Contests with Endogenous and Stochastic Entry*

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December 2011

Abstract

This paper studies imperfectly discriminatory contest with endogenous and stochastic entry, and its optimal (effort-maximizing) design. A fixed pool of potential bidders strategically decide whether to sink an entry cost and then vie for an indivisible prize. Applying Dasgupta and Maskin (1986), we establish the existence of symmetric equilibrium in this two-dimensional discontinuous game under a wide class of contest technologies, which includes all-pay auction as a limiting case. We show that a contest is subject to a "first best", which indicates the maximal amount of expected overall bid it could possibly elicit, regardless of the prevailing winner selection mechanism. For the subclass of Tullock contests, we further identify the conditions for the existence (non-existence) of a symmetric equilibrium with pure-strategy bidding upon entry. Based on these equilibrium analyses, we establish that a well structured Tullock contest is able to elicits this first best. Furthermore, we find that the discriminatory power of a Tullock contest non-monotonically affects the expected overall bid. Hence, a noisier contest, which provide weaker bidding incentives, may turn out to elicit more effort, which contrasts with the conventional wisdom in contest design. Our analysis further reveals that a contest designer should in general exclude potential bidders to elicit higher bids.

JEL Nos: C7, D8.

Keywords: Contest Design; Endogenous Entry; Entry Cost; Stochastic Entry

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*We are grateful to Atsu Amegashie, Masaki Aoyagi, Helmut Bester, Oliver Gürtler, Todd Kaplan, Dan Kovenock, John Morgan, Johannes Münster, Aner Sela, Roman Sheremeta, Randy Silvers, Ching-jen Sun, Samarth Vaidya, Cédric Wasser, Elmar Wolfstetter, participants of 2010 SAET Meetings and 2011 International Conference on Contests, Tournaments and Relative Performance Evaluation, and seminar participants at Deakin University and Free University of Berlin for helpful comments and suggestions. All errors remain ours.

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1 Introduction

Economic agents are often involved in contests. They expend costly effort to compete for a limited number of prizes, while their investments are usually non-refundable whether they win or lose. A wide variety of economic activities exemplify such competitions. They include rent seeking, lobbying, political campaigns, R&D races, competitive procurement, college admissions, ascents of organizational hierarchies, and movement in internal labor markets.

The literature on contests conventionally assumes that a fixed number (n) of bidders participate in the competition and the number is commonly known. Under this fixed-n paradigm, the majority of existing studies focus on the various aspects of bidders' ex post competitive activities, but abstract away from their ex ante participation incentives. In this paper, we complement these studies by examining a setting where a fixed pool of potential bidders are allowed to strategically decide whether to participate in a contest. They sink their bids only after entering the contest.

In our setting, participation incurs a nontrivial (fixed) cost. It allows a bidder to merely participate while it is unrelated to their chances of winning. Each bidder weighs his expected payoff in future competitions against the entry cost, and participates if and only if the former (at least) offsets the latter. As noted by Konrad (2009), entry cost, which can be explicitly sunk resources or foregone opportunities, is widespread in various competitive activities. To provide an analogy, while an air ticket enables Venus Williams to arrive at the Australian Open, it does not help her win the championship. Similarly, to participate in an R&D tournament, a research company may need to acquire some necessary laboratory equipment to gather project-specific information, or to turn down other profitable tasks, while its chances of winning depend on its subsequent creative input. The nontrivial entry cost leads potential bidders to participate in the contest randomly. The actual number of participants is uncertain. Participants take into account this uncertainty when placing their bids.

This entry-bidding game exhibits distinctive characteristics, which complement and enrich the existing literature in several aspects. First, this entry-bidding game exemplifies a discontinuous game with two-dimensional action space (Dasgupta and Maskin, 1986). With strategic and stochastic entry, the strategy of each potential bidder involves two elements: (1) whether to enter; and (2) how to bid upon entering.[1,2] The game thus distinguishes itself from standard contests that are typically identified as uni-dimensional discontinuous games. The conventional approaches to establishing equilibrium existence in contest models (e.g. Baye,

[1] The literature on contests recognizes that (Baye, Kovenock and de Vries 1994, and Alcalde and Dahm, 2010), a well-defined contest success function (e.g., Tullock contest) can be discontinuous at its origin, i.e., when all bidders bid zero.

[2] We assume that the entrants do not observe the number of entarnts. As will be discussed in the conclusion, there is no loss of generality for considering the optimal design of contests that do not disclose the actual number of participants.

Kovenock and de Vries 1994 and Alcalde and Dahm, 2010) or in auction settings do not encompass our settings.[3] This novel setting entails the application of Dasgupta and Maskin's (1986) general theorem on multi-dimensional discontinuous games. We establish that a symmetric mixed-strategy equilibrium always exists for a family of imperfectly discriminatory contests with general impact functions: each potential bidder enters with the same probability, and adopts the same (possibly mixed) bidding strategy upon entry.[4] This family of contest technology covers the Tullock contests as a subclass and all pay auction as a limiting case. To our knowledge, our analysis provides the first application of the existence theorem for multi-dimensional discontinuous games in contest literature.

Second, endogenous and stochastic entry leads to substantially more extensive strategic interactions in participants' bidding behaviors. The analysis in this novel setting sheds light on two classical questions in contest literature: (1) to what extent the equilibrium bidding strategies (conditional on their entry) can be solved for explicitly; and (2) under what conditions equilibria that do (or do not) involve pure-strategy bidding exist.[5] A participant, to place his bid, must form a rational belief on competing bidders' entry patterns and take into account all the possible contingencies that can be caused by the (endogenously determined) random entries. With this flavor, the general property of a bidder's overall expected payoff function cannot be readily discerned, and the results from existing studies do not carry over. For the class of Tullock contests, our analysis reveals the nature of the strategic decision problem, and identifies sufficient conditions under which participants do (or do not) randomize their bids upon entry. The results provide a foundation for the optimal design of contests with endogenous and stochastic entry.

Third, endogenous entry enriches contest design substantially. We first establish that for any entry-bidding game within the family of imperfectly discriminatory contests with general impact functions, there exists a unique "first best" level of total effort that can be induced. It imposes an upper limit for the amount of overall bid the game can possibly elicit, regardless of the prevailing winner selection mechanism (i.e. contest success functions). Based on the results from equilibrium analysis, we demonstrate that the "first best" can be implemented through a properly structured Tullock contest. It should be noted that the applicability of this "first best" level is not restricted to the family of contests we formally model in the paper,

[3] We elaborate upon this issue in Section 4.

[4] One should note that our two-dimensional strategy space of (entry, effort) cannot be reduced to a setting with single dimensional strategy of effort with a positive fixed cost. In our two dimensional setting, if no one enters the contest, no one wins. If everyone enters but exerts zero effort, every one incurs an entry cost and has an equal chance at winning. In the single dimensional setting, if everyone exerts zero effort, no one incurs any costs but has an equal chance of winning.

[5] It is well-known that a bidder's payoff maximization problem becomes irregular when the contest success function is excessively elastic to effort, e.g. when the discriminatory parameter r in a Tullock contest exceeds certain boundaries. Endogenous and stochastic entry compounds the complexity.

but also extends to any contest technology that induces symmetric entry. This means that the optimality of a Tullock contest holds even when a larger family of contest technology is considered. We investigate how various institutional elements of a contest affects its efficiency in enhancing total effort. The analysis witnesses the extensive strategic trade-offs, which are triggered uniquely by endogenous and stochastic entry. Two main observations are highlighted as follows.

- We demonstrate that a "noisier" winner selection mechanism, which provides a lower-power incentive in contests, may paradoxically elicit more effort. In a Tullock contest, the discriminatory parameter r in the impact function is conventionally interpreted as a measure of the level of precision in the winner selection mechanism. A greater r implies that a higher bid can be more effectively translated into a higher likelihood of winning, thereby increasing the marginal return to the bid and further incentivizing bidders. In our setting, a greater r gives rise to competing effects at two different layers. First, it gives rise to a trade-off between ex post bidding incentives and ex ante entry incentives. A greater r intensifies the competition on the one hand; while it leaves lesser rent to participants and restricts bidders' entry on the other. Second, there is a tension between the level of entry and the incentive of individual bidding. More active entry (under a smaller r) allows the contest to engage more bidders and tends to amplify their overall contributions; while it also leads individual participants to bid more prudently, as they anticipate lesser chance of winning. Thus, the expected overall bid may not vary monotonically with the size of r, and implementing the "first best" total effort may require a moderate incentives that balance the various countervailing forces.

- The contest literature states that the overall bid always increases with the number of bidders when their participation is deterministic. However, we demonstrate in our setting that the contest designer may prefer to limit competition by inviting only a subset of them for participation: a contest may elicit lesser effort, when a larger pool of potential bidders are involved.

The rest of the paper proceeds as follows. In Section 2, we discuss the relation of our paper to the relevant literature in the rest of this section. In section 3, we set up a generic entry-bidding game, and derive the "first best" level of overall bid that can be possibly elicited in the game. Section 4 carries out an equilibrium analysis of the entry-bidding game when a specific winner selection mechanism (i.e. Tullock contests) and discusses the possibility of implementing the "first best" through such a mechanism. Section 5 concludes this paper.

2 Relation to Literature

Our paper complements the literature on contests and auctions in various aspects.[6,7] Our paper primarily belongs to the literature on equilibrium existence in contests. Our paper provides a comprehensive and formal account of equilibrium existence in the entry-bidding game for a large family of contests, covers the Tullock contests as a subclass and all pay auction as a limiting case. Szidarovszky and Okuguchi (1997) establish the existence of pure-strategy equilibria when contestants have concave production functions. The existence and properties of the equilibria remain a nagging problem for contests with less well-behaved technologies. Baye, Kovenock and de Vries (1994) establish the existence of mixed-strategy equilibria in two-player Tullock contests with $r \geq 2$. Alcalde and Dahm (2010) further the literature by showing that *all-pay auction equilibria* exist under a wide class of contest success functions.[8] Both studies apply the results of Dasgupta and Maskin (1986) on uni-dimensional discontinuous games. Our paper contributes to this literature by introducing bidders' entry decisions while allowing the number of active bidders to be stochastic. These new flavours enrich our analysis by forming a two-dimensional discontinuous game, and provide a novel application of the general result of Dasgupta and Maskin (1986) on multi-dimensional discontinuous games in the contest literature.

The literature on contests with endogenous entry remains scarce. Higgins, Shughart, and Tollison (1985) in their pioneering work study a tournament model in which each rent seeker bears a fixed entry cost, and randomly participates in equilibrium. In an all-pay auction model, Kaplan and Sela (2010) provide a rationale for entry fees in contests. Besides the differing modeling choice and the diverging focus, Kaplan and Sela (2010) differ from the current paper in a few other aspects. First, they allow players to bear privately-known entry costs, while we assume that entry cost is uniform and commonly known. Second, they let participants know who else has entered, while we focus mainly on uninformed participants.[9]

Two recent experimental studies, Cason, Masters and Sherementa (2010) and Morgan, Orzen and Sefton (2010), also contribute to this research agenda by studying bidders' entries. Similar to Morgan, Orzen and Sefton's (2010) theoretical model, Fu and Lu (2010) also assume

[6] Our paper can also be related to the literature on standard oligopolistic competition. Our paper echoes the argument of Dixit and Shapiro (1986) and Shapiro (1989) on firms' behavior in oligopolistic markets. He shows that Bertrand competition, which is fiercer, can be more anti-competitive *ex post* than Cournot competition, which is *ex ante* more subdued, as the latter limits the contestability of the market and discourages entries. We focus on the issue of mechanism design in our particular context. In addition, the level of post-entry competition is a continuous variable and is considered as a strategic choice of the contest designer.

[7] Our paper also complements the literature on contests and auctions with entry. Its relation to auction literature will unfold in Section 4 when we compare the results to those of relevant auction studies.

[8] Wang (2010) also characterizes the equilibria in two-player asymmetric Tullock contests when r is large.

[9] We also discuss in the conclusion the ramifications of disclosure policy as an institutional element of contests.

that potential bidders enter sequentially, so neither setting involves stochastic participation.

A handful of papers have examined contests with stochastic participation. The majority of these studies, however, assume exogenous entry patterns. Myerson and Wärneryd (2006) examine a contest with an infinite number of potential entrants, whose entry follows a Poisson process. Münster (2006), Lim and Matros (2009) and Fu, Jiao and Lu (2011) assume a finite pool of potential contestants, with each contestant entering the contest with a fixed and independent probability.

The current study also contributes to the growing literature on contest design by exploring the optimal mechanism in a context with endogenous and stochastic entries.

First, our analysis complements the literature on the proper level of precision in evaluating bidding performance. Conventional wisdom says that a more precise evaluation mechanism incentivizes bidders. Gershkove, Li and Schweinzer (2009) and Giebe and Schweinzer (2011) provide two novel applications of this principles, and both espouse the merit of a more precise contest as incentive devices. A handful of studies, however, espouse low-powered incentives in contests and demonstrate that a less "discriminatory" contest can improve efficiency. One salient example is provided by Lazear (1989), who argues that excessive competition leads to sabotage. A more popular stream in the literature instead stresses the "handicapping" effect of the imprecise performance evaluation mechanism in (two-player) asymmetric contests. When contestants differ in their abilities, a noisier contest balances the playfield. This effect encourages weaker contestants to bid more intensely, and deters the stronger ones from shirking. O'Keeffe, Viscusi and Zeckhauser (1984) are among the first to formalize this logic. This rationale is further elaborated upon by Che and Gale (1997, 2000), Fang (2002), Nti (2004), Amegashie (2009), and Wang (2010). In a recent study, Epstein, Mealem and Nitzan (2011) contend that contest designers still prefer all-pay auctions to Tullock contests if they can strategically discriminate between bidders. In contrast to these studies, our paper adopts a M-player symmetric contest, and stresses the trade-off between *ex post* bidding incentives and *ex ante* entry incentives. Our paper is related to Cason, Masters and Sheremeta's (2010) experimental study in this aspect, which compares endogenous entries in all-pay auctions and lottery contests.

Our finding on efficient exclusion echoes a handful of pioneering studies by Baye, Kovenock and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003). These studies focus on heterogeneous contestants, and concern themselves with selecting (usually two) players of the "right types". Our result, however, obtains in a setting of homogenous players and concerns itself with creating a contest of the "right size". Dasgupta (1990) studies a two-stage procurement tournament. Bidders invest in cost reduction in the first stage, and place their bids in the second. Wider competition may diminish bidders' incentives to engage in R&D. Limiting the number of competing firms may or may not benefit the principal. None of these studies involves entry cost and endogenous entry. In contrast to these studies, an invited (potential) bidder in our setting has to decide whether to enter the subsequent contest, and the entry pattern in the equilibrium remains endogenous and stochastic.

3 Contest with Endogenous Entry

This section proceeds in three steps. First, we set up an entry-bidding game where a family of imperfectly discriminatory contests with general impact functions is adopted, and a fixed pool of potential bidders make costly and endogenous entry to a contest and bid for an indivisible prize. Note that this family of contest technology covers the popularly adopted Tullock contests as a subclass and all pay auction as a limiting case. Second, we establish that a symmetric entry-bidding equilibrium always exists in this generic game. Third, we demonstrate that the overall bid in such a game, regardless of the prevailing winner selection mechanism, is subject to an upper limit ("first best" level of effort).

3.1 Setup

A fixed pool of $M(\geq 2)$ identical risk-neutral potential bidders may compete for a prize of a common value $v > 0$. Potential bidders first decide simultaneously whether or not to participate in the competition. Each participant has to sink a fixed cost $\Delta > 0$ if he enters. Entry is irreversible, and the cost Δ cannot be recovered. These contest primaries can thus be denoted generically by (M, Δ, v).

Upon their entry, participants simultaneously submit their bids $x_i, i \in 1, 2, ..., N$, to compete for the prize v, where N denotes the number of entrants. We assume the number of entrants is not observed by the entrants.[10] A potential bidder i's strategy is thus denoted by an ordered pair $(q_i, \mu_i(x_i))$, where q_i is the probability he enters the contest, and the probability distribution $\mu_i(x_i)$ depicts his bidding strategy conditional on his entry. The distribution $\mu_i(x_i)$ reduces to a singleton when the participant does not randomize his bids.

Suppose that $N \geq 2$ potential bidders enter the contest. They simultaneously submit their bids $x_i, i = 1, 2, ..., N$, to compete for the prize v without knowing the number of entrants. Each participating bidder wins the prize with a probability $p_i(x_i, x_{-i})$, where $p_i(x_i, x_{-i})$ is mapping $p_i : \mathbb{R}^N_+ \to [0, 1]$. The probability of a participating bidder i winning the prize is given by

$$p^N(x_i, \mathbf{x}_{-i}) = \frac{g(x_i)}{\sum_{j=1}^N g(x_j)}, \text{ if } N \geq 2, \text{ and } \sum_{j=1}^N g(x) > 0, \tag{1}$$

where $g(\cdot)$ is a continuous impact function with $g(0) = 0$ and $g'(\cdot) > 0$. If all participants submit zero bids, i.e. $\sum_{j=1}^N g(x) = 0$, the winner is randomly picked from the participant.

[10] In this paper, as will be discussed in the conclusion it is optimal for the contest organizer not to reveal the number of entrants.

To the extent that only one bidder enters, he receives the prize v automatically, regardless of his bid. In the event that nobody enters, the designer keeps the prize. The probabilistic prize allocation mechanism (1) follows the setup of widely adopted ratio-form contest success function. It is axiomatized by Skaperdas (1996) and Clark and Riis (1998). Clark and Riis (1996) and Fu and Lu (2011) provide a micro foundation for contest success function (1) from a noisy ranking perspective. Clearly, this family of contest technology covers the Tullock contests as a subclass where $g(x_i) = x_i^r, r > 0$. In particular, when $r \to \infty$, all pay auction is obtained as a limiting case.

When bidding x_i, a participating bidder i bears a cost $c(x_i) = x_i^\alpha$ with $\alpha \geq 1$. When $N \geq 2$ bidders enter the contest, a participant i earns an expected payoff $\pi_i : \mathbb{R}_+^N \to \mathbb{R}$, which is given by

$$\pi_i(x_i, \mathbf{x}_{-i}) = \frac{g(x_i)}{\sum_{j=1}^N g(x_j)} v - x_i^\alpha.$$

As participants do not observe N, the actual number of participants, each participant is uncertain about the actual level of competition when placing his bid. He bids based on his rational belief about others' entry and bidding strategies. The solution concept of a subgame perfect equilibrium does not apply, as participants possess only imperfect information and no proper subgame is formed after entry. We simply use the concept of Nash equilibrium to solve the game. An equilibrium is a strategy combination $\times_{i=1}^M (q_i, \mu_i(x_i))$ of all contestants, which requires that the pair strategy $(q_i, \mu_i(x_i))$ of each potential bidder i maximizes his expected payoff based on his rational belief and others' strategy profile $\times_{j \neq i}(q_j, \mu_j(x_j))$.

Because of the symmetry among bidders, we, throughout the entire analysis, focus on symmetric equilibria where all potential bidders play the same strategy $(q^*, \mu^*(x))$. The existence of a symmetric equilibrium for a general impact function $g(\cdot)$ will be established later in Section 3.3.

It should be noted that though we start with a family of contest technology described by (1), our analyses of Section 3.4 on maximum total effort inducible and the optimal design of Section 4 would naturally extend to any family of contest technologies that induce symmetric entry.

3.2 Some Preliminaries

The following assumption focuses our the analysis on the most relevant case.

Assumption 1 $\frac{v}{M} < \Delta < v$.

Assumption 1 requires that the entry cost Δ be nontrivial but not prohibitively high. First, no entry is triggered if it costs more than the winner's purse. Second, the analysis becomes relatively trivial when entry involves little cost, in which case the institutional elements of the contest do not affect bidders' entry incentives significantly. Under the assumption, no equilibria exist where all potential bidders participate with certainty. They must randomize on their entry in a symmetric equilibrium.

We define two cutoff probabilities, which are used repeatedly throughout the analysis.

Definition 1 *Let* $\bar{q} \in (0,1)$ *be the unique solution to* $(1-(1-q)^M)v - Mq\Delta = 0$*, and* $q_0 \in (0,1)$ *be the unique solution to* $(1-q)^{M-1}v - \Delta = 0$*.*

Comparing the two cutoffs leads to the following.

Lemma 1 $q_0 < \bar{q}$*.*

Proof. See Appendix. ■

Let us discuss the implications of the two cutoffs briefly, although their implications unfold as the analysis proceeds. The entry-bidding game cannot trigger an equilibrium, where all potential bidders enter with a probability more than $\bar{q}$: they would otherwise end up with negative expected payoff in the game. In contrast, the cutoff q_0 defines a lower bound. To put it briefly, if there exists an equilibrium for a Tullock contest where all potential bidders enter with a probability less than q_0, participating bidders must randomize their bids upon entry. Its implications will be further elaborated upon in Section 4.

3.3 (Symmetric) Equilibrium Existence

The entry-bidding game exhibits two main characteristics. First, The strategy of each player involves two elements: entry and bidding. Second, a potential bidder's payoff can be discontinuous as the contest success function can be discontinuous. For instance, the payoff function is discontinuous at origin (see Baye, Kovenock and de Vries, 1994, and Alcalde and Dahm, 2010), i.e. when all participants bid zero.

The conventional approach (in auctions with endogenous entry) to establishing the existence of symmetric equilibria proceeds with two steps, which disentangles the two elements in each player's strategy and simplifies the analysis. In the first step, potential bidders are assumed to enter the competition with given (symmetric) entry probabilities. One establishes the existence of symmetric bidding equilibrium under each given entry probability q. Bidders' equilibrium payoff function in the auction $\pi(q)$ is obtained accordingly, which is typically continuous and monotonic in q. The second step identifies a (typically unique) entry probability, which equalizes potential bidder's expected payoff in the auction $\pi(q)$ and his entry cost.

As will be further illustrated in Section 4.1, this "disentangling" approach has only limited utility in our setting. The reasons are two-folds. First, existence of bidding equilibrium in contest conventionally relies on Dasgupta and Maskin's (1986) theorem on uni-dimensional games. The theorem, however, does not directly apply to games with an uncertain number of players, and the general existence of a symmetric bidding equilibrium under a given entry probability q thus cannot be readily verified. Second, a contest game with a fixed entry probability may not be well-behaved and the bidding strategy is not universally solvable (as will be seen in Section 4.2). It thus remains difficult to characterize the properties (e.g. continuity and monotonicity) of bidders' expected payoffs even if an equilibrium exists.

As a result, the entry-bidding game in general should be recognized as a two-dimensional discontinuous games, and thus it entails the application of Dasgupta and Maskin's (1986) equilibrium existence theorem for discontinuous games with multi-dimensional strategy space.

Theorem 1 *(a) For any impact function $g(\cdot)$ with $g(0) = 0$ and $g'(\cdot) > 0$, a symmetric equilibrium $(q^*, \mu^*(x))$ always exists. In the equilibrium, each potential bidder enters with a probability $q^* \in (0, \bar{q})$ and his bid follows a probability distribution $\mu^*(x)$. (b) Each potential bidder receives an expected payoff of zero in the entry-bidding equilibrium.*

Proof. See Appendix. ■

To our knowledge, Theorem 1 and its proof provide the first application of Dasgupta and Maskin's (1986) equilibrium existence result on two-dimensional discontinuous games in the literature on contests. Two remarks are in order.

First, the equilibrium existence result applies to broader contexts. We explicitly adopt ratio-form contest success functions to economize on our presentation and facilitate subsequent discussion on contest design. However, the proof of the theorem does not rely on the specific properties of ratio-form contest success functions and the particular form of bidding cost functions. The analysis can be readily adapted to contests with more broadly defined prize allocation mechanisms, such as those in Alcalde and Dahm (2010), by redefining the discontinuity set slightly.

Second, it should be noted that asymmetric equilibria always exist in the entry-bidding game, in which a subset of potential bidders stay inactive regardless, while the others enter either randomly or deterministically. We focus on symmetric equilibria for two reasons: (1) symmetric equilibria can be arguably viewed as a natural focal point of the game; and (2) the property of an asymmetric equilibrium where only a subset of $M'(< M)$ enter with a positive probability can be learned from the symmetric equilibrium of an entry-bidding game with a smaller pool of $M'(< M)$ potential bidders.

3.4 "First-Best" Level of Effort

Central to the contest literature is the question of how the institutional elements of the contest affect bidding efficiency. As Gradstein and Konrad (1999) argued, "... contest structures result from the careful consideration of a variety of objectives, one of which is to maximize the effort of contenders." We hereby demonstrate that there exists a "first best" level of total effort for each given contest primaries (M, Δ, v), which indicates the maximal amount of overall bids the contest could possibly elicit, regardless of the governing winner selection mechanism. It will be apparent that the "first best" upper bound for total effort would prevail for any family of contest technologies that induce symmetric entry across potential bidders.

Suppose that potential bidders enter with a probability $q \in (0,1)$ in a symmetric equilibrium. The prize v is given away with a probability $1-(1-q)^M$. Hence, bidders receive an expected overall rent of $[1-(1-q)^M]v$; while they on average incur entry cost $Mq\Delta$. The following fundamental equality must hold in a symmetric equilibrium:

$$[1-(1-q)^M]v \equiv Mq(\Delta + E(x^\alpha)), \tag{2}$$

where $E(x^\alpha)$ denotes the equilibrium expected effort costs of an entrant. The equilibrium expected overall bidding cost can then be identified without explicitly solving for it:

$$MqE(x^\alpha) = [1-(1-q)^M]v - Mq\Delta. \tag{3}$$

The convexity of cost function ($\alpha \geq 1$) further implies that the expected overall bid ($MqE(x)$) must be bounded from above:

$$(MqE(x)) = MqE[(x^\alpha)^{\frac{1}{\alpha}}] \leq Mq[E(x^\alpha)]^{\frac{1}{\alpha}}. \tag{4}$$

By (3) or (4), we further obtain

$$(MqE(x)) \leq [Mq]^{\frac{\alpha-1}{\alpha}}\{[1-(1-q)^M]v - Mq\Delta\}^{\frac{1}{\alpha}}. \tag{5}$$

The inequality (5) yields important implications: regardless of the equilibrium bidding strategy upon entry, the expected overall bids that a contest with primaries (M, Δ, v) could elicit in a symmetric equilibrium with entry probability q would never exceed the upper limit as given by RHS of (5).

Denote this upper boundary by

$$\overline{x}_T(q) \triangleq (Mq)^{\frac{\alpha-1}{\alpha}}\left\{[1-(1-q)^M]v - Mq\Delta\right\}^{\frac{1}{\alpha}}, \tag{6}$$

with $q \in (0,1)$. The function $\overline{x}_T(q)$ exhibits the following important properties.

Theorem 2 *(a) There exists a unique cutoff $\hat{q} \in (q_0, \bar{q})$, which uniquely maximizes $\overline{x}_T(q)$;*

(b) The function $\overline{x}_T(q)$ strictly increases with q when $q \in (0, \hat{q})$, and strictly decreases when $q \in (\hat{q}, 1)$.

Proof. See Appendix. ■

As stated by Theorem 2, the function $\overline{x}_T(q)$ varies non-monotonically with q and is uniquely maximized by $\hat{q} \in (q_0, \bar{q})$. This property implies that the overall bid that can be possibly elicited from the contest will never exceed $\overline{x}_T(\hat{q})$, regardless of the prevailing contest rules.

Definition 2 *Define* $\overline{x}_T^* \equiv \overline{x}_T(\hat{q})$*, which indicates the maximum amount of the overall bid a contest can elicit in a symmetric entry equilibrium.*

A few remarks are in order. First and foremost, the upper limit $\overline{x}_T^*$ is independent of the prevailing winner selection mechanism. The overall bid of any entry-bidding game with primaries (M, Δ, v) can never exceed this level whenever it leads to symmetric entry. As a result, a mechanism must be optimal as long it elicits the "first best" level of expected overall bid $\overline{x}_T^*$. Second, as implied by Theorem 2, the "first best" can be achieved *only if* the mechanism induces a symmetric entry probability of exactly $\hat{q}$. Third, as implied by (4) and (5), to achieve the first best, the mechanism must lead bidders to play a pure bidding strategy upon entry, except in the knife-edge case of $\alpha = 1$.

In light of Theorem 2, the contest design problem boils down to one simple question: Is there a winner selection mechanism that could implement the first best? That is, is there a winner selection mechanism that could induce an equilibrium with entry probability $\hat{q}$ and pure-strategy bidding? We discuss the possibility of implementing the first best $\overline{x}_T^*$ through properly structured contest rules subsequently.

4 Implementing "First Best" by Tullock Contests

To address the question we raised above, our analysis has yet to provide a more complete account of bidders' behaviors in the symmetric equilibria under particular winner selection mechanisms. It requires us to explore explicitly (1) under what conditions an equilibrium with pure-strategy bidding would (not) exist; and (2) to what extent the equilibrium of the entry-bidding game, i.e. equilibrium entry probability and bidding strategy upon entry, can be characterized explicitly.

For this purpose, we, throughout the rest of the analysis, focus on the most popularly adopted and relatively tractable Tullock contest success function, with $g(x) = x^r$, $r > 0$. When $N \geq 2$ bidders enter the contest, the probability of a participating bidder i winning the prize is given by

$$p_N(x_i, \mathbf{x}_{-i}) = \frac{x_i^r}{\sum_{j=1}^N x_j^r}, \text{ if } N \geq 2, \text{ and } \sum_{j=1}^N x_j^r > 0, \quad (7)$$

In this part, we identify the conditions under which pure-strategy bidding would (or would not) emerge, and solve for the equilibrium accordingly. We demonstrate that the first best $\overline{x}_T^*$ can be implemented by a properly structured Tullock contest.

4.1 Solving for Equilibrium

Consider an arbitrary potential bidder i who has entered the contest. Suppose that all other potential bidders play a strategy (q, x) with $x > 0$.[11] He chooses his bid x_i to maximize his expected payoff

$$\pi_i(x_i|\, q, x) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}[\frac{x_i^r}{x_i^r + (N-1)x^r}v - x_i^{\alpha}]. \tag{8}$$

Evaluating $\pi_i(x_i|\, q, x)$ with respect to x_i yields

$$\frac{d\pi_i(x_i|\, q, x)}{dx_i} = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{(N-1)rx_i^{r-1}x^r v}{[x_i^r + (N-1)x^r]^2} - \alpha x_i^{\alpha-1}. \tag{9}$$

Suppose that a symmetric equilibrium with pure-strategy bidding exists. The (pure) bidding strategy in the equilibrium can be solved for by the first order condition $\frac{d\pi_i}{dx_i}|_{x_i=x} = 0$ given the equilibrium entry probability q^*, and q^* can then be characterized by the zero payoff condition.[12] The following lemma fully characterizes such an equilibrium if it exists.

Lemma 2 *Suppose that a symmetric equilibrium (q^*, x^*) with pure-strategy bidding exists. In such an equilibrium, entry probability q^*must satisfy*

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) = \Delta. \tag{10}$$

Each participating bidder places a bid $x^* = [\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{N-1}{N^2}\frac{rv}{\alpha}]^{\frac{1}{\alpha}}$. The expected overall bid of the contest obtains as $x_T^* = Mq^*x^* = Mq^*[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}\frac{N-1}{N^2}\frac{rv}{\alpha}]^{\frac{1}{\alpha}}$.

Proof. See Appendix. ■

Lemma 2 depicts the main properties of a symmetric equilibrium with pure-strategy bidding, if it exists. We call equation (10) the break-even condition of the entry-bidding game with pure-strategy bidding, which yields the following.

Lemma 3 *(a) For any $r > 0$, there exists a unique $q^* \in (0, \overline{q})$ that satisfies the break-even condition (10). Hence, x^* is also uniquely determined for the given r.*

(b) The probability q^* strictly decreases with r.

[11] It is impossible to have all participating bidders bid zero deterministically in an equilibrium. When all others bid zero, a participating bidder would prefer to place an infinitely small positive bid, which allows him to win the prize with probability one.

[12] Note that this two-step procedure is only valid when the contest induces pure-strategy bidding in the equilibrium. As will be shown later, such an equilibrium would not exist if r is sufficiently high.

Proof. See Appendix. ■

Lemma 3 establishes a unique correspondence between r and (q^*, x^*). The symmetric equilibrium with pure bidding strategy must be unique for each given r, whenever it exists. However, the strategy profile (q^*, x^*) of Lemma 2 may not necessarily constitute an equilibrium for the reasons that will be explained below.

Consider an arbitrary participating bidder i's payoff maximization problem, when all others play the hypothetical symmetric strategy (q^*, x^*), as given by Lemma 2. Define

$$\tilde{\pi}_i(x_i) = \pi_i(x_i|q^*, x^*) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}(\frac{x_i^r}{x_i^r + (N-1)x^{*r}} v) - x_i^\alpha, \quad (11)$$

which is a participating bidder i's expected payoff in the contest when all other bidders play the strategy (q^*, x^*). The strategy profile (q^*, x^*) constitute an equilibrium if and only if x^* is a global maximizer of $\tilde{\pi}_i(x_i)$.

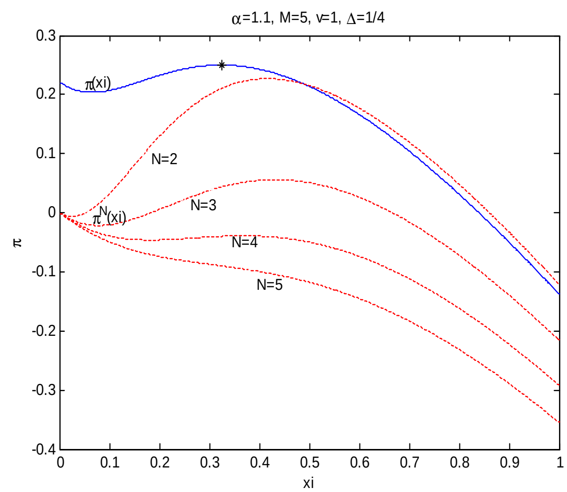

Figure 1

Note that his expected payoff $\tilde{\pi}_i(x_i)$ in the contest is the weighted sum of $\pi_i^N(x_i) = \frac{x_i^r}{x_i^r+(N-1)x^{*r}} v - x_i^\alpha$ over all possible N, i.e. $\tilde{\pi}_i(x_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \pi_i^N(x_i)$. Note that each individual component $\pi_i^N(x_i) = \frac{x_i^r}{x_i^r+(N-1)x^{*r}} v - x_i^\alpha$ is simply his expected payoff when he enters a contest in which he competes against $N-1$ other bidders deterministically and each of them bids x^*.

The equilibrium analysis is straightforward when $r \leq 1$. In that case, each component $\pi_i^N(x_i|\, q, x)$ is concave, and so is $\tilde{\pi}_i(x_i)$. The hypothetical equilibrium bid x^*, which is determined by the first order condition and the symmetry condition, must maximize $\tilde{\pi}_i(x_i)$. A strategy profile with all playing (q^*, x^*) must constitute the unique symmetric equilibrium.

When r exceeds 1, solving the game requires more cautions. It is well known in contest literature that the function $\pi_i^N(x_i|\, q, x)$ is no longer globally concave. Its maximization poses one long-lasting challenge in the equilibrium analysis of imperfectly discriminatory contests. The irregularity is further compounded in the current context. In addition to the concavity of $\pi_i^N(x_i)$, both the weights $\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}$ and the component $\pi_i^N(x_i)$ depend on (q^*, x^*), which is determined endogenously in the equilibrium. The general property of $\tilde{\pi}_i(x_i)$ cannot be readily discerned. Figure 1 graphically illustrates the nature of the maximization problem.

Despite the complexity, we derive the following *sufficient* conditions under which the payoff function $\tilde{\pi}_i(x_i)$ exhibit discernible regularity.

Lemma 4 *When $r \in (1, \alpha(1 + \frac{1}{M-2})]$, x^* is the unique inner local maximizer of $\tilde{\pi}_i(x_i)$ over $(0, \infty)$, i.e. $\tilde{\pi}_i(x^*) > \tilde{\pi}_i(x), \forall x \in (0, \infty)$ and $x \neq x^*$. There exists a unique $x_m \in (0, x^*)$ such that $\tilde{\pi}_i(x_i)$ decreases on $[0, x_m]$, increases on $[x_m, x^*]$, and then decreases on $[x^*, \infty)$.*

Proof. See Appendix. ■

When r remains in the range $(1, \alpha(1+\frac{1}{M-2})]$, the function $\tilde{\pi}_i(x_i)$ decreases first and is locally minimized at x_m. The corresponding curve eventually reaches a single peak at $x^* \in (x_m, \infty)$. Hence, x^* uniquely maximizes $\tilde{\pi}_i(x_i)$ for $x \in (0, \infty)$. Figure 1 in fact exemplifies one of such cases.

However, to establish x^* as the global maximizer, the boundary condition $\tilde{\pi}_i(x^*) \geq \tilde{\pi}_i(0)$ has to be satisfied as well. Recall that symmetric effort x^* is uniquely determined by setting the RHS of (9) equal to zero for $q = q^*$. A participating bidder's expected payoff in the contest when bidding x^*, i.e. $\tilde{\pi}_i(x^*)$, would amount to exactly Δ. However, the bidder automatically receives a reserve payoff $(1 - q^*)^{M-1}v$ from the contest by bidding zero: with a probability $(1-q^*)^{M-1}$, all other potential bidders stay out of the contest, and a rent of $(1-q^*)^{M-1}v$ will accrue to him automatically. The bidder has an incentive to bid x^* only if $(1 - q^*)^{M-1}v \leq \Delta$. The implication of this condition is straightforward: bidding $x^*(> 0)$ is rational only if it generates nonnegative additional return (when all others bid x^*) in excess of the reservation payoff from bidding zero. The condition essentially requires that the contest leaves sufficient rent to contenders and bidding x^* sufficiently rewards a bidder.

Recall the cutoff $q_0 \in (0, \overline{q})$ depicted by Definition 1, which uniquely satisfies $(1-q_0)^{M-1}v = \Delta$. The unique correspondence between r and (q^*, x^*), as determined by the break-even condition (10), allows us to obtain the following cutoff of r.

Definition 3 *Define $r_0 \in (\alpha(1 + \frac{1}{M-1}), 2\alpha]$ to be the unique solution to $\sum_{N=1}^{M} C_{M-1}^{N-1} q_0^{N-1}(1 - q_0)^{M-N} \frac{v}{N}(1 - \frac{N-1}{N}\frac{r_0}{\alpha}) = \Delta$.*

By Lemma 3(b), q^* is inversely related to r. The boundary condition $(1 - q^*)^{M-1}v \leq \Delta$ requires $q^* \geq q_0$, which would hold if and only if $r \leq r_0$. We then conclude the following.

Theorem 3 *A symmetric equilibrium with pure-strategy bidding does not exist if $r > r_0$. In the symmetric equilibria of the entry-bidding game, participating bidders must randomize their bids upon entry.*

Similar to contests with deterministic participation, pure-strategy bidding cannot be sustained when r is excessively large. Theorem 3 provides a sufficient condition under which randomized bidding must occur under endogenous entry. When $r > r_0$, bidding x^* upon entry is not a part of the best response of player i to (q^*, x^*) and the strategy profile (q^*, x^*) would not constitute an equilibrium. We then conclude the following.

Theorem 4 *For each $r \in (0, \min(r_0, \alpha(1+\frac{1}{M-2}))]$, the strategy profile (q^*, x^*), as characterized by Lemma 2, constitutes the unique symmetric equilibrium with pure-strategy bidding of the entry-bidding game.*

When r is bounded from above both r_0 and $\alpha(1 + \frac{1}{M-2})$, x^* is the global maximizer to $\tilde{\pi}_i(x_i)$, which establishes the strategy profile (q^*, x^*) as the unique symmetric equilibrium with pure-strategy bidding.

Theorem 4 imposes a (conservative) upper limit on r for pure-strategy bidding. The condition $r \leq \alpha(1 + \frac{1}{M-2})$ is sufficient but not necessary to establish x^* as the local maximizer of $\tilde{\pi}_i(x_i)$ for $x_i > 0$. These concerns however can be dismissed in more specific contexts with small numbers of potential bidders.

Corollary 1 *When M is small, i.e. $M = 2, 3$, a symmetric equilibrium with pure-strategy bidding exists if and only if $r \leq r_0$.*

In these instances, $(\alpha(1 + \frac{1}{M-2}), r_0]$ is empty, because $\alpha(1 + \frac{1}{M-2}) > r_0$ regardless of v and Δ. Whenever r falls below r_0, it automatically satisfies the condition $r \leq \alpha(1 + \frac{1}{M-2})$, which guarantees that x^* maximizes $\tilde{\pi}_i(x_i)$.

When M is larger, the cutoff r_0 may exceed $\alpha(1 + \frac{1}{M-2})$. It remains less than explicit how the equilibrium would behave if $\alpha(1 + \frac{1}{M-2}) < r_0$ and $r \in (\alpha(1 + \frac{1}{M-2}), r_0]$. Analytical difficulty prevents us from fully characterizing the property of $\tilde{\pi}_i(x_i)$ when r exceeds $\alpha(1 + \frac{1}{M-2})$. We resort to a numerical exercise to obtain further insights. For expositional efficiency, we postpone the presentation and discussion of these observations to Section 4.2.2 as they shed light on the optimal contest design problem explored in that section.

4.2 Optimal Contests

The results of equilibrium analysis allows us to explore the implementation of the first best $\bar{x}_T^*$ by a well structured Tullock contest. We focus on the level of precision of the winner selection mechanism, which is conventionally measured by the discriminatory power r in Tullock contests, as the primary instrument for a contest designer. Following the literature (e.g. Nti, 2004 and Gershkov, Li and Schweinzer, 2009), we let a designer choose the size of r and announce it to potential bidders as a part of the contest rules. An entry-bidding game ensues subsequently.

The level of precision in a contest is largely subject to the strategic choice of the contest designer. For instance, the designer can modify the judging criteria of the contest to suit her strategic goals, e.g. adjusting the weights of subjective component in contenders' overall ratings. Alternatively, she can vary the composition of judging committees (experts vs. non-experts).

A greater r implies that the mechanism better translates an increase in bid into additional likelihood of one's win. Hence, winner selection is related more closely to bidders' contributions, but less to various noisy factors. Hence, a greater r increases the marginal return of a bid and further incentivizes bidders. It is well known in the contest literature that both individual bids and overall bids strictly increase with r in a contest with deterministically M participants, whenever a pure-strategy equilibrium exists, i.e. $r \in [0, \alpha(1 + \frac{1}{M-1})]$. This conventional wisdom, however, is no longer straightforward in the current context.

Endogenous and stochastic entry gives rise to competing forces at two layers. First, it creates a trade-off between ex ante entry incentive and ex post bidding incentives. On the one hand, a more discriminatory contest compels participants to bid more, while on the other, the increasing dissipation of rent limits entry.[13] Second, it leads to the tension between the size of the overall contest and the incentive of individual bidding. More extensive participation (i.e. a higher q^*) does not necessarily improve the supply of bids in the contest. A higher q^* allows the contest to engage more bidders, which amplifies the sources of contribution and tends to increase the overall bid. It nevertheless limits each individual participant's incentive to bid, as one would anticipate more intense competition and therefore lesser return.

The optimum must balance these forces. As revealed by Theorem 2 and (2)-(5), the key to the optimal design problem is whether r can be set properly to implement the first best $\overline{x}_T^*$. The overall bid amounts to exactly $\overline{x}_T^*$ if the contest leads to a symmetric equilibrium with an entry probability of $\hat{q}$ and participants play a pure bidding strategy upon entry in the equilibrium, except in the knife-edge case of $\alpha = 1$.

In subsequent analysis, we focus on the implementation of first best in equilibria with pure-strategy bidding. The existing literature on contest has been limited on bidders' behaviors in equilibria that involve mixed strategy bidding, e.g. in Tullock contest with large but finite r. Due to the lack of handy solutions, it is difficult in the current context either (1) to explicitly solve an equilibrium when it involves mixed-strategy bidding, or (2) to simply verify

[13] As revealed by Lemma 3, q^* strictly decreases with r.

the existence of an equilibrium with mixed-strategy bidding under a particular entry pattern. The analytical difficulty thus prevents us from identifying the correspondence between the prevailing contest structure and the resultant equilibrium behaviors. However, we show that it is sufficient to focus on contests that induce pure-strategy bidding.

Before we proceed, recall the break-even condition $v\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N}[\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha}] = \Delta$, which defines the correspondence between r and q^* as given by Lemma 3.

Definition 4 *Let $r(\hat{q})$ be the unique solution of r to*

$$v\sum_{N=1}^{M} C_{M-1}^{N-1} \hat{q}^{N-1}(1-\hat{q})^{M-N}[\frac{1}{N} - \frac{N-1}{N^2}\frac{r}{\alpha}] = \Delta. \tag{12}$$

The fact $\hat{q} > q_0$ (Theorem 2(a)) and the definition of r_0 (Definition 3) lead to the following.

Lemma 5 *$r(\hat{q}) < r_0$.*

Proof. See Appendix. ■

We subsequently discuss the implementation of first best through choice of r in two cases according to the ranges of $r(\hat{q})$.

4.2.1 Optimal Contest When $r(\hat{q}) \leq \alpha(1+\frac{1}{M-2})$

When $r(\hat{q})$ falls below the cutoff $\alpha(1+\frac{1}{M-2})$, the optimal contest is straightforward. Recall by Lemma 4 that $r \leq \alpha(1+\frac{1}{M-2})$ is the sufficient condition under which a well-behaved payoff function $\tilde{\pi}(x_i)$ results. Lemma 5 further guarantees that the boundary condition $\tilde{\pi}(0) < \tilde{\pi}(x^*)$ holds. Hence, a contest with $r = r(\hat{q})$ must lead to a symmetric equilibrium with entry probability of exactly $\hat{q}$ and pure-strategy bidding. The following obtains.

Theorem 5 *Whenever $r(\hat{q}) \leq \alpha(1+\frac{1}{M-2})$, the contest designer can elicit the "first best" $\bar{x}_T^*$ by setting $r = r(\hat{q})$. It induces a symmetric equilibrium with pure-strategy bidding. Potential bidders enter the contest with a probability $\hat{q}$ in the symmetric equilibrium.*

Proof. See Appendix. ■

In this case, a contest with $r = r(\hat{q})$ elicits the "first best" $\bar{x}_T^* \equiv \bar{x}_T(q)$. By Theorem 2, the expected overall bid must strictly decrease when r deviates from $r(\hat{q})$. A higher-powered incentive would not pay off.

Comparison to Benchmark Case Our results run in sharp contrast to the conventional wisdom in contest literature. In a contest with a fixed number M of participants or free entry, a higher r provides stronger incentives to bidders, and elicits strictly higher bids whenever a pure-strategy equilibrium prevails, i.e. when $r \leq \alpha(1+\frac{1}{M-1})$. The size of r in our setting, however, affects the resultant equilibrium bid non-monotonically.

Despite the various trade-offs between conflicting forces, a softer *ex ante* incentive, i.e. a smaller r, may or may not be optimal. The optimal size of the parameter could either fall below or remain above the benchmark $\alpha(1+\frac{1}{M-1})$. In the left panel of Figure 2, the observations demonstrate the incidence of optimal "soft" incentives, with $r(\hat{q}) < \alpha(1+\frac{1}{M-1})$. In the right panel, the results illustrate the possibility of the opposite, with $r(\hat{q}) \in (\alpha(1+\frac{1}{M-1}), \alpha(1+\frac{1}{M-2}))$.

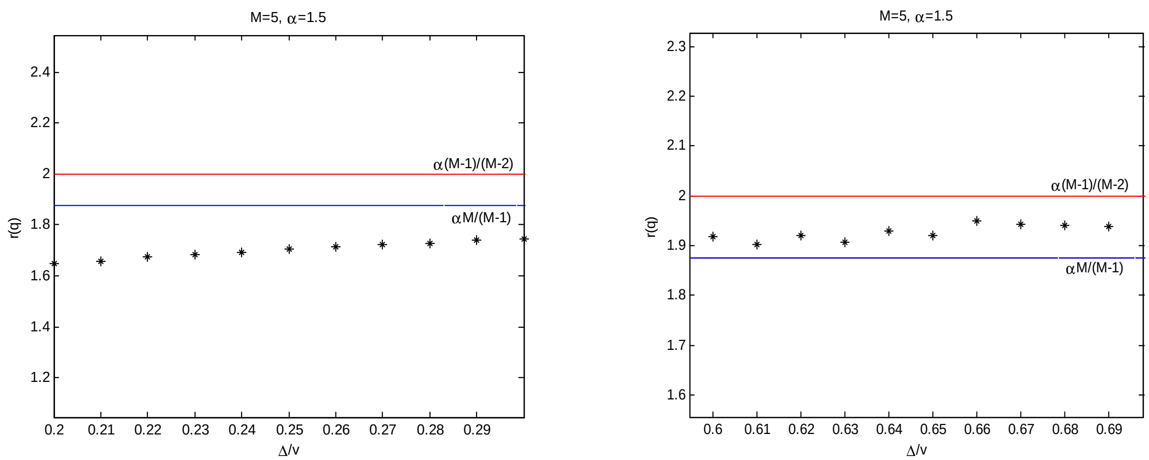

Figure 2

These observations also contrast the results of related studies in auction literature. A number of studies have been devoted to the optimal design of auctions with costly entry, including Menezes and Monteiro (2000), Levin and Smith (1994), Shi (2009), Moreno and Wooders (2010) and Lu (2009, 2010). They all espouse the optimality of a "softer" incentive: the optimal reserve price is always lower than in the free-entry benchmark. The insight from auction literature does not extend to our setting. The observations in Figure 2 demonstrate that the optimum does not necessarily requires a lower-powered incentive mechanism than the free-entry benchmark level $\alpha(1+\frac{1}{M-1})$.[14]

4.2.2 Optimal Contest when $r(\hat{q}) \in (\alpha(1+\frac{1}{M-2}), r_0)$

Setting r to $r(\hat{q})$ is optimal when it leads to pure-strategy bidding. The optimum, however, is less than straightforward when r exceeds the cutoff $\alpha(1+\frac{1}{M-2})$. As aforementioned, technical difficulty prevents us from drawing definitive conclusion on the property of bidders' expected

[14] In our setting, if entry does not involve fixed entry cost, all the M potential bidders will participate. The conventional wisdom in contest literature would apply, such that $r = \alpha(1+\frac{1}{M-1})$ would emerge as the optimum.

payoff function $\tilde{\pi}_i(x_i)$ when r exceeds $\alpha(1+\frac{1}{M-2})$. An equilibrium with pure-strategy bidding is not guaranteed when r is set $r(\hat{q})$, which thus casts doubt on the robustness of Theorem 5. This concern can be dismissed only in contests with small pools of potential participants.

Corollary 2 *When the contest is small, i.e., $M = 2, 3$, $r(\hat{q}) \leq \alpha(1+\frac{1}{M-2})$ must hold, and a symmetric equilibrium with pure-strategy bidding can always be induced by setting $r = r(\hat{q})$.*

In these cases, $\alpha(1+\frac{1}{M-2}) > r_0$, so the condition $r(\hat{q}) \leq \alpha(1+\frac{1}{M-2})$ is satisfied automatically. The condition, however, may not hold when M gets large. It thus remains to explore to what extent the first best can be robustly implemented when $r(\hat{q}) \in (\alpha(1+\frac{1}{M-2}), r_0)$.

The discussion in the rest of this subsection proceeds in two steps. We first present the observations from our numerical exercises, which shed light on the property of the function $\tilde{\pi}_i(x_i)$ when $r(\hat{q}) \in (\alpha(1+\frac{1}{M-2}), r_0)$. We then consider an alternative mechanism, which, with additional instruments, can implement the first best when $r(\hat{q}) \in (\alpha(1+\frac{1}{M-2}), r_0)$.

Numerical Exercises Recall that $r \leq \alpha(1+\frac{1}{M-2})$ is a sufficient but not necessary for pure-strategy bidding. It should be noted that pure-strategy bidding can still be induced by $r \in (\alpha(1+\frac{1}{M-2}), r_0)$. Our numerical exercises verify this possibility.

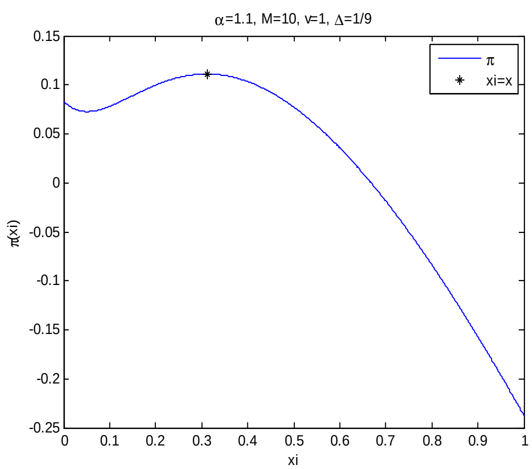

Figure 3

We normalize v to unity. The simulation is run over a large set of the parameters (α, M), which span the entire space of $[1, 2] \times \{4, 5, \ldots, 100\}$. For given (α, M), we let r vary over the entire range of $(\alpha(1+\frac{1}{M-2}), r_0]$ if $\alpha(1+\frac{1}{M-2}) < r_0$, and let Δ vary over the interval $(\frac{1}{M}, 1)$ as required by Assumption 1. We observe from our simulation results, with no exception, that all $\tilde{\pi}_i(x_i)$ demonstrates the property depicted by Lemma 4, and is uniquely maximized by x^*, despite that r exceeds $\alpha(1+\frac{1}{M-2})$. In all resultant figures, the curve is regularly shaped as described by Lemma 4. Figure 3 provides one example of them. The strategy profile (q^*, x^*)

constitutes the unique symmetric equilibrium with pure-strategy bidding in all the simulated settings.

Hence, in all the simulated settings, we can elicit the "first best" by setting $r = r(\hat{q})$, regardless of the value of $r(\hat{q})$. Based on these observations, we propose the following conjectures.

Conjecture 1 *(a) A symmetric equilibrium with pure bidding exists if $r \leq r_0$.*

(b) The first best overall bid $\bar{x}_T^*$ can always be induced in a unique symmetric equilibrium with pure-strategy bidding by setting r to $r(\hat{q})$.

We are unable to prove it analytically. However, all of our numerical exercises lend support to the claim. We leave it to future studies due to its technical difficulty.

Alternative Implementation of "First Best" The numerical exercises confirm the robustness of $r(\hat{q})$ as the optimum in a wide context. The doubt, however, has not been resolved completely. Despite the limitations of the analysis due to the technical difficulty, we demonstrate that the first best can be implemented under an alternative mechanism when $r(\hat{q}) \in (\alpha(1 + \frac{1}{M-2}), r_0)$ such that simply fine-tuning r may not be sufficient to induce $\bar{x}_T^*$ universally.

In particular, we allow the contest designer to charge an entry fee F to participating bidders and to commit to a prize schedule v, with $v = (v_1, \ldots, v_M)$. The winner's prize is allowed to be contingent on the actual number N of participating bidders, and v_N (the prize when N entrants show up) can be financed by the proceeds of entry fees. The designer is subject to a budget constraint $v_N \leq v + N \cdot F$. Apparently, to maximize overall expected bid, the contest designer is required to exhaust her revenue when topping up the prize purse. The contest designer thus chooses a combination of (r, F) to maximize the expected overall bid of the contest.

We first obtain the following.

Lemma 6 *Consider a winner-take-all Tullock contest with impact function $x^r, r \in (0, \alpha)$ and contingent prizes $v_N = v + N \times F$, where v_N is a winner prize that is contingent on the actual number of entrants and F (≥ 0) is a uniform entry fee. This contest induces a unique symmetric equilibrium. The equilibrium entry probability q^* satisfies*

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1 - q^*)^{M-N} \frac{v_N}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) = \Delta + F. \tag{13}$$

Each participating bidder, upon his entry, places a bid

$$x^* = [\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1 - q^*)^{M-N} \frac{N-1}{N^2}\frac{r v_N}{\alpha}]^{\frac{1}{\alpha}}. \tag{14}$$

Obviously, with $\frac{r}{\alpha} \leq 1$, the expected payoff function of an entrant is concave, which leads to a unique pure-strategy bidding equilibrium. We further obtain the following.

Lemma 7 *The equilibrium entry probability q^* strictly decreases with entry fee F, with $\lim_{F \to +\infty} q^* = 0$.*

Proof. See Appendix. ■

Lemmas 6 and 7 allow us to conclude the following.

Theorem 6 *When $r(\hat{q}) \in (\alpha(1 + \frac{1}{M-2}), r_0)$, for each given $r \in (0, \alpha)$ there exists a unique entry fee $F(> 0)$ such that contest depicted in lemma 6 induces pure-strategy bidding and an equilibrium entry probability $\hat{q}$. The contest elicits an expected overall bid of $\bar{x}_T^*$.*

Proof. See Appendix. ■

Entry fees F enter prize purses and the revenue is redistributed between entrants. Equations (2) to (5) of Section 3.4 must continue to hold. A combination of (r, F) thus must elicit an expected overall bid of exactly $\bar{x}_T(q)$ if it induces a symmetric equilibrium with entry probability of $q \in (0, 1)$ and also pure-strategy bidding upon entry. Hence, we conclude that the first best can always be implemented in a properly structured Tullock contest of Theorem 6 through an equilibrium with pure-strategy bidding. It should be noted that the optimal combination (r, F) is not unique. As enlightened by Lemma 7, a lower r can be complemented by a higher entry fee F to implement the first best effort.

4.3 Efficient Exclusion

The equilibrium analysis also allows us to investigate another classical question in the literature on contest design. We now allow the contest designer to invite only a subset of the M potential bidders for participation. The invited bidders decide whether to participate in the contest. The conventional wisdom holds that a contest elicits more effort when it involves a larger number of contestants. In what follows, we demonstrate that exclusion can improve the efficiency of the contest in our setting.

The expected overall bid of a contest (M, Δ, v) is bounded by the first best $\bar{x}_T^*$. It should be noted that the exact amount of $\bar{x}_T^*$ depends on the number of potential bidders who may enter the contest. Let M' be an arbitrary positive integer and let $\overline{x}_T^*(M')$ be the first best bid for a contest with M' potential bidders. The function $\overline{x}_T^*(M')$ exhibits the following property.

Lemma 8 *$\overline{x}_T^*(M')$ strictly decreases with M' for all M' that satisfies $\frac{v}{M'} < \Delta$.*

Proof. See Appendix. ■

Lemma 8 yields direct implications for the contest design: a contest may have a weaker potential of eliciting bid if it involves a larger pool of potential bidders. The contest designer, when she is able to structure the contest properly to implement the first $\overline{x}_T^*$, may get better off by excluding potential bidders. Define $M_0 \triangleq \min(M'| \frac{v}{M'} < \Delta)$ and assume $M_0 < M$. We obtain the following.

Theorem 7 *When the contest designer is allowed to exclude potential bidders, the optimal contest does not invite more than M_0 contestants.*

Theorem 7 demonstrates that exclusion improves bidding efficiency. By inviting M_0 of them, and adopting the optimal design discussed in Section 4.2, the contest designer elicits an overall bid $\overline{x}_T^*(M_0)$, which, by Lemma 8, is unambiguously more than what she can possibly achieve if she engages an M potential bidders. Our result thus provides an alternative rationale for shortlisting and exclusion in a setting with homogeneous bidders but endogenous entry. The logic resembles that on the optimal r. First, when the contest involves more potential bidders, each of them would enter less often and bid less (if he enters) when anticipating a more intense competition and expecting a smaller share of the rent. Second, extensive participation may lead to excessive rent dissipation because of the entry costs incurred, which tends to limit bidders' effort supply.

Theorem 7 provides only an upper bound for the possible optimum. It, however, does not pin down exactly how many bidders should be invited in the optimum. When the contest designer invites less than M_0 potential bidders, the overall bid of the contest can elicit would change indefinitely, and the efficiency of the contest may either improve or suffer.[15] The analysis for a contest with less than M_0 potential bidders is beyond the scope of the current paper, as Assumption 1 no longer holds in that setting.

5 Concluding Remarks

In this paper, we provide a thorough account of contests with endogenous and stochastic entries. We show the existence of a symmetric mixed-strategy equilibrium in which potential bidders randomly enter. We also provide a sufficient condition under which participants engage in pure bidding actions. Based on these equilibrium results, we identify relevant institutional elements in contest rules, and we demonstrate that analysis in this setting adds substantially to existing knowledge on optimal contest design.

While our study is one of the first to investigate the subtle and rich strategic interaction that occurs in contests with endogenous entries, our analysis reveals the enormous possibilities for

[15]Examples in specific settings are available from the authors upon request, which demonstrate that the overall bid may either decrease or increase.

future studies. Due to analytical difficulties, the open conjectures in Section 4 pose a challenge for future research on contests. However, the authors will attempt this, despite the technical difficulties.

In our analysis, we assume that the actual number of participants is unobservable to participating bidders. One natural question is whether the contest designer could improve the contest designer by disclosing the actual number of participating bidders when she can observe it. It deserves to be noted that there is no loss of generality in this aspect. The first best $\bar{x}_T^*$ would still hold when the number of entrants is revealed.[16] Hence, whenever a mechanism can successfully achieve the first best, it must (at least weakly) be optimal. Hence, disclosure would not improve bidding efficiency further when the designer has sufficient flexibility to structure the contest.[17]

Further, our setting (characterized by common entry cost, simultaneous entry and resultant stochastic entry) is only one way for modeling contests that involve endogenous entry. Other examples include the setting of Kaplan and Sela (2010). They consider all-pay auctions with privately-known entry costs. Fu and Lu (2010) assume that contestants enter sequentially. Various issues on optimal contest designs in these diverse settings remain open, and they deserve to be explored seriously in future efforts.

Appendix

Proof of Lemma 1

Proof. Let $f_1(q) = [1-(1-q)^M]v - Mq\Delta$, and $f_2(q) = (1-q)^{M-1}v - \Delta$. $\bar{q}$ (> 0) is defined as $f_1(\bar{q}) = 0$. The first order derivative of $f_1(q)$ is $f_1'(q) = Mf_2(q)$, which is a decreasing function of q. $f_1'(q)$ is positive when $q = 0$, and it is negative when $q = 1$.

q_0 is defined as $f_2(q_0) = 0$. Therefore, $f_1(q)$ increases on $[0, q_0]$, and decreases from $[q_0, 1)$. $f_1(q)$ thus has two zero points, i.e. $\{0, \bar{q}\}$, and $q_0 < \bar{q}$. ■

Proof of Theorem 1

Proof. Part (a) Existence of symmetric equilibria: Consider the following extended game. There are M contestants who simultaneously choose their two-dimensional actions, which are denoted by $a_i = (a_{i1}, a_{i2}) = (q_i, x_i) \in A$, $i = 1, 2, ..., M$, where the uniform action space $A = [0,1] \times [0, v^{1/\alpha}]$ is nonempty, convex and compact.

[16] The proof is similar to that of Section 3.4. A detailed proof is available from the authors.

[17] In a previous version of this paper, we showed in wide settings disclosing the actual number of contestants adversely affect the resultant bidding efficiency. We omit these results in this version of the paper, as the analysis is not directly related to the current context.

Let $\mathbf{k} = (k_1, k_2, ..., k_i, ..., k_N)$ where k_i is either 0 or 1. Let K to be the set of all possible $\mathbf{k}$. Similarly, we can define $\mathbf{k}_{-i}$ and K_{-i}, $i = 1, 2, ..., M$.

Given action profile $\mathbf{a} = \{a_1, a_2, ..., a_M\}$ of the M players, the payoff of player i is defined as

$$U_i(\mathbf{a}) = q_i\{[\sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} q_j^{k_j}(1-q_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x})]v - x_i^\alpha - \Delta\}, i = 1, 2, ..., M,$$

where $\Pr(i|\mathbf{k}_{-i}, \mathbf{x}) = \frac{g(x_i)}{g(x_i)+\sum_{j\neq i} k_j g(x_j)}$ if $g(x_i) + \sum_{j\neq i} k_j g(x_j) > 0$, and $\Pr(i|\mathbf{k}_{-i}, \mathbf{x}) = \frac{1}{1+\sum_{j\neq i} k_j}$ if $g(x_i) + \sum_{j\neq i} k_j g(x_j) = 0$. Note that $\Pr(i|\mathbf{k}_{-i}, \mathbf{x})$ equals to the winning probability of an entrant i when the entry status of others is denoted by $\mathbf{k}_{-i}$ and players' effort is $\mathbf{x}$ if they enter.

Note that this game is a symmetric game as defined by Dasgupta and Maskin (1986) in their Definition 7. We will apply their Theorem 6* in Appendix to establish the existence of symmetric equilibrium in mixed strategy.

In what follows, we show that for each i, the discontinuities of U_i are confined to a subset of a continuous manifold of dimension less than M as required by page 7 of Dasgupta and Maskin (1986). Following the notations on page 22 of Dasguspta and Maskin (1986). Let $Q = \{2\}, D(i) = 1$, and f_{ij}^1 to be an identity function. Following their $(A1)$ of page 22, we define manifold $A^*(i) = \{\mathbf{a} \in A|\exists j \neq i, \exists k \in Q, \exists d, 1 \leq d \leq D(i)$ such that $a_{jk} = f_{ij}^d(a_{ik})\}$. Clearly, $A^*(i)$ is of dimension less than M. The set of discontinuous points for $U_i(\mathbf{a})$ can be written as $A^{**}(i) = \{\mathbf{a} \in A|q_j x_j = 0, \forall j = 1, 2, ..., M; q_i > 0, x_i = 0; \exists j_0 \neq i,$ such that $q_{j_0} > 0$ and $x_{j_0} = 0\}$. Clearly, $A^{**}(i) \subset A^*(i)$, since any element in $A^{**}(i)$ must satisfy the following conditions: For $k = 2 \in Q, \exists j_0 \neq i$, such that $x_{j_0} = f_{ij}^1(x_{ik})$, i.e. $a_{j_0 2} = f_{ij}^1(a_{i2})$. According to their Theorem 6*, we need to verify the following conditions hold.

First, as constructed above, $U_i(\mathbf{a})$ is continuous except on a subset $A^{**}(i)$ of $A^*(i)$, where $A^*(i)$ is defined by $(A1)$.

Second, clearly, we have $\sum_i U_i(\mathbf{a}) = v[1 - \prod_i(1 - q_i)] - \sum_i q_i(x_i^\alpha + \Delta)$, which is continuous and thus upper semi-continuous.

Third, $U_i(\mathbf{a})$ clearly is bounded on $A = [0, 1] \times [0, v^{1/\alpha}]$.

Fourth, we verify that Property (α^*) of page 24 is satisfied. Define B^2 as the unit circle with the origin as its center, i.e. $B^2 = \{\mathbf{e} = (q, x) \mid q^2 + x^2 = 1\}$. Pick up any continuous density function $v(\cdot)$ on B^2 such that $v(\mathbf{e}) = 0$ iff $e_1 \leq 0$ or $e_2 \leq 0$. Note that $U_i(a_i, \mathbf{a}_{-i})$ is continuous in a_{i1} and lower semi-continuous in a_{i2}. $\forall \mathbf{a} = (\bar{a}_i, \mathbf{a}_{-i}) \in A^{**}(i)$, clearly we have that for any $\mathbf{e}$ such that $v(\mathbf{e}) > 0$ (i.e. $min(e_1, e_2) > 0$), $\liminf_{\theta\to 0^+} U_i(\bar{a}_i + \theta\mathbf{e}, \mathbf{a}_{-i}) > U_i(\bar{a}_i, \mathbf{a}_{-i})$ as $\theta > 0, e_2 > 0$ and $q_i > 0, x_i = 0$ in $\bar{a}_i$. This leads to that $\int_{B^2}[\liminf_{\theta\to 0^+} U_i(\bar{a}_i + \theta\mathbf{e}, \mathbf{a}_{-i})v(\mathbf{e})d\mathbf{e}] > U_i(\bar{a}_i, \mathbf{a}_{-i}), \forall \bar{a}_i \in A_i^{**}(i), \mathbf{a}_{-i} \in A_{-i}^{**}(\bar{a}_i)$, where $A_i^{**}(i)$ is the collection of all $\bar{a}_i$ of player i that appear in $A^{**}(i)$, $A_{-i}^{**}(\bar{a}_i)$ is the collection of others' actions $\mathbf{a}_{-i}$ such that $\mathbf{a} = (\bar{a}_i, \mathbf{a}_{-i}) \in A^{**}(i)$. This confirms that Property (α^*) holds for the above game.

Thus according to Theorem 6* of Dasgupta and Maskin (1986), there exists a symmetric mixed strategy equilibrium. Without loss of generality, we use $\mu_1(q)$ to denote the equilibrium probability measure of action q, and use $\mu_2(x)$ to denote the equilibrium probability measure of action x.

Next we show that for any strategy profile of players $\{(\mu_{i1}(q_i), \mu_{i2}(x_i))\}$. The players' payoffs are same from strategy profile of players that is defined as $\{(E_{\mu_{i1}} q_i, \mu_{i2}(x_i))\}$. The expected utility of player i from profile $\{(\mu_{i1}(q_i), \mu_{i2}(x_i))\}$ is

$$\begin{aligned}
E_{\mathbf{a}} U_i(\mathbf{a}) &= E_{q_i}\{E_{\mathbf{q}_{-i}} E_{\mathbf{x}}[q_i \sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} q_j^{k_j}(1-q_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x}))v - x_i^\alpha - \Delta]\} \\
&= E_{q_i}\{q_i E_{\mathbf{x}} E_{\mathbf{q}_{-i}}[\sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} q_j^{k_j}(1-q_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x}))v - x_i^\alpha - \Delta]\} \\
&= E_{q_i}\{q_i E_{\mathbf{x}}[\sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} (Eq_j)^{k_j}(1-Eq_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x}))v - x_i^\alpha - \Delta]\} \\
&= Eq_i \cdot E_{\mathbf{x}}[\sum_{\mathbf{k}_{-i}\in K_{-i}} (\prod_{j\neq i} (Eq_j)^{k_j}(1-Eq_j)^{1-k_j}) \Pr(i|\mathbf{k}_{-i}, \mathbf{x}))v - x_i^\alpha - \Delta]\}, \forall i. \quad (15)
\end{aligned}$$

The above result means that given others take strategy $(E_{\mu_1} q, \mu_2(x))$, the same strategy is also the best strategy for player i. Otherwise, $(\mu_1(q), \mu_2(x))$ would not be the optimal strategy for player i when others take the same strategy $(\mu_1(q), \mu_2(x))$. Therefore, $(E_{\mu_1} q, \mu_2(x))$ is a *symmetric equilibrium* for the above game.

It is easy to see that $(q^*, \mu^*(x)) = (E_{\mu_1} q, \mu_2(x))$ is a symmetric equilibrium for our original game based on the way the extended game is constructed. $U_i(\mathbf{a})$ equals player i's expected payoffs when he enters with probability q_i and exerts effort x_i when he enters, given that other bidder j enters with probability q_j and exerts effort x_j when he enters. This claim also holds when they adopt any other entry strategies with measure $\{\mu_{i1}(q), i = 1, 2, ..., M\}$ due to (15). According to (15), only the expected entry probabilities $\{E_{\mu_{i1}} q, i = 1, 2, ..., M\}$ count.

Note we must have $q^* = E_{\mu_1} q \in (0, 1)$. First, $q^* = E_{\mu_1} q = 0$ cannot be an entry equilibrium when $\Delta < v$ (Assumption 1). Second, $q^* = E_{\mu_1} q = 1$ cannot be an entry equilibrium when $\Delta > \frac{v}{M}$ (Assumption 1). The expected equilibrium payoff of players must be nonnegative. Thus we must have $(1 - (1 - E_{\mu_1} q)^M)v - M(E_{\mu_1} q)[\Delta + E_{\mu_2} x] \geq 0$. This leads to $(1 - (1 - E_{\mu_1} q)^M)v - M(E_{\mu_1} q)\Delta > 0$. Thus $q^* = E_{\mu_1} q < \bar{q}$ by Definition 1 and proof of Lemma 1.

Part (b): The equilibrium payoff cannot be negative. When $q^* = E_{\mu_1} q \in (0, 1)$, we must have the equilibrium payoffs of player to be zero as otherwise it cannot be an equilibrium as the player would enter with probability 1 and earn a positive payoff. ■

Proof of Theorem 2

Proof. Define an increasing transformation of $\bar{x}_T(q)$:

$$\Psi(q) = [\bar{x}_T(q)]^\alpha = (Mq)^{\alpha-1}\left\{[1 - (1 - q)^M]v - Mq\Delta\right\}$$

Note that $\Psi(q)|_{q=0} = 0$; and $\Psi(q)|_{q=1} = M^{\alpha-1}(v - M\Delta) < 0$ since $\frac{v}{M} < \Delta$ (Assumption 1). We have

$$\frac{d\Psi(q)}{dq} = f(q)\, q^{\alpha-2} M^{\alpha-1},$$

where

$$f(q) = (\alpha-1)\underbrace{\{[1-(1-q)^M]v - Mq\Delta\}}_{f_1(q)} + Mq\underbrace{[(1-q)^{M-1}v - \Delta]}_{f_2(q)}.$$

We have

$$f'(q) = Mv(1-q)^{M-2}[\alpha - (M+\alpha-1)q] - \alpha M\Delta.$$

Note that $f'(0) = \alpha Mv - \alpha M\Delta > 0$, $f'(1) = -\alpha M\Delta < 0$ and $f'(q)$ decreases with $q \in (0, \frac{\alpha}{M+\alpha-1}]$. Clearly, $f'(q) < 0$ when $q \in [\frac{\alpha}{M+\alpha-1}, 1]$. Then there exists a unique $q_c \in (0, \frac{\alpha}{M+\alpha-1})$, such that $f'(q_c) = 0$, which means q_c is the maximum point of $f(q)$. Since $f(0) = 0$, $f(q_c) > 0$ and $f(1) = (\alpha-1)v - \alpha M\Delta = \alpha(v - M\Delta) - v < 0$, then there must exist a unique $\hat{q} \in (q_c, 1)$, such that $f(\hat{q}) = 0$. Note that $f'(q) < 0$ on $(q_c, 1)$. Clearly, $f(q) > 0$ when $0 < q < \hat{q}$; and $f(q) < 0$ when $\hat{q} < q < 1$.

Since $\frac{d\Psi(q)}{dq}$ shares the same sign with $f(q)$, we have that $\frac{d\Psi(q)}{dq} > 0$ when $0 < q < \hat{q}$; and $\frac{d\Psi(q)}{dq} < 0$ when $\hat{q} < q < 1$. This implies $\hat{q} = \arg\max_q \Psi(q)$, i.e. $\hat{q} = \arg\max_q \bar{x}_T(q)$.

By the proof of Lemma 1, we know both $f_1(q)$ and $f_2(q)$ are positive when $q \in [0, q_0]$ and both are negative when $q > \bar{q}$. Thus the zero point $(\hat{q})$ of $f(q)$ must fall in $[q_0, \bar{q}]$. ■

Proof of Lemma 2

Proof. If a symmetric equilibrium with pure strategy bidding exists, according to the first order condition $\frac{d\pi_i(x_i)}{dx_i} = 0$ and the symmetry condition $x_i = x$, x^* must solve

$$\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{(N-1)rv}{N^2 x^*} - \alpha x^{*\alpha-1} = 0,$$

which yields

$$x^*(q) = [\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{N-1}{N^2}\frac{rv}{\alpha}]^{\frac{1}{\alpha}}.$$

The equilibrium expected payoff is

$$\begin{aligned}\pi^*(x^*(q), q) &= \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{v}{N} \\ &\quad -[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{N-1}{N^2}\frac{rv}{\alpha}] \\ &= \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N}\frac{v}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}).\end{aligned}$$

By entering the contest and submit the bid $x^*(q)$, every potential contestant i ends up with an expected payoff

$$\pi^*(x^*(q), q) - \Delta.$$

By Theorem 1 (b), each potential bidder receives a zero expected payoff for the equilibrium entry q^*, i.e. $\pi^*(x^*(q^*), q^*) = \Delta$.

The expected overall effort of the contest (x_T^*) obtains as

$$\begin{aligned} x_T^* &= Mq^*x^*(q^*) \\ &= Mq^*[\sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{N-1}{N^2} \frac{rv}{\alpha}]^{\frac{1}{\alpha}}. \end{aligned}$$

■

Proof of Lemma 3

Proof. By Lemma 2, q^* satisfies $F(q^*, r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) - \Delta = 0$. Apparently, $F(q^*, r)$ is continuous in and differentiable with both arguments. We first claim that $F(q^*, r)$ strictly decreases with q^*. Define $\pi_N = \frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha})$. Taking its first order derivative yields

$$\begin{aligned} \frac{F(q^*, r)}{dq^*} &= \sum_{N=1}^{M} C_{M-1}^{N-1}[(N-1)q^{*N-2}(1-q^*)^{M-N} - (M-N)q^{*N-1}(1-q^*)^{M-N-1}]\pi_N \\ &= \sum_{N=1}^{M} C_{M-1}^{N-1}(N-1)q^{*N-2}(1-q^*)^{M-N}\pi_N - \sum_{N=1}^{M} C_{M-1}^{N-1}(M-N)q^{*N-1}(1-q^*)^{M-N-1}\pi_N \\ &= (M-1)\{\sum_{N=2}^{M} C_{M-2}^{N-2} q^{*N-2}(1-q^*)^{M-N}\pi_N - \sum_{N=1}^{M-1} C_{M-2}^{N-1} q^{*N-1}(1-q^*)^{M-N-1}\pi_N\} \\ &= (M-1)\sum_{N=1}^{M-1} C_{M-2}^{N-1} q^{*N-1}(1-q^*)^{M-N-1}\left(\pi_{N+1} - \pi_N\right), \end{aligned}$$

which is obviously negative because $\pi_N = \frac{1}{N}\left[1 - \left(1 - \frac{1}{N}\right)\frac{r}{\alpha}\right] v \geq 0$ and it monotonically decreases with N.

When all other potential contestants play $q = 0$, a potential contestant receives a payoff $v - \Delta > 0$, and he must enter with probability one. When all others play $q = \overline{q}$, a participating contestant receives negative expected payoff if he enters by Definition 1 and Lemma 1 ($(1 - \overline{q})^{M-1}v < \Delta$), which cannot constitute an equilibrium either. Hence, a unique $q^* \in (0, \overline{q})$ must exist that solves $\pi^*(x^*, q) = \Delta$. Each potential contestant is indifferent between entering and staying inactive when all others play the strategy. This constitutes an equilibrium.

Moreover, $F(q^*, r)$ strictly decreases with r. Since it also strictly decreases with q^*, the part (b) of the lemma is then verified. ■

Proof of Lemma 4

Proof. Denote $k_i = x_i^\alpha$, $k^* = x^{*\alpha}, t = \frac{r}{\alpha} \in (0, 1 + \frac{1}{M-2}]$, then $\tilde{\pi}_i(x_i)$ can be rewritten as

$$\tilde{\pi}_i(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{k_i^t}{k_i^t + (N-1)k^{*t}} v - k_i,$$

Evaluating $\tilde{\pi}_i$ with respect to k_i yields

$$\frac{d\tilde{\pi}_i}{dk_i} = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{(N-1)tk_i^{t-1}k^{*t}v}{[k_i^t + (N-1)k^{*t}]^2} - 1.$$

Note

$$k^* = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{N-1}{N^2} tv.$$

To verify that k^* is the global maximizer of $\tilde{\pi}_i(k_i)$ given that all other participants exert the same effort. Define $p_i(k_i, \mathbf{k}_{-i}; N) = \frac{k_i^t}{k_i^t + (N-1)k^{*t}}$. One can verify $\xi_N(k_i) = \left.\frac{\partial^2 p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i^2}\right|_{k_{-i}=k^*} = \frac{-(t+1)k_i^t + (t-1)(N-1)k^{*t}}{[k_i^t + (N-1)k^{*t}]^3} tk_i^{t-2}(N-1)k^{*t}$. It implies that $\Phi_N(k_i) = \left.\frac{\partial p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i}\right|_{k_{-i}=k^*}$ is not monotonic: It is positive if $k_i^t < \frac{t-1}{t+1}(N-1)k^{*t}$, and negative if $k_i^t > \frac{t-1}{t+1}(N-1)k^{*t}$. Clearly $\frac{t-1}{t+1}(N-1) \le 1$ if and only if $t \le \frac{N}{N-2}$. Because $t \le 1 + \frac{1}{M-2}$, we must have $\frac{t-1}{t+1}(N-1) < 1$ for all $N \le M$.

Let $\Phi(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{\partial p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i}|_{k_{-i}=k^*}$, and $\xi(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{*N-1}(1-q^*)^{M-N} \frac{\partial^2 p_i(k_i, \mathbf{k}_{-i}; N)}{\partial k_i^2}|_{k_{-i}=k^*}$. The above results imply that $k_i^t > \frac{t-1}{t+1}(N-1)k^{*t}$ when $k_i = k^*$ for all $N \le M$, which means that $\xi(k_i)|_{k_i=k^*} < 0$. This leads to that $\left.\frac{d^2\tilde{\pi}_i(k_i)}{dk_i^2}\right|_{k_i=\mathbf{k}_{-i}=k^*} = v\,\xi(k_i)|_{k_i=k^*} < 0$. Hence, $k_i = k^*$ must be at least a local maximizer of when $k_{-i} = k^*$.

Since when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$, $\xi_N(k_i) > 0$ for all $N \le M$, we have $\xi(k_i) > 0$ when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$, which means that $\Phi(k_i)$ increases when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$. Similarly, $\xi(k_i) < 0$ when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$, which means that $\Phi(k_i)$ decreases when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$. We next show that there exists a unique $k' \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$ such that $\Phi(k_i)$ increases (decreases) if and only if $k_i < (>)\ k'$. For this purpose, it suffices to show that there exists a unique $k' \in ([\frac{t-1}{t+1}]^{1/t}k^*, [\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$, such that $\xi(k') = 0$.

First, such k' must exist by continuity of $\xi(k_i)$. As have been revealed, $\xi(k_i) > 0$ when $k_i < [\frac{t-1}{t+1}]^{1/t}k^*$; and $\xi(k_i) < 0$ when $k_i > [\frac{t-1}{t+1}(M-1)]^{1/t}k^*$.

Second, the uniqueness of k' can be verified as below. We have

$$\begin{aligned}
&\left.\frac{\partial^3 p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i^3}\right|_{k_{-i}=k^*}\\
=\;& t(N-1)k^{*t}\left\{\begin{array}{c}(t-2)k_i^{t-3}\frac{-(t+1)k_i^t+(t-1)(N-1)k^{*t}}{[k_i^t+(N-1)k^{*t}]^3}\\ +k_i^{t-2}\frac{-t(t+1)k_i^{t-1}[k_i^t+(N-1)k^{*t}]-3tk_i^{t-1}[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]}{[k_i^t+(N-1)k^{*t}]^4}\end{array}\right\}\\
=\;& \frac{t(N-1)k^{*t}k_i^{t-3}}{[k_i^t+(N-1)k^{*t}]^3}\left\{\begin{array}{c}(t-2)[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]\\ +\frac{-t(t+1)k_i^t[k_i^t+(N-1)k^{*t}]-3tk_i^t[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]}{[k_i^t+(N-1)k^{*t}]}\end{array}\right\}\\
=\;& \frac{t(N-1)k^{*t}k_i^{t-3}}{[k_i^t+(N-1)k^{*t}]^3}\left\{\begin{array}{c}(t-2)[-(t+1)k_i^t+(t-1)(N-1)k^{*t}]\\ +\frac{2tk_i^t}{[k_i^t+(N-1)k^{*t}]}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}]\end{array}\right\}.
\end{aligned}$$

Recall $\xi_N(k_i)=\frac{-(t+1)k_i^t+(t-1)(N-1)k^{*t}}{[k_i^t+(N-1)k^{*t}]^3}tk_i^{t-2}(N-1)k^{*t}$. We then have

$$\begin{aligned}
&\left.\frac{\partial^3 p_i(k_i,\mathbf{k}_{-i};N)}{\partial k_i^3}\right|_{k_{-i}=k^*}\\
=\;& (t-2)k_i^{-1}\xi_N(k_i)\\
&+\frac{2t^2(N-1)k^{*t}k_i^{2t-3}}{[k_i^t+(N-1)k^{*t}]^4}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}].
\end{aligned}$$

We now claim $[(t+1)k_i^t-(2t-1)(N-1)k^{*t}]$ is negative for all $k_i\le[\frac{t-1}{t+1}(M-1)]^{1/t}k^*$. A detailed proof is as follows. From $k_i\le[\frac{t-1}{t+1}(M-1)]^{1/t}k^*$, we have $(t+1)k_i^t\le(t-1)(M-1)k^{*t}$. To show $(t+1)k_i^t-(2t-1)(N-1)k^{*t}<0$, it suffices to show $(t-1)(M-1)<(2t-1)(N-1)$ when $N=2$, which requires $t<1+\frac{1}{M-3}$. This holds as $t\le 1+\frac{1}{M-2}$.

We thus have at any $k_i\in([\frac{t-1}{t+1}]^{1/t}k^*,[\frac{t-1}{t+1}(M-1)]^{1/t}k^*)$ such that $\xi(k_i)=0$, $\xi(k_i)$ must be locally decreasing, because $\frac{\partial\xi(k_i)}{\partial k_i}=(t-2)k_i^{-1}\sum_{N=1}^{M}C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}\xi_N(k_i)+\sum_{N=1}^{M}C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}A_N(k_i)=(t-2)k_i^{-1}\xi(k_i)+\sum_{N=1}^{M}C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}A_N(k_i)=\sum_{N=1}^{M}C_{M-1}^{N-1}q^{*N-1}(1-q^*)^{M-N}A_N(k_i)<0$ as $A_N(k_i)=\frac{2t^2(N-1)k^{*t}k_i^{2t-3}}{[k_i^t+(N-1)k^{*t}]^4}[(t+1)k_i^t-(2t-1)(N-1)k^{*t}]<0$.

We are ready to show the uniqueness of k' by contradiction. Suppose that there exists more than one zero points k' and k'' with $k'\neq k''$ for $\xi(k_i)$. Because $\xi(k_i)$ must be locally decreasing, then there must exist at least another zero point $k'''\in(k',k'')$ at which $\xi(k_i)$ is locally increasing. Contradiction thus results. Hence, such a zero point k' of $\xi(k_i)$ must be unique.

Recall $\Phi(k_i)$ increases (decreases) if and only if $k_i<(>)\;k'$ and it reaches its maximum at k'. Note $\frac{\partial\tilde{\pi}_i(k_i)}{\partial k_i}=v\Phi(k_i)-1$ and $\Phi(0)=0$. Therefore $\frac{\partial\tilde{\pi}_i(k_i)}{\partial k_i}|_{k_i=0}<0$. Thus $\frac{\partial\tilde{\pi}_i(k_i)}{\partial k_i}$ has exactly two zero points with the smaller one (k_s) being the local minimum point of $\tilde{\pi}_i(k_i)$. Note $k_i=k^*$ must be a zero point for $\frac{\partial\tilde{\pi}_i(k_i)}{\partial k_i}$ by definition. Since $k_i=k^*$ is a local maximum point of $\tilde{\pi}_i(k_i)$, it is higher than other zero point (k_s) of $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ which is a local minimum point of $\tilde{\pi}_i(k_i)$.

Note $x_m = (k_s)^{1/\alpha}$ is the unique local minimum of $\tilde{\pi}_i(x_i)$, and note $x^* = (k^*)^{1/\alpha}$ is the unique inner local maximum of $\tilde{\pi}_i(x_i)$. Note $x_m < x^*$. The results of Lemma 4 are shown. ■

Proof of Lemma 5

Proof. Proof of Lemma 3 has shown that $F(q,r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} \frac{v}{N}(1-\frac{N-1}{N}\frac{r}{\alpha}) - \Delta$ decreases with both q and r. Thus $F(q,r) = 0$ uniquely defines r as a decreasing function of q. Since $F(q_0, r_0) = 0$ and $\hat{q} > q_0$, we must have $r(\hat{q}) < r_0$. ■

Proof of Theorem 5

Proof. According to Lemma 5, Theorem 4 thus means that contest $r(\hat{q})$ would induce entry equilibrium $\hat{q}$ and pure-strategy bidding whenever $r(\hat{q}) \le \alpha(1 + \frac{1}{M-2})$. Since we have a pure-strategy bidding, an overall effort of $\bar{x}_T(\hat{q})$ clearly is induced at the equilibrium.

Consider any other $r \neq r(\hat{q})$. If r induces equilibrium entry $q(r)$ and pure-strategy bidding, then the total effort induced is $\bar{x}_T(q(r))$. Note that by Lemma 3, equilibrium $q(r)$ decreases with r. Thus $r \neq r(\hat{q})$ means $q(r) \neq \hat{q}$. $\bar{x}_T(q)$ is single peaked at $\hat{q}$ according to Theorem 1. Thus for any $r \neq r(\hat{q})$, we must have $\bar{x}_T(q(r)) < \bar{x}_T(\hat{q})$. If r induces equilibrium entry $q(r)$ and mixed-strategy bidding, then the total expected effort induced is strictly lower than $\bar{x}_T(q(r))$ when $\alpha > 1$, based on the arguments deriving this boundary in Section 3.4. Therefore the total effort induced must be strictly lower than $\bar{x}_T(\hat{q})$. ■

Proof of Lemma 7

Proof. Entry equilibrium q^* from (13) is the solution of

$$\Phi(q) = \sum_{N=0}^{M-1} C_{M-1}^{N} q^N (1-q)^{(M-1)-N} \zeta_{N+1} = \Delta, \tag{16}$$

where $\zeta_N = \frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) - \frac{N-1}{N}\frac{r}{\alpha} \cdot F, \forall N \ge 1$. Note $\frac{v}{N}(1 - \frac{N-1}{N}\frac{r}{\alpha}) = v[\frac{1}{N}(1 - \frac{r}{\alpha}) + \frac{1}{N^2}\frac{r}{\alpha}]$ decreases with N and $\frac{N-1}{N}\frac{r}{\alpha}$ increases with N. Thus ζ_N strictly decreases with N.

$$\begin{aligned}\Phi'(q) &= (M-1)\sum_{N=0}^{M-2} C_{M-2}^{N} q^N (1-q)^{(M-2)-N}(\zeta_{N+2} - \zeta_{N+1}) \\ &< 0.\end{aligned}$$

Since $\Phi'(q) < 0, \forall q > 0$ and $\Phi'(F) < 0, \forall F \ge 0$, we must have $\frac{dq}{dF} < 0$. When F is big, only ζ_1 is positive and ζ_N, $N \ge 2$ are very negative. Thus $\Phi(q) = \Delta$ means a very small q. ■

Proof of Theorem 6

Proof. First, when $F = 0$, clearly r $(< r(\hat{q}))$ induces a higher q $(> \hat{q})$ based on similar arguments in the proof of Lemma 7. For this $r \in (0, \alpha)$, Lemma 7 means that there exists a unique entry fee F (> 0) such that the lemma 6 contest induces equilibrium entry $\hat{q}$. Note that for the Lemma 6 contest with impact function $x^r, r \in (0, \alpha)$ and entry fee F, equation (2) still holds. Since the contest induces entry $\hat{q}$ and pure-strategy bidding, it must induces an expected overall bid of $\bar{x}_T^*$. ■

Proof of Lemma 8

Proof. By definition $\bar{x}_T^*\left(M'\right) = \bar{x}_T(\hat{q}(M'); M')$.

By Envelope Theorem, $\frac{d\bar{x}_T(\hat{q}(M');M')}{dM'} = \frac{\partial \bar{x}_T(q;M')}{\partial M'}|_{q=\hat{q}(M')}$. We have

$$\begin{aligned}
& \frac{\partial \bar{x}_T(q; M')}{\partial M'}|_{q=\hat{q}(M')}. \\
= \; & \partial \left[(M'\hat{q}(M'))^{\frac{\alpha-1}{\alpha}} \left\{ [1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta \right\}^{\frac{1}{\alpha}} \right] /\partial M' \\
= \; & \frac{\alpha-1}{\alpha} M'^{-\frac{1}{\alpha}} \left[\hat{q}(M')\right]^{\frac{\alpha-1}{\alpha}} \left\{ [1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta \right\}^{\frac{1}{\alpha}} \\
& + \frac{1}{\alpha} (M'\hat{q}(M'))^{\frac{\alpha-1}{\alpha}} \left\{ [1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta \right\}^{\frac{1}{\alpha}-1} \\
& \times [-(1-\hat{q}(M'))^{M'} v \ln(1-\hat{q}(M')) - \hat{q}(M')\Delta],
\end{aligned}$$

which has the same sign as

$$\lambda = (\alpha - 1) \left\{ [1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta \right\} + M'[-(1-\hat{q}(M'))^{M'} v \ln(1-\hat{q}(M')) - \hat{q}(M')\Delta].$$

Because $-\ln(1-\hat{q}(M')) < \frac{\hat{q}(M')}{1-\hat{q}(M')}$, we have $M'[-(1-\hat{q}(M'))^{M'} v \ln(1-\hat{q}(M')) - \hat{q}(M')\Delta] < \hat{q}(M')[M'(1-\hat{q}(M'))^{M'-1}v - M'\Delta]$. Hence, $\lambda < (\alpha-1)\left\{ [1-(1-\hat{q}(M'))^{M'}]v - M'\hat{q}(M')\Delta \right\} + \hat{q}(M')[M'(1-\hat{q}(M'))^{M'-1}v - M'\Delta] = 0$ (by the definition of $\hat{q}(M')$). We then have $\frac{d\bar{x}_T(\hat{q}(M');M')}{dM'} < 0$. ■

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