A Micro Foundation of Generalized Multi-Prize Lottery Contests: A Noisy Ranking Perspective*

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Abstract

This paper proposes a multi-prize noisy-ranking contest model. Contestants are ranked in descending order based on their perceived outputs, and rewarded by their ranks. A contestant's perceivable output increases with his autonomous effort, but is subjected to random perturbation. Under plausible conditions, the stochastic equivalence of the model presented in this paper with the family of (winner-take-all and multi-prize) lottery contests built upon ratio-form contest success functions is established. The model thus provides a plausible microfoundation for this family of often-studied contests. In addition, the approach taken in this paper reveals a common thread that connects a broad class of seemingly disparate competitive activities (such as rent-seeking contests, patent race, research tournament, and auctions with pre-auction investments) and unifies them through a common performance evaluation rule.

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1 Introduction

A wide class of competitive activities can be viewed as contests, in which all participants forfeit scarce resources in order to compete for a limited number of prizes. The amount of effort exerted directly affects whether or not one wins, but measurement errors, subjective biases and randomness in the production processes may influence the outcomes as well.

A wide variety of theoretical frameworks of contest/tournament models is available in the economic literature. They exhibit diverse technical characteristics and mirror various contexts.[1] Central to formally modelling contests is a mechanism that picks the winners and distributes the prizes. In "imperfectly discriminatory contests", where factors other than effort may influence whether one wins or loses, the selection mechanism is conventionally represented by a contest success function, which maps contestants' effort entries into the likelihood of every contestant winning each prize.

The lion's share of the existing literature concerns itself with winner-take-all contests. One, perhaps the most, widely adopted approach is the lottery contest model that assumes a ratio-form contest success function, with the Tullock contest model as its most popular special case: The likelihood that a contestant i wins P_i is given by the ratio of the output of his effort to the total output supplied by the entire cohort, i.e., $P_i = g_i(x_i)/\sum_{j=1}^{n} g_j(x_j)$, where the output production function $g_i(x_i)$ is usually an increasing function of effort x_i.[2]

This framework provides an intuitive and tractable specification for the winning probabilities as functions of effort when other random factors may be involved in determining the winner. However, a ratio-form contest success function does not directly apply to a multi-prize contest setting. In order to model the widely observed competitive events where contestants vie for more than one prize,[3] Clark and Riis (1996b and 1998a) have introduced a clever "generalization" of the basic Tullock contest model that allows a block of prizes to be distributed. By adopting ratio-form success functions as its building block, this "**multiple-winner nested contest model**" hypothetically conducts a series of conditionally independent (single-winner) "lotteries" and allows each of them

[1] Examples of applications include college admissions, influential politics, sports, war and conflict, internal labor market competition, etc. See Konrad (2007) for a thorough survey of economic studies on contests.

[2] Recent applications of ratio-form contest success functions can be seen in Wärneryd (2000), Yildirim (2005), Morgan and Várdy (2006) and Hann, Hui, Lee and Png (2008), among many others.

[3] For instance, in internal labor markets, firms often set aside a number of bonus packages to reward top performing workers. Employees may compete to fill multiple vacancies higher up in the organizational hierarchy.

to "draw" one prize recipient until all the prizes are given away.[4]

The nested contest model offers the most reasonable and convenient, as well as the most prominent, alternative so far in determining multiple recipients of prizes in imperfectly discriminatory contests.[5,6] Nevertheless, the nature of this process requires further exploration: the economic activities that underlie each single "lottery", as well as the entire selection process, remain in a "black box". Furthermore, while Clark and Riis (1996b) point out that their procedure is "one of many reasonable alternatives" to select winners, its popularity naturally raises the following question: Is this nested model "the one" for imperfectly discriminatory multi-prize contests in the family of lottery contests? Are there any other reasonable ways to model the winning probabilities in multi-prize lottery contests? More fundamentally, how are the "sequential" lottery contest and winner-take-all lottery contest intrinsically connected? This paper aims to address these issues.

As Konrad (2007) has pointed out in a thorough survey of economic studies on contests, a contest can be naturally regarded as a competitive event where contestants expend costly efforts to "get ahead of their rivals". Based on this natural notion, a contest requires the contest organizer to (at least partially) "order" the contestants based on a ranking system. While a ranking rule should therefore underpin the winning mechanism, it remains less than explicit for the family of lottery contests, especially in multi-prize settings.

This paper proposes a multi-prize contest model that selects prize recipients through a noisy ranking of contestants. This framework borrows its technical form directly from consumers' discrete choice econometric model (i.e. McFadden's general extreme value model), but has a different economic concern. In particular, the model involves a fixed number of economic agents (contestants) who produce their output out of their input (effort entries). Specifically, one's observed output is the sum of a deterministic component (a strictly increasing function of effort) and a noise term that could arise from numerous sources, e.g. observation error or perturbation in production. The

[4]As a result, the conditional probability that a remaining contestant will be selected in the next "draw" is independent of the effort entries of contestants selected in previous "draws".

[5]Besides the studies conducted by Clark and Riis (1996b, 1998a), the application of lottery contest models in multiple-winner settings have been discussed by Amegashie (2000), Yates and Heckelman (2001), Szymanski and Valletti (2005) and Fu and Lu (2007 and 2008).

[6]Another approach to modelling multiple-winner contests is the multiple-prize all-pay auction model. A handful of studies have contributed to this research agenda, including Barut and Kovenock (1998), Moldovanu and Sela (2001), Moldovanu, Sela, and Shi (2007) and Siegel (2007).

decision maker ranks these contestants by their observed outputs in descending order.[7] That is, the higher the observed output, the better a contestant's rank. As a result, given a set of effort entries by contestants, and any (simultaneous) realization of noise terms, a complete ranking arises. The decision maker assigns each agent a prize of his rank accordingly.[8]

We establish that this model (uniquely) generates an outcome that is stochastically identical to that of a generalized multi-prize lottery contest. For any given effort entry and production functions, the ex ante likelihood of every possible prize distribution outcome perfectly coincides with that in the multiple-winner nested contest model, which seemingly requires a sequential lottery process.[9] Furthermore, as will be illustrate later, our ranking model is based on a "*favorable extreme value ranking*" (hereafter referred to as **FEV ranking**). These results rationalize the convenient specification of the multi-prize distribution rule assumed by Clark and Riis (1996b and 1998a): The winning probabilities specified by the "sequential" lottery contest model literally reflect one statistical property of this underlying (simultaneous) ranking system. As a winner-take-all lottery is indeed a special case of the nested contest model (where only the first draw is of interest), the winner-take-all lottery contest is thus integrated with a multiple-prize nested contest through a (uniquely) hidden ranking system.

A handful of papers have worked to probe the micro-foundation of winner-take-all contest success functions. In a seminal paper, Skaperdas (1996) has shown that the ratio-form contest function is the only alternative that satisfies a number of axiomatic properties. This axiomatic foundation, as pointed out by Skaperdas (1996) and Clark and Riis (1996a and 1997), reveals the connection between the contest model, the probabilistic choice models (See Luce and Suppes,

[7] To provide an analogy, this scenario mirrors a standard moral hazard setting, in which the decision maker (an employer) cannot observe or verify the effort supplied by contestants (employees). Hence, he ranks the perceived performance of contestants in order to determine their compensation and other rewards.

[8] According to Clark and Riis (1998a) and Fu and Lu (2007), this prize allocation rule maximizes the amount of individual effort in a multi-prize lottery contest (multiple-winner nested contest). As this paper will establish the stochastic equivalence between our noisy ranking model and a multi-prize lottery contest, this prize allocation rule also maximizes the expected output in our framework. This type of rank-ordered prize distribution rule has also been discussed in studies by Glazer and Hassin (1988), Barut and Kovenock (1998), Moldovanu and Sela (2001), etc.

[9] The ex ante likelihood that a contestant is ordered on the l-th rank is equivalent to the probability that a contestant is selected for the l-th draw in a multiple-winner nested contest.

1965) and the discrete choice econometric models.[10,11] To our best knowledge, the current paper is the first one that explores the microeconomic underpinning of the generalized (multi-prize) lottery contest model, and investigates its connection to other theoretical frameworks.

This paper is closely related to the literature that attempts to bridge different contest modelling approaches. Baye and Hoppe (2003) reveal the strategic equivalence of research tournament models (Fullerton and McAfee, 1999), Patent Race models (Dasgupta and Stiglitz, 1980) and winner-take-all Tullock Contest models. Hirshleifer and Riley (1992) have derived *single-winner* ratio-form contest success functions through contests based on noisy ranking. In addition to allowing for a more generalized prize structure and output production technologies, this paper further expands this family of models and finds that it further accommodates auctions with pre-investment (Tan, 1992, Piccione and Tan, 1996 and Bergemann and Välimäki, 2006).

The technical isomorphism leads to the following, more fundamental, questions: (1) Why can different classes of contests be unified under the same umbrella (ratio-form contest success functions)? (2) To what extent can such isomorphism continue to hold? What kind of competitive activities can be abstracted as lottery contests? (3) Ultimately, how should the economic fundamentals behind this popular ratio-form contest success functions, and the popularity of this functional form, be interpreted?

This paper introspectively scrutinizes the noisy-ranking model in order to address these questions. A rationale for these issues unfolds as the economic interpretation of the technical approach (McFadden, 1973 and 1974) is developed: Although these models visualize different circumstances, they all embody the FEV ranking. The statistical nature of the model presented here (FEV ranking) allows us to deduce one naturally plausible economic interpretation for the particular competitive activities covered within the framework of ratio-form success functions. It strongly indicates that the evaluation mechanism underlying these contests essentially honors the most favorable shock that is realized on each contestant's performance, i.e., "the best shot" of each contestant. Such an evaluation rule is explicitly witnessed in many real-life competitive events and conforms to a

[10] Assuming unmeasurable psychological factors, this literature investigates randomized choices of decision makers (consumers) that result from stochastic ranking. Among others, McFadden (1973 and 1974) has demonstrated the econometric implementation of modelling revealed choice among discrete alternatives while adopting a probabilistic choice model.

[11] Clark and Riss (1996a) have pointed out the equivalence of a random choice model and lottery contests in the winner-takes-all case.

natural regularity. This hidden link uncovered in our analysis enables us not only to connect a wide variety of observationally detached competitive activities, but also to explore in depth the unobserved common thread that runs through these contests and imposes a conceptual limit on the scope of this unity.

The rest of the paper is organized as follows. In Section 2, the model is set up, the analysis is completed and the implications of this model are briefly discussed. Section 3 elaborates upon the economic implications of the results, and reinforces the argument by presenting the "dual" problem within our benchmark model and other applications of FEV ranking in alternative settings (auctions). Section 4 provides some concluding remarks.

2 A Multi-Prize Noisy-Ranking Contest Model

2.1 Setup

A multi-prize noisy-ranking contest model is proposed. $I \geq 2$ contestants, indexed by $i \in \mathbf{I} \triangleq \{1, 2, ..., I\}$, simultaneously submit their effort entries $\mathbf{x} = (x_1, \ldots x_I)$, to compete for $L \in \{1, 2, ..., I\}$ prizes. The contest organizer observes a noisy signal (y_i) about contestant i's output and evaluates their performance through this signal. Following McFadden (1973 and 1974), we assume that the noisy signal (y_i) is described by

$$\log y_i = \log g_i(x_i) + \varepsilon_i, \quad \forall i \in \mathbf{I}, \tag{1}$$

where the deterministic strictly increasing function $g_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ measures the impact of contestant i's effort x_i,[12] and the additive noise term ε_i reflects the randomness in the production process or the imperfection of the observation and evaluation process. $g_i(\cdot)$ is named as the production function of contestant i. Define $\mathbf{g} \triangleq \{g_i(\cdot), i \in \mathbf{I}\}$, which denotes the set of technologies. The idiosyncratic noises $\boldsymbol{\varepsilon} \triangleq \{\varepsilon_i(\cdot), i \in \mathbf{I}\}$ are independently and identically distributed. The additive-noise ranking model (1) is equivalent to a multiplicative-noise ranking model

$$y_i = g_i(x_i)\tilde{\varepsilon}_i, \ \forall i \in \mathbf{I}, \tag{2}$$

where the noise term $\tilde{\varepsilon}_i$ is defined as $\tilde{\varepsilon}_i \triangleq \exp \varepsilon_i$.

The L prizes are ordered by their values, with $V_1 \geq V_2 \geq \ldots \geq V_L$. Each contestant is assumed to be eligible for one prize at the most. As contestants' outputs accrue to the benefits of the contest

[12] Define $\log g_i(x_i) = -\infty$ if $g_i(x_i) = 0$.

organizer, the contest organizer ranks these contestants based on their performance evaluations (i.e., perceivable output $\log y_i$) in descending order. Prizes are allocated among contestants based on their ranks, given the availability of the prizes. That is, the contestant who contributes the highest perceivable output y_i receives V_1, the contestant who contributes the second highest perceivable output then receives V_2, and so on, until all the prizes are given away.[13]

When $L = 1$, the model degenerates into a winner-take-all contest, with the top-ranked contestant being the only winner. When $L \geq 2$, a multi-prize contest follows, which requires a more complete ranking among contestants in order to implement its prize distribution rule. For any given effort entries $\mathbf{x}$, a complete ranking among contestants immediately results from any realization of the noise terms $\boldsymbol{\varepsilon}$. A fair tie breaking rule is assumed. The probability of a contestant i winning a prize V_l is simply given by the probability that he is ranked at the l-th position. This setup therefore embraces the notion that a contest is a competitive event where contestants compete to "get ahead of others" (Konrad, 2007).[14]

While this model imposes virtually no restrictions on the technology $g_i(\cdot)$ and the number of prizes L, it follows McFadden (1973 and 1974) in assuming the random component ε_i to be drawn from a type I extreme-value (maximum) distribution. When the cumulative distribution function of ε_i is denoted by $F(\cdot)$, then we have

$$F(\varepsilon_i) = e^{-e^{-\varepsilon_i}}, \varepsilon_i \in (-\infty, +\infty), \quad \forall i \in \mathbf{I}, \tag{3}$$

and the density function is

$$f(\varepsilon_i) = e^{-\varepsilon_i - e^{-\varepsilon_i}}, \varepsilon_i \in (-\infty, +\infty), \ \forall i \in \mathbf{I}. \tag{4}$$

The performance evaluation mechanism underlying this formulation will be discussed in Section 3, which reveals the economic implications of this seemingly peculiar distribution. Note that when ε_i follows a type I extreme-value (maximum) distribution, then $\tilde{\varepsilon}_i \triangleq \exp \varepsilon_i$ must follow a Weibull (maximum) distribution.

[13] The optimality of this prize allocation rule has been explained in footnote 8.

[14] This family of contest models includes Lazear and Rosen (1981), Nalebuff and Stiglitz (1983), etc. These models link the top ranked contestant to a unique prize, while they differ in the output technology, the formulation of the random component, and therefore the probability of a contestant winning the *single* prize given the effort entries.

2.2 The Equivalence to Lottery Contests

In this subsection, it will be shown that this noisy-ranking model is stochastically equivalent to the family of lottery contests (winner-take-all lottery contests and multiple-winner nested contests).

In our setting, given the effort entries $\mathbf{x}$, a contestant i is ranked ahead of another j, if and only if

$$\begin{aligned} \log g_i(x_i) + \varepsilon_i &\geq \log g_j(x_j) + \varepsilon_j \\ &\Leftrightarrow \varepsilon_j \leq \varepsilon_i + \log \frac{g_i(x_i)}{g_j(x_j)}. \end{aligned}$$

A contestant i would be top ranked if and only if

$$\varepsilon_j \leq \varepsilon_i + \log \frac{g_i(x_i)}{g_j(x_j)}, \forall j \in \mathbf{I} \backslash \{i\}.$$

In the setup of McFadden (1973 and 1974), the decision maker cares about the top-ranked choice. The results established by McFadden (1973 and 1974) are therefore adapted to our contest setting.

Lemma 1 *For any given* $\mathbf{x} \geq 0$ *such that* $\sum_{j \in \mathbf{I}} g_j(x_j) > 0$, *the ex ante likelihood that a contestant* i *achieves the top rank is*

$$p(i|\mathbf{x}) = \frac{g_i(x_i)}{\sum_{j \in \mathbf{I}} g_j(x_j)}, \quad \forall i \in \mathbf{I}. \tag{5}$$

The proof is omitted as it is available from McFadden (1973 and 1974). By Lemma 1, the probability of a contestant being top ranked can be expressed as the ratio of his output $g_i(x_i)$ to the sum of outputs contributed by all contestants. This winning probability coincides exactly with the popularly assumed ratio-form contest success function of winner-take-all lottery contests, provided that each contestant i produces the deterministic component of his output through a technology $g_i(x_i)$.

When $L \geq 2$, the model is a multi-prize contest. Furthermore, it should be noted that a multi-prize contest is sensible only if $I \geq 3$.[15] To fully describe a multi-prize contest, the probability of each contestant winning each prize has to be completely characterized. To this end, the probabilities of all possible complete (when $L \geq I - 1$) or partial (when $L \leq I - 2$) ranking must be explored. In order to accomodate all these possibilities, we next study the complete ranking of all contestants for a given set of effort entries $\mathbf{x}$.

[15] When $I = 2$ and $L = 2$, the contest simply boils down to a winner-take-all contest with a unique prize $V = V_1 - V_2$.

Suppose that K $(1 \leq K \leq I-2)$ contestants are ranked from top 1 to top K by the amount of y_i. Let i_k indicate the index of the k-th ranked contestant. Define $\mathbf{I}_K = \{i_k, k = 1, ..., K\}$, which is the index set of the top ranked K contestants. We thus have $y_{i_1} \geq y_{i_2} \geq \cdots \geq y_{i_K} \geq y_j$, $\forall j \in \Omega_{K+1} \triangleq \mathbf{I} \backslash \mathbf{I}_K$. Next, the conditional probability of a contestant $n \in \Omega_{K+1}$ being the $(K+1)$-th ranked is calculated. This probability is denoted by $p(n|\mathbf{N}_K, \mathbf{x}, Y_K)$, where $\mathbf{N}_K = (i_1, ..., i_K)$ denotes the sequence of the top K-ranked contestants, and $Y_K = (y_{i_1}, ..., y_{i_K})$ denotes the sequence of the observed outputs of the top K-ranked contestants.

Since ε_i are i.i.d., the conditional cumulative distribution function of ε_j, $\forall j \in \Omega_{K+1}$ is described by

$$\begin{aligned} F(\varepsilon_j|\mathbf{N}_K, \mathbf{x}, Y_K) &= F(\varepsilon_j|y_j < y_{i_K}) \\ &= e^{-e^{-\varepsilon_j}} / e^{-e^{-\bar{\varepsilon}_j}}, \varepsilon_j \in (-\infty, \bar{\varepsilon}_j),\ \forall j \in \Omega_{K+1}, \end{aligned} \tag{6}$$

where $\bar{\varepsilon}_j \equiv \log y_{i_K} - \log g_j(x_j)$, $\forall j \in \Omega_{K+1}$. It therefore yields the density function

$$f(\varepsilon_j|\mathbf{N}_K, \mathbf{x}, Y_K) = e^{-\varepsilon_j - e^{-\varepsilon_j}} / e^{-e^{-\bar{\varepsilon}_j}}, \varepsilon_j \in (-\infty, \bar{\varepsilon}_j),\ \forall j \in \Omega_{K+1}. \tag{7}$$

As implied by (6) and (7), the conditional distribution of $\varepsilon_j, \forall j \in \Omega_{K+1}$, only depends on the minimum of $\{y_{i_k}, k = 1, ..., K\}$, i.e., y_{i_K}, because y_i are ranked in descending order. The following then results.

Lemma 2 *For any given effort entries $\mathbf{x} \geq 0$ such that $\sum_{j \in \mathbf{N}} g_j(x_j) > 0$, the probability that a contestant $n \in \Omega_{K+1}$ is the $(K+1)$-th ranked, conditional on that contestants $i_1, i_2, ..., i_K$ are respectively ranked from top 1 to top K, can be expressed as*

$$p(n|\mathbf{N}_K, \mathbf{x}) = \frac{g_n(x_n)}{\sum\limits_{j \in \Omega_{K+1}} g_j(x_j)}, \quad \forall n \in \Omega_{K+1}. \tag{8}$$

Proof. We first calculate $p(n|\mathbf{N}_K, \mathbf{x}, Y_K)$, which denotes the probability that a contestant $n \in \Omega_{K+1}$ is the $(K+1)$-th ranked conditioning on that contestants $\mathbf{N}_K = (i_1, i_2, ..., i_K)$ are respectively ranked from top 1 to top K and their observed outputs are Y_K. Note that $\varepsilon_n + \log g_n(x_n) - \log g_j(x_j) \leq$

$\bar{\varepsilon}_j, \forall \varepsilon_n \in (-\infty, \bar{\varepsilon}_n)$, $\forall j, n \in \Omega_{K+1}, j \neq n$. We thus have

$$
\begin{aligned}
& p(n|\mathbf{N}_K, Y_K, \mathbf{x}) \\
= \; & \Pr(\varepsilon_j \leq \varepsilon_n + \log g_n(x_n) - \log g_j(x_j),\ \forall j \in \Omega_{K+1}, j \neq n.) \\
= \; & \int_{-\infty}^{\bar{\varepsilon}_n} [\Pi_{j\in\Omega_{K+1}, j\neq n} F(\varepsilon_n + \log g_n(x_n) - \log g_j(x_j)|\mathbf{N}_K, \mathbf{x}, Y_K)] f(\varepsilon_n|\mathbf{N}_K, \mathbf{x}, Y_K) d\varepsilon_n \\
= \; & \int_{-\infty}^{\bar{\varepsilon}_n} [\Pi_{j\in\Omega_{K+1}, j\neq n} e^{-e^{-(\varepsilon_n + \log g_n(x_n) - \log g_j(x_j))}} / e^{-e^{-\bar{\varepsilon}_j}}] e^{-\varepsilon_n - e^{-\varepsilon_n}} / e^{-e^{-\bar{\varepsilon}_n}} d\varepsilon_n \\
= \; & (\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \int_{-\infty}^{\bar{\varepsilon}_n} [\Pi_{j\in\Omega_{K+1}, j\neq n} e^{-e^{-(\varepsilon_n + \log g_n(x_n) - \log g_j(x_j))}}] e^{-\varepsilon_n - e^{-\varepsilon_n}} d\varepsilon_n \\
= \; & (\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \int_{-\infty}^{\bar{\varepsilon}_n} \exp[-\varepsilon_n - e^{-\varepsilon_n} \cdot (1 + \sum_{j\in\Omega_{K+1}, j\neq n} \frac{g_j(x_j)}{g_n(x_n)})] d\varepsilon_n. \qquad (9)
\end{aligned}
$$

Let $\lambda_{n,\Omega_{K+1}} = \log(1 + \sum_{j\in\Omega_{K+1}, j\neq n} \frac{g_j(x_j)}{g_n(x_n)}) = \log(\sum_{j\in\Omega_{K+1}} \frac{g_j(x_j)}{g_n(x_n)})$, then

$$
\begin{aligned}
& p(n|\mathbf{N}_K, Y_K, \mathbf{x}) \\
= \; & (\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \int_{-\infty}^{\bar{\varepsilon}_n} \exp[-\varepsilon_n - e^{-(\varepsilon_n - \lambda_{n,\Omega_{K+1}})}] d\varepsilon_n \\
= \; & (\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \exp(-\lambda_{n,\Omega_{K+1}}) \int_{-\infty}^{\bar{\varepsilon}_n - \lambda_{n,K}} \exp[-\varepsilon'_n - e^{-\varepsilon'_n}] d\varepsilon'_n \\
= \; & (\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}) \exp(-\lambda_{n,\Omega_{K+1}}) \exp[-e^{-(\bar{\varepsilon}_n - \lambda_{n,K})}] \\
= \; & [g_n(x_n) \diagup \sum_{j\in\Omega_{K+1}} g_j(x_j)] \cdot \{(\Pi_{j\in\Omega_{K+1}} \exp[e^{-\bar{\varepsilon}_j}]) \exp[-e^{-(\bar{\varepsilon}_n - \lambda_{n,\Omega_{K+1}})}]\} \\
= \; & [g_n(x_n) \diagup \sum_{j\in\Omega_{K+1}} g_j(x_j)] \cdot \exp\{(\sum_{j\in\Omega_{K+1}} e^{-\bar{\varepsilon}_j}) - e^{-(\bar{\varepsilon}_n - \lambda_{n,\Omega_{K+1}})}\}. \qquad (10)
\end{aligned}
$$

Note that

$$
\begin{aligned}
& (\sum_{j\in\Omega_{K+1}} e^{-\bar{\varepsilon}_j}) - e^{-(\bar{\varepsilon}_n - \lambda_{n,\Omega_{K+1}})} \\
= \; & (\sum_{j\in\Omega_{K+1}} e^{-(\varepsilon_{n_K} + \log g_{n_K}(x_{n_K}) - \log g_j(x_j))}) \\
& - \exp\{-[\varepsilon_{n_K} + \log g_{n_K}(x_{n_K}) - \log g_n(x_n) - (\log(\sum_{j\in\Omega_{K+1}} g_j(x_j)) - \log g_n(x_n))]\} \\
= \; & \frac{e^{-\varepsilon_{n_K}}}{g_{n_K}(x_{n_K})} \{\sum_{j\in\Omega_{K+1}} g_j(x_j) - \sum_{j\in\Omega_{K+1}} g_j(x_j)\} \\
= \; & 0. \qquad (11)
\end{aligned}
$$

(10) and (11) give

$$
p(n|\mathbf{N}_K, Y_K, \mathbf{x}) = g_n(x_n) \diagup \sum_{j\in\Omega_{K+1}} g_j(x_j). \qquad (12)
$$

(12) is a very strong result as it states that $p(n|\mathbf{N}_K, Y_K, \mathbf{x})$ does not depend on Y_K. Aggregating over all possible Y_K, we must have

$$p(n|\mathbf{N}_K, \mathbf{x}) = g_n(x_n) \diagup \sum_{j \in \Omega_{K+1}} g_j(x_j), n \in \Omega_{K+1}, K = 1, ..., I-2, \tag{13}$$

which completes the proof.

Q.E.D. ■

The implications of Lemma 2 are the following. Firstly it reveals that, given the top K ranked contestants, the conditional probability of a contestant being ranked as the next is completely independent of $(x_{i_1}, \ldots, x_{i_K})$, the effort entries of these top K ranked contestants. Secondly, it shows that the conditional probability $p(n|\mathbf{N}_K, \mathbf{x})$ can be conveniently written as a ratio-form contest success function $g_n(x_n) \diagup \sum_{j \in \Omega_{K+1}} g_j(x_j)$, which thus mimics a single-winner lottery among the set of contestants who are ranked worse than level K.

Let the sequence $\{i_k\}_{k=1}^I$ denote a complete ranking among the I contestants, where i_k is the index of the $k-$th ranked contestant. Combining Lemma 1 and Lemma 2, the following can therefore be concluded.

Corollary 1 *For any given effort entries $\mathbf{x} \geq 0$ such that $g_i(x_i) > 0, \forall i \in \mathbf{I}$, the ex ante likelihood of any complete ranking outcome $\{i_k\}_{k=1}^I$ can be expressed as*

$$p(\{i_k\}_{k=1}^I) = \Pi_{k=1}^I \frac{g_{i_k}(x_{i_k})}{\sum_{k'=k}^I g_{i_{k'}}(x_{i_{k'}})}. \tag{14}$$

Corollary 1 states that the ex ante likelihood of a complete ranking can be expressed as the cumulative product of the conditional probability $p(i_k|\mathbf{N}_{k-1}, \mathbf{x}) = g_{i_k}(x_{i_k}) / \sum_{k'=k}^I g_{i_{k'}}(x_{i_{k'}})$ that the contestant i_k is ranked as the top among all contestants $\{i_k, i_{k+1}, ..., i_I\}$.[16] As mentioned earlier, the L prizes are awarded to the L contestants who contribute the highest y_is, respectively, based on their ranks. Thus, a prize distribution outcome is represented by the subsequent $\{i_k\}_{k=1}^L$ of $\{i_k\}_{k=1}^I$, where i_k denotes the index of the contestant who is ranked at the k-th position and receives V_k. The probability of a prize distribution outcome $\{i_k\}_{k=1}^L$ is therefore determined in light of Corollary 1.

[16] This property was first non-constructively proposed by Luce and Suppes (1965) as a hypothetical decision rule. It was first used in the econometrics literature by Beggs, Cardell and Hausman (1981). To our knowledge, it has not been applied in the multi-prize contest literature.

Theorem 1 *For any given effort entries* $\mathbf{x} \geq 0$ *such that* $g_i(x_i) > 0, \forall i \in \mathbf{I}$*, the ex ante likelihood of any prize distribution outcome* $\{i_k\}_{k=1}^L$, $L \geq 1$ *can be expressed as*

$$p(\{i_k\}_{k=1}^L) = \Pi_{k=1}^L \frac{g_{i_k}(x_{i_k})}{\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})}. \tag{15}$$

Note that Theorem 1 does not cover the case where $g_i(x_i) = 0$ for some contestants. Since ties are fairly broken in our noisy ranking model, these contestants will be ranked among the bottom-most in a random manner. From these results, the stochastic equivalence between our noisy-ranking model and a generalized multiple-winner nested contest model that follows Clark and Riis (1996b and 1998a) can be concluded. Clark and Riis (1996b and 1998a) extend winner-take-all Tullock contests to allow for a block of prizes to be allocated among contestants. The selection mechanism is conveniently illustrated as a sequential lottery process. Contestants simultaneously submit their one-shot effort entries $\mathbf{x}$ and the recipient of each prize is selected through a lottery among all remaining candidates represented by a ratio-form contest success function. As each contestant is eligible for one prize at the most, the recipient of a prize is immediately removed from the pool of candidates who are eligible for the next draw. This procedure is repeated until all the prizes are given away. If Ω_m is used to represent the index set of all remaining contestants for the m-th draw for the m-th prize V_m, then for any contestant $j \in \Omega_m$, he wins prize V_m with a probability of $\frac{f_j(x_j)}{\sum_{i \in \Omega_m} f_i(x_i)}$ if $\sum_{i \in \Omega_m} f_i(x_i) > 0$. Here $f_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is the output function of contestant i, which is assumed to be strictly increasing with effort outlay x_i. To the extent that $\sum_{i \in \Omega_m} f_i(x_i) = 0$, i.e., $f_i(x_i) = 0, \forall i \in \Omega_m$, prizes are randomly given away. Thus, the prize distribution outcome of this nested contest is determined by a series of M independent lotteries if M prizes are available. This nested contest is reduced to a standard winner-take-all lottery contest when only one prize is available. This special case of single-prize contests is well studied by Skaperdas (1996) and Clark and Riis (1996a, 1997) among others.

Let $C(\mathbf{I}, \mathbf{g}(\cdot), \mathbf{V})$ denote a multiple-winner nested Tullock contest with contestants $\mathbf{I}$, output functions $\mathbf{g}(\cdot)$ and prizes $\mathbf{V}$. The vector $\mathbf{V} = (V_1, \ldots, V_L)$ represents the ordered set of L prizes with $V_1 \geq V_2 \geq \ldots \geq V_L$. Each contestant i is endowed with an output production technology $f_i(x_i) = g_i(x_i)$.

An intriguing question naturally arises: does there exist another distribution of the noise term ε_i that could deliver the ratio-form winning probability as given by (15)? The answer is in the affirmative when $N = 2$. Hirshleifer and Riley (1992) have shown one such example, which implies that extreme value type I (**minimum**) distribution could also lead to this equivalence when only two contestants are involved.[17] However, the answer becomes negative for $I \geq 3$. This result is highlighted in the following Theorem.

Theorem 2 *When $I \geq 3$ and $L \geq 1$, the benchmark noisy ranking model (1) is equivalent to the generalized multiple-winner nested contest model $C(\mathbf{I},\ \mathbf{g}(\cdot),\ \mathbf{V})$ if and only if ε_i follows a type I extreme-value (maximum) distribution.*

Proof. Suppose the equivalence holds for $L \geq 2$, then we must have that the equivalence also holds for $L' = L - 1$ for the following reasons. Given that (15) holds for $L \geq 2$, we thus obtain the probability of any prize distribution outcome $\{i_k\}_{k=1}^{L}$. The probability of prize distribution outcome $\{i_k\}_{k=1}^{L'}$ is then observed: While the ranks of first L' prize recipients are fixed as the same as those in $\{i_k\}_{k=1}^{L}$, the L-th winner can be any one among those who is not among $\{i_k\}_{k=1}^{L'}$. According to (15), we must have $p(\{i_k\}_{k=1}^{L'}) = \sum_{i_L \in I \backslash \{i_k\}_{k=1}^{L'}} p(\{i_k\}_{k=1}^{L'}, i_L) = \Pi_{k=1}^{L'} \frac{g_{i_k}(x_{i_k})}{\sum\limits_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})}$. This result means that as long as the benchmark noisy ranking model (1) is equivalent to the generalized multiple-winner nested contest model for a particular $L \geq 2$ under our ranking rule, the equivalence must also hold for $L' = 1$.

Yellot (1977) has shown some nice results, which imply that when there are more than three contestants ($I \geq 3$) and $L' = 1$, the winning probabilities take the form of (15) if and only if ε_i follows a type I extreme-value (maximum) distribution. Specifically, combining his Lemma 1 (p.116), Definition 3 (p.120) and Theorem 5 (p.135) leads to the uniqueness of type I extreme-value (maximum) distribution for ε_i, which leads to the ratio-form winning probability (15) for $L' = 1$.

Q.E.D. ■

Thus, Theorem 2 establishes the (unique) stochastic equivalence of the noisy-ranking contest model with a generalized multiple-winner nested contest that is built upon ratio-form contest success functions.

Consequently, the "sequential" lottery process conveniently "visualizes" a hidden noisy ranking process, while the contest itself does not rely on a sequentially implemented selection mechanism. Indeed, a noisy ranking system under the guise of a lottery process has been "uncovered" by

[17] This result can also be derived from the race-type contest setting presented in Section 3.1. More details will be provided at a later point.

Theorem 2! From the perspective of noisy ranking, it is clear that this framework represents the most natural generalization of winner-take-all lottery contests in the context of multi-prize competition. We conclude that the the "**multiple-winner nested contest model**" and winner-take-all lotteryy contest can be integrated into a unifying framework through a unique underlying ranking system. Hence, the "**multiple-winner nested contest model**" is indeed a very natural choice for modeling imperfectly discriminatory multi-prize contests.

Clark and Riis (1998a), as well as Fu and Lu (2008), have provided a complete solution for the multiple-winner nested contests when contestants are symmetric. These results, by Theorem 2, also solve the equilibrium of the noisy-ranking model (1) when contestants are assumed to be identical.

3 A Micro Foundation of Ratio-Form Contest Success Functions

This section further explores the economic implications of our model. As mentioned in Section 1, this paper is inspired by and closely linked to the study conducted by Baye and Hoppe (2003). They have established the strategic equivalence of research tournament (Fullerton and McAfee, 1999), Patent Race (Dasgupta and Stiglitz, 1980) and a winner-take-all Tullock contest. Our results follow in the vein of these pioneering studies, and will allow us to further expand the family of competitive activities that can be unified within an integral framework. On top of this, our results enable us to address a more fundamental question: Why could these seemingly disparate models be unified?

The ties are revealed as the setup of our noisy ranking model is closely scrutinized. The perceivable output ($\log y_i$) is assumed to contain a deterministic and a random component, which follows an extreme value type I (maximum) distribution. Following Lazear and Rosen (1981), the observed output ($\log y_i$) of a contestant could be interpreted as the sum of its expectation and a random shock. Type I Extreme value distributions (Gumbel) are the limiting distributions of the maximum or minimum of a large collection of i.i.d. random observations from the same arbitrary continuous distribution on support $(-\infty, \infty)$. The type I extreme value (maximum) distribution is pertinent in a circumstance where (only) the maximum value of a collection of random shocks is of interest. By assuming this distribution, together with the fact that contestants are ranked in descending order, the model therefore depicts a selection mechanism where the performance of each contestant is ordered by the most favorable shocks to his observed performance. This ranking system is named a "*favorable extreme value ranking*" (**FEV Ranking**).

One possible intuitive framework for this ranking rule is that the decision maker (the contest organizer) honors the "best shot" of each contestant's repeated attempts when their performance is subject to random perturbation.[18] This winning rule can be a man-made one. For instance, weight lifters are ranked in the Olympic Games by their most successful tries. More plausibly, this winning rule captures a natural regularity that is common in many real-world competitive events: it represents the situations where only the best performance is observable to the decision maker. To provide an analogy of this point, an architect would submit only his best idea to a design competition. A lawyer will proffer only the most favorable evidence in court, while the strongest case prevails.[19] Indeed, on many occasions, only the best performance of a contestant is observed. The seemingly peculiar type I extreme value (maximum) distribution adopted in Section 2.1 in fact captures the essence of a broad class of competition activities. Our benchmark model thus approximates a performance evaluation scheme that is based on the "best luck" of contestants.

In light of the isomorphism between our model and contests built upon ratio-form success functions, this argument sheds light on the hidden mechanism in the black box of the family of lottery contests. A lottery contest model could also be understood as a representation of a noisy ranking system that orders contestants based on their best performances. The prevalence of such an evaluation scheme, as has been argued above, thus supports the extensive application of lottery contest models, and strengthens the plausibility of this family of models.

In the rest of this section, the proposed argument is dialectically elaborated upon. First, the "dual" problem is presented to our original model, i.e., a multi-prize race model. Its equivalence to a multiple-winner nested contest is established. A FEV ranking is intuitively "uncovered" that confirms and reinforces our argument. In addition, FEV ranking is shown to be present in an alternative type of contests, which are directly adapted from models of auctions with pre-investment, and the economic implications of this ranking system in this setup are illustrated. Finally, "the antithesis" to our argument is presented. A model is provided that cannot be abstracted as a standard lottery contest, as the FEV ranking is missing.

[18] This visually intuitive interpretation is not the unique explanation for this ranking system. More details can be found in Section 3.2.

[19] This court judging rule is extracted from Baye, Kovenock and de Vries (2005).

3.1 The "Dual" Problem: A Multi-Prize Race Model

Baye and Hoppe (2003) have established the strategic equivalence of a patent race and a standard Tullock contest. In this subsection, we show that "FEV ranking" is also "uncovered" beneath models of racing competitions, i.e., the type of competitive events where participants are better rewarded by accomplishing a specific task faster than others.

A generalized racing model is first proposed that allows for more than one prize. The framework of Dasgupta and Stiglitz (1980) is adopted in this regard. Each of I contestants chooses a lump-sum effort x_i. A contestant i would accomplish a task (e.g. making a scientific discovery) by the time t_i with a probability (i.e. a Weibull minimum distribution) of

$$\Psi(t_i|x_i) = 1 - e^{-z_i(x_i)t_i}, \quad x_i, t_i \geq 0, \tag{16}$$

where $z_i(x_i)$ represents the hazard rate of contestant i, i.e., the conditional probability of accomplishing this task between t_i and time $t_i + \Delta t_i$. Conditional on effort entry $\mathbf{x}$, t_is are i.i.d. The hazard rate $z_i(x_i)$ is a strictly increasing function of the expenditure x_i. Define $\mathbf{z}(\cdot) \triangleq (z_i(\cdot))$.

Diverging from Dasgupta and Stiglitz (1980), we allow for a number of contestants to receive tangible rewards. Assume that $L \in \{1, 2, ..., I\}$ prizes (denoted by $\mathbf{V} = (V_1, V_2, ..., V_L)$) are to be awarded to contestants. That is, the contestant who finishes first receives prize V_1, the contestant who finishes second receives V_2, and so on.[20] Given effort entries $\mathbf{x}$, each conditional realization of (t_i) determines the ranking of contestants and the prize distribution outcome accordingly. Based on the very nature of a race, this type of competition may also be intuitively interpreted as a noisy-ranking contest: contestants are ranked in ascending order based on the amount of time they spend on accomplishing the given task, and a contestant is better rewarded for the realization of a smaller t_i.

Denote by $R(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$ this multi-prize race model and $C(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$ a multiple-winner nested Tullock contest with contestants $\mathbf{I}$, technology $\mathbf{z}(\cdot)$ and prizes $\mathbf{V}$. The following result will first be presented and its micro foundation will later be built.

Theorem 3 *A multi-prize race $R(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$ is stochastically equivalent to a multiple-winner nested Tullock contest $C(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$.*

[20] For instance, imagine a number of firms that are engaged in process R&D competition that is not perfectly patentable. The earlier a firm discovers the secret of a cost-reduction technology, the higher its accumulated profit.

Proof. For given expenditure entries $\mathbf{x}$ such that for $\sum\limits_{j\in\mathbf{I}} z_j(x_j) > 0$, a firm i could leapfrog all others with a probability of

$$\begin{aligned}\Pr(t_j &\geq t_i,\ j\in\mathbf{I},\ j\neq i) = \int_0^\infty z_i(x_i) e^{-t_i \sum\limits_{j\in\mathbf{I}} z_j(x_j)} dt_i \\ &= \frac{z_i(x_i)}{\sum\limits_{j\in\mathbf{I}} z_j(x_j)},\quad \forall i\in\mathbf{I}, \end{aligned} \tag{17}$$

which perfectly mimics the odds of winning in a generalized Tullock lottery contest with (increasing) output functions $z_i(x_i)$.

Suppose $\tilde{K}$ $(1\leq \tilde{K}\leq I-2)$ contestants are ranked as the first to the $\tilde{K}$-th in ascending order according to (t_i), with contestant i_k ranked as the k-th one. Define $\tilde{\mathbf{I}}_{\tilde{K}} = \{i_k, k=1,...,\tilde{K}\}$. We thus have $t_{i_1}\leq t_{i_2}\leq\cdots\leq t_{i_{\tilde{K}}}\leq t_j$, $\forall j\in\tilde{\Omega}_{\tilde{K}+1} = \mathbf{I}\backslash\tilde{\mathbf{I}}_{\tilde{K}}$. Next, consider the conditional probability of a contestant $n\in\tilde{\Omega}_{\tilde{K}+1}$ being the $(\tilde{K}+1)$-th ranked. This probability is denoted by $q(n|\tilde{\mathbf{N}}_{\tilde{K}},\mathbf{x},T_{\tilde{K}})$, where $T_{\tilde{K}} = (t_{i_1},...,t_{i_{\tilde{K}}})$, $\tilde{\mathbf{N}}_{\tilde{K}} = (i_1,...,i_{\tilde{K}})$. This conditional probability is simply

$$\begin{aligned} q(n|\tilde{\mathbf{N}}_{\tilde{K}},\mathbf{x},T_{\tilde{K}}) &= \Pr(t_n\leq t_j,\ j\in\tilde{\Omega}_{\tilde{K}+1}\ ,\ j\neq n|t_n\geq t_{i_{\tilde{K}}}) \\ &= \int_{t_{i_{\tilde{K}}}}^\infty z_n(x_n)\exp(-t_n \sum_{j\in\tilde{\Omega}_{\tilde{K}+1}} z_j(x_j))dt_n \diagup \exp(-t_{i_{\tilde{K}}}\sum_{j\in\tilde{\Omega}_{\tilde{K}+1}} z_j(x_j)) \\ &= \frac{z_n(x_n)}{\sum\limits_{j\in\tilde{\Omega}_{\tilde{K}+1}} z_j(x_j)}, \forall n\in\tilde{\Omega}_{\tilde{K}+1}. \end{aligned} \tag{18}$$

This strong result states that $q(n|\tilde{\mathbf{N}}_{\tilde{K}},\mathbf{x},T_{\tilde{K}})$ does not depend on $T_{\tilde{K}}$. Aggregating over all possible $T_{\tilde{K}}$, we must have that conditioning on contestants $i_1, i_2, ..., i_K$ being respectively ranked from top 1 to top K, the probability that a contestant $n\in\tilde{\Omega}_{\tilde{K}+1}$ is the $(\tilde{K}+1)$-th ranked is

$$q(n|\mathbf{N}_K,\mathbf{x}) = z_n(x_n)\diagup\sum_{j\in\Omega_{K+1}} z_j(x_j), n\in\tilde{\Omega}_{\tilde{K}+1}, K=1,...,I-2. \tag{19}$$

It can therefore be seen from (18) and (19) that the resulting prize distribution outcome is stochastically equivalent to that of a multiple-winner nested contest with output functions $z_i(x_i)$.

Q.E.D. ■

The Source of Equivalence: FEV Ranking in the Dual Model

Theorem 3 states the stochastic equivalence of our multi-prize race model and a nested lottery contest model of Section 2 where more than one prize is available. A micro foundation remains to be laid for this equivalence. It will now be shown that the argument proposed for our benchmark model of Section 2 continues to apply and that a selection mechanism that implements a FEV ranking is also hidden beneath the race model.

Theorem 4 *A multiple-winner race $R(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$ is equivalent to a descending-order noisy-ranking contest (1) with the set of output functions $\mathbf{z}(\cdot)$ and the noises $\boldsymbol{\varepsilon}$ that are individually and independently distributed following an extreme value type I (maximum) distribution.*

This result may not be very surprising, as the equivalence of the multi-prize race model and a multiple-winner nested contest model has already been established. A dedicated technical proof will not be laid out; instead, the reasoning will be presented in the following discussion. The hidden ties that connect all these models will surface as we set out to establish the result.

It is worth noting that t_i (the time the contestant i takes to finish a given task) can be modelled as the product of two multiplicatively separable components as follows

$$t_i = h_i(x_i)q_i, \ \forall i \in \mathbf{I}, \tag{20}$$

where $t_i, q_i \in (0, \infty)$ and $h_i(x_i) \triangleq z_i^{-1}(x_i)$. In other words, t_i is jointly determined by the deterministic component $h_i(x_i)$, which depends only on one's effort entry, and a stochastic term q_i. Obviously, the function $h_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ strictly decreases with one's effort. As suggested by simple statistical facts, t_i follows a Weibull (minimum) distribution of (16) if and only if q_i follows a Weibull (minimum) distribution with c.d.f. $1 - e^{-q_i}$. Under this assumption, model (16) is equivalent to model (20).

Model (20) can be further equivalently expressed as

$$\log t_i = \log h_i(x_i) + \eta_i, \ \forall i \in \mathbf{I}, \tag{21}$$

where i.i.d idiosyncratic noises $\eta_i \equiv \log q_i$ follow an extreme value type I extreme-value (minimum) distribution.[21] The c.d.f. and p.d.f. of η_i can respectively be expressed as follows,

$$\Phi(\eta_i) = 1 - e^{-e^{\eta_i}}, \eta_i \in (-\infty, +\infty), \quad \forall i \in \mathbf{I}, \text{ and} \tag{22}$$

$$\varphi(\eta_i) = e^{\eta_i - e^{\eta_i}}, \quad \eta_i \in (-\infty, +\infty), \ \forall i \in \mathbf{I}. \tag{23}$$

A closer look reveals that an ascending-order model (21) is in fact equivalent to the benchmark framework set up in Section 2.1. Note that model (21) can be equivalently expressed as

$$\log \widetilde{y}_i = \log z_i(x_i) + \xi_i, \ \forall i \in \mathbf{I}, \tag{24}$$

[21] The extreme value type I (minimum) distribution is also known as a "log-Weibull" distribution.

where $\widetilde{y}_i = t_i^{-1}$ and $\xi_i = \log q_i^{-1}$. Note that when $\eta_i = \log q_i$ follows an extreme value type I extreme-value (**minimum**) distribution, ξ_i must follow an extreme value type I extreme-value (**maximum**) distribution as given by (3). Based on simple statistical facts, the extreme value type I (maximum) distribution is simply the inverse of its "minimum" counterpart (the extreme value type I (minimum) distribution). Consequently, ranking (t_i) of model (21) in ascending order is equivalent to ranking (t_i^{-1}) of model (24) in descending order. In short, the race model (21) is the "dual" of the benchmark model proposed in Section 2.1. This observation immediately results in Theorem 4, which reveals why our multi-prize race model is equivalent to a multiple-winner nested contest (Theorem 3). Clearly, the above equivalence result means that our race type model is equivalent to a generalized multiple-winner nested contest model if and only if q_i in (20) follows a Weibull (minimum) distribution.

Model (21) thus provides a micro foundation for model (16). The multi-prize race model, as well as its "tweak" model (21), is underpinned by the same "ordering mechanism" as the model presented in Section 2.1. Both represent an evaluation mechanism that honors "the most favorable shock", which lays a common foundation for all the equivalence results we have established.

In order to see this, note that the random term η_i in (21) follows an extreme-value (minimum) type I distribution, which is also known as "log-Weibull (minimum)" distribution. A Weibull (minimum) distribution describes the timing of "the minimum" from a collection of random samples from an arbitrary distribution with a support on $[0, \infty)$.[22] This fact, together with the ascending-order ranking rule, naturally corresponds to a selection mechanism that honors "the best luck", when the output is "bad" to the decision maker, and contestants get ahead of others by contributing less of their observed outputs: Under such circumstances, "the most favorable shock" is seen by the realized minimum. A race directly exhibits such characteristics: One secures a more favorable rank by accomplishing one's task as quickly as possible, i.e., by making t_i as "small" as possible. Particularly in a R&D race towards a successful innovation, a competing party's entry, i.e., the innovation time, depends on its first successful (parallel) experiment.

This setup is derived by manipulating a race model. However, it is worth noting that this model could also be applied to other competitive events, where contestants win by reducing the amount of their "devalued" outputs, e.g., pollution.

[22] This is the reason that it is the inverse of the extreme value type I maximum distribution.

3.2 FEV Ranking in Auction Models

The nature and function of "FEV ranking" are further elaborated upon through an auction model with pre-investment, which is adapted from Tan (1992) and Piccione and Tan (1996). A buyer is to procure one unit of a novel good. I firms compete for this contract. The production cost Y_i of each firm is randomly drawn from a continuous distribution with the c.d.f $F(y)$ on the fixed support $[\underline{y}, \overline{y}]$. However, a firm i could invest a costly R&D effort x_i to reduce its production cost, which makes its production cost randomly drawn from the distribution

$$G_i(y_i) = \Pr(Y_i < y_i) = 1 - [1 - F(y_i)]^{h_i(x_i)}, \tag{25}$$

where $h_i(x_i)$ is a strictly increasing function of x_i. That is, the greater the amount of effort exerted, the more likely the realized production cost is to be low. After the R&D investment, firms bid to compete for this contract, and the lowest bidder wins. Hence, in a monotonic Bayesian equilibrium, the firm with the lowest realized cost receives the contract.

For any given set (y_i), the outcome of the contest is determined by ranking y_i in ascending order. We define variables $Z_i = \log \frac{1}{[1-F(Y_i)]}, i = 1, 2, ..., I$. Obviously, Z_i represents a uniform monotonic transformation of Y_i across all i. Thus, ranking the realizations of Z_is in ascending order is equivalent to the competition rule of the original model.

Theorem 5 *The auction model (25) with pre-investment is equivalent to a lottery contest and the benchmark contest model (1) of FEV-ranking .*

Again, a dedicated proof is not provided, but the work is laid out below to demonstrate the connection of this model to a lottery contest. This objective is fulfilled by finding the distribution function of the random variable Z_i. We have

$$\begin{aligned}
\Pr(Z_i &< z_i) = \Pr(\log \frac{1}{[1 - F(Y_i)]} < z_i) \\
&= \Pr(\frac{1}{[1 - F(Y_i)]} < e^{z_i}) \\
&= \Pr(F(Y_i) < 1 - e^{-z_i}) \\
&= \Pr(Y_i < F^{-1}(1 - e^{-z_i})).
\end{aligned} \tag{26}$$

By the definition of $G_i(y_i)$, we obtain

$$\begin{aligned}
\Pr(Z_i &< z_i) = 1 - [1 - F(F^{-1}(1 - e^{-z_i}))]^{h_i(x_i)} \\
&= 1 - e^{-h_i(x_i)z_i},
\end{aligned} \tag{27}$$

which indicates that Z_i follows a Weibull (minimum) distribution. As the realizations of Z_is are to be ranked in ascending order, the model setup is therefore equivalent to a standard race model with a hazard rate of $h_i(x_i)$. By the previous argument, it further boils down to a lottery contest, and can be rewritten by simple manipulation as a FEV ranking model. As a result, a firm i would win the contract with an ex ante likelihood of $p_i = \frac{h_i(x_i)}{\sum_{j\in\mathbf{I}} h_j(x_j)}$. It deserves to be noted that the likelihood of winning does not depend on the underlying distribution $F(\cdot)$.

An alternative auction model with pre-investment would deliver the same outcome. Let I firms compete for a procurement contract. The buyer chooses the most valuable product. Initially, the value Y_i of each firm's product is distributed as $F(y)$ on a common support. Each firm i could invest a costly R&D effort x_i to enhance the value of its product, which makes y_i randomly drawn from the distribution

$$G_i(y_i) = \Pr(Y_i < y_i) = [F(y_i)]^{h_i(x_i)}, \tag{28}$$

where $h_i(x_i)$ is a strictly increasing function of x_i. That is, the greater the effort, the more likely that a higher value will be realized. It is apparent that this setting simply represents the dual model of (25).

This type of investment-auction model has been adopted and studied in a number of scholarly papers (Tan (1992), Piccione and Tan, 1996) on auctions, which are detailed in the survey conducted by Bergemann and Välimäki (2006).[23] The technical isomorphism thus allows us to connect economic findings from different fields and integrate these diverse literatures.

However, two remarks must be made. Firstly, in this auction model with pre-investment, the investment impacts the distribution of Y_i through the power term $h_i(x_i)$, while $F(\cdot)$ can be any continuous cumulative distribution function. It is a curiosity as to how this popular and convenient setting can be economically interpreted. Secondly, the economic link between this type of pre-investment competition model and our benchmark FEV ranking model remains to be explained.

Its economic interpretation naturally emerges from the research tournament model proposed by Fullerton and McAfee (1999). In the research tournament model, each firm i hires n_i scientists to conduct R&D and each scientist can come up with an idea. The value of the idea follows a continuous distribution with the common c.d.f $F(\cdot)$. The firm picks the most valuable idea to

[23] It should be noted that these papers assume symmetric investment and identical technology as they mainly study the bidding strategies of bidders in the auction stage. They focus on different strategic aspects of the game from the literature on regularly structured contests.

compete with other firms. Denote the value of the best idea by Y_i. Apparently, Y_i follows a continuous distribution with the c.d.f $G_i(y_i) = \Pr(Y_i \leq y_i) = [F(y_i)]^{n_i}$. As shown by Baye and Hoppe (2003), as well as our previous analysis, a firm i wins with a probability $\frac{n_i}{\sum_{j=1}^{I} n_j}$. This research tournament, by its rule, thus directly exemplifies a contest that honors the best shot of each firm, and ranks competing firms according to the realized favorable extreme values! This further explains why the likelihood of winning in Fullerton and McAfee (1999) does not depend on the underlying distribution $F(\cdot)$. As illustrated in our analysis, in the resulting model (27) in the form of extreme value distribution, $F(\cdot)$ is rather irrelevant.

3.3 The "Antithesis": An Example of Non-Lottery Contests

So far we have proposed a micro foundation, i.e., a ranking system, which underpins a wide range of contests. This permits us to connect varieties of seemingly disparate models on the one hand, while it imposes a limit on this unity on the other: This family of contests may not include competitive events that do not honor "the most favorable shocks" when picking the winners.

To illustrate this point, a contest model is provided that hosts a different performance evaluation rule. One salient example is the noisy ranking contest model suggested by Hirshleifer and Riley (1992). Two contestants simultaneously submit their effort entries x_1 and x_2, and they are ranked by their composite output $q_i x_i$, where q_i is a random variable that follows a Weibull (**minimum**) distribution with c.d.f. $F(q_i) = 1 - e^{-aq_i}$. The contestant with the higher output wins, thus output needs to be ranked in descending order. It can be easily verified that given the set of effort entries, the ex ante winning odd of a contestant is exactly identical to a standard Tullock success function $\frac{x_i}{x_1+x_2}, i = 1, 2$.

However, the equivalence of this model with a lottery contest does not hold when there are more than two contestants. Consider a more generalized variation of this model. Assume that the deterministic component of the composite output takes the form of $q_i g_i(x_i)$, where $g_i(x_i)$ is a strictly increasing function of the effort outlay x_i. Obviously, linear technology $g_i(x_i) = x_i$ is a special case of this setting. When $I = 3$, and when only one prize is available, contestant 1 wins with a probability of

$$\begin{aligned} P_1 &= 1 - \frac{g_2(x_2)}{g_1(x_1)+g_2(x_2)} - \frac{g_3(x_3)}{g_1(x_1)+g_3(x_3)} \\ &\quad + \frac{g_2(x_2)g_3(x_3)}{g_1(x_1)g_3(x_3)+g_2(x_2)g_3(x_3)+g_1(x_1)g_2(x_2)}. \end{aligned} \tag{29}$$

The proof is provided in the Appendix.

This setting results in a well defined contest success function, while it appears in a different form from lottery contests. The source of this dichotomy is not difficult to technically detect as we look at the distribution of the noise term. The Weibull (minimum) distribution indicates the distribution of the incidence of the "minimum" among a collection of shocks. By referring to (20), readers would immediately realize that this model is no different from our race model except for the winning rule. One wins in a race by a smaller observed output $q_i x_i$ in model (20). By way of contrast, a contestant in Hirshleifer and Riley's model (1992) wins by a larger output. Consequently, this model is underpinned by an "unfavorable extreme value" ranking. Literally, the decision maker does prefer a higher output and ranks contestants by observed outputs in descending order. However, he does not honor the best shot, but ranks the performances based on their "weakest links".

This mechanism contradicts the one beneath our model (21) and represents different economic activities from those that underlie lottery contests. It may mirror these contests when the worst performance matters (the most) for a win and contestants compete by improving their own blindsights. A close analogy is high-profile board game competitions such as Chess Olympic Championships. A loss is often perceived to be a result of making the most harmful misplay despite of other marvelous moves. This dichotomy in the underlying performance evaluation mechanism thus drives this observed disparity when the number of participants exceeds two, and excludes this type of contest from the family of models that can be represented as standard lottery contests.

4 Concluding Remarks

This paper has set forth a multi-prize contest model that links its prize distribution outcome to the ranking of contestants based on their noisy performances. The performance of a contestant is modelled as the sum of a deterministic output of his spontaneous effort and a random component. Contestants exert their one-shot effort simultaneously, and the ordered prizes are awarded to the best performers by their ranks.

It has been found that if the contestants are evaluated and ranked by their "most favorable shocks" in a collection of attempts, our noisy-ranking model delivers exactly the same success functions as a lottery contest. Stochastic equivalence can therefore be established between our noisy-ranking contest model and the family of lottery contests. The implications of this result are multi-fold. Firstly, it provides an alternative interpretation of lottery contests, in particular, the multiple-winner nested contest model (Clark and Riis, 1996b and 1998a): a noisy ranking system can be uncovered beneath its literally sequential lottery process. Secondly, this result illuminates a hidden common thread that connects a wide variety of seemingly disparate contests within the framework of ratio-form contest success functions: underlying all these contests is a common winning mechanism that honors contestants' most favorable shocks! Thus, this result provides a behavioral foundation that underpins the family of commonly adopted lottery contest models. Finally, our result nevertheless imposes a limit on the boundary of this broad class of models: the family of contests that can be united in the framework of lottery contests may not include competition schemes that do not honor "the most favorable shocks" on contestants' performance.

5 Appendix: The Proof of the "Antithesis"

Here, it is proven that when there are three contestants ($I = 3$), the noisy-ranking contest model presented in Section 3.2 does not deliver a standard lottery contest.

Following Hirshleifer and Riley (1992), we utilize a formulation with multiplicative noise term:

$$y_i = q_i g_i(x_i), \tag{A.1}$$

where the q_i follows a Weibull minimum distribution with c.d.f. $1 - e^{-q_i}$. (A.1) can be equivalently expressed as

$$\log y_i = \log g_i(x_i) + \log q_i. \tag{A.2}$$

This distribution of $\varepsilon_i \triangleq \log q_i$ is a type I extreme-value (minimum) distribution. The c.d.f. and p.d.f. of ε_i are thus

$$\begin{aligned} F(\varepsilon_i) &= 1 - \exp(-e^{\varepsilon_i}), \text{ and} && \text{(A.3)} \\ f(\varepsilon_i) &= e^{\varepsilon_i - e^{\varepsilon_i}}. && \text{(A.4)} \end{aligned}$$

Consider the case of three contestants ($I = 3$). Given effort x_i, contestant 1 wins with the following probability

$$\begin{aligned} & \int_{-\infty}^{+\infty} [\Pi_{j=2,3} F(\varepsilon_1 + \log g_1(x_1) - \log g_j(x_j))] f(\varepsilon_1) d\varepsilon_1 \\ = & \int_{-\infty}^{+\infty} [(1 - \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_2(x_2)}))(1 - \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_3(x_3)})] e^{\varepsilon_1 - e^{\varepsilon_1}} d\varepsilon_1 \\ = & \; 1 - \int_{-\infty}^{+\infty} \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_2(x_2)}) \cdot e^{\varepsilon_1 - e^{\varepsilon_1}} d\varepsilon_1 \\ & - \int_{-\infty}^{+\infty} \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_3(x_3)}) \cdot e^{\varepsilon_1 - e^{\varepsilon_1}} d\varepsilon_1 \\ & + \int_{-\infty}^{+\infty} \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_2(x_2)}) \cdot \exp(-e^{\varepsilon_1 + \log g_1(x_1) - \log g_3(x_3)}) \cdot e^{\varepsilon_1 - e^{\varepsilon_1}} d\varepsilon_1 \end{aligned}$$

$$
\begin{aligned}
&= 1-\int_{-\infty}^{+\infty}\exp(\varepsilon_1-e^{\varepsilon_1}(1+\frac{g_1(x_1)}{g_2(x_2)}))d\varepsilon_1-\int_{-\infty}^{+\infty}\exp(\varepsilon_1-e^{\varepsilon_1}(1+\frac{g_1(x_1)}{g_3(x_3)}))d\varepsilon_1\\
&\quad+\int_{-\infty}^{+\infty}\exp(\varepsilon_1-e^{\varepsilon_1}(1+\frac{g_1(x_1)}{g_2(x_2)}+\frac{g_1(x_1)}{g_3(x_3)}))d\varepsilon_1\\
&= 1-\int_{-\infty}^{+\infty}\exp(\varepsilon_1-e^{\varepsilon_1+\log(1+\frac{g_1(x_1)}{g_2(x_2)})})d\varepsilon_1-\int_{-\infty}^{+\infty}\exp(\varepsilon_1-e^{\varepsilon_1+\log(1+\frac{g_1(x_1)}{g_3(x_3)})})d\varepsilon_1\\
&\quad+\int_{-\infty}^{+\infty}\exp(\varepsilon_1-e^{\varepsilon_1+\log(1+\frac{g_1(x_1)}{g_2(x_2)}+\frac{g_1(x_1)}{g_3(x_3)})})d\varepsilon_1\\
&= 1-\frac{g_2(x_2)}{g_1(x_1)+g_2(x_2)}-\frac{g_3(x_3)}{g_1(x_1)+g_3(x_3)}\\
&\quad+\frac{g_2(x_2)g_3(x_3)}{g_1(x_1)g_3(x_3)+g_2(x_2)g_3(x_3)+g_1(x_1)g_2(x_2)}.
\end{aligned}
$$

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