Directed Search on the Job, Heterogeneity, and Aggregate Fluctuations

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In models of search on the job (e.g. Burdett and Mortensen 1998, Burdett and Coles 2003, Delacroix and Shi 2006, Shi 2009), employed and unemployed workers search the labor market for job openings. Workers who are unemployed are willing to accept any job that makes them better off than enjoying leisure and continue searching. Workers who are employed are willing to accept any job that offers them more than their current job. On the other side of the market, firms are indifferent between opening jobs that offer different wages, as firms that offer higher wages can fill their jobs faster and retain their workers for a longer period of time. The extent of search frictions determines how quickly workers move from unemployment to the top of the wage-offer distribution and the shape of the wage-offer distribution itself. Overall, models of search on the job provide an equilibrium theory of workers' transitions between employment, unemployment and across employers, and, simultaneously, a theory of wage inequality. Because of these properties, models of search on the job are a useful (and popular) tool for studying the labor market.

When the search process is assumed to be random, these models are difficult to use for studying the aggregate dynamics of the labor market. This is because workers and firms need to forecast the dynamics of the entire distribution of workers across employment states (unemployment and employment at different wages) in order to solve their problems. For example, a firm needs to forecast the dynamics of the distribution in order to compute the probability of filling a job opening that offers a certain wage, as well as to compute its survival probability once it is filled. Similarly, a worker needs to forecast the dynamics of the distribution in order to compare the value of being unemployed with the value of a job offer. Mathematically, solving for the equilibrium of these models outside of the steady state requires solving a system of functional equations in which the unknown functions (the agents' value and policy functions) have a function (the distribution of workers) as one of their arguments. Solving such system is a daunting task both analytically and computationally.

In contrast, when the search process is directed, these models are easy to use for the aggregate dynamics of the labor market. Building on Shi (2009), in Menzio and Shi (2009a, 2009b), we develop a rather general model of directed search on the job with aggregate productivity shocks. For this model, we establish the existence of an equilibrium in which agents do not need to forecast the evolution of the distribution of workers in order to solve their problem. Agents only need to forecast the evolution of the aggregate productivity shock. Mathematically, solving for this equilibrium amounts to solving a system of functional equations in which the unknown functions (the agents' value and policy functions) depend only on a one-dimensional argument (the aggregate shock). Solving such a system is just as easy as solving the equilibrium of a standard representative-agent model. We shall refer to this as a Block Recursive Equilibrium (BRE).

In this paper, we first present our model of directed search on the job. We outline the proof of the existence of an equilibrium in which the agents' value and policy functions depend on the aggregate state of the economy only through the aggregate shock, and not on the entire distribution of workers across employment states. Then, we explain why this type of equilibrium does exist when the search process is directed and it does not when the search process is random. Finally, we generalize the existence of this type of equilibrium to a version of our model in which workers are ex-ante heterogeneous with respect to some observable characteristic such as education or skill.

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I. Model

We consider an economy populated by a continuum of workers with measure 1, and by a continuum of firms with positive measure. Each worker is endowed with an indivisible unit of labor. Each worker maximizes the expected sum of periodical utilities $\sum_{t=0}^{\infty} \beta^t \upsilon(c_t)$, where β is the discount factor and υ is the periodical utility function with $\upsilon' \in [\underline{\upsilon}', \overline{\upsilon}']$, $\underline{\upsilon}' > 0$, and $\upsilon'' \leq 0$. Each firm operates a constant return to scale technology that turns 1 unit of labor into $y + z$ units of output. The first component of productivity, y, is common to all firms and its value lies in the set Y. The second component of productivity, z, is specific to a firm-worker pair and its value lies in the set Z. Each firm maximizes the expected sum of periodical profits discounted at the factor β. The labor market where firms and workers meet is organized in a continuum of submarkets indexed by $x \in [\underline{x}, \overline{x}] = X$, the lifetime utility offered by firms to workers in such submarket. In submarket x, the ratio of the number of vacancies to the number of workers who are looking for jobs is given by the tightness $\theta(x) \geq 0$ and is determined in equilibrium.

At the beginning of each period, the state of the economy can be summarized by the triple $(y, u, g) = \psi$. The first element of ψ is the aggregate component of productivity. The second element is the measure of unemployed workers, $u \in [0, 1]$. The third element is a function $g : X \times Z \to [0, 1]$, with $g(V, z)$ denoting the measure of workers who are employed at jobs that give them the lifetime utility $\tilde{V} \leq V$ and that have an idiosyncratic component of productivity $\tilde{z} \leq z$.

Every period is divided into four stages: separation, search, matching and production. During the separation stage, an employed worker is forced to move into unemployment with probability $\delta \in (0, 1]$. Also, during the separation stage, an employed worker has the option to voluntarily move into unemployment. During the search stage, workers (both employed and unemployed) choose in which submarket to apply for a job, and firms choose how many vacancies to create and where to locate them. The cost of maintaining a vacancy is $k > 0$. Both workers and firms take the tightness $\theta(x)$ parametrically.

During the matching stage, the workers and the vacancies in submarket x come together through a frictional matching process. In particular, a worker meets a vacancy with probability $p(\theta(x))$, where p is a strictly increasing and concave function such that $p(0) = 0$ and $p'(0) > 0$. Similarly, a vacancy meets a worker with probability $q(\theta(x))$, where q is a strictly decreasing and convex function such that $q(\theta) = p(\theta)/\theta$, $q(0) = 1$, $q'(0) < 0$, and $p(q^{-1}(.))$ concave. When a vacancy and a worker meet, the firm that owns the vacancy offers to the worker an employment contract that gives him the lifetime utility x. If, off the equilibrium path, the worker rejects the offer, he returns to his previous employment position. If the worker accepts the offer, the two parties form a new match with match-specific productivity z_0.

In the production stage, an unemployed worker produces and consumes $b > 0$ units of output. A worker employed at a job z produces $y + z$ units of output and consumes w of them, where w is specified by the worker's labor contract. At the end of the production stage, Nature draws next period's aggregate component of productivity, $\hat{y}$, from the probability distribution $\Phi_y(\hat{y}|y)$, and next period's idiosyncratic component of productivity, $\hat{z}$, from the distribution $\Phi_z(\hat{z}|z)$. The draws of the idiosyncratic component of productivity are independent across matches.

We consider two alternative specifications of the contractual environment: (i) Dynamic Contracts. In this environment, the firm commits to an employment contract that specifies the worker's wage as a function of the history of realizations of the idiosyncratic component of the match, z, the history of realizations of the aggregate state of the economy, ψ, and the history of realizations of a lottery that is drawn at the beginning of every production stage. This environment generalizes the contractual environment considered by Burdett and Coles (2003) and Shi (2009) to an economy with stochastic productivity. (ii) Fixed-Wage Contracts. In this environment, the firm commits to a wage that remains constant throughout the entire duration of the employment relationship. This constant wage is allowed to depend only on the outcome of a lottery that is drawn at the beginning of the employment relationship. This environment is similar to the one considered by Burdett and Mortensen (1998).

Notice that, in general, the contracting problem of the firm is a complicated sequence problem in which the history upon which wages are contingent grows to infinity over time. However, following the literature on dynamic contracts, we can rewrite this problem recursively by using the worker's lifetime utility as an auxiliary value function.

II. Block Recursive Equilibrium

We denote as $U(y)$ the lifetime utility of a worker who is unemployed, given that the aggregate component of productivity is y. We denote as $J(V, y, z)$ the profits of a firm that employs a worker, given that the employment contract is worth the lifetime utility V to the worker, and the idiosyncratic component of the firm-worker match is z. The value functions U and J are measured at the beginning of the production stage. We denote as $R(V, y)$ the lifetime utility of a worker at the beginning of the search stage, given that the worker's current employment position is worth V at the production stage. We denote as $\theta(x, y)$ the equilibrium tightness of submarket x. Finally, we denote with $\Phi_\psi(\hat{\psi}|\psi)$ the transition probability function for the aggregate state of the economy ψ.

Now, we are in the position to define a Block Recursive Equilibrium for the environment with dynamic contracts. The reader can find the definition of a BRE for the environment with fixed-wage contracts in Menzio and Shi (2009).

DEFINITION 1: *A Block Recursive Equilibrium is a list of value and policy functions (θ, R, m, U, J, c) together with a transition probability function Φ_ψ. These functions satisfy the following conditions:*
(i) For all $(V, \psi) \in X \times \Psi$,

$$R(V, y) = \max_x p(\theta(x, y))x + (1 - p(\theta(x, y)))V,$$

and m is the associated policy function;
(ii) For all $\psi \in \Psi$,

$$U(y) = \upsilon(b) + \beta \mathbb{E} R(U(\hat{y}), \hat{y});$$

(iii) For all $(V, \psi, z) \in X \times \Psi \times Z$,

$$J(V, y, z) = \max_{w,d,\hat{V}} y + z - w + \beta \mathbb{E}\left[\begin{array}{c}(1 - d(\hat{y}, \hat{z}))\times \\ (1 - \tilde{p}(\hat{y}, \hat{z}))J(\hat{V}(\hat{y}, \hat{z}), \hat{y}, \hat{z})\end{array}\right],$$

subject to the constraints

$$V = \upsilon(w) + \beta \mathbb{E}\left[\begin{array}{c} d(\hat{y}, \hat{z})R(U(\hat{y}), \hat{y}) \\ +(1 - d(\hat{y}, \hat{z}))\hat{V}(\hat{y}, \hat{z})\end{array}\right],$$

$$d(\hat{y}, \hat{z}) = \{1 if U(y) > V(y, z), \delta else\},$$

where $\tilde{p}(\hat{y}, \hat{z}) = p(\theta(m(\hat{V}(\hat{y}, \hat{z})), \hat{y}))$ and c is the associated policy function;
(iv) For all $(x, \psi) \in X \times \Psi$,

$$k \geq q(\theta(x, y))J(x, y)$$

and $\theta(x, y) \geq 0$, with complementary slackness;
(v) For all $\psi \in \Psi$, Φ_ψ is consistent with the transition probability of the exogenous variables, y and z, and with the policy functions m and c.

The interpretation of the equilibrium conditions (i)-(v) is straightforward. Condition (i) insures that, given the tightness of each submarket x, the worker chooses where to apply for a job to maximize his lifetime utility. Condition (iii) insures that, given the optimal search strategy of the worker, the firm chooses the employment contract to maximize its profits and to provide the worker with the promised lifetime utility V. Condition (iv) insures that the number of vacancies created in each submarket x maximizes the profits of the firm. Finally, condition (v) insures that the law of motion for the aggregate state of the economy is consistent with the stochastic process for productivity and with the agents' policy functions. Overall, conditions (i)-(v) insure that, in a BRE just like in a standard recursive equilibrium, the choices of each agent are optimal given the other agents' choices.

However, unlike in a standard recursive equilibrium, in a BRE the agents' value and policy functions (θ, R, m, U, J, c) depend on the aggregate state of the economy only through the aggregate component of productivity, y, and not through the whole distribution of workers across different employment states, (u, g). This implies that, in order to solve for a BRE, one needs to solve a system of functional equation in which the argument of the unknown functions depend on three real numbers (the individual state variables, V and z, and the driving force of aggregate uctuations, y) and not on an entire function (the distribution of workers across employment states). We refer to this equilibrium as a Block Recursive Equilibrium because it is a recursive equilibrium and the block of equilibrium conditions that describe the agents' value and policy functions can be solved before having solved the law of motion of the aggregate state of the economy.

III. Existence of Block Recursive Equilibrium

While we can define a Block Recursive Equilibrium for any model of search on the job, there is no reason to believe that such equilibrium would generally exist. In fact, in models of random search on the job, it is easy to establish that no equilibrium is block recursive. However, for our model of directed search on the job, we can establish the following result.

THEOREM 2: *A Block Recursive Equilibrium exists.*

In Menzio and Shi (2009), we provide the proof of this theorem. There, we also provide a detailed characterization of the qualitative properties of the value and policy functions in a BRE. For example, we prove that the tightness function is strictly decreasing in the offered value, the profit function is strictly decreasing in the promised value, and the search policy function is increasing in the worker's current value.

Here we sketch the proof of the existence of a BRE. Take an arbitrary profit function, J, that depends on the aggregate state of the economy, ψ, only through the aggregate component of productivity, y, and not through the distribution of workers across employment states, (u, g). Given J, the tightness function θ that solves the equilibrium condition (iv) depends on ψ only though y. Intuitively, since the value of filling a vacancy in submarket x does not depend on the distribution of workers and the cost of creating a vacancy is constant, the equilibrium probability of filling a vacancy in submarket x, and hence the tightness of submarket x, must be independent from the distribution of workers.

Given θ, the search value function R that solves the equilibrium condition (i) depends on the aggregate state of the economy, ψ, only though y. Intuitively, since the probability that a worker meets a vacancy in submarket x and the value to the worker of meeting a vacancy in submarket x are both independent from the distribution of workers, so is the value of searching across the various submarkets. In turn, given R, the unemployment value function U that solves the equilibrium condition (ii) depends only on ψ only through y. Intuitively, since the unemployment benefit and the value of searching are both independent from the distribution of workers, so is the value of unemployment.

Finally, if we insert J, θ, R and U in the RHS of the equilibrium condition (iii), we obtain an update for the profit function. First, we can prove that the updated profit function, TJ, depends on the aggregate state of the economy only through the aggregate component of productivity. Intuitively, TJ does not depend on (u, g) because the output of the match in the current period, the probability that the match survives until the next production stage, and the value to the firm of the match at the next production stage are all independent from the distribution of workers. This property implies that the operator T maps the set of firm's value functions that are independent from the distribution of workers into itself. Second, we can prove that, under our assumptions on the utility and the matching function, the equilibrium operator T admits a fixed point J^*. Given J^*, we can construct the value and policy functions θ^*, R^*, m^*, U^* as described above. Taken together, the functions $(\theta^*, R^*, m^*, U^*, J^*, c^*)$ satisfy the equilibrium conditions (i)-(iv) and depend on the aggregate state of the economy ψ only through y. Hence, they constitute a Block Recursive Equilibrium.

Directed search is necessary for the existence of a BRE. In general, the probability that a firm fills a vacancy offering the lifetime utility x depends on the number of workers who apply for the vacancy (i.e. on the inverse of the tightness), and on the fraction of these applicants that would be willing to fill the vacancy if it was offered to them. When the search process is directed, workers choose where to apply for a job and, hence, the fraction of the applicants that are willing to fill the vacancy if it was offered to them is always equal to one. Therefore, when the search process is directed, the probability that a firm fills a vacancy only depends on the number of applicants. This property guarantees that, if the value to the firm from filling a vacancy is independent from the distribution of workers, then the applicant-to-vacancy ratio that equates the cost and the benefit of creating a vacancy must be independent from the distribution of workers as well. In turn, if the applicant-to-vacancy ratio does not depend on the distribution, then all of the agents' value and policy functions will not depend on it.

In contrast, when the search process is random, workers cannot choose where to apply for a job and the probability that a firm fills a vacancy offering the lifetime utility x depends both on the number of applicants and on the distribution of workers across employment states. This property implies that, even if the value to the firm from filling a vacancy did not depend on the distribution of workers, the applicant-to-vacancy ratio that equates the cost and the benefit of creating a vacancy would depend on it. In turn, if the applicant-to-vacancy ratio does depend on the distribution, so do the agents' value and policy functions. Hence, when search is random, a Block Recursive Equilibrium does not exist.

IV. Ex-ante Heterogeneous Workers

For some empirical applications, it might be necessary to use a version of the model in which workers are ex-ante heterogeneous with respect to some observable characteristic such as education or skill. Therefore, it would be reassuring to know that such a version of the model still admits a Block Recursive Equilibrium.

Consider an economy populated by workers who are ex-ante heterogeneous and denote the worker's type by i, $i = 1, 2...I$. For concreteness, assume that a worker's type affects his labor productivity. Specifically, we assume that a worker of type i produces $y + p_i + z$ units of output when employed on a job with idiosyncratic productivity z. In this economy, a submarket is indexed by the vector $\mathbf{x}$, where the i-th component of $\mathbf{x}$ is the lifetime utility offered by the firm to a worker of type i. Denote as $\theta(\mathbf{x})$ the vacancy-to-applicant ratio in submarket $\mathbf{x}$, and as $\phi(\mathbf{x})$ the distribution of applicants across different types. Denote as $U(i, y)$, $R(i, V, y)$ and $m(i, V, y)$ the value and policy functions of a worker of type i. Similarly, denote as $J(i, V, z, y)$ the profit function of a firm employing a worker of type i.

THEOREM 3: *Let $(\theta_i, R_i, m_i, U_i, J_i, c_i)$ be a BRE for the economy with ex-ante homogeneous workers of type i. There exists a BRE for the economy with ex-ante heterogeneous workers in which $\theta(\underline{x}, ...x_i, ...\underline{x}) = \theta_i(x_i)$, $R(i, V, y) = R_i(V, y)$, $m(i, V, y) = (\underline{x}, ...m_i(V, y), ...\underline{x})$, $U(i, y) = U_i(y)$, $J(i, V, y, z) = J_i(V, y, z)$, and $c(i, V, y, z) = c_i(V, y, z)$.*

Theorem 3 establishes the existence of a BRE for the economy with ex-ante heterogeneous workers. This BRE is the stratification of the BRE for the economy with ex-ante homogeneous workers of type $i = 1, 2, ...I$. Specifically, in this BRE, the value and policy functions of a worker of type i are the same as the value and policy functions of a worker in the BRE for an economy with ex-ante homogeneous workers of type i. Similarly, the value and policy functions of a firm employing a worker of type i are the same as the value and policy functions of a firm in an economy with ex-ante homogeneous workers of type i.

There is a simple intuition behind Theorem 3. In an economy with ex-ante heterogeneous workers, the labor market becomes endogenously segmented by worker's type, in the sense that any of the active submarkets is visited by one type of worker only. Given that the labor market is segmented, it is clear that the existence of a BRE in an economy with ex-ante homogeneous workers implies the existence of a BRE in an economy with ex-ante heterogeneous workers. To understand why the labor market is segmented, suppose that there is a submarket $\mathbf{x} = (x_1, x_2)$ that is visited by workers of type 1 and 2. Also suppose that the profits of the firm are higher if it fills the vacancy with a worker of type 1 rather than 2. Then, the firm's expected profits from filling a vacancy must be greater in submarket $\mathbf{x}' = (x_1, \underline{x})$ than in submarket $\mathbf{x}$, because no worker of type 2 applies for a job in submarket $\mathbf{x}'$. In turn, this implies that the vacancy-to-applicant ratio in submarket $\mathbf{x}'$ must be greater than in submarket $\mathbf{x}$. But if $\theta(\mathbf{x}')$ is greater than $\theta(\mathbf{x})$, a worker of type 1 is better off applying for a job in submarket $\mathbf{x}'$ rather than in submarket $\mathbf{x}'$. This contradicts the conjecture that submarket $\mathbf{x}$ is visited by workers of type 1 and 2.

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Proof of Theorem 3

In order to keep the exposition as simple as possible, we prove the theorem for $I = 2$. The proof for $I \geq 3$ is similar.

Theorem 1 guarantees that there exists at least one BRE for the economy with ex-ante homogeneous workers of type i, $i = 1, 2$. Let $(\theta_i, R_i, m_i, U_i, J_i, c_i)$ be any BRE for the economy with ex-ante homogeneous types of type i. For all $y \in Y$ such $\theta_i(\underline{x}, y) \geq \theta_{-i}(\underline{x}, y)$, denote as $\underline{x}_i(y)$ the unique solution for x to the equation $\theta_i(x, y) = \theta_{-i}(\underline{x}, y)$.

Let $(\theta, \phi, R, m, U, J, c)$ denote a candidate BRE for the economy with ex-ante heterogeneous agents. Fix an arbitrary $y \in Y$ and, without loss in generality, suppose that $\theta_1(\underline{x}, y) \geq \theta_2(\underline{x}, y)$. Now, choose the market tightness function, θ, and the distribution of applicants, ϕ, as follows. For all $(x_1, x_2) \in X \times (\underline{x}, \overline{x}]$, let $\theta(x_1, x_2) = \min\{\theta_1(x_1), \theta_2(x_2)\}$ and let $\phi(x_1, x_2) = (1, 0)$ if $\theta_i(x_i) < \theta_{-i}(x_{-i})$ and $\phi(x_1, x_2) = (0, 1)$ if $\theta_i(x_i) > \theta_{-i}(x_{-i})$. For all $x_1 \in X$, let $\theta(x_1, \underline{x}) = \theta_1(x_1)$ and $\phi(x_1, \underline{x}) = (1, 0)$. That is, for $x_2 > \underline{x}$, we set the tightness of submarket (x_1, x_2) to the minimum between the tightness of submarket x_1 in an economy with ex-ante heomogeneous workers of type 1, and the tightness of submarket x_2 in an economy with ex-ante homogeneous workers of type 2. For $x_2 = \underline{x}$, we set the tightness of submarket (x_1, x_2) to be the minimum between the tightness of submarket x_1 in an economy with ex-ante heomogeneous workers of type 1. Notice that, in the previous expressions, we have omitted the dependance of various functions on y. We shall do the same in the remainder of the proof.

Next, choose the search value function, R, the search policy function, m, the profit function, J, and the unemployment value function, U, as follows. For all $V \in X$ and $i = 1, 2$, let $R(i, V) = R_i(V)$. For all $V \in X$, let $m(1, V) = (m_1(V), \underline{x})$ and $m(2, V) = (\underline{x}, m_2(V))$. For all $(V, z) \in X \times Z$ and $i = 1, 2$, let $J(i, V, z) = J_i(V, z)$ and $c(i, V, z) = c_i(V, z)$. For $i = 1, 2$, let $U(i, y) = U_i(y)$. In words, the lifetime utility of worker i in an economy with ex-ante heterogeneous workers is set equal to the lifetime utility of a worker in an economy with ex-ante homogeneous workers of type i. Similarly, the profits of a firm from employing a worker i in an economy with ex-ante heterogeneous workers are equal to the profits of a firm in an economy with ex-ante homogeneous workers of type i.

Now, we verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium conditions (i)-(iv) and, hence, it is a BRE for the economy with ex-ante heteorogeneous workers. First, we verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium condition (iv). Consider a submarket $(x_1, x_2) \in X^2$ such that either $\theta_1(x_1) \leq \theta_2(x_2)$ or $x_2 = \underline{x}$. In this case, we have

$$\begin{aligned} &q(\theta(x_1, x_2)) \textstyle\sum_{i=1}^{2} \phi_i(x_1, x_2) J(i, x_i, z_0) = q(\theta_1(x_1)) J_1(x_1, z_0) \leq k, \\ &\theta(x_1, x_2) = \theta_1(x_1) \geq 0, \end{aligned} \tag{1}$$

with complementary slackness. The first line in (1) denotes as $\phi_i(x_1, x_2)$ the i-th component of the vector $\phi(x_1, x_2)$ and makes use of the equations $\phi(x_1, x_2) = (1, 0)$, $J(i, x_i, z_0) = J_i(x_i, z_0)$, the second line makes use of the equation $\theta(x_1, x_2) = \theta_1(x_1)$, and both lines use the fact that $(\theta_1, R_1, m_1, U_1, J_1, c_1)$ is a BRE. The inequalities in (1) imply that the equilibrium condition (iv) is satisfied for all $(x_1, x_2) \in X^2$ such that either $\theta_1(x_1) \leq \theta_2(x_2)$ or $x_2 = \underline{x}$. Using a similar argument, we can prove that the equilibrium condition (iv) is satisfied for all other submarkets.

Next, we verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium condition (i). Consider an arbitrary $x_1 \in X$. For all $x_2 \in (\underline{x}, \overline{x}]$, the tightness of submarket (x_1, x_2) is $\theta(x_1, x_2) \leq \min\{\theta_1(x_1), \theta_2(x_2)\}$. For $x_2 = \underline{x}$, the tightness of submarket (x_1, x_2) is $\theta(x_1, x_2) = \theta_1(x_1)$. Since these results hold for an arbitrary x_1, we have that

$$\begin{aligned} &\max_{(x_1, x_2) \in X^2} p(\theta(x_1, x_2)) (x_1 - V) \\ &= \max_{x_1 \in X} p(\theta_1(x_1))(x_1 - V) \\ &= R_1(V) = R(1, V), \end{aligned} \tag{2}$$

where the third line makes use of the fact that $(\theta_1, R_1, m_1, U_1, J_1, c_1)$ is a BRE. Moreover, we have that

$$p(\theta(m(1, V)))\,(m_1(1, V) - V) = p(\theta_1(m_1(V)))\,(m_1(V) - V)\,, \tag{3}$$

where $m_1(1, V)$ denotes the first component of the vector $m(1, V)$. Taken together, equations (2) and (3) imply that the equilibrium condition (i) is satisfied for all $V \in X$ and $i = 1$. Using a similar argument, we can prove that the equilibrium condition (i) is satisfied also for $i = 2$. Moreover, notice that the distribution of applicants ϕ across types is consistent with the worker's equilibrium search strategy m.

Finally, it is straightforward to verify that $(\theta, \phi, R, m, U, J, c)$ satisfies the equilibrium conditions (ii) and (iii). Hence, $(\theta, \phi, R, m, U, J, c)$ is a BRE for the economy with ex-ante heterogeneous agents.