All-Pay Contests*

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August 2008

Abstract

The paper studies a class of games, "All-Pay Contests", which captures general asymmetries and sunk investments inherent in scenarios such as lobbying, competition for market power, labor-market tournaments, and R&D races. Players compete for one of several identical prizes by choosing a score. Conditional on winning or losing, it is weakly better to do so with a lower score. This formulation allows for differing production technologies, costs of capital, prior investments, attitudes towards risk, and conditional and unconditional investments, among others. I provide a closed-form formula for players' equilibrium payoffs, and analyze player participation. A special case of contests is multi-prize, complete-information all-pay auctions.

*I am indebted to Ed Lazear and Ilya Segal for their continuous guidance and encouragement. I thank Jeremy Bulow for many beneficial discussions. I thank Larry Samuelson and four referees for very detailed comments. I thank Julio González-Díaz, Matthew Jackson, Sunil Kumar, Jonathan Levin, Stephen Morris, Marco Ottaviani, Kareen Rozen, Yuval Salant, Andrzej Skrzypacz, Balázs Szentes, Michael Whinston, Robert Wilson, Eyal Winter, and seminar participants at Chicago GSB, CSIO-IDEI, Cornell, Hebrew University, Northwestern, NYU Stern, Penn State, Stanford, Tel-Aviv, Toronto, and Yale for helpful comments and suggestions.

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1 Introduction

In many settings, economic agents compete by making irreversible investments before the outcome of the competition is known. Lobbying activities, research and development races, and competitions for promotions, to name a few, all have this property.

This type of competition has been widely studied in the literature. In the classic all-pay auction with complete information (henceforth: all-pay auction), for example, rivals incur a cost of bidding that is the same whether they win or lose, but may differ in their valuation for winning the single prize. The all-pay auction has been used to model rent-seeking and lobbying activities (Hillman & Samet (1987), Hillman & Riley (1989), Baye, Kovenock & de Vries (1993)), competitions for a monopoly position (Ellingsen (1991)), waiting in line (Clark & Riis (1998)), sales (Varian (1980)), and R&D races (Dasgupta (1986)). Variations of the all-pay auction have been used to model competitions for multiple prizes (Clark & Riis (1998) and Barut & Kovenock (1998)), the effect of lobbying caps (Che & Gale (1998, 2006) and Kaplan & Wettstein (2006)), and R&D races with endogenous prizes (Che & Gale (2003)). While this literature has produced interesting results, the models considered are often restrictive in some or all of the following dimensions: the types of asymmetries across players, the number of prizes, the number of players, and the degree of irreversibility of the investments.

The goal of this paper is to better understand competitions in which contestants are asymmetrically positioned and make irreversible investments. To this end, I investigate all-pay contests (henceforth: contests). In a contest, each player chooses a costly "score", and the players with the highest scores obtain one prize each (relevant ties can be resolved using any tie-breaking rule). Thus, ex-post, each player can be in one of two states: winning or losing. Conditional on winning or losing, a player's payoff decreases weakly and continuously with his chosen score; choosing a higher score entails a player-specific cost, which may differ across the two states. The primitives of the contest are commonly known. This captures players' knowledge of the asymmetries among them. Consequently, equilibrium payoffs represent players' "economic rents", in contrast to "information rents" that arise in models of competition with private information.[1] Contests are defined in Section 2.

The generality of players' cost functions allows for differing production technologies, costs of capital, and prior investments, among others. Moreover, contests allow for non-ordered cost functions. These arise when different competitors are disadvantaged relative to others in different regions of the competition (see the example of Section 1.1 below). In addition, state-dependent costs accommodate both sunk and conditional investments, player-specific risk attitudes, and

[1] Such models typically assume ex-ante identical players (for example, Moldovanu & Sela (2001, 2006) and Kaplan, Luski, Sela, & Wettstein (2002)). An exception is the work by Parreiras & Rubinchik (2006), who allow for asymmetry between players and provide a partial characterization of equilibrium.

player- and score-dependent valuations for a prize.[2]

When all investments are unconditional, each player is characterized by his valuation for a prize, which is the payoff difference between the two states, and a weakly increasing, continuous cost function that determines his cost of choosing a score independently of the state. I refer to such contests as *separable contests*. Separable contests nest many models of competition that assume a deterministic relation between effort and prize allocation.[3] For example, the models of Che & Gale (2006) and Kaplan & Wettstein (2006) are two-player, single-prize separable contests.[4] Single- and multi-prize all-pay auctions are separable contests with linear costs, in which asymmetries among players are captured only by differences in valuations for a prize.

Section 3 begins the analysis by identifying an unambiguous ranking of players. Such a ranking is not immediately obvious, because players' costs may not be ordered. Players are ranked in decreasing order of their *reach*, where a player's reach is the highest score he can choose without obtaining a negative payoff if he wins a prize with certainty.

The key result of the paper is Theorem 1, which provides a full characterization of players' expected equilibrium payoffs in a "generic" contest for m prizes. Expected payoffs are determined by reaches, *powers*, and the *threshold*. The threshold is the reach of player $m+1$. A player's power equals his payoff from winning a prize with certainty when choosing a score equal to the threshold. Theorem 1 shows that under "generic" conditions - checked using players' powers - each player's expected payoff equals the higher of his power and zero. Thus, a generic contest has the same payoffs in all equilibria. Theorem 1 also shows that the number of players who obtain strictly positive expected payoffs equals the number of prizes. The derivation of the payoff characterization does not rely on solving for an equilibrium.

The payoff result implies that a player's expected payoff does not depend on his cost when he loses. For example, a player's expected payoff in a modified complete-information first-price auction in which he pays a strictly positive fraction of his bid if he loses does not depend on the

[2]González-Díaz (2007) also allows for non-linear, state-dependent costs, but accommodates only a single prize and ordered costs. His techniques and results do not generalize to non-ordered costs or multiple prizes.

[3]Other models assume a probabilistic relation between players' efforts and prize allocation. The classic examples are Tullock's (1980) "lottery model", in which players are symmetric and each player's probability of winning the single prize is proportional to the player's share of the total expenditures, and Lazear & Rosen's (1981) two-player tournaments. Recent contributions to this large, single-prize literature that accommodate a degree of asymmetry include Cornes & Hartley (2005) and Szymanski & Valletti (2005). The analysis of probabilistic models typically focuses on pure-strategy equilibria by using first-order conditions. Functional form assumptions are made "[...] to ensure the existence of pure-strategy equilibria and first-order conditions characterizing these equilibria" (Szymanski & Valletti (2005)).

[4]Although Che & Gale (2006) have $N \geq 2$ players, they assume strictly ordered costs so only two players participate in equilibrium.

size of the fraction. When the fraction equals 1 for all players, we have an all-pay auction.

Section 3.2 discusses contests that are not generic. In such contests, players' payoffs in at least one equilibrium are specified by the payoff characterization, but payoffs in other equilibria may be different. Perturbing a contest that is not generic leads to a generic contest.

Section 4 examines equilibrium participation. Participation is related to the ordering of players' costs. Theorem 2 shows that when players' costs (appropriately normalized) are strictly ordered, at most $m + 1$ players participate in any equilibrium. This is why precisely $m + 1$ players participate in an all-pay auction with distinct valuations. In contrast, non-ordered costs may lead more than two players to participate even when there is only one prize.[5]

Section 5 concludes by briefly discussing implications of the analysis for rent dissipation and comparative statics. The Appendix contains an example of a contest with multiple equilibria, the proofs of Corollary 1 and Theorem 2, and a technical lemma.

I begin with an example that illustrates the payoff result and some of its implications.

1.1 An Example

Three risk-neutral firms compete for one monopoly position allocated by a government official. Each firm chooses how much to invest in lobbying activities, and this investment leads to a score, which can be interpreted as the amount of influence the firm has achieved over the government official. The firm with the highest score obtains the monopoly position, which carries a monetary value of 1, but all firms pay the lobbying costs associated with their respective scores. Firms 1 and 2 have better "lobbying technologies" than firm 3, because they have better lobbyists, are located closer to the government official, or have lower costs of capital. Firm 3 has an initial advantage, due to prior investments or reputation.

Figure 1 depicts players' cost functions, where player $i \in \{1, 2, 3\}$ corresponds to firm i. Firm 3's initial advantage is captured by its initial marginal cost, $\gamma \geq 0$, which is low relative to the other firms' marginal costs. In contrast, firm 3's cost for high scores is high relative to those of the other two firms. Thus, players' costs are not ordered. Cost functions are commonly known, and relevant ties are broken randomly.

The costs of choosing 1 for players 1,2, and 3 are $K < 1$, 1, and $L > 1$, respectively. Consequently, players 2 and 3 would never choose a score higher than 1, since that would cost them more than the value of the prize. Player 1 can therefore guarantee himself a payoff

[5] When more than two players participate, general asymmetric costs significantly complicate equilibrium analysis. For this reason, a non-crossing property is the standard assumption in the literature. The few papers that analyze mixed-strategy equilibria with more than two participants make assumptions, in addition to non-crossing, that lead to limited asymmetry among participating players (see Baye et al. (1996), Clark & Riis (1998), and González-Díaz (2007), who showed that the analysis of Baye et al. (1996) generalizes to non-linear costs).

arbitrarily close to $1 - K$ by choosing a score slightly higher than 1, so $1 - K$ is a lower bound on his expected payoff in any equilibrium.

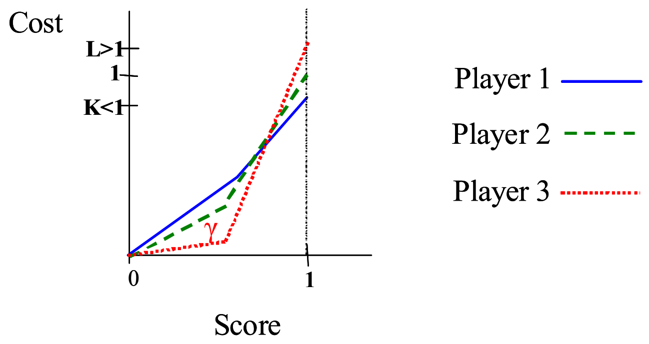

Figure 1: Players' costs

Player 1 does not, however, choose scores greater or equal to 1 with certainty in equilibrium, since players 2 and 3 would best-reply by choosing scores lower than 1, in which case player 1 would be better off choosing a lower score. In fact, player 1 must employ a mixed strategy in any equilibrium. It may therefore seem plausible that such a strategy could give player 1 an equilibrium payoff higher than $1 - K$.

Theorem 1 shows that the expected equilibrium payoff of player 1 is *exactly* $1-K$. Similarly, players 2 and 3 can guarantee themselves no more than 0; the payoff characterization shows that this is exactly their equilibrium payoff. This implies that aggregate equilibrium expenditures equal K.

For low, strictly positive values of γ, all three players must participate (invest) in equilibrium. This is shown in Section 4. Each player contributes to aggregate expenditures and wins the prize with strictly positive probability. This participation behavior results from the non-ordered nature of players' cost functions.[6]

Now consider a variant of the contest, in which player 3 has 0 marginal cost up to a score whose cost for player 1 is at least 1. This represents a very large initial advantage for player 3. In this case, there is a pure-strategy equilibrium in which player 3 wins with certainty and no player invests. Thus, it may be that no player invests in a contest for a valuable prize.

Regardless of the value of γ, precisely one player receives a strictly positive expected payoff. This too follows from the payoff characterization, since there is only one prize.

Section 5 discusses the effects of changes in competition structure. For example, the addition of player 3 from Figure 1 to a contest that includes only players 1 and 2 changes neither expected payoffs nor expected aggregate expenditures, but changes individual expenditures for low, strictly

[6]Unlike in the all-pay auctions of Baye et al. (1996), participation by more than two players in a contest for one prize does not rely on players' valuations being identical.

positive values of γ, because all three players participate. Thus, the addition of a player may change equilibrium behavior, without changing players' payoffs or aggregate expenditures. In contrast, lowering the prize's value can lead to a positive payoff for player 3, making him the only player who obtains a positive expected payoff.

2 The Model

In a *contest*, n players compete for m homogeneous prizes, $0 < m < n$. The set of players $\{1, \ldots, n\}$ is denoted by N. Players compete by each choosing a score, simultaneously and independently. Player i chooses a score $s_i \in S_i = [a_i, \infty)$, where $a_i \geq 0$ is his *initial score*. Positive initial scores capture starting advantages, or "head starts", without allowing players to choose lower scores. This may eliminate equilibria involving weakly dominated strategies (see Example 4 in Siegel (2007)). Each of the m players with the highest scores wins one prize. In case of a relevant tie, any procedure may be used to allocate the tie-related prizes among the tied players.

Player i has preferences over lotteries whose outcomes are pairs (s_i, W_i), where s_i is the player's score and W_i indicates whether he obtains a prize ($W_i = 1$) or not ($W_i = 0$). These preferences are represented by a Bernoulli utility function. Because W_i equals 0 or 1, this function can be written as $W_i v_i(s_i) - (1 - W_i) c_i(s_i)$, where $v_i : S_i \to \mathbb{R}$ is *player i's valuation for winning* and $c_i : S_i \to \mathbb{R}$ is *player i's cost of losing*. The primitives of the contest are commonly known. Thus, given a profile of scores $s = (s_1, \ldots, s_n)$, $s_i \in S_i$, *player i's payoff* is

$$u_i(s) = P_i(s) v_i(s_i) - (1 - P_i(s)) c_i(s_i)$$

where $P_i : \times_{j \in N} S_j \to [0,1]$, *player i's probability of winning*, satisfies

$$P_i(s) = \begin{cases} 0 & \text{if } s_j > s_i \text{ for } m \text{ or more players } j \neq i \\ 1 & \text{if } s_j < s_i \text{ for } N - m \text{ or more players } j \neq i \\ \text{any value in } [0,1] & \text{otherwise} \end{cases}$$

such that $\sum_{j=1}^{n} P_j(s) = m$.

Note that a player's probability of winning depends on all players' scores, but his valuation for winning and cost of losing depend only on his chosen score.

I make the following assumptions.

A1 *v_i and $-c_i$ are continuous and non-increasing.*

A2 *$v_i(a_i) > 0$ and $\lim_{s_i \to \infty} v_i(s_i) < c_i(a_i) = 0$.*

A3 $c_i(s_i) > 0$ *if* $v_i(s_i) = 0$.

Assumption A1 means that, conditional on winning or losing, a lower score is weakly preferable.[7] This represents an "all-pay" component. Assumption A2 means that with the initial score winning is better than losing, so prizes are valuable, but losing with the initial score is preferable to winning with sufficiently high scores. Assumption A3 means that if winning with score s_i is as good as losing with the initial score, then winning with score s_i is strictly better than losing with score s_i. This last condition stresses the all-pay nature of contests. It is not satisfied by complete-information first-price auctions, for example, since a player pays nothing if he loses, and is therefore indifferent between losing and winning with a bid that equals his valuation for the prize. But the condition *is* met when an all-pay element is introduced, e.g., when every bidder pays some strictly positive fraction of his bid whether he wins or not, and only the winner pays the balance of his bid.

The formulation allows the difference between a player's valuation for winning and his cost of losing to depend on his chosen score. For example, in a competition for promotions in which a higher score is achieved by investing in managerial skills, such skills are costly to acquire and may increase the value associated with a promotion. Or, it may be that the value of the prize for player i is fixed at V_i but some costs are only borne if the player wins, so his costs are c_i^W when he wins and c_i^L when he loses. In this case, $v_i(s_i) = V_i - c_i^W(s_i)$ and $c_i = c_i^L$, so $u_i(s) = P_i(s)\left(V_i - c_i^W(s_i)\right) - (1 - P(s))\, c_i^L(s_i)$. When thinking about the outcome of a contest in monetary terms, contests can capture players' risk attitudes. In the previous setting, for example, we can let $v_i(s_i) = f\left(V_i - c_i^W(s_i)\right)$ and $c_i = f\left(c_i^L\right)$ for some strictly increasing f such that $f(0) = 0$.

One subclass of contests that is of particular interest is *separable contests*. In a separable contest, every player i's preferences over lotteries with outcomes (s_i, W_i) depend only on the marginal distributions of the lotteries.[8] This implies that the effect of winning or losing on a player's Bernoulli utility is additively separable from that of the score, i.e., $v_i(s_i) = V_i - c_i(s_i)$ and $u_i(s) = P_i(s)\, V_i - c_i(s_i)$ for $V_i = v_i(a_i) > 0$.[9] If we interpret payoffs as money, the value

[7] The "non-increasing" part of Assumption A1 can be relaxed to requiring that (i) v_i be non-increasing where it is non-negative, i.e., if $y > x$, then $v_i(x) \geq 0$ and $v_i(y) \geq 0$ imply $v_i(x) \geq v_i(y)$, (ii) v_i cross 0 once from above, i.e., if $y > x$, $v_i(x) < 0$ implies $v_i(y) < 0$, and (iii) $c_i \geq 0$ and if $y > x$, then $c_i(x) > 0$ implies $c_i(y) > 0$.

[8] For example, in a separable contest player i is indifferent between a lottery in which $(a_i, 1)$ occurs with probability $\frac{1}{2}$ and $(s_i, 0)$ occurs with probability $\frac{1}{2}$, and a lottery in which $(a_i, 0)$ occurs with probability $\frac{1}{2}$ and $(s_i, 1)$ occurs with probability $\frac{1}{2}$ (for any $s_i \geq a_i$), because the marginal distribution of the first coordinate is the same in both lotteries, and the marginal distribution of the second coordinate is the same in both lotteries.

[9] Indeed, if player i's Bernoulli utility is not additively separable, then $v_i(a_i) + c_i(a_i) \neq v_i(s_i) + c_i(s_i)$ for some $s_i > a_i$. But then the player is not indifferent between the two lotteries described in the previous footnote, even though their marginal distributions are the same. The converse is also true: if the player i's Bernoulli utility

$c_i(s_i)$ could be thought of as player i's cost of choosing score s_i, which does not depend on whether he wins or loses, and V_i could be thought of as player i's *valuation for a prize*, which does not depend on his chosen score. All expenditures are unconditional, and players are risk neutral. The example of Section 1.1 depicts a separable contest with three players and non-linear costs. The lobbying games of Kaplan & Wettstein (2006) and Che & Gale (2006) are two-player separable contests. Separable contests with linear costs are the single- and multi-prize complete-information all-pay auction (Hillman & Samet (1987), Hillman & Riley (1989), Clark & Riis (1998)).[10]

3 Payoff Characterization

The following concepts are key in analyzing the payoffs of players in equilibrium.

Definition (i) Player i's *reach* r_i is the highest score at which his valuation for winning is 0. That is, $r_i = \max\{s_i \in S_i | v_i(s_i) = 0\}$. Re-index players in (any) decreasing order of their reach, so that $r_1 \geq r_2 \geq \ldots \geq r_n$.

(ii) Player $m+1$ is the *marginal player.*

(iii) The *threshold* T of the contest is the reach of the marginal player: $T = r_{m+1}$.

(iv) Players i's *power* w_i is his valuation for winning at the threshold. That is, $w_i = v_i(\max\{a_i, T\})$. In particular, the marginal player's power is 0.

In a separable contest, a player's reach is the highest score he can choose by expending no more than his valuation for a prize. In the example of Section 1.1, players are indexed in decreasing order of their reach, player 2 is the marginal player, and the threshold is 1. Player 1's power is $1 - K > 0$, player 2's power is 0, and player 3's power is $1 - L < 0$. In an m-prize all-pay auction, player i is the player with the ith highest valuation, and his power equals his valuation less that of the marginal player.

Theorem 1 below characterizes players' equilibrium payoffs in contests that meet the following two conditions.

Generic Conditions (i) *Power Condition* - The marginal player is the only player with power 0. (ii) *Cost Condition* - The marginal player's valuation for winning is strictly decreasing at the

is additively separable, it is immediate that his preferences over lotteries with outcomes (s_i, W_i) depend only on the marginal distributions of the lotteries.

[10] Formally, $v_i(s_i) = V_i - s_i, c_i(s_i) = s_i, a_i = 0$, and ties are resolved by randomizing uniformly, where V_i is bidder i's valuation for a prize.

threshold, i.e., for every $x \in [a_{m+1}, T)$, $v_{m+1}(x) > v_{m+1}(T) = 0$.[11]

I refer to a contest that meets the Generic Conditions as a *generic contest*. The separable contest in the example of Section 1.1 is generic, because the marginal player's costs are strictly increasing at the threshold and only he has power 0. An m-prize all-pay auction meets the Cost Condition because costs are strictly increasing. If the Power Condition is met, i.e., the valuation of the marginal player is different from those of all other players, the all-pay auction is generic. Contests that do not meet the Generic Conditions can be perturbed slightly to meet them. Perturbing the marginal player's valuation for winning around the threshold leads to a contest that meets the Cost Condition. Doing the same for all players with power 0 generates a contest that meets the Power Condition. Note that in a generic contest, players in $N_W = \{1, \ldots m\}$ ("winning players") have strictly positive powers, and players in $N_L = \{m+1, \ldots, n\}$ ("losing players") have non-positive powers.

I now state the main result of the paper.

Theorem 1 *In any equilibrium of a generic contest, the expected payoff of every player equals the maximum of his power and* 0*.*

An immediate implication of Theorem 1 is that in a generic contest players in N_W have strictly positive expected payoffs and players in N_L have expected payoffs of zero. Equivalently, a player obtains a strictly positive expected payoff in a generic contest if and only if his reach is strictly higher than the threshold.

Players' equilibrium strategies may be mixed, so players in N_W may obtain a prize with probability smaller than 1, and players in N_L may obtain a prize with strictly positive probability. It is only *expected payoffs* that are positive for players in N_W, and 0 for players in N_L. In the example of Section 1.1, $N_W = \{1\}$ and $N_L = \{2, 3\}$. The contest is generic, so player 1's payoff is $1 - K$, and those of players 2 and 3 are 0. In an m-prize generic all-pay auction in which player i's value is V_i, the payoff of every player i is $\max\{V_i - V_{m+1}, 0\}$.[12]

I use the following notation in the proof of Theorem 1. $P_i(\cdot)$, player i's probability of winning, and $u_i(\cdot)$, player i's utility, are expanded to mixed strategies. A mixed strategy G_i of player i is a cumulative probability distribution that assigns probability 1 to his set of pure strategies S_i. When a strategy profile $G = (G_1, \ldots, G_n)$ is specified, $P_i(x)$ is shorthand for player i's probability of winning when he chooses $x \geq a_i$ with certainty and all other players play according to G, and similarly for $u_i(x)$. For an equilibrium $(G_1, \ldots, G_n)$, denote by $u_i = u_i(G_i)$ *player i's equilibrium payoff*. Note that in equilibrium best responses are chosen with probability 1. The

[11] In a separable contest, because $v_{m+1}(x) = V_{m+1} - c_{m+1}(x)$, the cost condition is that for every $x \in [a_{m+1}, T)$, $c_{m+1}(x) < c_{m+1}(T) = V_{m+1}$.

[12] Players' payoffs in a multi-prize all-pay auction in which all players have different valuations were first derived by Clark & Riis (1998).

phrase "player i beats player j" refers to player i choosing a strictly higher score than player j does. For a set I, denote by $|I|$ the cardinality of I.

Proof of Theorem 1. Choose a generic contest and an equilibrium $G = (G_1, \ldots, G_N)$ of the contest.

Least Lemma *A player's expected payoff in G is at least the maximum of his power and 0.*

Proof. Every player i can guarantee himself a payoff of 0 by choosing his initial score, a_i (recall that $v_i(a_i) > 0$ and $c_i(a_i) = 0$). It therefore suffices to consider players with strictly positive power, all of whom are in N_W. In equilibrium, no player chooses scores higher than his reach with a strictly positive probability, since choosing such scores leads to a negative payoff (by assumptions A1 and A3). So, by choosing $\max\{a_i, T+\varepsilon\}$ for $\varepsilon > 0$, a player i in N_W beats all $N - m$ players in N_L with certainty. This means that for every player i in N_W

$$u_i \geq v_i(\max\{a_i, T+\varepsilon\}) \underset{\varepsilon \to 0}{\to} v_i(\max\{a_i, T\}) = w_i$$

by continuity of v_i. ■

Tie Lemma *Suppose that in G two or more players have an atom at a score x, i.e., choose x with strictly positive probability. Then players who have an atom at x either all win with certainty or all lose with certainty when choosing x.*

Proof. Denote by N' the set of players who have an atom at x, with $|N'| \geq 2$. Denote by E the strictly positive-probability event that all players in N' choose x. Denote by $D \subseteq E$ the event in which a relevant tie occurs at x, i.e., the event in which m' prizes are divided among the $|N'|$ players in N', with $1 \leq m' < |N'|$. Suppose D has strictly positive probability. Then, conditional on D, at least one player i in N' can strictly increase his probability of winning to 1 by choosing a score slightly higher than x, regardless of the tie-breaking rule. Since i chooses x with strictly positive probability, $x \leq r_i$ so $v_i(x) > -c_i(x)$ (by assumptions A1-A3). Thus, by continuity of v_i and c_i, player i would be strictly better off by choosing a score slightly higher than x. Therefore, D has probability 0. This implies that $P(E) = P(E^L) + P(E^W)$, where $E^L \subseteq E$ is the event that at *least* m players in $N \backslash N'$ choose scores strictly higher than x, $E^W \subseteq E$ is the event that at *most* $m - |N'|$ players in $N \backslash N'$ choose scores strictly higher than x, and $P(A)$ denotes the probability of event A. By independence of players' strategies, either E^L or E^W have probability 0, otherwise D would have strictly positive probability. Suppose that $P(E) = P(E^L)$. Independence of players' strategies now implies that, without conditioning on E, at least m players in $N \backslash N'$ choose scores strictly higher than x with probability 1, so $P_i(x) = 0$ for every player i in N'. Similarly, if $P(E) = P(E^W)$ then $P_i(x) = 1$ for every player i in N'. ■

Several players may have an atom at the same score in equilibrium; the Tie Lemma only rules out ties in which at least one player wins with a strictly positive probability that is less than 1. That no such ties arise in equilibrium helps establish which players have an expected payoff of 0.

Zero Lemma *In G, at least $n-m$ players have best responses with which they win with probability 0 or arbitrarily close to 0. These players have an expected payoff of at most 0.*

Proof. Denote by J a set of some $m+1$ players. Denote by $\tilde{S}$ the union of the best-response sets of the players in J, and by $s_{\inf}$ the infimum of $\tilde{S}$. Consider three cases: (1) two or more players in J have an atom at $s_{\inf}$, (2) exactly one player in J has an atom at $s_{\inf}$, and (3) no player in J has an atom at $s_{\inf}$.

Case 1: Denote by $N' \subseteq J$ the set of players in J who have an atom at $s_{\inf}$. It cannot be that $P_i(s_{\inf}) = 1$ for every player i in N': because any player in $J \backslash N'$ chooses scores strictly higher than $s_{\inf}$ with probability 1, even if the players in $N \backslash J$ choose scores strictly lower than $s_{\inf}$ with probability 1, only

$$m - |(J \backslash N')| = m - (m + 1 - |N'|) = |N'| - 1 > 0$$

prizes are divided among the $|N'|$ players in N'. Thus, the Tie Lemma shows that $P_i(s_{\inf}) = 0$ for every player i in N'.

Case 2: Denote by i the only player in J with an atom at $s_{\inf}$. $P_i(s_{\inf}) = 0$, since all m players in $J \backslash \{i\}$ choose scores strictly higher than $s_{\inf}$ with probability 1.

In cases (1) and (2), $P_i(s_{\inf}) = 0$ for some player $i \in J$ who has an atom at $s_{\inf}$, so $s_{\inf}$ is a best response for this player at which he wins with probability 0.

Case 3: By definition of $s_{\inf}$, there exists a player i in J with best responses $\{x_n\}_{n=1}^{\infty}$ that approach $s_{\inf}$. Since $1 \geq 1 - P_i(x_n) \geq \prod_{j \in J \backslash \{i\}} (1 - G_j(x_n))$, no player has an atom at $s_{\inf}$, and G is right-continuous, as n tends to infinity $P_i(x_n)$ approaches 0.

Because J was a set of any $m+1$ players, at least $n-m$ players have best responses with which they win with probability 0 or arbitrarily close to 0, and therefore have an expected payoff of at most 0. ■

The Least Lemma, the Tie Lemma, and the Zero Lemma hold regardless of the Generic Conditions. The Least Lemma and the Power Condition show that the m players in N_W have strictly positive expected payoffs. Therefore, the Least Lemma and the Zero Lemma imply that under the Power Condition the $n-m$ players in N_L obtain expected payoffs of 0. Using this fact, I show that players in N_W obtain at most their power.

Threshold Lemma *The players in N_W have best responses that approach or exceed the threshold, and therefore have an expected payoff of at most their power.*

Proof. By the Power Condition, players in $N_L \backslash \{m+1\}$ have strictly negative powers. Their reaches, and therefore the supremum of their best responses, are strictly below the threshold. Consequently, there is some $s_{\text{sup}} < T$ such that $G_i(x) = 1$ for every player i in $N_L \backslash \{m+1\}$ and every score $x > s_{\text{sup}}$. This implies that every player i in N_W chooses scores that approach or exceed the threshold, i.e., has $G_i(x) < 1$ for every $x < T$. Otherwise, for some s in (s_{sup}, T), $G_i(s) = 1$ for all but at most $m-1$ players in $N \backslash \{m+1\}$. But then the marginal player could win with certainty by choosing a score in $(\max\{a_{m+1}, s\}, T)$ (note that $a_{m+1} < T$); because of the Cost Condition, this would give him a strictly positive payoff, a contradiction (recall that the marginal player is in N_L and therefore has an expected payoff of 0).
Take a player i in N_W. Because $G_i(x) < 1$ for every $x < T$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of best responses for player i that approach some $z_i \geq T$. Since x_n is a best response for player i, who has a strictly positive payoff by the Least Lemma and the Power Condition, $v_i(x_n) > 0$. So, by assumptions A1 and A2, $v_i(x_n) > -c_i(x_n)$. By continuity of v_i, we have

$$u_i = u_i(x_n) = P_i(x_n)\, v_i(x_n) - (1 - P_i(x_n))\, c_i(x_n) \leq v_i(x_n) \underset{x_n \to z_i}{\to} v_i(z_i) \leq v_i(T) = w_i$$

so every player in N_W obtains at most his power. ■

The Least Lemma and the Threshold Lemma, which relies on the Generic Conditions, show that players in N_W have expected payoffs equal to their power. We have seen that players in N_L have expected payoffs of 0. Since $N_L \cup N_W = N$, the expected payoff of every player equals the maximum of his power and 0. ■

3.1 Discussion of the Payoff Characterization

Equilibrium payoffs in generic contests depend only on players' valuations for winning at the threshold, even though the equilibria generally depend on players' valuations for winning and costs of losing at all scores up to the threshold.[13] From an applied perspective, only the reach of each player and valuations for winning at a single score, the threshold, need to be computed. In particular, players' costs of losing do not affect payoffs. This means, for example, that a player's expected payoff in an all-pay auction does not change if instead of paying his entire bid he pays only a strictly positive fraction of his bid in advance and the rest only if he wins (as long as the fraction is specified in advance).

The payoff result implies that the number of players who obtain positive expected payoffs equals the number of prizes. That no more than m players obtain positive payoffs for every

[13] Equilibria may also include multiple atoms at various scores and multiple gaps in players' best response sets (see Example 4 and page 23 of Siegel (2007)).

realization of players' strategies (and tie-breaking randomization, if necessary) follows from the definition of contests. The payoff result shows that this is also true in expectation.

As the example of Section 1.1 illustrates, contests do not, in general, have pure-strategy equilibria.[14] Existence of an equilibrium (in pure or mixed strategies) is not immediately obvious, since payoffs are discontinuous in pure strategies, of which there is a continuum. Simon & Zame's (1990) result shows that an equilibrium exists for *some* tie-breaking rule. The following corollary of their result and the Tie Lemma above, whose proof is in the Appendix, shows that an equilibrium exists for *any* tie-breaking rule.

Corollary 1 *Every contest has a Nash equilibrium.*

The payoff result does not rely on equilibrium uniqueness: Example 3 in the Appendix describes a generic separable contest and two equilibria of the contest. Moreover, these equilibria lead to different allocations of the prize and different aggregate expenditures, so standard revenue equivalence techniques cannot be used to compare players' payoffs across equilibria.

3.2 Contests That Are Not Generic

Although the payoff result does not apply to contests that are not generic, it implies the following.

Corollary 2 *Every contest (generic or not) has at least one equilibrium in which every player's payoff is the maximum of his power and* 0*.*

This "upper-hemicontinuity" result can be proved by considering a sequence of generic contests that "approach" the original contest, and an equilibrium for each contest in the sequence. Every limit point (in the weak* topology) of the resulting sequence of equilibria is an equilibrium of the original contest in which payoffs are given by the payoff result.[15]

The payoff result also holds for contests in which all players are identical, even though such contests do not meet the Power Condition (all players have power 0). This is because in any equilibrium of any contest, identical players have identical payoffs,[16] and the Zero Lemma, which does not require the Generic Conditions, shows that at least one player has payoff 0. Therefore,

[14] Pure-strategy equilibria arise when players with positive power have head starts sufficiently large to dissuade weaker players from participating. Such equilibria do not arise in all-pay auctions, regardless of the difference in players' valuations. The payoff result applies to both pure- and mixed-strategy equilibria.

[15] Probability measures on the Borel subsets of the compact metric set $[a_i, r_i + \varepsilon]$ are regular (for any $\varepsilon > 0$), so the set of these probability measures is weak* compact and has at least one limit point (see, for example, Dunford & Schwartz (1988)).

[16] Suppose players i and $j \neq i$ are identical, and consider an equilibrium G. Player i can choose scores slightly above the supremum of player j's best responses, beating player j for sure and beating the other players at least

Corollary 3 *In any equilibrium of a contest in which all players are identical, all players have a payoff of* 0.

When players are not identical and the contest is not generic, the payoff of a player in some equilibrium may be very close to his valuation for winning at his initial score, even if his power is very low. Example 1 below shows this when the Power Condition fails; Example 2 below shows this when the Cost Condition fails.

3.2.1 Example 1 - The Power Condition Fails and the Payoff Result Does Not Hold

Consider the following three-player separable contest for one prize of common value 1, which is a modification of the example from Section 1.1. Players' costs are

$$c_1(x) = \begin{cases} (1-\alpha)x & \text{if } 0 \le x \le h \\ (1-\alpha)h + \left(1 + \frac{\alpha h}{1-h}\right)(x-h) & \text{if } x > h \end{cases}$$

$$c_2(x) = \begin{cases} (1-\varepsilon)x & \text{if } 0 \le x \le h \\ (1-\varepsilon)h + \left(1 + \frac{\varepsilon h}{1-h}\right)(x-h) & \text{if } x > h \end{cases}, c_3(x) = \begin{cases} \gamma x & \text{if } 0 \le x \le h \\ \gamma h + L(x-h) & \text{if } x > h \end{cases}$$

for some small α, ε in $(0,1)$, small $\gamma \ge 0$, h in $(0,1)$, and $L > 0$. Regardless of the value of L, the threshold is 1 and the Power Condition is violated (since at least two players have power 0). Costs are strictly increasing at 1 for all players, so the Cost Condition is met. It is straightforward to verify that for any h in $(0,1)$, there exists some $\beta > 0$ and $M > 0$ such that if $\alpha, \varepsilon, \gamma < \beta$ and $L > M$, then (G_1, G_2, G_3) is an equilibrium, for

$$G_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ (1-\varepsilon)h + \frac{\gamma}{1-\alpha}\left(\frac{x}{h} - 1\right) & \text{if } 0 \le x \le h \\ (1-\varepsilon)h + \left(1 + \frac{\varepsilon h}{1-h}\right)(x-h) & \text{if } h < x \le 1 \\ 1 & \text{if } x > 1 \end{cases}$$

$$G_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ (1-\alpha)h & \text{if } 0 \le x \le h \\ (1-\alpha)h + \left(1 + \frac{\alpha h}{1-h}\right)(x-h) & \text{if } h < x \le 1 \\ 1 & \text{if } x > 1 \end{cases}, G_3(x) = \begin{cases} x/h & \text{if } x \le h \\ 1 & \text{if } x > h \end{cases}$$

The top part of Figure 2 below depicts players' costs. The bottom part depicts players' atoms and densities in the equilibrium (G_1, G_2, G_3).

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as often as player j does with any best response. This implies that player i's payoff in G is at least as high as that of player j, by applying a continuity argument and a reasoning similar to the one used in the proof of Theorem 2.

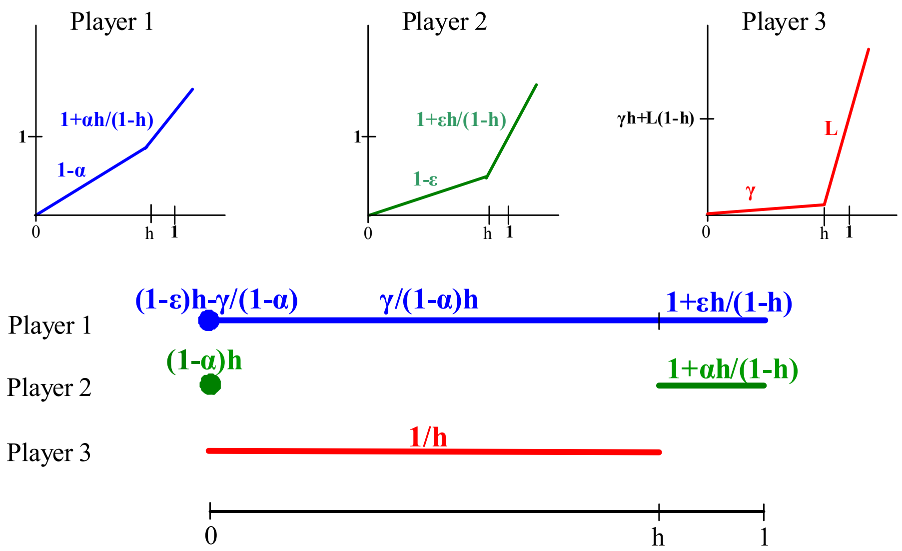

Figure 2: Cost functions and the equilibrium (G_1, G_2, G_3) of Example 1

Player 3's power is $1 - \gamma h - L(1-h)$ (his valuation for the prize less his cost of choosing the threshold), so as L increases player 3's power becomes arbitrarily low. But as h tends to 1 and ε, α, and γ tend to 0, for any value of $L > M$ player 3 wins with near certainty, and his payoff $(1-\varepsilon)(1-\alpha)h^2 - \gamma h$ approaches the value of the prize. While the equilibrium of Figure 2 may seem robust, a slight change in player 1's or 2's valuation for the prize leads to a generic contest and destroys the equilibrium (since Theorem 1 then applies).

3.2.2 Example 2 - The Cost Condition Fails and the Payoff Result Does Not Hold

When the Cost Condition fails the marginal player cannot obtain a strictly positive payoff on some interval of scores leading up to the threshold, so competition may stop before the threshold is reached. Consider the following two-player separable contest for one prize of common value 1. Players' costs are $c_1(x) = bx$ for some $b < 1$, and

$$c_2(x) = \begin{cases} \frac{x}{d} & \text{if } 0 \le x < d \\ 1 & \text{if } d \le x \le 1 \\ 2x - 1 & \text{if } \quad x > 1 \end{cases}$$

for some d in $(0,1)$. Player 1's reach is $\frac{1}{b} > 1$, player 2's reach is 1, the threshold is 1, and players' powers are $w_1 = 1 - b > 0$ and $w_2 = 0$, so the Power Condition holds. But the Cost Condition fails, because $c_2^{-1}(c_2(r_2)) = [d, 1]$. As a result, (G_1, G_2) is an equilibrium in which player 1 has a payoff of $1-bd>w_1$, for

$$G_1(x)=\begin{cases}0 & \text{if} \quad x<0\\ \frac{x}{d} & \text{if} \quad 0\le x\le d\\ 1 & \text{if} \quad x>d\end{cases},G_2(x)=\begin{cases}0 & \text{if} \quad x<0\\ 1-bd+bx & \text{if} \quad 0\le x\le d\\ 1 & \text{if} \quad x>d\end{cases}$$

As b approaches 1, player 1's power approaches 0. But for any value of b, as d approaches 0 player 1's payoff approaches 1, the value of the prize. Note, however, that $\left(G_1,\tilde{G}_2\right)$ is an equilibrium in which both players' payoffs equal their powers, for

$$\tilde{G}_2(x)=\begin{cases}0 & \text{if} \quad x<0\\ 1-b+bx & \text{if} \quad 0\le x\le 1\\ 1 & \text{if} \quad x>1\end{cases}$$

4 Participation

A player *participates* in an equilibrium of a contest if with strictly positive probability he chooses scores associated with strictly positive costs of losing. Players with strictly negative powers that are disadvantaged everywhere with respect to the marginal player do not participate in any equilibrium. This is the content of Theorem 2, which is proved in the Appendix.

Theorem 2 *In a generic contest, if the normalized costs of losing and valuations for winning for the marginal player are, respectively, strictly lower and weakly higher than those of player $i>m+1$, that is*

$$\frac{c_{m+1}(\max\{a_{m+1},x\})}{v_{m+1}(a_{m+1})}<\frac{c_i(x)}{v_i(a_i)} \text{ for all } x\in S_i \text{ such that } c_i(x)>0$$

and

$$\frac{v_{m+1}(\max\{a_{m+1},x\})}{v_{m+1}(a_{m+1})}\ge\frac{v_i(x)}{v_i(a_i)} \text{ for all } x\in S_i$$

then player i does not participate in any equilibrium. In particular, if these conditions hold for all players in $N_L\backslash\{m+1\}$, then in any equilibrium only the $m+1$ players in $N_W\cup\{m+1\}$ may participate.

Theorem 2 shows that in a generic all-pay auction only players $1,\ldots,m+1$ may participate.[17] And because players' cost functions in all-pay auctions are strictly increasing and all initial scores

[17] Baye et al. (1996) showed that more than two players may participate in certain non-generic, single-prize all pay auctions.

equal 0, players $1, \ldots, m+1$ do indeed participate.[18] This explains why in all-pay auctions with distinct valuations precisely the $m+1$ players with the highest valuations place strictly positive bids with strictly positive probability.

In contrast, the example of Section 1.1 shows that a player in $N_L \setminus \{m+1\}$ may participate if he has a local advantage with respect to the marginal player. Indeed, the proof of the Threshold Lemma shows that players 1 and 2 choose scores that approach or exceed the threshold, and so participate in any equilibrium. Suppose player 3 did not participate. Then, players 1 and 2 would have to play strategies that make all scores in $(0, T)$ best responses for both of them.[19] For low values of $\gamma > 0$, player 3 could then obtain a strictly positive payoff by choosing a low score, contradicting the payoff result. Thus, player 3 must also participate in any equilibrium, even though his expected equilibrium payoff is 0.

5 Concluding Remarks

All-pay contests capture general asymmetries among contestants, and allow for both sunk and conditional investments. The paper has provided a closed-form formula for players' expected payoffs in generic contests, and analyzed players' participation. The main insight is that reach and power are the right variables to focus on when examining contests.

Additional, seemingly complicated questions become simple when this insight in put to use. Consider for example the issue of rent dissipation, which is central to the rent-seeking literature. In a separable contest for m prizes of value V, aggregate equilibrium expenditures are simply mV less players' payoffs. As winning players' powers approach 0, which happens when their costs at the threshold approach V, rent dissipation is complete. As winning players' powers approach V, which happens when their costs at the threshold approach 0, no rent is dissipated.

The addition of a player to a contest never lowers the threshold, and therefore makes existing players weakly worse off. If the new player's reach is below the existing threshold, existing players' payoffs do not change, and the new player has a payoff of 0. The addition of a prize makes player $m+2$ the marginal player, and this lowers the threshold and makes existing players better off. In contrast, making prizes more valuable may make players worse off, because it raises the threshold. Further analysis of these issues, which have implications for contest design, is left for future work.

[18] The m players in N_W participate because, as shown in the proof of the Threshold Lemma, they choose scores that approach or exceed the threshold. If the marginal player does not choose scores that approach or exceed the threshold, any player in N_W can win with certainty and increase his payoff by choosing a score strictly below the threshold, contradicting the Threshold Lemma.

[19] Apply Lemma 1 in the Appendix to the contest that includes only players 1 and 2.

A Appendix

A.1 Example 3

The example depicts a separable contest and two equilibria of the contest, in which different players participate and in which aggregate expenditures differ. Let $n = 4$ and $m = 1$. I extend the example of Section 1.1 by adding player 4 , without constructing the contest explicitly. I will demonstrate the existence of two equilibria, G and G', such that only players 1,2, and 3 participate in G, and only players 1,2, and 4 participate in G'. Different player participation implies different allocations, since a player chooses a costly score only if he has a strictly positive probability of winning by doing so.

Begin with an equilibrium $G = (G_1, G_2, G_3)$ of the example of Section 1.1. When γ is small and positive, all three players participate in G as shown in Section 4. Let $t_3 = \inf \{x : G_3(x) = 1\}$. Add player 4 with $V_4 = 1$ and continuous, strictly increasing costs c_4 that are lower than those of player 3 below t_3 and equal to them starting from t_3. That is, $\forall x \in (0, t_3) : c_4(x) \in (0, c_3(x))$, and $\forall x \geq t_3 : c_4(x) = c_3(x)$.

There exist such functions c_4, for which an equilibrium is for players 1,2, and 3 to play G, and for player 4 not to participate. Indeed, assume that player 4 does not participate. Since player 3's equilibrium payoff is 0 (his power is negative), he obtains at most 0 by choosing any score when players 1 and 2 play G_1 and G_2, respectively, and he wins a prize under G if and only if he beats players 1 and 2. By Lemma 1 below, there are no atoms in $(0, T)$ in G so $\forall x \in (0, T) : P_3(x) = G_1(x) G_2(x)$. Player 4, who considers joining in when the others are playing G, must beat players 1,2, and 3 to win, i.e., $\forall x \in (0, T) : P_4(x) = G_1(x) G_2(x) G_3(x)$. Let $x \in (0, t_3)$ (note that $t_3 < T$), which implies that $G_3(x) < 1$. If $u_3(x) = 0$, then $P_3(x) > 0$ because $c_3(x) > 0$. Thus, $P_4(x) < P_3(x)$ and $P_4(x) - c_3(x) < P_3(x) - c_3(x) = u_3(x) = 0$. If $u_3(x) < 0$, since $P_4(x) \leq P_3(x)$ again $P_4(x) - c_3(x) < 0$. Thus, there exist continuous, non-decreasing functions c_4 such that $\forall x \in (0, t_3) : c_4(x) \in (0, c_3(x))$ and $P_4(x) - c_4(x) < 0$. For such functions, it is a best response for player 4 not to participate when the other players play G. Since G is an equilibrium of the contest that includes only players 1, 2, and 3, we have an equilibrium.

Maintaining the same cost functions, consider now an equilibrium $G' = \left(G'_1, G'_2, G'_4\right)$ of the contest that includes only players 1,2, and 4. As in the example of Section 1.1, all three players must participate in G'. When player 3 is added to the contest and doesn't participate, this remains an equilibrium. Indeed, player 4's payoff is zero in G' (his power is negative), and at every score player 3's costs are weakly higher than those of player 4 whereas his probability of winning is weakly lower than that of player 4.

If aggregate expenditures under the two equilibria are the same, multiply the valuation and cost of player 4 by some strictly positive $d \neq 1$. This does not change the equilibria of the contest, but changes aggregate expenditures in G'.

A.2 Proof of Corollary 1

Consider a contest C and the restricted contest C', in which every player i chooses scores in $S'_i = [a_i, K]$, for $K = \max_{i \in N} r_i < \infty$. Any equilibrium of C' is an equilibrium of C, since scores higher than K are strictly dominated by a_i for every player i. Thus, it suffices to show that C' has an equilibrium.

To do this, consider $S^* = \times_{i \in N} S_i' \setminus \{(s_1, \ldots, s_n) | \exists i \neq j : s_i = s_j\}$. That is, S^* is the set of n-tuples of distinct strategies. Players' payoffs are bounded and continuous on S^*, which is dense in $\times_{i \in N} S_i'$. So, following Simon & Zame (1990) page 864, there exists some tie-breaking rule, which may be score dependent, such that C' has a mixed-strategy equilibrium G when this tie-breaking rule is used. Denote the game in which this tie-breaking rule is used by $\tilde{C}$, and player i's payoff in the equilibrium G by $\tilde{u}_i$. To complete the proof, it suffices to show that G is an equilibrium of C'. I do this in two steps.

Step 1: A G_i-measure 1 of best responses for player i in $\tilde{C}$ gives player i the same payoff $\tilde{u}_i$ in C' as it does in $\tilde{C}$ when all other players play according to G. Indeed, choosing a score s_i at which no player has an atom gives player i the same payoff regardless of the tie-breaking rule. Since the number of atoms in G is countable, it is enough to show that when choosing G_i-atoms in C' player i obtains $\tilde{u}_i$. Consider a G_i-atom s_i. If only i has an atom at s_i then a tie occurs with probability 0 so player i obtains $\tilde{u}$ in C'. If there are multiple atoms at s_i, by the Tie Lemma the tie is never binding regardless of the tie-breaking rule, so again player i obtains $\tilde{u}_i$ in C'.

Step 2: No score gives player i a payoff in C' higher than $\tilde{u}_i$. The only scores s_i to check are those at which player i does not have an atom and another player does. Consider such a score s_i, and denote player i's payoff in C' when choosing s_i by $u_i'(s_i)$. By choosing scores slightly higher than s_i in $\tilde{C}$, player i can obtain a payoff of at least $u_i'(s_i) - \varepsilon$, for any $\varepsilon > 0$. Thus, $u_i'(s_i) \leq \tilde{u}_i$.

A.3 Proof of Theorem 2

Since dividing a player's Bernoulli utility by $v_i(a_i) > 0$ does not change his strategic behavior, it suffices to prove the result for contests in which $v_i(a_i) = 1$ for every player i. Choose an equilibrium G of such a contest, and suppose player $i > m + 1$ that meets the conditions of the proposition participated in G. Let $t_i = \inf\{x : G_i(x) = 1\} < T$ and and let $\tilde{t}_i = \max\{a_{m+1}, t_i\}$. Then $\tilde{t}_i < T$, $P_i(t_i) < 1$ (because the m players in N_W choose scores that approach or exceed the threshold, as shown in the proof of the Threshold Lemma), and for every $\delta > 0 : P_{m+1}(\tilde{t}_i + \delta) \geq P_i(t_i)$ since by choosing $(\tilde{t}_i + \delta)$ player $m+1$ beats player i for sure and beats the other players at least as often as player i does. Therefore, since for every $\delta > 0$ such that $\tilde{t}_i + \delta < r_{m+1} = T$ we have

$$v_{m+1}(\tilde{t}_i + \delta) > 0 \geq -c_{m+1}(\tilde{t}_i + \delta)$$

we obtain

$$\begin{gathered} u_{m+1} \geq P_{m+1}(\tilde{t}_i + \delta) v_{m+1}(\tilde{t}_i + \delta) - \left(1 - P_{m+1}(\tilde{t}_i + \delta)\right) c_{m+1}(\tilde{t}_i + \delta) \geq \\ P_i(t_i) v_{m+1}(\tilde{t}_i + \delta) - (1 - P_i(t_i)) c_{m+1}(\tilde{t}_i + \delta) \end{gathered}$$

Now, by definition of participation, $c_i(t_i) > 0$, so $c_i(t_i) > c_{m+1}(\tilde{t}_i)$. Since $P_i(t_i) < 1$ and $v_{m+1}(\max\{a_{m+1}, x\}) \geq v_i(x)$ for all $x \in S_i$, by continuity of v_{m+1} and c_{m+1} player $m+1$ can choose $\tilde{t}_i + \delta$ for sufficiently small $\delta > 0$ such that

$$P_i(t_i) v_{m+1}(\tilde{t}_i + \delta) - (1 - P_i(t_i)) c_{m+1}(\tilde{t}_i + \delta) >$$

$$P_i(t_i)\, v_i(t_i) - (1 - P_i(t_i))\, c_i(t_i) = u_i(t_i) \geq 0$$

so $u_{m+1} > 0$, which contradicts the payoff result because $w_{m+1} = 0$.

A.4 Statement and Proof of Lemma 1

Lemma 1 *In any equilibrium G of a contest with strictly decreasing valuations for winning, strictly increasing costs of losing, and initial scores of* 0*, (1) G is continuous on $(0, T)$, and (2) every score in $(0, T)$ is a best response for at least two players.*

Proof. Suppose that a player i had an atom at $x \in (0, T)$. Because players $1, \ldots, m+1$ choose scores that approach or exceed the threshold (see the argument in footnote 18), $P_j(x) < 1$ for every player j, and, in particular, $P_i(x) < 1$. If $P_i(x) = 0$, then player i would be better off by choosing 0. Therefore $P_i(x) \in (0, 1)$. Now, consider a player $j \neq i$. Because player i has an atom at x and $P_i(x) \in (0, 1)$, player j would be better off choosing scores slightly above x than choosing scores slightly below x. To see this, suppose player j chose x and all other players played according to G, with some randomization as the tie-breaking rule at x. Then player i would still win with a probability in $(0, 1)$ when choosing x, so j would be better off choosing a slightly higher score, by a reasoning similar to that of the Tie Lemma. Thus, no player $j \neq i$ chooses scores in some region below x (regardless of the tie-breaking rule), and, by the Tie Lemma, no player $j \neq i$ has an atom at x. Therefore, player i would be better off by choosing a score slightly below x. This shows that G is continuous on $(0, T)$. For (2), note that if $x \in (0, T)$ is not a best response for player i, then continuity of G implies that the same is true for scores in some neighborhood of x. Therefore, if only one player had a best response at x, he could choose scores slightly lower than x and win with the same probability, making him better off. Suppose no player had a best response at x. Then there would be a gap in the union of players' best response sets. Continuity of G on $(0, T)$ implies that the top of the gap cannot be below the threshold. And because valuations for winning are strictly decreasing and $G_i(T) = 1$ for every player i in N_L, players in N_W do not have best responses above the threshold. So, $G_i(T) = 1$ for every player i. Now, every player i in $\{1, \ldots, m+1\}$ has $G_i(x) < 1$ because he chooses scores that approach or exceed the threshold. So, the top of the gap must be the threshold, and players $\{1, \ldots, m+1\}$ must each have an atom there, contradicting the Tie Lemma. This shows that x is a best response for at least two players. ■

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