$p^* = \frac{\mu(1-\lambda)}{\lambda(1-\theta^*)+\mu(1-\lambda)} \tag{19}$$

together with the following clearing algorithm for market m^*:

$$\begin{aligned} \omega(\chi, n) &= \max \tilde{\omega} \\ s.t. \quad \tilde{\omega} &\leq \max_{\chi} \left[1 - \sum_{j=1}^{n-1} \omega(\chi, j)\right] \\ S^{n-1}(i) - \tilde{\omega} &\int a^n(i,\chi)\, dD(\chi) \geq 0 \end{aligned} \tag{20}$$

and the following equilibrium objects:

1. Supply decisions

$$s(i,m,t)=\begin{cases}1 & \text{if } i\leq\lambda\\ 1 & \text{if } i>\lambda,\ p(m)\geq p^* \text{ and t}<\mu\\ 0 & \text{if } i>\lambda,\ p(m)<p^* \text{ and t}<\mu\\ 1 & \text{if } i>\lambda,\ p(m)\geq 1 \text{ and t}\geq\mu\\ 0 & \text{if } i>\lambda,\ p(m)<1 \text{ and t}\geq\mu\end{cases} \tag{21}$$

leading to consumption decisions

$$c_1(t)=\begin{cases}p^*\int_0^1\eta(i,m^*)\,di & \text{if } t<\mu\\ p^*\int_0^\lambda\eta(i,m^*)\,di & \text{if } t\geq\mu\end{cases}$$
$$c_2(t)=\begin{cases}0 & \text{if } t<\mu\\ (1-\lambda) & \text{if } t\geq\mu\end{cases} \tag{22}$$

2. Aggregate supply

$$S(i,m)=\begin{cases}1 & \text{if } i\leq\lambda\\ 1 & \text{if } i>\lambda,\ p(m)\geq 1\\ \mu & \text{if } i>\lambda,\ p(m)\in[p^*,1)\\ 0 & \text{if } i>\lambda,\ p(m)<p^*\end{cases} \tag{23}$$

3. Demand decisions

$$d(m,\theta)=\begin{cases}\frac{w(\theta)}{p^*} & \text{if } m=m^* \text{and } \theta>\theta^*\\ 0 & \text{otherwise}\end{cases} \tag{24}$$

and acceptance rules

$$\chi(i,m,\theta)=x(i,\theta) \tag{25}$$

leading to consumption

$$c_1(\theta)=\begin{cases}w(\theta) & \text{if } \theta<\theta^*\\ 0 & \text{if } \theta\geq\theta^*\end{cases}$$
$$c_2(\theta)=\begin{cases}0 & \text{if } \theta<\theta^*\\ \frac{w(\theta)}{p^*}\frac{\mu(1-\lambda)}{\lambda(1-\theta)+\mu(1-\lambda)} & \text{if } \theta\geq\theta^*\end{cases} \tag{26}$$

4. Aggregate demand

$$D(C,m)=\begin{cases}\frac{1}{p^*}\int_{\{\theta:x(\cdot,\theta)\in C\}}w(\theta) & \text{if } m=m^*\\ 0 & \text{otherwise}\end{cases} \tag{27}$$

5. Allocation functions

$$a(i,\chi,m)=\begin{cases}\frac{\chi(i)\mathbb{I}(i\leq\lambda)}{\int_{i<\lambda}\chi(i)di} & \text{if } p(m)<p^* \text{ and } \int_{i<\lambda}\chi(i)\,di>0\\ 0 & \text{if } p(m)<p^* \text{ and } \int_{i<\lambda}\chi(i)\,di=0\\ \frac{\chi(i)[\mathbb{I}(i\leq\lambda)+\mu\mathbb{I}(i>\lambda)]}{\int_{i<\lambda}\chi(i)di+\int_{i>\lambda}\mu\chi(i)di} & \text{if } p(m)\in[p^*,1) \text{ and } \int\chi(i)\,di>0\\ \frac{\chi(i)}{\int\chi(i)di} & \text{if } p(m)\geq 1 \text{ and } \int\chi(i)\,di>0\\ 0 & \text{if } p(m)\geq p^* \text{and } \int\chi(i)\,di=0\end{cases} \tag{28}$$

6. Probability of selling

$$\eta\left(i,m\right)=\begin{cases}1 & \text{if } m=m^{*}\text{and } i>\lambda\\ \int_{\theta^{*}}^{\frac{i}{\lambda}}\frac{1}{\lambda(1-\theta)+\mu(1-\lambda)}\frac{w(\theta)}{p}d\theta & \text{if } m=m^{*}\text{and } i\in\left[\lambda\theta^{*},\lambda\right]\\ 0 & \text{if } m=m^{*}\text{and } i<\lambda\theta^{*}\\ 0 & \text{if } m\neq m^{*}\end{cases} \tag{29}$$

Proposition 1. *Equations (18)-(29) describe an equilibrium.*

Proof. .

1. Seller optimization.

   Selling probabilities (29) imply that Sellers will be able to sell all assets $i\in[\lambda,1]$ and a fraction $\eta\left(i,m^{*}\right)<1$ of assets $i\in(\lambda\theta^{*},\lambda]$ in market m^{*}, and nothing else. A necessary and sufficient condition for a solution to program (1) is that in market m^{*} distressed Sellers supply the maximum possible amount of assets $i\in[\lambda\theta^{*},1]$ and non-distressed sellers supply the maximum possible amount of assets $i\in[\lambda\theta^{*},\lambda]$ and no assets $i\in(\lambda,1]$. Imposing robustness then implies (21). The level of consumption (22) then follows from the budget constraint.

2. Supply.

   (23) follows from aggregating (21) over all Sellers.

3. Buyer optimization.

   The optimality of acceptance rule (25) is immediate. The only other possible rule would be to accept all assets, which would result in a higher fraction of lemons.

   Define

$$\tau\left(m,\theta\right)\equiv\begin{cases}\frac{\int q(i)a(i,\chi(\cdot,m,\theta),m)di}{p(m)A(I,\chi(\cdot,m,\theta),m)} & \text{if } A\left(I,\chi\left(\cdot,m,\theta\right),m\right)>0\\ 0 & \text{otherwise}\end{cases} \tag{30}$$

   and let

$$\tau^{\max}\left(\theta\right)\equiv\max_{m}\tau\left(m,\theta\right)$$
$$M^{\max}\left(\theta\right)\equiv\arg\max_{m}\tau\left(m,\theta\right)$$

   Buyer optimization implies that

   (a) if $\tau^{\max}\left(\theta\right)<1$, then

$$d\left(m,\theta\right)=0\quad\forall m$$

   (b) if $\tau^{\max}\left(\theta\right)>1$, then

$$\int_{m\in M^{\max}(\theta)}\frac{d\left(m,\theta\right)}{p\left(m\right)}dm=w\left(\theta\right)$$

   and

$$d\left(m,\theta\right)=0\quad\forall m\notin M^{\max}\left(\theta\right)$$

Using equation (28),

$$\tau(m,\theta)=\begin{cases}\frac{1}{p(m)}\frac{\mu(1-\lambda)}{\lambda(1-\theta)+\mu(1-\lambda)} & \text{if } p(m)\geq p^{*}\\ 0 & \text{otherwise}\end{cases}$$

so for all Buyers

$$\tau^{\max}(\theta)=\frac{1}{p^{*}}\frac{\mu(1-\lambda)}{\lambda(1-\theta)+\mu(1-\lambda)}$$

and

$$M^{\max}(\theta)=\{m\in M \quad s.t. \quad p(m)=p^{*}\}$$

Together with condition (19), this implies that types $\theta<\theta^{*}$ have $\tau^{*}(\theta)<1$ so $d(m,\theta)=0$ is optimal for them and types $\theta>\theta^{*}$ spend their entire endowment on markets where $p(m)=p^{*}$; within these they are indifferent so buying just in market m^{*} is optimal, which implies (24). The level of consumption (26) then follows from the budget constraint.

4. Demand.

   (27) follows from aggregating (24) over all Buyers.

5. Allocation function.

   In all markets except m^{*} demand is zero, so for any clearing algorithm, equation (16) implies

$$S^{n}(i,m)=S^{n-1}(i,m) \quad \forall i,n,m\neq m^{*}$$

   for all clearing algorithms. By induction, this implies that

$$a^{n}(i,\chi,m)=a^{1}(i,\chi,m) \quad \forall i,n,m\neq m^{*}$$

   and therefore

$$a(i,\chi,m)=a^{1}(i,\chi,m)=\begin{cases}\frac{\chi(i)S(i)}{\int\chi(i)S(i)di} & \text{if } \int\chi(i)S(i)\,di>0\\ 0 & \text{otherwise}\end{cases} \qquad \forall i,m\neq m^{*} \tag{31}$$

   In market m^{*}, condition (18) together with (23) and (27), implies that $\omega(\chi,1)=1 \quad \forall\chi$, so (31) holds as well. (31) and (23) together imply (28).

6. Selling probabilities.

   (29) follows from direct application of formula (17).

□

In equilibrium, all trades take place in market m^{*}, where the clearing algoritm is (20). This algorithm says that all demand is cleared in the first round whenever possible, and if it's not possible because supply is insufficient, all acceptance rules are rationed in the same proportion in each round until they are all cleared. In equilibrium, supply and demand are such that everyone indeed clears in the first round, receiving a representative sample of the assets that they accept. Clearing in the first round makes this an attractive market for Buyers because, as the example in table 1 illustrates and Lemma 1 below proves, any Buyer faces more adverse selection if higher-θ Buyers have been cleared before him.

Sellers, for their part, are indifferent regarding what algorithm is used to clear trades: they just care about the price and the probability of selling. Therefore they supply the same assets in all markets that have the same price.

All markets besides m^* have zero demand, so no matter what the clearing algorithm, a Buyer in those markets would receive a representative sample of the assets he accepts, just as in market m^*. This means that Buyers are indifferent between buying in market m^* or in other markets where the price is also p^*, but sticking to m^* is one of the optimal choices.

Buying at prices other than p^* is never optimal for Buyers. At prices lower than p^*, the supply includes only lemons, so Buyers prefer to stay away, whereas at prices above p^*, the supply of assets is exactly the same as at p^* but the price is higher.

This does not settle the question of whether a Buyer chooses to buy at all. Consider a Buyer of type θ. The sample of assets he accepts includes all the high quality assets that are supplied, which are $\mu(1-\lambda)$, as well as all lemons with indices $i \in (\theta\lambda, \lambda]$, which total $\lambda(1-\theta)$. Therefore the terms of trade (in terms of $t = 2$ goods per $t = 1$ good spent) for Buyer θ if he buys at price p^* are:

$$\tau(\theta) = \frac{1}{p^*} \frac{\mu(1-\lambda)}{\lambda(1-\theta) + \mu(1-\lambda)}$$

Condition (19) implies that the terms of trade for type θ^* are $\tau(\theta^*) = 1$, which leave him indifferent. Buyers with $\theta > \theta^*$ get $\tau(\theta) > 1$, so they spend all their endowment buying assets and Buyers with $\theta < \theta^*$ would get $\tau(\theta) < 1$, so they prefer not to buy at all.

Because $\beta(t) = 0$ for distressed Sellers, they will supply all their assets in market m unless they can sell them for sure in markets where the price is higher than $p(m)$. The probability of selling asset i in market m^* is given by the ratio of the total allocation of that asset across of Buyers to the supply of that asset. For high quality assets, the supply is $\mu(1-\lambda)$ and Buyer θ (with $\theta \geq \theta^*$) obtains $\frac{w(\theta)}{p} \frac{\mu(1-\lambda)}{\lambda(1-\theta)+\mu(1-\lambda)}$ units, so the probability of selling the asset is given by the left-hand-side of equation (18). This equation implies that Sellers can sell high quality assets for sure in market m^* and therefore do not supply them in markets with prices below p^*. For assets $i \in (\lambda\theta^*, \lambda]$, the supply is 1 and Buyer θ obtains $\frac{w(\theta)}{p} \frac{1}{\lambda(1-\theta)+\mu(1-\lambda)}$ as long as $\theta \in \left(\theta^*, \frac{\lambda}{i}\right)$; lower types demand nothing and higher types reject asset i. This implies the selling probability (29). Notice that the selling probability is continuous. Lemons with indices just below λ fool almost all Buyers into thinking they are likely to be high quality and therefore Sellers are able to sell them with high probability; assets with indices just above $\lambda\theta^*$ fool very few Buyers and are sold with low probability.

The equilibrium described above is essentially unique. In order to prove this, I first establish a series of preliminary results.

Lemma 1. *Consider an arbitrary market m and suppose $i < j$. Then in any equilibrium the residual supplies after n rounds of clearing in market m satisfy*

$$\frac{S^n(j)}{S^{n-1}(j)} \leq \frac{S^n(i)}{S^{n-1}(i)}$$

Proof. By equation (16) and (14)

$$\begin{aligned}\frac{S^n(i)}{S^{n-1}(i)} &= 1 - \frac{\int \frac{\chi(i)S^{n-1}(i)}{\int \chi(i)S^{n-1}(i)di}\omega(\chi,n)\,dD(\chi)}{S^{n-1}(i)} \\ &= 1 - \int \frac{\chi(i)}{\int \chi(i)\,S^{n-1}(i)\,di}\omega(\chi,n)\,dD(\chi)\end{aligned}$$

Therefore

$$\frac{S^n(j)}{S^{n-1}(j)} - \frac{S^n(i)}{S^{n-1}(i)} = \int \frac{\chi(i)-\chi(j)}{\int \chi(k)\,S^{n-1}(k)\,dk}\omega(\chi,n)\,dD(\chi) \tag{32}$$

Given their information, each Buyer has three feasible acceptance rules:

$$\begin{aligned}\chi(i) &= x(i,\theta) \\ \chi(i) &= 1 \\ \chi(i) &= 1 - x(i,\theta)\end{aligned}$$

The last rule will never be used in equilibrium, because it implies accepting only assets that are known to be lemons. This means that all Buyers will either accept all assets or accept only those for which they observe $x = 1$. Under either of these rules, $\chi(i) \leq \chi(j)$, so the right hand side of (32) must be nonpositive. □

Lemma 1 states that as the rounds of clearing progress, high-quality assets leave the pool at a greater rate than lemons. This implies that, other things being equal, Buyers prefer to trade in markets where their trades will clear sooner rather than later. In fact, the following result implies that, given an acceptance rule, the best terms of trade that a Buyer can obtain in a market are if his trades clear in the first round so that he obtains a representative sample of the acceptable assets supplied to that market.

Lemma 2. *Let*

$$\tau(\chi,m) \equiv \begin{cases} \frac{\int q(i)a(i,\chi,m)di}{p(m)A(I,\chi,m)} & \textit{if } A(I,\chi,m) > 0 \\ 0 & \textit{otherwise}\end{cases}$$

$$\tau^{\max}(\chi,m) \equiv \begin{cases} \frac{1}{p(m)}\frac{\int q(i)\chi(i)S(i,m)di}{\int \chi(i)S(i,m)di} & \textit{if } \int \chi(i)\,S(i,m)\,di > 0 \\ 0 & \textit{otherwise}\end{cases}$$

In any equilibrium,

$$\tau(\chi,m) \leq \tau^{\max}(\chi,m) \quad \forall \chi, m$$

with equality if $\omega(\chi,1) = 1$.

Proof. If if $A(I,\chi,m) = 0$, then the result is immediate. Otherwise, using equation (15) we can write:

$$\tau(\chi,m) = \frac{\int q(i)\left[\sum_{n=1}^{\infty}\omega(\chi,n)\,a^n(i,\chi,m)\right]di}{p(m)\sum_{n=1}^{\infty}\omega(\chi,n)\,A^n(I,\chi,m)} \tag{33}$$

For any n,

$$A^n\left(I,\chi,m\right)=\begin{cases}1 & \text{if } \int\chi\left(i\right)S^{n-1}\left(i,m\right)di>0\\ 0 & \text{otherwise}\end{cases}$$

Letting $\bar{n}$ be the highest value of n such that $A^n\left(I,\chi,m\right)=1$, we can rewrite (33) as:

$$\begin{aligned}\tau\left(\chi,m\right)&=\frac{1}{p\left(m\right)}\int q\left(i\right)\left[\sum_{n=1}^{\bar{n}}\frac{\omega\left(\chi,n\right)}{\sum_{n=1}^{\bar{n}}\omega\left(\chi,n\right)}a^n\left(i,\chi,m\right)\right]di\\ &=\frac{1}{p\left(m\right)}\sum_{n=1}^{\bar{n}}\left[\frac{\omega\left(\chi,n\right)}{\sum_{n=1}^{\bar{n}}\omega\left(\chi,n\right)}\int_{\lambda}^{1}a^n\left(i,\chi,m\right)di\right]\\ &=\frac{1}{p\left(m\right)}\sum_{n=1}^{\bar{n}}\left[\frac{\omega\left(\chi,n\right)}{\sum_{n=1}^{\bar{n}}\omega\left(\chi,n\right)}\frac{\int_{\lambda}^{1}\chi\left(i\right)S^{n-1}\left(i,m\right)di}{\int\chi\left(i\right)S^{n-1}\left(i,m\right)di}\right]\end{aligned}$$

Lemma 1 implies that the term

$$\frac{\int_{\lambda}^{1}\chi\left(i\right)S^{n-1}\left(i,m\right)di}{\int\chi\left(i\right)S^{n-1}\left(i,m\right)di}$$

is weakly decreasing in n and therefore the right hand side is maximized at $\omega\left(\chi,1\right)=1$, where $\tau\left(\chi,m\right)=\tau^{\max}\left(\chi,m\right)$. □

Lemma 2 places an upper bound on the terms of trade than can be obtained in a market given an acceptance rule. It is also possible to compute an upper bound for a given Buyer who can choose among all his feasible acceptance rules.

Lemma 3. *Let*

$$\tau\left(\theta,m\right)\equiv\max_{\chi\text{ feasible for }\theta}\tau\left(\theta,m\right)$$

In any equilibrium

$$\tau\left(\theta,m\right)\leq\tau^{\max}\left(\chi_\theta,m\right)$$

where χ_θ is defined by equation (25)

Proof. By definition, there is some acceptance rule χ that is feasible for θ that attains the maximum. Therefore

$$\begin{aligned}\tau\left(\theta,m\right)&=\tau\left(\chi,m\right)\\ &\leq\tau^{\max}\left(\chi,m\right)\\ &\leq\tau^{\max}\left(\chi_\theta,m\right)\end{aligned}$$

where the first inequality follows from Lemma 2 and the second holds because χ_θ maximizes $\frac{\int q(i)\chi(i)S(i,m)di}{\int\chi(i)S(i,m)di}$ among feasible acceptance rules. □

Knowing the upper bound on the terms of trade a Buyer can obtain in a given market m is useful because if one can find a market m' where a Buyer can obtain better terms of trade than $\tau\left(\theta,m\right)$, then implies that

Buyer θ will not buy from market m. Using this fact, the following result establishes that in equilibrium all trades take place at the same price.

Lemma 4. *In equilibrium there is trade at only one price*

Proof. Assume the contrary, suppose there is trade at p_H and p_L. If Buyers are willing to buy in a market m where $p = p_L$, then it means that distressed Sellers are willing to sell some assets $i > \lambda$ at a price p_L. In a robust solution to problem (1), this means that if $t < \mu$ and $i > \lambda$, then $s(i, m, t) = 1$, i.e. all distressed sellers supply the maximum amount of any asset $i > \lambda$ in all markets where $p > p_L$. This implies that in any market m where $p(m) \in (p_L, p_H]$.

$$S(i, m) = \begin{cases} \mu & \text{if } i > \lambda \\ 1 & \text{if } i \leq \lambda \end{cases}$$

Fix any θ and take a market m such that acceptance rule χ_θ is cleared in the first round and $p(m) \in (p_L, p_H)$. In such a market

$$\begin{aligned} \tau(\chi_\theta, m) &= \tau^{\max}(\chi, m) \\ &> \tau(\theta, m') \text{ for any } m' \text{ such that } p(m') = p_H \end{aligned}$$

The first equality follows from Lemma 2 and the inequality follows from Lemma 3, the fact that supply is the same in markets m and m' and the fact that $p(m) < p_H$. Therefore Buyer θ will not buy from any market where the price is p_H. Since this applies to all θ, there can be no trade at p_H. $\square$

Next I show that in any equilibrium where there is trade at price p^*, Sellers are able to sell all their high-quality assets.

Lemma 5. *Define*

$$\eta(i, p) \equiv \int\limits_{m:p(m)=p} \eta(i, m)\, dm$$

In any equilibrium where there is trade at p^* *,* $\eta(i, p^*) = 1$ *for all* $i > \lambda$.

Proof. Assume the contrary. Since no feasible acceptance rule for any Buyers distinguish between different high quality assets, $\eta(i, p^*) < 1$ for some $i > \lambda$ implies $\eta(i, p^*) < 1$ for all $i > \lambda$. By Lemma 4, there is no trade at any other price, which means that a fraction of high-quality assets held by distressed Sellers remains unsold. Therefore in a robust solution to program (1), distressed Sellers will supply $s(i, m, t) = 1$ for all m for $i > \lambda$, which implies $S(i, m) = \mu$ for $i > \lambda$ and $p(m) \leq 1$. Now take any Buyer θ and any market m' with $p(m') < p^*$ which clears acceptance rule χ_θ in the first round. The terms of trade for buying in market

m' are

$$\begin{aligned}
\tau\left(\chi_\theta, m'\right) &= \frac{1}{p\left(m'\right)} \frac{\mu\left(1-\lambda\right)}{\int_{\theta\lambda}^{\lambda} S\left(i, m'\right) di + \mu\left(1-\lambda\right)} \\
&\geq \frac{1}{p\left(m'\right)} \frac{\mu\left(1-\lambda\right)}{\int_{\theta\lambda}^{\lambda} S\left(i, m\right) di + \mu\left(1-\lambda\right)} \\
&> \frac{1}{p^*} \frac{\mu\left(1-\lambda\right)}{\int_{\theta\lambda}^{\lambda} S\left(i, m\right) di + \mu\left(1-\lambda\right)} \\
&= \tau^{\max}\left(\chi_\theta, m\right) \\
&\geq \tau\left(\theta, m\right) \quad \text{for any } m \text{ s.t. } p\left(m\right) = p^*
\end{aligned}$$

where the first inequality follows from the fact that robust solutions to program (1) imply $S\left(i, m\right) \leq S\left(i, m'\right)$ if $p\left(m\right) < p\left(m'\right)$, for all i; the second from $p\left(m'\right) < p^*$ and the last from Lemma 3. This implies that Buyer θ prefers to buy from market m' rather than from any market where $p\left(m\right) = p^*$. Since this is true for all θ, it contradicts the assumption that there is trade at p^*. □

Now I can use the above results to prove that the equilibrium is essentially unique.

Proposition 2. *In any equilibrium, the price and allocations are those of the equilibrium described by equations (18)-(29).*

Proof. Let p^* and θ^* be defined by equations (18) and (19).

Suppose there was an equilibrium where trade took place at price $p_H > p^*$. For any market m where $p = p_H$, supply satisfies $S\left(i, m\right) = 1$ for $i \leq \lambda$ and $S\left(i, m\right) \leq \mu$ for $i > \lambda$. Therefore

$$\tau\left(\theta, m\right) \leq \frac{1}{p_H} \frac{\mu\left(1-\lambda\right)}{\lambda\left(1-\theta\right) + \mu\left(1-\lambda\right)}$$

so the lowest θ that may be willing to buy is θ_H, defined by

$$\frac{1}{p_H} \frac{\mu\left(1-\lambda\right)}{\lambda\left(1-\theta_H\right) + \mu\left(1-\lambda\right)} = 1$$

Equation (19) implies $\theta_H > \theta^*$.

The maximum measure of high-quality assets that Buyer θ can get is

$$\frac{\mu\left(1-\lambda\right)}{\lambda\left(1-\theta\right) + \mu\left(1-\lambda\right)} \frac{w\left(\theta\right)}{p_H}$$

which means that the probability of selling a high-quality asset is at most

$$\int_{\theta_H}^{1} \frac{1}{\lambda\left(1-\theta\right) + \mu\left(1-\lambda\right)} \frac{w\left(\theta\right)}{p_H} d\theta$$

but because $p_H > p^*$ and $\theta_H > \theta^*$, equation (18) implies this is less than 1, which contradicts Lemma (5).

Suppose now that there was an equilibrium where trade took place at $p_L < p^*$. For any market m where $p(m) \in (p_L, 1)$, supply satisfies $S(i,m) = 1$ for $i \leq \lambda$ and $S(i,m) = \mu$ for $i > \lambda$. This is true in particular for markets where acceptance rule χ_θ is cleared in the first round, so for ant $p \in (p_L, 1)$, Buyer θ can find a market where the terms of trade are

$$\tau = \frac{1}{p} \frac{\mu(1-\lambda)}{\lambda(1-\theta) + \mu(1-\lambda)}$$

Therefore, in order for trade to only take place at p_L, it must be that all Buyers with $\theta > \theta_L$ obtain terms of trade of at least

$$\frac{1}{p_L} \frac{\mu(1-\lambda)}{\lambda(1-\theta) + \mu(1-\lambda)}$$

in markets where $p = p_L$, where θ_L is defined by

$$\frac{1}{p_L} \frac{\mu(1-\lambda)}{\lambda(1-\theta_L) + \mu(1-\lambda)} = 1$$

For this to happen, it must be that all traders with $\theta > \theta_L$ have their trades cleared in the first round. This would demand a total of

$$\int_{\theta_L}^{1} \frac{\mu(1-\lambda)}{\lambda(1-\theta) + \mu(1-\lambda)} \frac{w(\theta)}{p_L} d\theta$$

high-quality assets, but because $p_L < p^*$ and $\theta_L < \theta^*$, equation (18) implies this is more than $\mu(1-\lambda)$, which is the total supply of high-quality assets, so not all trades can clear in the first round.

This means that in any equilibrium, all trades take place at $p = p^*$ and high-quality assets can be sold for sure. The rest of the equilibrium objects follow immediately. $\square$

Fire Sales

The term "fire sales" is sometimes used to refer to situations where a trader's urgency for funds leads him to sell assets at prices that are far below their usual price. In the current model, something like a fire-sale would arise if p^* were decreasing in μ, so that when more Sellers are distressed, asset prices fall. From equations (18) and (19), one can compute:

$$\frac{dp^*}{d\mu} = \frac{\lambda(1-\lambda)}{[\lambda(1-\theta^*) + \mu(1-\lambda)]^2} \left[1 - \theta^* - \mu \frac{\frac{\lambda(1-\theta^*)}{\mu^2(1-\lambda)} p^* + \frac{1}{p^*} \int_{\theta^*}^1 \frac{(1-\lambda)w(\theta)}{[\lambda(1-\theta)+\mu(1-\lambda)]^2} d\theta}{\frac{\lambda}{\mu(1-\lambda)} p^* + \frac{1}{p^*} \frac{w(\theta^*)}{\lambda(1-\theta^*)+\mu(1-\lambda)}} \right] \tag{34}$$

$$= \frac{\lambda(p^*)^2}{\mu^2(1-\lambda)} \left[1 - \theta^* - \mu \frac{\frac{\lambda(1-\theta^*)}{\mu^2(1-\lambda)} p^* + \frac{1}{p^*} \int_{\theta^*}^1 \frac{(1-\lambda)w(\theta)}{[\lambda(1-\theta)+\mu(1-\lambda)]^2} d\theta}{\frac{\lambda}{\mu(1-\lambda)} p^* + \frac{1}{p^*} \frac{w(\theta^*)}{\lambda(1-\theta^*)+\mu(1-\lambda)}} \right] \tag{35}$$

Inspection of equation (34) leads to the following result, which relates the possibility of fire sales to the joint distribution of wealth and expertise.

Proposition 3. *1.* $\lim_{w(\theta^*)\to\infty} \frac{dp^*}{d\mu} = \frac{\lambda(p^*)^2}{\mu^2(1-\lambda)} (1-\theta^*) > 0$

2. If $w(\theta^*) = 0$, *then* $\frac{dp^*}{d\mu} = -\frac{1}{\mu} \int_{\theta^*}^1 \frac{(1-\lambda)w(\theta)}{[\lambda(1-\theta)+\mu(1-\lambda)]^2} d\theta < 0$

3. If $w(\theta) = w$ for all θ, then $\frac{dp^*}{d\mu} = 0$

In general, there are two opposing effects when more Sellers become distressed. On the one hand, distressed Sellers are the only ones who are willing to sell high-quality assets. Other things being equal, this should improve the pool of assets being sold and thus lead to higher, not lower, prices. This is the effect emphasized by Uhlig (2010). On the other hand, more distressed Sellers mean that more assets are being offered for sale. Given that the more expert Buyers have exhausted their wealth, it is necessary to resort to less expert Buyers. These less expert Buyers are aware that they are less clever at filtering out the lemons so, other things being equal, they will make up for this by only entering the market if prices are lower.

Proposition 3 shows that which effect dominates (locally) depends on the density of wealth at the equilibrium cutoff level of expertise. If $w(\theta^*)$ is high, this means that a large amount of wealth would enter the market if the cutoff level of expertise was lowered slightly. In this case, the direct selection effect dominates and prices rise, meaning there are no fire sales. Instead when $w(\theta)$ is low, cutoff level of expertise needs to fall a lot in order to attract sufficient wealth to buy the extra units supplied. In this case, the changing-threshold effect dominates and prices fall. Interestingly, for the special case where wealth is evenly distributed across all levels of expertise, the price is the same for any μ, so both effects cancel out.

The model is useful for exploring the relationship among other theories of fire-sales in the existing literature. One class of theories (Shleifer and Vishny 1992, 1997, Kiyotaki and Moore 1997) emphasizes that the marginal buyer of an asset can be a second-best user with diminishing marginal product. If first-best users need to sell more units, asset prices will fall along the marginal-product curve of second-best users. Evidence consistent with this pattern has been documented by Pulvino (1998) in the market for used aircraft. But financial assets are not aircraft. The holder of a financial asset does not need to use his expertise and/or complementary assets in order to extract value from it, so the idea of a second-best user does not naturally fit fire-sales in financial markets. However, the current model illustrates that expertise may be relevant in the trade itself, and moving along a gradient of expertise can induce to fire-sale effects.

A second class of theories (Allen and Gale 1994, 1998, Acharya and Yorulmazer 2008) relies on the notion of cash-in-the-market pricing. There is a given amount of purchasing-power available, so if more units are to be sold, the price must fall. But these class of models typically leave unanswered the question of why buyers with deep pockets (for instance, rich individuals) stay out of the market. The current model provides an explanation for buyers staying out of the market: even though there are good deals available for those who have expertise, those who do not have expertise are rationally worried that they are not able to select the deals among all the assets on offer. In other words, given their expertise, buying from this market does not provide excess returns, even though it does for experts.

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