Sequential All-Pay Auctions with Head Starts and Noisy Outputs*

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Abstract

We study a sequential (Stackelberg) all-pay auction with two contestants who are privately informed about a parameter (ability) that affects their cost of effort. Contestant 1 (the first mover) exerts an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then exerts an effort in the second period. Contestant 2 wins the contest if his effort is larger than or equal to the effort of contestant 1; otherwise, contestant 1 wins. We characterize the unique subgame perfect equilibrium of this sequential all-pay auction and analyze the use of head starts to improve the contestants' performance. We also study this model when contestant 1 exerts an effort in the first period which translates into an observable output but with some noise. We study two variations of this model where contestant 1 either knows or does not know the realization of the noise before she chooses her effort. Contestant 2 does not know the realization of the noise in both variations. For both variations, we characterize the subgame perfect equilibrium and investigate the effect of a random noise on the contestants' performance.

Keywords: Sequential all-pay auctions, head starts, noisy outputs.

JEL classification: D44, O31, O32

*This paper is a combination of our papers "Sequential all-pay auctions with head starts" and "Sequential all-pay auctions with noisy outputs".

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1 Introduction

Consider an individual such as a building constructor who needs a service from a service provider . In many cases this individual will ask constructor A for a price quote and then will reveal this quote to constructor B and ask for a lower quote. If he refuses to give a lower quote then the job will be given to constructor A. The constructors or service providers invest time and effort to correctly evaluate the job at hand in order to produce a competitive but still profitable price quote. This scenario can be likened to a sequential contest between two contestants which is the focus of this paper. Indeed, in many contest settings, effort choices are made sequentially rather than simultaneously. The differences between simultaneous and sequential contests have been addressed in the literature by several researchers.[1] Baik and Shogren (1992), Leininger (1993) and Morgan (2003) investigated the question of which form of contest, sequential or simultaneous, naturally arises in competitive situations. They studied two-player models where contestants compete in the (generalized) Tullock contest and each contestant is able to choose between two dates to make their efforts. If the contestants choose different dates, a sequential contest occurs, but if they choose the same date the contest will be a simultaneous one. They all showed that sequential contests may arise endogenously in equilibrium.[2] Despite these important findings, while numerous studies have dealt with simultaneous all-pay auctions (all-pay contests) only a few have focused on sequential all-pay auctions. The purpose of this paper is to fill this gap in the literature by studying a sequential all-pay auction with heterogeneous contestants under incomplete information.

In the all-pay auction each player submits a bid (effort) and the player who submits the highest bid wins the contest, but, independently of success, all players bear the cost of their bids. All-pay auctions have been studied either under complete information where each player's type (valuation for winning the contest or ability) is common knowledge[3] or under incomplete information where each player's type is private

[1] Dixit (1987) studied a sequential Tullock contest and examined whether the ability to commit to an effort choice before other contestants choose their effort while assuming that they can then observe this choice is advantageous or not. Linster (1993) analyzed two-player sequential Tullock contests and showed that if the stronger player is the first (second) mover in the sequential contest the players' total effort is larger (smaller) than in the simultaneous contest.

[2] Hamilton and Slutsky (1990) Deneckere and Kovenock (1992) and Mailath (1993) studied sequential oligopoly games and showed that sequential choices of quantities in a Cournot competition can be the equilibrium outcome of non-cooperative play.

[3] All-pay auctions under complete information have been studied, among others, by Hillman and Samet (1987), Hillman and

information and only the distribution from which the players' types is drawn is common knowledge.[4] Most studies dealing with sequential all-pay auctions assume a two-stage contest under complete information. Leininger (1991) modeled a patent race between an incumbent and an entrant as a sequential asymmetric all-pay auction under complete information, and Konrad and Leininger (2007) characterized the equilibrium of the all-pay auction under complete information in which a group of players choose their effort 'early' and the other group of players choose their effort 'late'. The assumption of incomplete information complicates the analysis of the sequential all-pay auction but also makes it more relevant and interesting.

In this work, we study a sequential all-pay auction under incomplete information where the ability of each contestant is private information. We consider first a sequential all-pay auction with two contestants where contestant 1 (the first mover) exerts an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then exerts an effort in the second period. Contestant 2 wins the contest if his effort is larger than or equal to the effort of contestant 1; otherwise, contestant 1 wins.[5] This particular type of sequential contest where the players' outputs are observable in any stage of the contest has various applications, including sport contests such as athletics and gymnastics, political races in which the candidates confront each other by a sequence of speeches, and court trials when the lawyers of both sides make their final speeches. Moreover, in R&D and other market races it is sometimes the case that the incumbent observes the output of the leader and only then decides how much effort to put in. In all these cases, the players in the later stages have some advantage because they have observed their opponents' outputs in the previous stages. Similarly, in our model contestant 2 has an obvious advantage over contestant 1. For this reason contestant 1 exerts a relatively low effort and sometimes, depending on the distribution of his opponent's abilities, he might even prefer not to participate in the contest at all (it is worth noting

Riley (1989), Leininger (1991), Baye et al. (1993, 1996), Che and Gale (1998, 2000) and Siegel (2009)).

[4] All-pay auctions under incomplete information have been studied, among others, by Hillman and Riley (1989), Amann and Leininger (1996), Krishna and Morgan (1997), Gavious et al. (2003), Moldovanu and Sela (2001, 2006) and Moldovanu et al. (2010).

[5] The concept of Stackelberg games in which players choose their strategies sequentially was introduced and analyzed also by computer scientists such as Garg and Narahari (2008), Luh et al. (1984) and others. All these authors impose a hierarchical decision-making structure on a simultaneous game to describe sequential choices of strategies. The solution concept they use is a Stackelberg equilibrium where the leaders use "secure strategy" that guarantees them a minimal payoff while the followers use an optimal response strategy.

that this feature of our model can explain why players sometimes choose to stay out of a contest). Given the low effort of contestant 1 in the first period as well as the rules of the contest according to which contestant 2 needs only to equalize the effort of contestant 1 in order to win, we have a relatively low expected total effort as well as a low expected highest effort. However, a designer who wishes to maximize the expected total effort or the expected highest effort can change the rules of the sequential all-pay auction to make it more profitable by explicitly or implicitly favoring contestant 1 over contestant 2. In other words, he can give contestant 1 a head start.

There are numerous examples of real-life sequential contests in which the players who play in the first stage are given a head start. Suppose, for example, that Microsoft Corporation is the first company to produce a hardware product. Then, if Apple Inc. wants to produce a competitive product, in order to convince customers to buy this new product it has to be either better or cheaper than the Microsoft product. In that case, Microsoft is exogenously given a head start. However, head starts can also be given endogenously. For example, a common situation often occurs in the labor market when an applicant gets a job and then any new applicant is required to be better in order to win his place. Thus contests with head starts may raise the contestants' expected total effort or alternatively their expected highest effort. Kirkegaard (2009), for example, studied asymmetric all-pay auctions with head starts under incomplete information where players simultaneously choose their efforts. He showed that the total effort increases if the weak contestant is favored with a head start, but if the contestants are sufficiently heterogenous, then in some cases the weak contestant should be given both a head start and a handicap.[6] Corns and Schotter (1999) demonstrated by theoretical and empirical arguments that a head start in the form of a price preference policy that is given to a subset of the firms might not only benefit that subset but can actually lower the purchasing cost of the government. In our sequential all-pay auction therefore we wish to demonstrate that a head start can not only benefit one of the players but can also enhance the overall expected performance of the players. Since in our setting, contestant 2 has an advantage over contestant 1 because of the timing of their play, we assume that contestant 1 is given a multiplicative head start which is exogenously determined . That is, contestant 2 will win the contest if his effort x_2 is larger or equal to tx_1, where x_1 is the effort of contestant 1 and

[6] Siegel (2010) provided an algorithm that constructs the unique equilibrium in simultaneous all-pay auctions with head starts in which players do not choose weakly-dominated strategies.

t is a constant larger than 1.[7] We provide sufficient conditions under which by imposing a head start for contestant 1 the designer of the contest can significantly increase the expected efforts of both contestants, particularly the expected total effort as well as the expected highest effort. The optimal head start can be high enough such that several types of contestant 1 will win for sure since no type of contestant 2 will want to participate. As such, head starts may also play the role of a winning bid in a sequential all-pay auction when contestant 1 has an incentive to participate independently of the distribution of his opponent's type. Finally, head starts improve the inherent inefficiency of the sequential all-pay auction. The probability that a low ability contestant wins against a high ability contestant in a contest with a head start is lower than in a contest without any head start.

So far in our sequential all-pay auction, efforts translate deterministically into observable outputs such that the contestant who made the highest effort is also the one with the highest output and this contestant wins the contest. However, in real-life contests, the relationship between the contestant's effort and her observable output is usually not deterministic. Rather, it is frequently the case that there is some noise in the process that maps efforts into measured outputs. Contests with outputs which are not deterministically determined by efforts have received some attention in the literature. For example, Lazear and Rosen (1981) considered a contestant's output to be a stochastic function of the unobservable effort and the identity of the most productive agent to be determined by an external shock. This model is known in the literature as a rank-order tournament and was later extended and generalized by several authors, e.g., Green and Stokey (1983), Nalebuff and Stiglitz (1983), Rosen (1986), Krishna and Morgan (1998) and Akerlof and Holden (2008). The all-pay auction under complete information is actually the limiting case of the rank-order tournament when the noise approaches zero. In the rest of this paper, similarly to the rank-order tournament, we assume that the output is a stochastic function of the effort, but in contrast to the rank-order tournament model, we analyze sequential all-pay auctions under incomplete information. Thus the novelty of this part of the paper lies in the fact that we combine incomplete information and noisy outputs in the same model. This combination is natural in several environments but is quite complex for a theoretical

[7] This multiplicative head start was chosen for the sake of convenience and may not necessarily be the optimal form of a head start.

analysis.[8]

Formally, contestant 1 exerts an effort x_1 in the first period, this effort translates with some noise into an output that player 2 observes. Thus contestant 2 observes a noisy output of contestant 1's effort, $x_1 + t$, where t is the noise term. Then, contestant 2 exerts an effort x_2 in the second period, and wins the contest if her effort is larger than or equal to the noisy output of contestant 1, i.e., $x_2 \geq x_1 + t$; otherwise, contestant 1 wins. We assume that the random noise t is uniformly distributed on an interval $[-k, k]$ where k describes the magnitude of the random noise and determines its variance.[9] The smaller the value of k is, the higher is the contest's accuracy.

We present two variations of the model with noisy outputs. In the first one we assume that both contestants do not know the realization of the noise when they exert their effort. We show that when the magnitude of the noise, k, increases, then in equilibrium less types of contestant 1 will exert a positive effort in the contest. If the magnitude of the random noise is sufficiently high, contestant 1 will have no incentive to exert any positive effort since anyway she wins with zero effort. Thus, we focus on a more interesting case where the magnitude of the random noise is relatively low (k goes to zero). We show that the marginal effect of the magnitude of the random noise, k, on the contestants' strategies goes to zero when k goes to zero. Thus, we conclude that the equilibrium behavior in the sequential all-pay auction is robust under the existence of a small noise.

In the second variation of our model with noisy outputs we assume that contestant 1 knows the realization of the noise when exerting her effort, while contestant 2 does not. We thus assume that contestant 1 has more information about the contest than contestant 2. This assumption describes contests in which the first mover has the opportunity to gather information about the contest environment before exerting an effort. This commonly occurs in market situations when one firm identifies the market earlier than the other firm which enables her to evaluate correctly the connection between the effort and the observed output.

[8] Ederer (2010), for example, also studies a two-stage model that combines incomplete information and noisy outputs. However, he assumes that players do not know their types before they make their choices in the first stage while in our model each player has private information on her type at the outset, from the beginning of the contest.

[9] The model can be studied for any symmetric distribution of noise but then a closed-form expression for the subgame perfect equilibrium bid function cannot be derived.

We show that a positive realization of noise decreases contestant 1's equilibrium effort for any type who exerts a positive effort while a negative realization increases it with respect to the contest without any noise. Moreover, in equilibrium, the probability that contestant 1 will exert a positive effort in the contest decreases in the absolute value of the realization of the noise. Therefore, we conclude that a positive realization of noise decreases the expected output of contestant 1 with respect to the contest without the noise. The effect of a negative realization of the noise is however ambiguous since, on the one hand, it increases the effort of contestant 1 for any type who exerts a positive effort, but on the other, it decreases the incentive to exert a positive effort in the contest. It is worth noting that a positive and a negative realizations of the noise with the same absolute value do not have the same effect on the contestants' outputs. However, similarly to the case when both contestants do not know the realization of the noise, we show that the marginal effect of the magnitude of the random noise, k, on the contestants' strategies goes to zero when k goes to zero. Hence, independently of the information of the contestants on the random noise, the equilibrium behavior in the sequential all-pay auction is robust under the existence of a small noise. Furthermore, while in several works on sequential contests (see, for example, Yildirim 2005 and Ederer 2010) it is shown that the information revelation policy can affect the contestants' choices before and after the release of information, in our model, if the realization of the noise is sufficiently small, then it does not matter how the information about it is released.

The rest of the paper is organized as follows: Section 2 presents the sequential all-pay auction with and without head starts . Section 3 presents the sequential all-pay auction with noisy outputs. Section 4 concludes. All proofs are in the Appendix.

2 The sequential all-pay auction with head starts

We consider first a sequential all-pay auction with two risk neutral contestants where contestant 1 (the first mover) exerts an effort in the first period, while contestant 2 (the second mover) observes the effort of contestant 1 and then exerts an effort in the second period. Contestant 2 wins the contest if her effort (x_2) is larger than or equal to the effort of contestant 1 (x_1); otherwise, contestant 1 wins. Both contestants' valuation for the prize is 1. The cost of an effort x_i to player i, is $\frac{x_i}{a_i}$ where $a_i \geq 0$ is the ability (or type) of contestant i which is private information to i.[10] Contestants' abilities are drawn independently. Contestant i's ability is drawn from the interval $[0,1]$ according to a distribution function F_i which is common knowledge. We assume that $F_i, i=1,2$ has a positive and continuous density function $F_i' > 0$.

We begin the analysis by considering the equilibrium effort function of contestant 2 in the second period. We assume that if both contestants make the same effort then contestant 2 is the winner. Therefore contestant 2 makes the same effort as contestant 1 as long as his type a_2 is larger than or equal to the effort of contestant 1; otherwise he stays out of the contest. Formally, the equilibrium effort of contestant 2 is given by:

$$b_2(a_2;a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_2 < b_1(a_1) \\ b_1(a_1) & \text{if} \quad b_1(a_1) \leq a_2 \leq 1 \end{cases}$$

where we assume that contestant 1 uses a strictly monotonic equilibrium effort function $b_1(a_1)$. Contestant 1 with ability a_1 chooses an effort b_1 that solves the following optimization problem:

$$\max_{b_1} \left\{ F_2(b_1(a_1)) - \frac{b_1(a_1)}{a_1} \right\} \tag{1}$$

The F.O.C. is then

$$F_2'(b_1(a_1))b_1'(a_1) - \frac{1}{a_1}b_1'(a_1) = 0 \tag{2}$$

and the S.O.C. is

$$F_2''(b_1(a_1))(b_1'(a_1))^2 + F_2'(b_1(a_1))b_1''(a_1) - \frac{1}{a_1}b_1''(a_1) = F_2''(b_1(a_1))(b_1'(a_1))^2 < 0$$

Note that if F_2 is convex, the S.O.C does not hold and then $b_1(a_1) = 0$ for all a_1 is the solution of the maximization problem (1). In the following we assume that F_2 is concave (F_1 is not necessarily concave). Then the S.O.C. holds and in equilibrium, the effort of contestant 1 with type a_1 is given by

$$b_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_1 \leq \tilde{a} \\ (F_2')^{-1}\left(\frac{1}{a_1}\right) & \text{if} \quad \tilde{a} \leq a_1 \leq 1 \end{cases} \tag{3}$$

where the cutoff $\tilde{a}$ is defined by $\tilde{a} = \frac{1}{F_2'(0)}$. If $F_2'(x) \to \infty$ when $x \to 0$ then $\tilde{a} = 0$. This cutoff depends on the distribution of the second player's ability. If $F_2'(0)$ is a finite number then types $0 \leq a_1 \leq \tilde{a}$ do not find it optimal to exert a positive effort. As was mentioned above, for the class of convex distribution functions

[10] An equivalent interpretation is that a_i is player's i valuation for the prize and his cost is equal to her bid.

we have $\tilde{a}=1$ such that all types of contestant 1 choose to stay out of the contest (in the following we will solve this problem by providing an incentive, a head start, for contestant 1 to participate in the contest). However, if contestant 2's distribution function F_2 is concave, we have a real competition in the sequential all-pay auction even without head starts.

The expected efforts of contestants 1 and 2 are

$$
\begin{aligned}
TE_1 &= \int_{\tilde{a}}^{1} b_1(a_1)\, F_1'(a_1) da_1 = \int_{\tilde{a}}^{1} (F_2')^{-1}(\frac{1}{a_1}) F_1'(a_1) da_1 \\
TE_2 &= \int_{\tilde{a}}^{1} \left(\int_{b_1(a_1)}^{1} b_2(a_2; a_1)\, F_2'(a_2) da_2 \right) F_1'(a_1) da_1 \\
&= \int_{\tilde{a}}^{1} \left[1 - F_2((F_2')^{-1}(\frac{1}{a_1})) \right] (F_2')^{-1}(\frac{1}{a_1}) F_1'(a_1) da_1
\end{aligned}
$$

Note that contestant 2 exerts the same effort as contestant 1 or else exerts an effort of zero. Therefore the expected highest effort is equal to the expected effort of contestant 1 and is given by

$$HE = \int_{\tilde{a}}^{1} (F_2')^{-1}(\frac{1}{a_1}) F_1'(a_1) da_1 \tag{4}$$

The expected total effort is given by

$$TE = TE_1 + TE_2 = \int_{\tilde{a}}^{1} \left[2 - F_2((F_2')^{-1}(\frac{1}{a_1})) \right] (F_2')^{-1}(\frac{1}{a_1}) F_1'(a_1) da_1 \tag{5}$$

Finally, we define the efficiency (Eff) of the contest as the probability that the contestant with the higher ability (valuation) wins the contest. If contestant 1 wins the contest it is necessarily true that $a_2 < b_1(a_1) \leq a_1$. However, if $b_1(a_1) \leq a_2 < a_1$ contestant 2 wins the contest although he has a lower type. We thus have

$$
\begin{aligned}
Eff &= \int_{\tilde{a}}^{1} \left(\int_{0}^{b_1(a_1)} F_2'(a_2)\, da_2 + \int_{a_1}^{1} F_2'(a_2)\, da_2 \right) F_1'(a_1)\, da_1 \\
&= \int_{0}^{1} \left(F_2 \left((F_2')^{-1}(\frac{1}{a_1}) \right) + 1 - F_2(a_1) \right) F_1'(a_1)\, da_1
\end{aligned}
$$

Example 1 *Consider a sequential all-pay auction with two contestants whose abilities are distributed according to the distribution functions $F_1(x) = F_2(x) = x^{0.5}$. By (3), the equilibrium effort function of contestant 1 in the sequential all-pay auction is*

$$b_1(a_1) = \frac{a_1^2}{4} \quad \textit{for all } a_1 \geq 0$$

Therefore by (4) the expected highest effort is given by

$$HE = \int_{0}^{1} \frac{a_1^2}{4} \frac{1}{2\sqrt{a_1}} da_1 = 0.05$$

and by (5) the expected total effort is

$$TE = \int_0^1 \left(2 - \sqrt{\frac{a_1^2}{4}}\right) \frac{a_1^2}{4} \frac{1}{2\sqrt{a_1}} da_1 = \frac{23}{280} \approx 0.0821$$

We also have

$$Eff = \int_0^1 \left(\frac{a_1}{2} + 1 - \sqrt{a_1}\right) \frac{1}{2\sqrt{a_1}} da_1 = \frac{2}{3}$$

In Example 1, the contestants' expected highest effort as well as their expected total effort are significantly lower than in the standard all-pay auction where both contestants simultaneously choose their efforts. In the next subsection we change the rules of the sequential all-pay auction by adding a head start to improve the contestants' performances in the contest.

2.1 Head start

In our sequential all-pay auction, contestant 2 has an advantage over contestant 1 because of the timing of their play. Thus, contestant 1's effort is relatively low and sometimes, depending on the distribution of contestant 2's abilities, will choose to stay out of the contest. In that case there is no real competition. Thus we examine whether the contestants' performance can be enhanced by using a head start for contestant 1. By introducing a head start we may also improve the inherent inefficiency of the sequential all-pay contest. The probability that contestant 1 with a high ability wins against contestant 2 with a low ability is higher with a head start. In our model, unlike in the symmetric all-pay auction, we can distinguish between the contestants based on their position - first or second, even if they are ex-ante symmetric (their types are drawn from the same distribution). Thus we may give the headstart to the first player.

We want the head start to be independent of the contestant's effort and therefore we introduce a multiplicative head start. We therefore assume that contestant 2 will win the contest if her effort x_2 is larger than or equal to tx_1 where x_1 is the effort of contestant 1 and t is a constant larger than 1. The equilibrium effort of contestant 2 is then given by

$$\beta_2(a_2; a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_2 < t\beta_1(a_1) \\ t\beta_1(a_1) & \text{if} \quad t\beta_1(a_1) \le a_2 \le 1 \end{cases}$$

where we assume that contestant 1 uses a strictly monotonic equilibrium effort function $\beta_1(a_1)$. Contestant 1 with ability a_1 solves the following optimization problem:

$$\max_{\beta_1} \left\{ F_2(t\beta_1(a_1)) - \frac{\beta_1(a_1)}{a_1} \right\} \tag{6}$$

The F.O.C. is

$$F_2'(t\beta_1(a_1))t\beta_1'(a_1) - \frac{1}{a_1}\beta_1'(a_1) = 0$$

and the S.O.C. is

$$F_2''(t\beta_1(a_1))(t\beta_1'(a_1))^2 + F_2'(t\beta_1(a_1))t\beta_1''(a_1) - \frac{1}{a_1}\beta_1''(a_1) = F_2''(t\beta_1(a_1))(t\beta_1'(a_1))^2 < 0$$

Thus, if F_2 is concave, the equilibrium effort of contestant 1 with type a_1 is given by

$$\beta_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_1 \le \widehat{a} \\ \frac{1}{t}\left(F_2'\right)^{-1}\left(\frac{1}{ta_1}\right) & \text{if} \quad \widehat{a} \le a_1 \le a^* \\ \frac{1}{t} & \text{if} \quad a^* \le a_1 \le 1 \end{cases} \tag{7}$$

where $\widehat{a}$ is defined as $\widehat{a} = \frac{1}{tF_2'(0)}$ and a^* is the minimum between 1 and the solution to the following equation

$$t\beta_1(a) = 1 \Rightarrow a^* = \min\left\{1, \frac{1}{tF_2'(1)}\right\}$$

Note that $a^* \ge \widehat{a}$ since F_2' is a decreasing function and from the concavity of F_2 we also know that $F_2'(0) > 1$. Furthermore, if $1 \le t \le \frac{1}{F_2'(1)}$, then $a^* = 1$ and only when $t > \frac{1}{F_2'(1)}$ does there exist a cutoff type $0 < a^* < 1$ and an interval of types $a^* \le a_1 \le 1$ who exert the effort $\beta_1(a_1) = \frac{1}{t}$ and win for sure (this serves as a winning bid).

The expected efforts of contestants 1 and 2 are given by

$$\begin{aligned} TE_1(t) &= \int_{\widehat{a}}^{a^*} \frac{1}{t}\left(F_2'\right)^{-1}\left(\frac{1}{ta_1}\right) F_1'(a_1)da_1 + \int_{a^*}^{1} \frac{1}{t} F_1'(a_1)da_1 \\ TE_2(t) &= \int_{\widehat{a}}^{a^*} \left[1 - F_2\left(\left(F_2'\right)^{-1}\left(\frac{1}{ta_1}\right)\right)\right]\left(F_2'\right)^{-1}\left(\frac{1}{ta_1}\right) F_1'(a_1)da_1 \end{aligned}$$

The expected total effort is therefore

$$\begin{aligned} TE(t) &= TE_1(t) + TE_2(t) \\ &= \int_{\widehat{a}}^{a^*} \left[\frac{1}{t} + 1 - F_2((F_2')^{-1}(\frac{1}{ta_1}))\right](F_2')^{-1}(\frac{1}{ta_1})F_1'(a_1)da_1 + \int_{a^*}^{1} \frac{1}{t}F_1'(a_1)da_1 \end{aligned} \tag{8}$$

Note that the expected effort of contestant 1 is not always higher than the expected effort of contestant 2 as was the case without a head start and therefore the expected highest effort is not equal to the expected effort of contestant 1. The expected highest effort is given by

$$
\begin{aligned}
HE(t) &= \int_0^1 \int_0^1 \max\left\{\beta_1(a_1), \beta_2(a_2; a_1)\right\} F_2'(a_2)\, da_2 F_1'(a_1)\, da_1 \qquad (9) \\
&= \int_{\widehat{a}}^{a^*} \left[F_2((F_2')^{-1}(\frac{1}{ta_1}))\right] \frac{1}{t}(F_2')^{-1}(\frac{1}{ta_1}) F_1'(a_1) da_1 \\
&+ \int_{\widehat{a}}^{a^*} \left[1 - F_2((F_2')^{-1}(\frac{1}{ta_1}))\right] (F_2')^{-1}(\frac{1}{ta_1}) F_1'(a_1) da_1 + \int_{a^*}^1 \frac{1}{t} F_1'(a_1) da_1
\end{aligned}
$$

The first term describes those types of contestant 2 who choose to stay out of the contest ($0 \leq a_2 < t\beta_1(a_1)$) in which case the highest effort is equal to that of contestant 1, $\beta_1(a_1) = \frac{1}{t}(F_2')^{-1}(\frac{1}{ta_1})$. The second term describes those types of contestant 2 who equalize the effort of contestant 1 multiplied by t in which case the highest effort is equal to $t\beta_1(a_1) = (F_2')^{-1}(\frac{1}{ta_1})$. The last term describes those types of contestant 1 who win for sure by choosing the winning bid.

Example 2 *Consider a sequential all pay auction with two contestants where $F_1(x) = F_2(x) = x^{0.5}$. By (7), the equilibrium effort function of contestant 1 is given by*

$$
\beta_1(a_1) = \begin{cases} \frac{1}{t}(F_2')^{-1}(\frac{1}{a_1 t}) = \frac{ta_1^2}{4} & \text{if} \quad 0 \leq a_1 \leq \min\left\{\frac{2}{t}, 1\right\} \\ \frac{1}{t} & \text{if} \quad \min\left\{\frac{2}{t}, 1\right\} < a_1 \leq 1 \end{cases}
$$

The expected total effort is given by

$$
\begin{aligned}
TE &= \int_0^{\min\left\{\frac{2}{t},1\right\}} \left(\frac{a_1^2 t}{4}\right) \frac{1}{2\sqrt{a_1}} da_1 + \int_{\min\left\{\frac{2}{t},1\right\}}^1 \left(\frac{1}{t}\right) \frac{1}{2\sqrt{a_1}} da_1 \\
&+ \int_0^{\min\left\{\frac{2}{t},1\right\}} \left(\int_{\frac{a_1^2 t^2}{4}}^1 \left(\frac{a_1^2 t^2}{4}\right) \frac{1}{2\sqrt{a_2}} da_2\right) \frac{1}{2\sqrt{a_1}} da_1
\end{aligned}
$$

Figure 1 presents the expected total effort as a function of t.

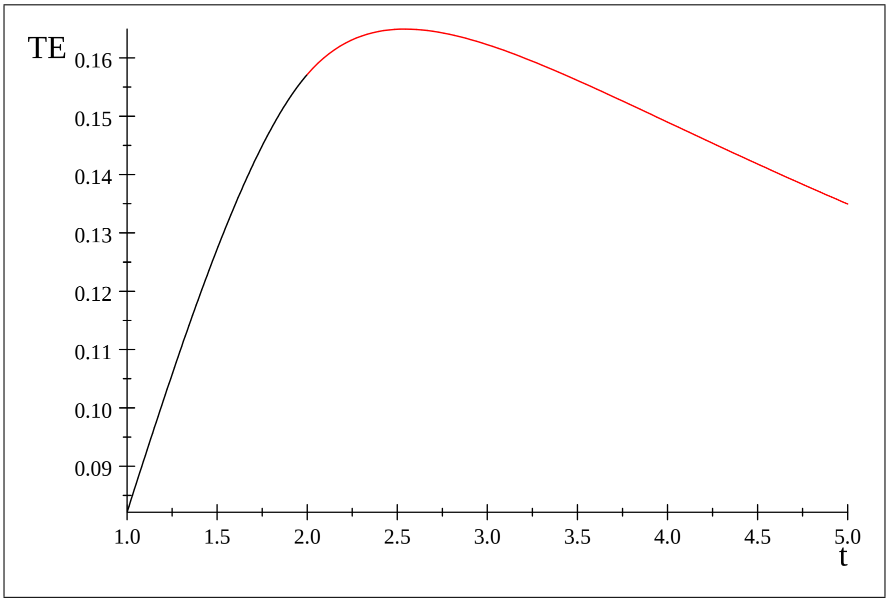

Figure 1: expected total effort as a function of the headstart

The optimal head start that yields the highest expected total effort in the sequential all-pay auction is therefore

$$t_{total} = \frac{7}{4}\left(199 - 5\sqrt{1561}\right) = 2.541\,9$$

and the expected total effort is then

$$TE(t_{total}) = 0.16492$$

The expected highest effort is

$$\begin{aligned} HE &= \int_0^{\min\left\{\frac{2}{t},1\right\}} \left(\int_0^{\frac{a_1^2 t^2}{4}} \left(\frac{a_1^2 t}{4}\right) \frac{1}{2\sqrt{a_2}} da_2 + \int_{\frac{a_1^2 t^2}{4}}^1 \left(\frac{a_1^2 t^2}{4}\right) \frac{1}{2\sqrt{a_2}} da_2 \right) \frac{1}{2\sqrt{a_1}} da_1 \\ &\quad + \int_{\min\left\{\frac{2}{t},1\right\}}^1 \frac{1}{t} \frac{1}{2\sqrt{a_1}} da_1 \end{aligned}$$

Figure 2 presents the expected highest effort as a function of t.

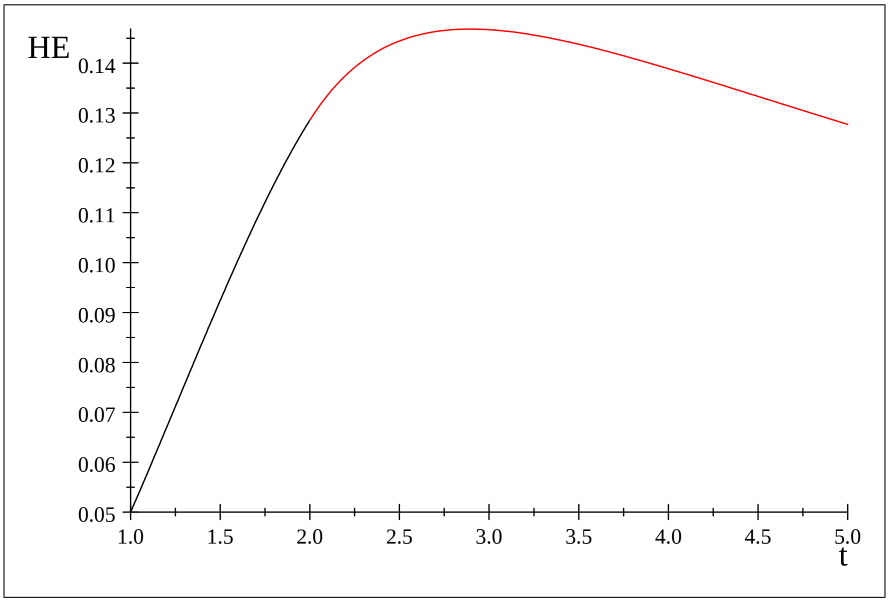

Figure 2: expected highest effort as a function of the headstart

The optimal head start that yields the highest expected highest effort in the sequential all-pay auction is therefore

$$t_{high} = \frac{1}{\left(\frac{1}{180}\sqrt{10}\sqrt{317} + \frac{7}{36}\sqrt{2}\right)^{2}} = 2.8945$$

and the expected highest effort is then

$$HE(t_{high}) = 0.1468$$

From Examples 1 and 2 we can see that the optimal head start significantly increases the contestants' expected highest effort as well as their expected total effort.

Below we discuss the equilibrium behavior of the contestants when the distribution function of contestant 2's types is convex rather than concave (again, there is no restriction on the distribution of contestant 1's types). When F_2 is convex and a head start $t > 1$ is given to contestant 1 then the equilibrium effort of contestant 2 is once again

$$\beta_2(a_2; a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_2 < t\beta_1(a_1) \\ t\beta_1(a_1) & \text{if} \quad t\beta_1(a_1) \leq a_2 \leq 1 \end{cases}$$

while the equilibrium effort of contestant 1 is given by

$$\beta_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \leq a_1 < \frac{1}{t} \\ \frac{1}{t} & \text{if} \quad \frac{1}{t} \leq a_1 \leq 1 \end{cases}$$

Note that when F_2 is convex and a head start is given to contestant 1 some of contestant 1's types participate in the contest. In this case the expected total effort and the expected highest effort are the same and are both equal to contestant 1's expected effort. The optimal head start is then t that maximizes $\frac{1}{t}\left(1 - F_1\left(\frac{1}{t}\right)\right)$.

We now turn to examine the conditions under which a head start is beneficial in the sequential all-pay auction with a concave F_2. Namely we identify the conditions on the distribution of the contestants' abilities that ensure that a head start increases the expected highest effort or the expected total effort. The following condition is required for establishing the effects of a head start on contestant 1's equilibrium effort. Let $b_1(a_1) = (F_2')^{-1}\left(\frac{1}{a_1}\right)$ represent the equilibrium effort function of contestant 1 in the sequential all-pay auction without a head start, when $\tilde{a} \le a_1 \le 1$.

Condition 1 *The function $b_1(a_1)$ is strictly convex for all $\tilde{a} \le a_1 \le 1$.*

If Condition 1 is satisfied[11], any head start t close to 1 increases the expected effort of contestant 1 since then, for $t > 1$ and $\tilde{a} \le a_1 \le 1$ we have $b_1(a_1) < \frac{1}{t} b_1(ta_1) = \beta_1(a_1)$. Given that without any head start, the expected highest effort is equal to the expected effort of contestant 1, we obtain the following result about the positive effect of a head start on the expected highest effort in the contest.

Proposition 1 *If Condition 1 holds, then the expected highest effort in the sequential all-pay auction with a head start $1 < t \le \frac{1}{F_2'(1)}$ is higher than the expected highest effort in the sequential all-pay auction without any head start.*

Proof. See Appendix. ■

We now examine the effect of a head start on the expected effort of contestant 2. On the one hand, the effort of every type of contestant 1 increases when a head start is given and therefore contestant 2 should also increase his effort if he wants to win the contest. But, on the other hand, by giving a head start to contestant 1, low types of contestant 2 will prefer to stay out of the contest since the minimal effort which is required from them in order to win is relatively high.

[11] Note that if this condition holds then the density function $F_2'(x)$ is convex. This follows by taking the derivative w.r.t. a of both sides of the equality $F_2'(b_1(a)) = \frac{1}{a}$. We get $-a^2 F_2''(b_1(a))\, b_1'(a) = 1$. Taking the derivative w.r.t. a of both sides of this equality and rearranging yields the following equality $b_1''(a) = \frac{\left(2F_2''(b_1(a)) + aF_2'''(b_1(a))b_1'(a)\right)b_1'(a)}{-aF_2''(b_1(a))}$ and since by our assumptions $F_2''(b_1(a)) < 0$ and $b_1'(a) > 0$ we conclude that $b_1''(a) > 0 \Rightarrow F_2'''(b_1(a)) > 0$.

The following conditions are required for establishing the effect of a head start on the effort of contestant 2.

Condition 2 *The function $G(x) = (1 - F_2(x))\,x$ is concave.*[12]

If condition 2 holds then we can define the argument that maximizes $G(x)$ on the interval $[0,1]$,

$$x^* = \arg\max_{x\in[0,1]} G(x)$$

Condition 3 *The highest equilibrium effort of contestant 1 (the effort of type $a_1 = 1$) in the contest without a head start is lower than x^*. Formally,*

$$b_1(1) = (F_2')^{-1}(1) < x^*$$

Using conditions 1, 2 and 3 we obtain a positive effect of a relatively small head start on the expected effort of contestant 2 as well.

Proposition 2 *If Conditions 1,2 and 3 hold, then for $t > 1$ sufficiently close to 1, the expected effort of contestant 2 increases in t.*

Proof. See Appendix. ■

Note that all the three conditions 1,2 and 3 hold for a large class of distribution functions including, for example, every concave distribution function of the form $F(x) = x^{\gamma}, 0 < \gamma < 1$. The combination of Proposition 1 and Proposition 2 yields the result that the use of a head start in the sequential all pay auction is beneficial for a designer who wishes to maximize the expected total effort.

Proposition 3 *If Conditions 1,2 and 3 hold, then the expected total effort in the sequential all-pay auction with a head start $t > 1$ which is sufficiently close to 1 is higher than the expected total effort in the two-player sequential all-pay auction without any head start.*

By Proposition 3, a head start $t > 1$ that is sufficiently close to 1 increases the expected highest effort as well as the expected total effort. However, we cannot conclude that the optimal head start for a designer

[12] The *failure (or hazard) rate* of F is given by the function $\lambda(x) \equiv F'(x) / [1 - F(x)]$. F is said to have an increasing failure rate (IFR) if $\lambda(x)$ is increasing in x. The IFR condition implies Condition 2.

who wishes to maximize the expected highest or total effort is close to 1. Note that for $1 < t \leq \frac{1}{F_2'(1)}$ the effort of every type of contestant 1 is higher than in the contest without a head start. However, for $t > \frac{1}{F_2'(1)}$ the effort of low types of contestant 1 is higher than in the contest without a head start but the effort of the high types in the contest with a head start is not necessarily higher than their efforts in the contest without a head start. In this case, the head start serves as a winning bid and therefore some high types will choose the winning bid but not any bid above it as they might have done without the head start. Nevertheless, as we can see from Example 2, the optimal head starts (that induce the highest expected total effort and the highest expected highest effort) might be obtained for $t > \frac{1}{F_2'(1)}$ although such a head start does not necessarily increase the effort of all possible contestants' types.

Finally, for t close enough to 1, we can express the efficiency in terms of the headstart by (in this case it is still true that $t\beta_1(a_1) \leq a_1$ for all $\widehat{a} \leq a_1 \leq 1$)

$$\begin{aligned} Eff(t) &= \int_{\widehat{a}}^{1} \left(\int_{0}^{t\beta_1(a_1)} F_2'(a_2)\, da_2 + \int_{a_1}^{1} F_2'(a_2)\, da_2 \right) F_1'(a_1)\, da_1 \\ &= \int_{\widehat{a}}^{1} \left(F_2\left((F_2')^{-1}\left(\frac{1}{ta_1} \right) \right) + 1 - F_2(a_1) \right) F_1'(a_1)\, da_1 \end{aligned}$$

We can see that the efficiency is increasing with t (recall that $\widehat{a}$ is decreasing with t). Thus, we conclude that a headstart also improves the efficiency of the contest.

3 The sequential all-pay auction with noisy outputs

We consider now a sequential all-pay auction with two risk neutral contestants where contestant 1 (the first mover) exerts an effort x_1 in the first period, while contestant 2 (the second mover) observes an output of $x_1 + t$ where t represents a random noise that is drawn from a uniform distribution on the interval $[-k, k]$, $0 \leq k \leq \frac{1}{2}$ and this information is common knowledge. The value of k determines the variance of the random noise and the smaller the value of k is the higher is the contest's accuracy. Contestant 2 exerts an effort x_2 in the second period, and wins the contest if the effort x_2 is larger than or equal to $x_1 + t$; otherwise, contestant 1 wins. The valuation of both contestants for the prize is 1. An effort x_i costs $\frac{x_i}{a_i}$ where $a_i \geq 0$ is the ability (or type) of contestant i which is private information to i. Contestant i's ability is drawn independently from the interval $[0, 1]$ according to a cumulative distribution function F_i which is common knowledge. We assume that $F_i, i = 1, 2$ has a positive and continuous density function $F_i' > 0$. Since the ability of the players is distributed on $[0, 1]$ we can assume that the output is limited to this interval and therefore we assume that if $x_1 + t \leq 0$, then contestant 2 observes an output of zero while if $x_1 + t \geq 1$ she observes an output of 1.

3.1 Symmetric information

Assume that both contestants do not know the realization of the noise t when exerting their effort. If contestant 1 exerts an effort of $b_1(a_1)$ in the first period, contestant 2 observes a noisy output of $b_1(a_1) + t$. Then contestant 2's equilibrium strategy is given by

$$b_2(a_2) = \begin{cases} 0 & \text{if} \quad a_2 < b_1(a_1) + t \\ b_1(a_1) + t & \text{if} \quad a_2 \geq b_1(a_1) + t \end{cases} \tag{10}$$

In the following, we assume that $k \leq \frac{1}{2}$ and that F_2 is concave. Then we can show (see the proof of Proposition 4) that contestant 1's equilibrium strategy satisfies $k \leq b_1(a_1) \leq 1-k$. In that case, contestant's 1 maximization problem is given by

$$\max_{b_1} \left\{ \int_{-k}^{k} F_2(b_1(a_1) + t) \frac{1}{2k} dt - \frac{b_1(a_1)}{a_1} \right\}$$

The F.O.C. is therefore

$$\int_{-k}^{k} \frac{1}{2k} F_2'(b_1(a_1) + t) \, dt - \frac{1}{a_1} = 0 \tag{11}$$

The S.O.C. is

$$\int_{-k}^{k} \frac{1}{2k} F_2''(b_1(a_1) + t) \, dt < 0 \tag{12}$$

If F_2 is concave then the S.O.C. holds everywhere. Thus, according to the above analysis, contestant 1's equilibrium strategy is as follows:

Proposition 4 *In the sequential all-pay auction, for every concave distribution function F_2, the equilibrium strategy of contestant 1 is given by $b_1(a_1) = 0$ for all $0 \leq a_1 < a_1^*$, and for all $a_1 \geq a_1^*$ it is implicitly defined by*

$$\frac{1}{2k} F_2(b_1(a_1) + k) - \frac{1}{2k} F_2(b_1(a_1) - k) = \frac{1}{a_1} \tag{13}$$

The cutoff type a_1^* is implicitly defined by

$$\frac{1}{2k}\int_{-k}^{k} F_2\left(b_1(a_1^*)+t\right)dt - \frac{b_1(a_1^*)}{a_1^*} = \frac{1}{2k}\int_{0}^{k} F_2\left(t\right)dt \tag{14}$$

where $b_1(a_1^*)$ is implicitly defined by (13).

Proof. See Appendix. ■

In the following we use Proposition 4 to illustrate the contestants' behavior in a sequential all-pay auction.

Example 3 *Assume a sequential all-pay auction where contestant 2's type is distributed according to $F_2(x) = x^{0.5}$. By (13), contestant 1's equilibrium strategy is implicitly given by*

$$\frac{1}{2k}\left(\sqrt{b_1+k}-\sqrt{b_1-k}\right) = \frac{1}{a_1}$$

Thus, contestant 1's equilibrium strategy is explicitly given by

$$b_1(a_1) = \begin{cases} 0 & \text{if} \quad 0 \le a_1 < a_1^* \\ \frac{a_1^2}{4}+\frac{k^2}{a_1^2} & \text{if} \quad a_1^* \le a_1 \le 1 \end{cases}$$

The cutoff type a_1^* is defined by (14)

$$\frac{1}{2k}\int_{-k}^{k}\left(\frac{a_1^2}{4}+\frac{k^2}{a_1^2}+t\right)^{\frac{1}{2}}dt - \left(\frac{a_1}{4}+\frac{k^2}{a_1^3}\right) = \frac{1}{2k}\int_{0}^{k} t^{\frac{1}{2}}\,dt$$

Therefore

$$a_1^* = c\sqrt{k}$$

where c is the solution to the equation

$$4c^3 - 3c^4 + 4 = 0 \Rightarrow c \simeq 1.6372$$

Note that contestant 1's effort is increasing in a_1 for all $a_1 \ge \sqrt{2k} = 1.4142\sqrt{k}$ and therefore for all $a_1 \ge a_1^*$. Moreover, for a given type a_1 who exerts a positive effort, the effort is increasing in k. Note also that a_1^* approaches zero when $k \to 0$. In that case, contestant 1 will exert an effort of $b_1(a_1) = \frac{a_1^2}{4}$. Finally, note that $a_1^* \le 1$ iff $k \le \left(\frac{1}{1.6372}\right)^2 = 0.37308$. That is, if $k > 0.37308$ contestant 1, independent of her type, exerts an effort of $b_1 = 0$.

Our goal in the following is to examine the effect of the size of the interval of the random noise, k, on the contestants' behavior, and particularly on the expected highest output in the contest. The expected highest output in our model is equal to contestant 1's expected output which is given by

$$TE_1 = \int_{a_1^*}^{1} \left(\frac{1}{2k} \int_{-k}^{k} (b_1(a_1) + t)\, dt \right) f_1(a_1) da_1 = \int_{a_1^*}^{1} b_1(a_1) f_1(a_1) da_1 \tag{15}$$

where $b_1(a_1)$ is defined by (13). If k is sufficiently large, no type of contestant 1 will exert a positive effort (i.e., $a_1^* = 1$). The following example illustrates contestant 1's expected effort in the contest.

Example 4 *Consider a sequential all-pay auction with two contestants whose types are distributed according to $F_1(x) = F_2(x) = x^{0.5}$. Then, by (15), the expected highest effort is given by*

$$TE_1 = \int_{1.637\sqrt{k}}^{1} \left(\frac{a_1^2}{4} + \frac{k^2}{a_1^2} \right) \frac{1}{2\sqrt{a_1}} da_1 = \frac{1}{20} - \frac{k^2}{3} - 0.01236 k^{\frac{5}{4}}$$

Figure 3 depicts the expected highest effort as a function of k.

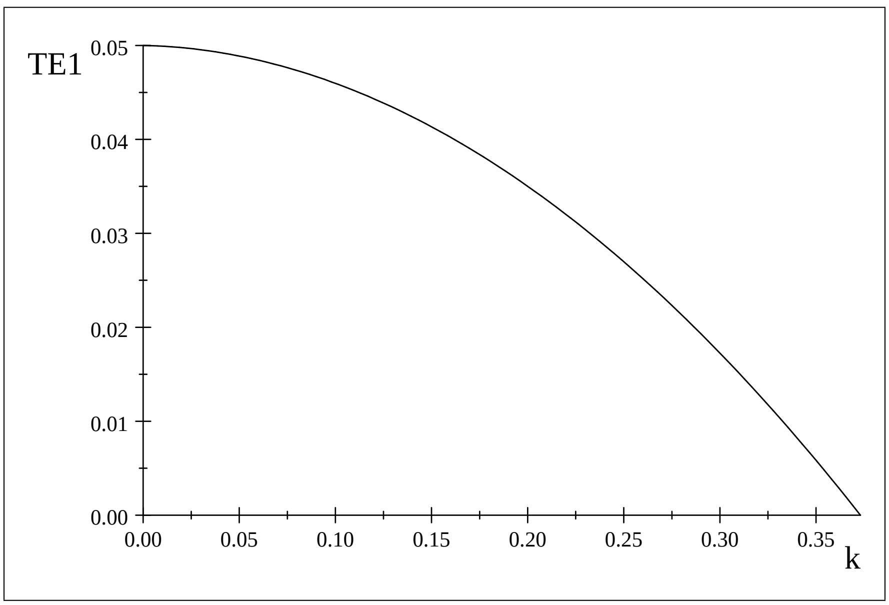

Figure 3: The expected highest effort as a function of k

We next show that the cutoff type a_1^* increases in k for any (concave) distribution of contestant 2's types.

Proposition 5 *The ex-ante probability that contestant* 1 *will exert a positive effort in the sequential all-pay auction decreases in the magnitude of the random noise, i.e.,*

$$\frac{da_1^*}{dk} > 0$$

Proof. See Appendix. ■

The effect of the magnitude of the random noise on the contestants' expected efforts is ambiguous. On the one hand, from the above proposition, it decreases the ex-ante probability that contestant 1 will exert a positive effort, but, on the other, in some cases it increases the effort of contestant 1 for any given type (as in Example 3 where $b_1(a_1)$ is increasing in k). However, the following result shows that if the magnitude of the random noise k is small enough, then it has no effect on the expected effort of contestant 1.

Proposition 6 *In the sequential all-pay auction, if $F_2'(x) \to \infty$ when $x \to 0$, the marginal effect of the magnitude of the random noise, k, on the expected highest output is zero when k approaches zero, i.e.,*

$$\lim_{k \to 0} \frac{dTE_1}{dk} = 0$$

Proof. See Appendix. ■

Note that the condition in the above proposition, $F_2'(x) \to \infty$ when $x \to 0$, holds, for example, for all concave distribution functions of the form $F(x) = x^{\gamma}, 0 < \gamma < 1$. Moreover, since contestant 2 only equalizes the output of contestant 1 the effect of the random noise is similar on both contestants. Thus, we conclude that a relatively small noise in the sequential all-pay contest does not result in a dramatic change in the contestants' output. In other words, the sequential all-pay auction is robust under a small noise with respect to the contestants' outputs.

3.2 Asymmetric information

In many market situations the first player to arrive at the market gathers the available information and can successfully evaluate the connection between her effort and the observable output. Assume therefore that contestant 2 does not know the value of the realization of the noise t which is uniformly distributed on the interval $[-k, k]$, while contestant 1 knows the realization of t before she exerts her effort. Note, however, that contestant 2's behavior will not be change when contestant 1 knows the realization of t. Then, as in the previous section, the equilibrium strategy of contestant 2 is given by

$$b_2(a_2) = \begin{cases} 0 & \text{if} \quad a_2 < b_1(a_1) + t \\ b_1(a_1) + t & \text{if} \quad a_2 \geq b_1(a_1) + t \end{cases}$$

Given a noise of t, contestant 1 with ability a_1 solves the following optimization problem:

$$\max_{b_1}\left\{F_2(b_1(a_1)+t)-\frac{b_1(a_1)}{a_1}\right\} \tag{16}$$

The F.O.C. is then

$$F_2'(b_1(a_1)+t)-\frac{1}{a_1}=0 \tag{17}$$

The S.O.C. is

$$F_2''(b_1(a_1)+t)<0 \tag{18}$$

We assume again that F_2 is concave. Thus, contestant 1's equilibrium strategy is as follows:

Proposition 7 *In the sequential all-pay auction, for every concave distribution function F_2, the equilibrium strategy of contestant* 1 *is given by*

$$b_1(a_1)=\begin{cases} 0 & \text{if} \quad a_1<\overline{a} \\ (F_2')^{-1}(\frac{1}{a_1})-t & \text{if} \quad a_1\geq\overline{a} \end{cases} \tag{19}$$

If $0\leq t\leq (F_2')^{-1}(1)$, the cutoff type $\overline{a}=a^*$ is determined by

$$(F_2')^{-1}(\frac{1}{a^*})-t=0\Leftrightarrow a^*=\frac{1}{F_2'(t)} \tag{20}$$

and if $t<0$, the cutoff type $\bar{a}=a^{**}$ is determined by

$$F_2((F_2')^{-1}(\frac{1}{a^{**}}))-\frac{1}{a^{**}}\left((F_2')^{-1}(\frac{1}{a^{**}})-t\right)=0 \tag{21}$$

Finally, if $t>(F_2')^{-1}(1)$ then $\overline{a}=1$

Note that $b_1(a_1)+t<1$ for all $a_1\leq 1$ and therefore the maximization problem (16) is well defined. Moreover, if $t>(F_2')^{-1}(1)$ then all types of contestant 1 exert a zero effort. If $t<0$, the cutoff $\bar{a}=a^{**}$ is the type whose expected payoff is equal to zero when a positive effort is exerted. Finally, given the realization of the noise t, contestant 1's equilibrium effort is (weakly) increasing in her type.

By Proposition 7, a positive noise decreases contestant 1's output and a negative noise increases it with respect to the situation without any noise. The noise, either negative or positive, increases the cutoff, that is, it decreases the ex-ante probability that contestant 1 will exert a positive effort. Thus, a positive noise necessarily decreases contestant 1's expected output. However, the effect of a negative noise on contestant 1's expected output is ambiguous since, on the one hand, it increases the effort, but, on the other, it increases the probability that contestant 1 will exert a zero effort. Note that if contestant 1 exerts a positive effort when t is positive as well as when t is negative with the same absolute value, then by (19) her effort when the noise is negative is higher by $2t$ than when the noise is positive. However, a positive noise and a negative noise, even if they have the same absolute value, by (20) and (21) will affect differently contestant 1's decision whether or not to exert a positive effort. The following result provides a condition on the distribution function of contestant 2's types for which a negative noise encourages a larger set of contestant 1's types to exert a positive effort in the contest than a positive noise with the same absolute value. Thus, this result also provides a condition according to which a negative noise is better than a positive one with the same absolute value from the viewpoint of a designer who wishes to maximize the expected highest output.

Proposition 8 *In a sequential all-pay contest with* $t \leq (F_2')^{-1}(1)$, *assume that the following condition holds*

$$\frac{F_2(t)}{F_2'(t)} \geq 2t \tag{22}$$

Then, if contestant 1 exerts a positive effort with a positive noise t, *she also exerts a positive effort with a negative noise of* $-t$, *i.e., we have*

$$a^{**} \leq a^*$$

In that case, a negative noise of $-t$ *yields a higher expected output of contestant* 1 *than a positive noise of* t.

Proof. See Appendix. ■

Condition (22) is satisfied in particular for all concave distribution functions of the form $F_2(t) = t^{\gamma}$,for $0 < \gamma \leq \frac{1}{2}$. Thus, for all these distribution functions, a negative noise yields a higher expected output of contestant 1 than a positive noise with the same absolute value.

The expected highest output, given a noise of t, is equal to the expected output of contestants 1 which is given by

$$TE_1(t) = \int_{\bar{a}}^{1} ((F_2')^{-1}(\frac{1}{a_1}) - t)F_1'(a_1)da_1 \tag{23}$$

If t is uniformly distributed on the interval $[-k,k]$ and $k \leq (F_2')^{-1}(1)$, the expected output of contestant 1 is given by

$$\begin{aligned} TE_1 &= \int_0^k \left(\int_{a^*}^1 b_1(a_1) F_1'(a_1) da_1 \right) \frac{1}{2k} dt + \int_{-k}^0 \left(\int_{a^{**}}^1 b_1(a_1) F_1'(a_1) da_1 \right) \frac{1}{2k} dt \\ &= \int_0^k \frac{1}{2k} \left(\int_{a^*}^1 (F_2')^{-1}(\frac{1}{a_1}) F_1'(a_1) da_1 - t(1-F_1(a^*)) \right) dt \\ &\quad + \int_{-k}^0 \frac{1}{2k} \left(\int_{a^{**}}^1 (F_2')^{-1}(\frac{1}{a_1}) F_1'(a_1) da_1 - t(1-F_1(a^{**})) \right) dt \end{aligned} \tag{24}$$

In the following example, we illustrate the equilibrium strategy and the effect of a random noise on the expected highest output in a sequential all-pay auction.

Example 5 *Consider a sequential all-pay auction where both contestants' types are distributed according to $F(x) = x^{0.5}$. Then, by (19), (20) and (21) for $0 \leq t \leq \frac{1}{4}$, contestant 1's equilibrium bid is given by*

$$b_1(a_1) = \begin{cases} \frac{a_1^2}{4} - t & \text{if } a_1 \geq 2\sqrt{t} \\ 0 & \text{if } a_1 < 2\sqrt{t} \end{cases}$$

and for $t < 0$, it is given by

$$b_1(a_1) = \begin{cases} \frac{a_1^2}{4} - t & \text{if } a_1 \geq 2\sqrt{-t} \\ 0 & \text{if } a_1 < 2\sqrt{-t} \end{cases}$$

By (24), the expected output of contestants 1 is

1) for $t \geq 0$

$$TE_1(t \geq 0) = \int_{2\sqrt{t}}^1 (\frac{a_1^2}{4} - t) \frac{1}{2\sqrt{a_1}} da_1 = \frac{4}{5}\sqrt{2} t^{\frac{5}{4}} - t + \frac{1}{20}$$

and

$$\frac{dTE_1}{dt} = \sqrt{2} t^{\frac{1}{4}} - 1$$

Then for all $0 \leq t \leq \frac{1}{4}$, we have $\frac{dTE_1}{dt} \leq 0$; that is, any positive noise decreases the expected output of contestant 1 compared to the case without any noise.

2) for $t < 0$

$$TE_1(t < 0) = \int_{2\sqrt{-t}}^1 (\frac{a_1^2}{4} - t) \frac{1}{2\sqrt{a_1}} da_1 = \frac{1}{20} - \frac{6}{5}\sqrt{2}(-t)^{\frac{5}{4}} - t$$

and

$$\frac{dTE_1}{dt} = \frac{3}{2}\sqrt{2}(-t)^{\frac{1}{4}} - 1$$

Thus, for all $t > -0.04939$, $\frac{dTE}{dt} \leq 0$ and for all $t < -0.04939$, $\frac{dTE}{dt} \geq 0$. In other words, a small negative noise increases contestant 1' expected output, and a large negative noise decreases it. We plot the expected highest effort as a function of the realization of the noise t in Figure 4.

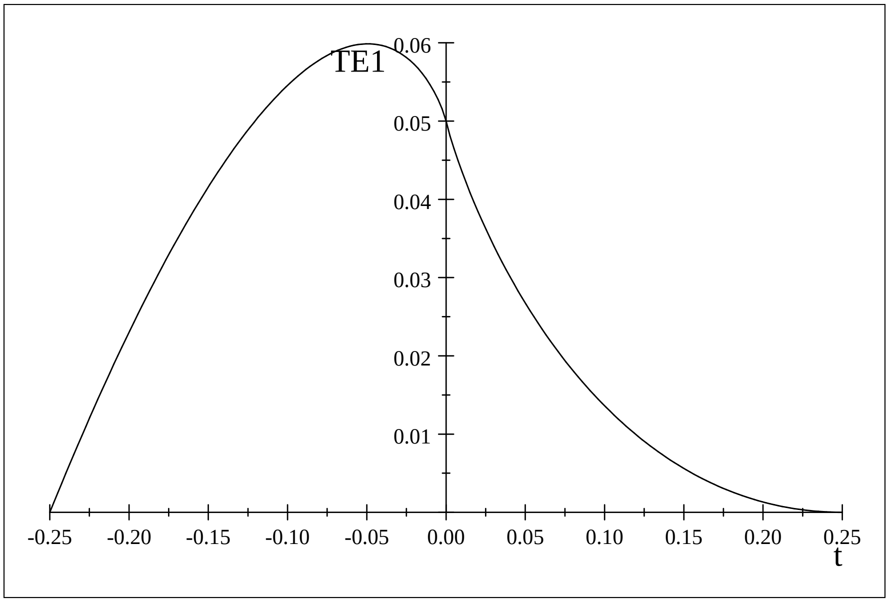

Figure 4: The expected highest effort as a function of t

If t is uniformly distributed on $[-k, k]$ for $k \leq \frac{1}{4}$ then

$$\begin{aligned} TE_1 &= \int_0^k TE_1 (t \geq 0) \frac{1}{2k} dt + \int_{-k}^0 TE_1 (t < 0) \frac{1}{2k} dt \\ &= \frac{1}{2k} \left(\int_0^k \left(\frac{4}{5}\sqrt{2} t^{\frac{5}{4}} - t + \frac{1}{20} \right) dt + \int_{-k}^0 \left(\frac{1}{20} - \frac{6}{5}\sqrt{2} (-t)^{\frac{5}{4}} - t \right) dt \right) = \frac{1}{20} - \frac{4}{45}\sqrt{2} k^{\frac{5}{4}} \end{aligned}$$

In Figure 5 we can see that the expected effort of contestant 1 decreases in the magnitude of the random noise k.

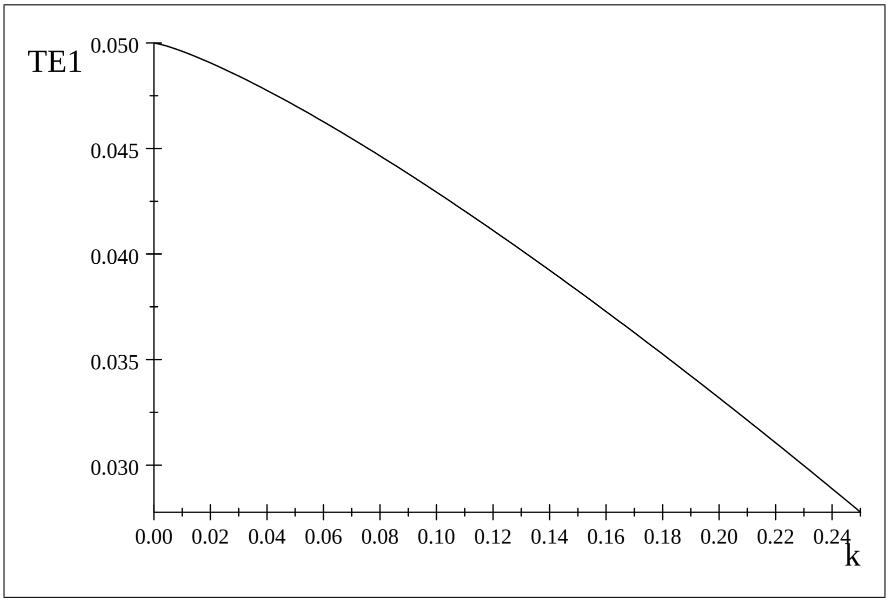

Figure 5: The expected highest effort as a function of k

In the above example, although the random noise is symmetrically distributed around zero, the equilibrium output of contestant 1 is not symmetric, i.e., $b_1(a_1, t) \neq b_1(a_1, -t)$ and, in particular the expected highest effort is not symmetric around zero as can be seen in Figure 4. In the following, we show that when the magnitude of the noise k is small enough the effect of the negative noises will be positive and will balance the negative effect of the positive noises such that the overall effect of random noise on contestant 1's expected effort will be zero.

Proposition 9 *In the sequential all-pay auction, if $F_2'(x) \to \infty$ when $x \to 0$, the marginal effect of the magnitude of the random noise, k, on the expected highest output is zero when k approaches zero, i.e.,*

$$\lim_{k \to 0} \frac{dTE_1}{dk} = 0.$$

Proof. See Appendix. ■

Proposition 9 demonstrates that the sequential all-pay auction is robust under a small noise in contestant 1's output when she knows the realization of the noise before she exerts the effort. Thus, by Propositions 6 and 9 we can conclude that with either symmetric or asymmetric information on the realization of random noise, a relatively small noise has no effect on the expected highest effort.

4 Concluding remarks

We presented a model of two-players sequential all-pay auction with incomplete information. We characterized the equilibrium behavior of the contestants and derived expressions for the expected total and highest efforts. Then we analyzed the implications of using a head start mechanism in which the first mover is favored over the second one. This head start, on the one hand, encourages the first mover to exert higher efforts but, on the other, may cause the second mover to withdraw from the contest. We demonstrated that in our model the allocation of head starts increases the expected highest effort as well as the expected total effort. It seems natural to generalize the sequential all-pay auction to the case with $n > 2$ contestants. In this generalized model (without head starts) contestants arrive one by one, contestant $j, j \leq n$, wins if his effort is larger than or equal to the efforts of all the contestants in the previous periods and his effort is larger than the efforts of all the contestants in the following periods. However, the characterization of the subgame perfect equilibrium of the generalize model with and without head starts is not tractable for all forms of the distribution functions of the contestants' types.

We also established the existence of a subgame perfect equilibrium in the sequential all-pay auction with noisy outputs. We showed that when the noise is uniformly distributed around zero, this auction is robust in the sense that the marginal effect of small noises on the contestants' expected highest effort is zero. In other words, in a sequential all-pay auction, small noises do not have a dramatic effect on the contestants' output with respect to the contest without any noise. Owing to the complexity of the environment we focused here on a specific distribution of the random noise, namely, the uniform distribution. However, we conjecture that our results will hold for other distributions of random noise as long as they are symmetrically distributed around zero.

We assumed for simplicity that contestant 1's output is subject to a random noise. If we will assume that contestant 2's output is also subject to a random noise and that the noises are independent random variables then the nature of our results do not change. This is true since contestant 2's behavior is characterized by a cutoff. Contestant 2 will exert a positive effort iff his type is larger than or equal to the observable output of contestant 1 plus a constant. This follows from the fact that his expected payoff will still be linear in his effort. Thus we can implicitly define contestant 1's equilibrium effort function and derive similar results.

5 Appendix

5.1 Proof of Proposition 1

The expected highest effort in the two-players model without a head start is equal to contestant 1's expected effort, while the expected highest effort in the two-player model with a head start is larger than or equal to contestant 1's expected effort. Thus, in order to prove that a head start increases the expected highest effort it is sufficient to show that a head start increases contestant 1's expected effort. However, what we actually show is even stronger. We show that for every type of contestant 1 who made a positive effort when there was no head start, this effort increases when a head start is given. Therefore we show that

$$\beta_1(a_1) \geq b_1(a_1) \text{ for all } 0 \leq a_1 \leq 1 \text{ and } 1 \leq t \leq \frac{1}{F_2'(1)}$$

Note that if Condition 1 holds then since $b_1(a_1)$ is increasing in a_1 and $\tilde{a} \geq 0$ then for all $t > 1$,

$$\beta_1(a_1) = \frac{1}{t}(F_2')^{-1}\left(\frac{1}{ta_1}\right) > (F_2')^{-1}\left(\frac{1}{a_1}\right) = b_1(a_1)$$

Moreover the lowest type of contestant 1 who is active in the two-player model with a head start is lower than the lowest active type of contestant 1 in the two-player model without any head start. Formally, $\widehat{a} = \frac{1}{tF_2'(0)} \leq \frac{1}{F_2'(0)} = \widetilde{a}$ for any $t \geq 1$. Thus, we have

$$\begin{aligned}
\beta_1(a_1) &= \frac{1}{t}(F_2')^{-1}\left(\frac{1}{ta_1}\right) > (F_2')^{-1}\left(\frac{1}{a_1}\right) = b_1(a_1) \quad \text{for all } \widetilde{a} \leq a_1 \leq 1 \\
\beta_1(a_1) &= \frac{1}{t}(F')^{-1}\left(\frac{1}{ta_1}\right) > b_1(a_1) = 0 \quad \text{for all } \widehat{a} \leq a_1 \leq \widetilde{a} \\
\beta_1(a_1) &= b_1(a_1) = 0 \quad \text{for all } 0 \leq a_1 \leq \widehat{a}
\end{aligned}$$

and the expected effort of contestant 1 with a head start t is higher than his expected effort without any head start. *Q.E.D.*

5.2 Proof of Proposition 2

The expected effort of contestant 2 given an effort $\beta_1(a_1, t) > 0$ of contestant 1 is

$$E_2(t, a_1) = (1 - F_2(t\beta_1(a_1, t)))\, t\beta_1(a_1, t)$$

The expected effort of contestant 2 is then

$$TE_2(t) = \int_0^1 E_2(t, a_1) F_1'(a_1) da_1 = \int_{\widehat{a}}^1 E_2(t, a_1) F_1'(a_1) da_1 > 0$$

The function $t\beta_1(a_1, t) = (F_2')^{-1}\left(\frac{1}{a_1 t}\right)$ is increasing in a_1 as well as in t. By Condition 3 we know that $(F_2')^{-1}(1) < x^*$. Therefore we obtain that, for $t > 1$ close enough to 1 and for all $a_1 \leq 1$,

$$t\beta_1(a_1, t) \leq t\beta_1(a_1 = 1, t) = (F_2')^{-1}\left(\frac{1}{t}\right) < x^*$$

Thus by Condition 2 we have

$$\frac{dE_2(t, a_1)}{dt} > 0$$

So far we showed that given a type $\widehat{a} \leq a_1 \leq 1$ of contestant 1 that exerts a positive effort, the expected effort of contestant 2 increases in t as long as t is sufficiently close to 1. By Condition 1, the interval of types of contestant 1 who exert a positive effort increases in t, i.e., $\frac{d\widehat{a}}{dt} = \frac{d}{dt}\left(\frac{1}{tF_2'(0)}\right) \leq 0$ and therefore, if t is sufficiently close to 1 we established that

$$\frac{d}{dt} TE_2(t) = \frac{d}{dt} \int_0^1 E_2(t, a_1) F_1'(a_1) da_1 = \frac{d}{dt} \int_{\widehat{a}}^1 E_2(t, a_1) F_1'(a_1) da_1 > 0$$

Q.E.D.

5.3 Proof of Proposition 4

We wish to characterize the equilibrium effort function $b_1(a_1)$ of contestant 1 when the equilibrium effort function of contestant 2 is given by (10). We divide our analysis into the following three cases: 1. $b_1(a_1) < k$ 2. $b_1(a_1) > 1 - k$ and 3. $k \leq b_1(a_1) \leq 1 - k$.

1) Assume first that $b_1(a_1) < k$. Then contestant $1'$s maximization problem is given by

$$\max_{b_1} \left\{ \int_{-k}^{-b_1} F_2(0) \frac{1}{2k} dt + \int_{-b_1}^{k} F_2(b_1 + t) \frac{1}{2k} dt - \frac{b_1}{a_1} \right\} = \max_{b_1} \left\{ \int_{-b_1}^{k} F_2(b_1 + t) \frac{1}{2k} dt - \frac{b_1}{a_1} \right\}$$

The F.O.C. is given by

$$\int_{-b_1}^{k} \frac{1}{2k} F_2'(b_1 + t) dt - \frac{1}{a_1} = 0$$

Thus,

$$\frac{1}{2k} F_2(b_1 + k) = \frac{1}{a_1} \tag{25}$$

The S.O.C. is given by

$$\frac{1}{2k}F_2'(0) + \int_{-b_1}^{k} \frac{1}{2k}F_2''(b_1+t)\,dt = \frac{1}{2k}F_2'(b_1+k)$$

Since $F_2'(b_1+k) > 0$, the S.O.C. does not hold and therefore the maximum is never achieved at an internal effort $b_1 \in (0,k)$.

2) Assume now that $b_1(a_1) > 1-k$. Then, contestant $1'$s maximization problem is given by

$$\max_{b_1}\left\{\int_{-k}^{1-b_1} F_2(b_1+t)\frac{1}{2k}dt + \int_{1-b_1}^{k} \frac{1}{2k}dt - \frac{b_1}{a_1}\right\}$$

The F.O.C. is

$$\frac{1}{2k}\int_{-k}^{1-b_1} F_2'(b_1+t)\,dt - \frac{1}{a_1} = 0$$

Thus,

$$\frac{1}{2k}(1-F_2(b_1-k)) = \frac{1}{a_1} \tag{26}$$

Let $a_1 = 1$. Then if $b_1(1) > 1-k$ we obtain

$$\frac{1}{2k}(1-F_2(b_1(1)-k)) < \frac{1}{2k}(1-F_2(1-2k)) < \frac{1}{2k}(1-(1-2k)) = 1$$

The second inequality is due to our assumption that F_2 is concave. This inequality contradicts equation (26) and therefore $b_1(1) < 1-k$, which implies by the monotonicity of b_1 that $b_1(a_1) < 1-k$ for all $a_1 \leq 1$.

3) When $k \leq b_1(a_1) \leq 1-k$, contestant's 1 maximization problem is given by

$$\max_{b_1}\left\{\int_{-k}^{k} F_2(b_1+t)\frac{1}{2k}dt - \frac{b_1}{a_1}\right\}$$

The F.O.C. is therefore

$$\int_{-k}^{k} \frac{1}{2k}F_2'(b_1+t)\,dt - \frac{1}{a_1} = 0$$

The S.O.C. is

$$\int_{-k}^{k} \frac{1}{2k}F_2''(b_1+t))dt < 0$$

If F_2 is concave then the S.O.C. holds everywhere. Thus, the equilibrium strategy of contestant 1, $b_1(a_1)$, is implicitly determined by

$$\frac{1}{2k}F_2(b_1+k) - \frac{1}{2k}F_2(b_1-k) = \frac{1}{a_1}$$

where $b_1(a_1)$ is a an increasing function. The cutoff type a_1^* is the type who is indifferent between an effort of zero and an effort given by (13). Therefore it is given by

$$\frac{1}{2k}\int_{-k}^{k} F_2\left(b_1(a_1^*)+t\right)dt - \frac{b_1(a_1^*)}{a_1^*} = \frac{1}{2k}\int_{0}^{k} F_2\left(t\right)dt$$

where $b_1(a_1^*)$ is implicitly defined by (13). By the analysis in case (1) it is possible that an interval of types will find it optimal to exert $b_1 = k$. We show next, however, that no such interval exists. Denote by $\hat{a}_1$ the type who by equation (13) is supposed to exert an effort of $b_1\left(\hat{a}_1\right) = k$. Thus $\hat{a}_1$ is the solution to

$$\frac{1}{2k}F_2\left(2k\right) = \frac{1}{a_1}$$

or

$$\hat{a}_1 = \frac{2k}{F_2\left(2k\right)}$$

Since F_2 is concave, $\hat{a}_1$ is between 0 and 1. If contestant 1 with type a_1 exerts an effort of $b_1 = 0$, then her expected payoff is $\pi_{a_1}\left(0\right) = \frac{1}{2k}\int_0^k F_2\left(t\right)dt$, while if she exerts an effort of k her expected payoff is $\pi_{a_1}\left(k\right) = \frac{1}{2k}\int_{-k}^{k} F_2\left(k+t\right)dt - \frac{k}{a_1}$. Recall that we already showed that she will never exert an effort strictly between 0 and k. The difference between these expected payoffs is given by

$$\begin{aligned}
\Delta(a_1) &= \pi_{a_1}\left(0\right) - \pi_{a_1}\left(k\right) = \frac{1}{2k}\int_0^k F_2\left(t\right)dt - \frac{1}{2k}\int_{-k}^{k} F_2\left(k+t\right)dt + \frac{k}{a_1} \\
&= \frac{k}{a_1} - \frac{1}{2k}\int_k^{2k} F_2\left(t\right)dt > \frac{k}{a_1} - \frac{1}{2}F_2\left(2k\right)
\end{aligned}$$

By the definition of $\hat{a}_1$, we obtain that $\Delta(a_1)$ is positive for all $a_1 < \hat{a}_1$. Thus, all types of contestant 1 that are smaller than $\hat{a}_1$ will exert an effort of $b_1 = 0$. Therefore no type $a_1 < \hat{a}_1$ can be indifferent between an effort of zero and an effort given by (13) (since by the monotonicity of the effort function given in (13), $b_1\left(a_1\right) < k$). Therefore $a_1^* \geq \hat{a}_1$. Finally, all $a_1 \in [\hat{a}_1, a_1^*]$ prefer an effort of zero over an effort given by (13). This follows from the fact that the L.H.S. of equation (14) is constant while the R.H.S. is increasing in a_1

$$\frac{d}{da_1}\left(\frac{1}{2k}\int_{-k}^{k} F_2\left(b_1(a_1)+t\right)dt - \frac{b_1(a_1)}{a_1}\right) = \frac{b_1'\left(a_1\right)}{2k}\int_{-k}^{k} F_2'\left(b_1(a_1)+t\right)dt - \frac{b_1'\left(a_1\right)}{a_1} + \frac{b_1\left(a_1\right)}{a_1^2} = \frac{b_1\left(a_1\right)}{a_1^2} > 0$$

Therefore all $a_1 \in [\hat{a}_1, a_1^*]$ prefer an effort of zero. Note that all types which are larger than a_1^* will exert an effort according to (13), and particularly, since $b_1\left(a_1\right)$ is a monotonically increasing function, all the positive efforts are larger than k. *Q.E.D.*

5.4 Proof of Proposition 5

Recall that a_1^* is implicitly defined as the solution to the equation

$$\frac{1}{2k}\int_{-k}^{k} F_2\left(b_1(a_1)+t\right)dt - \frac{b_1(a_1)}{a_1} - \frac{1}{2k}\int_0^k F_2\left(t\right)dt = 0$$

where $b_1(a_1^*)$ is implicitly defined by (13). Therefore

$$\frac{da_1^*}{dk} = -\frac{\left(-\frac{1}{2k^2}\int_{-k}^{k} F_2\left(b_1(a_1)+t\right)dt + \frac{1}{2k}F_2\left(b_1(a_1)+k\right) + \frac{1}{2k}F_2\left(b_1(a_1)-k\right) + \frac{1}{2k^2}\int_0^k F_2\left(t\right)dt - \frac{1}{2k}F_2\left(k\right)\right)}{b_1'\left(a_1\right)\left(\frac{1}{2k}\int_{-k}^{k} F_2'\left(b_1(a_1)+t\right)dt - \frac{1}{a_1}\right) + \frac{b_1(a_1)}{a_1^2}}$$

Note that by equation (13) the denominator is positive. Moreover, we have

$$\frac{1}{2k^2}\int_0^k F_2\left(t\right)dt - \frac{1}{2k}F_2\left(k\right) < 0$$

and

$$2k\left(\frac{1}{2}\left(F_2\left(b_1(a_1)+k\right) + F_2\left(b_1(a_1)-k\right)\right)\right) < \int_{-k}^{k} F_2\left(b_1(a_1)+t\right)dt$$

This last inequality is true since F_2 is concave. Therefore, we conclude that $\frac{da_1^*}{dk} > 0$. *Q.E.D.*

5.5 Proof of Proposition 6

If $F_2'\left(x\right) \to \infty$ when $x \to 0$, then in the limit when k goes to zero we have the following equilibrium strategy for player 1

$$\frac{1}{2k}F_2\left(b_1(a_1)+k\right) - \frac{1}{2k}F_2\left(b_1(a_1)-k\right) = \frac{1}{a_1} \Rightarrow \tag{27}$$
$$b_1(a_1) = \left(F_2'\right)^{-1}\left(\frac{1}{a_1}\right) \quad \text{for all } a_1 \geq 0$$

Note that if $F_2'\left(0\right)$ is finite then an interval of types $a_1 \in \left[0, \frac{1}{F_2'(0)}\right]$ will exert zero effort since the solution to the maximization problem

$$\max_s \left\{F_2(b_1(s)) - \frac{b_1\left(s\right)}{a_1}\right\}$$

is $b_1 = 0$. But when $F_2'\left(x\right) \to \infty$ when $x \to 0$, all positive types find it optimal to exert a positive effort in the limit when k goes to zero.

By (23) we have

$$\frac{dTE_1}{dk} = \int_{a_1^*}^{1} \frac{db_1}{dk} f_1(a_1)da_1 - \frac{da_1^*}{dk} b_1\left(a^*\right) f_1(a^*)$$

Using the implicit condition (13) we obtain that

$$\lim_{k\to 0}\frac{db_1}{dk}=\lim_{k\to 0}-\frac{\left(\frac{1}{2k}(F_2'(b_1+k)+F_2'(b_1-k))-\frac{1}{2k^2}(F_2(b_1+k)-F_2(b_1-k))\right)}{\frac{1}{2k}(F_2'(b_1+k)-F_2'(b_1-k))}$$

Since $F_2^{'}(b_1)=\lim_{k\to 0}\left(\frac{1}{2k}(F_2(b_1+k)-F_2(b_1-k))\right)$ and $F_2^{''}(b_1)=\lim_{k\to 0}\left(\frac{1}{2k}(F_2'(b_1+k)-F_2'(b_1-k))\right)$, then

$$\lim_{k\to 0}\frac{db_1}{dk}=\frac{\frac{1}{k}(F_2'(b_1)-F_2'(b_1))}{F_2''(b_1)}=0$$

Moreover, from the above we know that $\lim_{k\to 0}a_1^*(k)=0$ if $F_2'(x)\to\infty$ when $x\to 0$, and thus we obtain that

$$\lim_{k\to 0}\frac{dTE_1}{dk}=0$$

Q.E.D.

5.6 Proof of Proposition 8

Given a positive realization and a negative realization of the noise with the same absolute value we let $v=|t|=|-t|$; Then, by (20) when $t>0$, a^* is determined by

$$(F_2')^{-1}(\frac{1}{a^*})=v$$

and by (21) when $t<0$, a^{**} is determined by

$$F_2((F_2')^{-1}(\frac{1}{a^{**}}))-\frac{1}{a^{**}}\left((F_2')^{-1}(\frac{1}{a^{**}})+v\right)=0$$

If we replace a^{**} by a^* in the L.H.S. of the last equation we obtain

$$F_2(v)-F_2'(v)2v \tag{28}$$

Thus, if (28) is positive, contestant 1 with type a^* has a positive expected payoff when the realization of the noise is $-v$. Since the expected payoff of contestant 1 increases in her type, and the expected payoff of type a^{**} is zero when the realization of the noise is $-v$, we obtain that $a^{**}\leq a^*$. *Q.E.D.*

5.7 Proof of Proposition 9

The derivative of (24) is

$$\begin{aligned}\frac{dTE_1}{dk} &= \frac{1}{2k}\left(\int_{a^*}^{1}(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 - k(1-F_1(a^*(k)))\right)\\ &\quad -\int_0^k \frac{1}{2k^2}\left(\int_{a^*}^{1} b_1(a_1)F_1'(a_1)da_1\right)dt\\ &\quad +\frac{1}{2k}\left(\int_{a^{**}}^{1}(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 + k(1-F_1(a^{**}(-k)))\right)\\ &\quad -\int_{-k}^{0} \frac{1}{2k^2}\left(\int_{a^{**}}^{1} b_1(a_1)F_1'(a_1)da_1\right)dt\end{aligned}$$

Thus,

$$\lim_{k\to 0}\frac{dTE_1}{dk} = \lim_{k\to 0}\left(\begin{array}{c}\frac{1}{2k}\left(\int_{a^*}^{1}(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 + \int_{a^{**}}^{1}(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1\right) - \frac{1}{2}\left(F_1(a^{**}(-k)) - F_1(a^*(k))\right)\\ -\frac{1}{2k^2}\left(\int_0^k\left(\int_{a^*}^{1} b_1(a_1)F_1'(a_1)da_1\right)dt + \int_{-k}^{0}\left(\int_{a^{**}}^{1} b_1(a_1)F_1'(a_1)da_1\right)dt\right)\end{array}\right)$$

When $F_2'(x)\to\infty$ when $x\to\infty$ then $\lim_{k\to 0}a^*(k)\to 0$ and $\lim_{k\to 0}a^{**}(k)\to 0$. Therefore,

$$\begin{aligned}\lim_{k\to 0}\frac{dTE_1}{dk} &= \lim_{k\to 0}\left(\frac{1}{k}\int_0^1(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 - \frac{1}{2k^2}\int_{-k}^{k}\left(\int_0^1 b_1(a_1)F_1'(a_1)da_1\right)dt\right)\\ &= \lim_{k\to 0}\frac{1}{k}\int_0^1(F_2')^{-1}(\frac{1}{a_1})F_1'(a_1)da_1 - \lim_{k\to 0}\frac{1}{k}\left(\int_0^1 b_1(a_1)F_1'(a_1)da_1\right)dt = 0\end{aligned}$$

Q.E.D.

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