Market Segmentation and Product Steering: Supplementary Appendix

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This Supplementary Appendix extends Theorem 1 to correlated valuations (Section SA.1), non-identical distributions (Section SA.2), continuous valuations (Section SA.3), and the case that the seller offers two products in each market (the aggregate market $\bar{\mu}$ being the uniform distribution over $[0,1]^n$; Section SA.4).

SA.1 Correlated Valuations

Our model in Section I supposes that a consumer's valuation for one product is statistically independent of his valuations for other products. Naturally, in certain applications a consumer's valuations for different products may be correlated; for instance, books by the same author, or from the same genre, might be valued similarly.

We present a simple generalization of the model that allows for correlation between valuations. Replace the definition of the aggregate market $\bar{\mu}$ in (I) by

$$\bar{\mu}(\mathbf{v}) = f(v_1) \prod_{k=2}^{n} \Big(t\delta^{v_{k-1}}(v_k) + (1-t) f(v_k) \Big), \quad \forall \mathbf{v} \in X^n,$$

where $t \in [0,1)$ and $\delta^x \in \Delta X$ denotes the Dirac measure centered on $x \in X$. Thus, the valuation vector corresponds to a Markov chain. With probability t, the valuation for product k coincides with the one for product $k-1$; with probability $1-t$, the valuation

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for product k has distribution f, the distribution of the first product. The interpretation is that adjacent products are similar, so that consumers may have similar valuations. The correlation between the valuations, captured by t, can be arbitrarily strong; we only exclude perfect correlation. The baseline model assumed $t = 0$ (no correlation).

We now show that Theorem 1, as stated in Section II, extends to this model. Lemma 1 and thus the first sentence of the theorem obviously still hold. To prove the second sentence of the theorem, we only need to show that Lemma 3 still holds.

We present an adapted proof of Lemma 3. The broad idea is as follows. Because the correlation is imperfect, the aggregate market can still be segmented such that, independently for each product k, the distribution of valuations is either equal to a given g_i or some residual. In contrast to the original proof of Lemma 3, the residual now depends on the valuation for product $k-1$. We then show that the seller always prefers to offer a product for which the distribution of valuations is g_i.

Proof of Lemma 3. Let $(x_i, c_i) \in \tilde{S}$. Analogously to the proof of Lemma 3 for the original model, choose $\lambda \in (0,1)$ such that

$$\lambda g_i(x) \leq t\delta^y(x) + (1-t)f(x), \quad \forall x, y \in X,$$
$$\lambda g_i(x) \leq f(x), \quad \forall x \in X,$$

and define $h(\cdot \mid v_0) \in \Delta X$, and, for every $y \in X$, $h(\cdot \mid y) \in \Delta X$, as

$$h(\cdot \mid v_0) := \frac{1}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i,$$
$$h(\cdot \mid y) := \frac{t}{1-\lambda}\delta^y + \frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i.$$

Finally, set $p^* := x_m$ if $c_i = 0$; set $p^* := x_i$ if $c_i = \overline{u}(x_i)$.

Next, we present a market segmentation τ supported on 2^n markets. The markets in the support of τ are indexed by superscript $\mathbf{a} \in \{\mathtt{g}, \mathtt{h}\}^n$. The notation $\zeta(a_k)$ will also be used and means 1 if $a_k = \mathtt{g}$ and 0 if $a_k = \mathtt{h}$. Set

$$\tau(\mu^{\mathbf{a}}) := \prod_k \Big(\zeta(a_k)\lambda + (1-\zeta(a_k))(1-\lambda)\Big), \quad \forall \mathbf{a} \in \{\mathtt{g}, \mathtt{h}\}^n.$$

Market $\mu^{\mathbf{a}}$ is given by

$$\mu^{\mathbf{a}}(\mathbf{v}) := \frac{\prod_k \Big(\zeta(a_k)\lambda g_i(v_k) + (1-\zeta(a_k))(1-\lambda)h(v_k \mid v_{k-1})\Big)}{\tau(\mu^{\mathbf{a}})}$$
$$= \prod_k \Big(\zeta(a_k) g_i(v_k) + (1-\zeta(a_k))h(v_k \mid v_{k-1})\Big), \quad \forall \mathbf{v} \in X^n.$$

Then τ is a market segmentation:

$$\sum_{\mathbf{a}} \tau(\mu^{\mathbf{a}})\mu^{\mathbf{a}}(\mathbf{v}) = f(v_1)\prod_{k=2}^{n}\Big(t\delta^{v_{k-1}}(v_k) + (1-t)f(v_k)\Big) = \bar{\mu}(\mathbf{v}), \quad \forall \mathbf{v} \in X^n.$$

Next, consider any market $\mu^{\mathbf{a}}$. If $a_k = \mathtt{g}$ for any $k \in \{1,\ldots,n\}$, then

$$\begin{aligned}
\mu_k^{\mathbf{a}}(x) &= \sum_{\mathbf{v}:v_k=x} \mu^{\mathbf{a}}(\mathbf{v}) \\
&= \sum_{\mathbf{v}:v_k=x} \prod_{k'} \Big(\zeta(a_{k'}) g_i(v_{k'}) + (1-\zeta(a_{k'}))h(v_{k'} \mid v_{k'-1})\Big) \\
&= \sum_{v_1,\ldots,v_{k-1}} \prod_{k'<k} \Big(\zeta(a_{k'}) g_i(v_{k'}) + (1-\zeta(a_{k'}))h(v_{k'} \mid v_{k'-1})\Big) g_i(x) \\
&= g_i(x), \quad \forall x \in X.
\end{aligned}$$

Hence,

$$\max_p p \sum_{x\geq p} \mu_k^{\mathbf{a}}(x) = \max_p p \sum_{x \geq p} g_i(x) = p^* \sum_{x \geq p^*} g_i(x) = x_i. \tag{SA.1}$$

In the following, we show that if $a_k = \mathtt{h}$ for any $k \in \{1,\ldots,n\}$, then

$$x_i \geq \max_p p \sum_{x \geq p} \mu_k^{\mathbf{a}}(x). \tag{SA.2}$$

For $k = 1$,

$$\begin{aligned}
\mu_k^{\mathbf{a}}(x) &= \sum_{\mathbf{v}:v_k=x} \mu^{\mathbf{a}}(\mathbf{v}) \\
&= \sum_{\mathbf{v}:v_k=x} \prod_{k'} \Big(\zeta(a_{k'}) g_i(v_{k'}) + (1-\zeta(a_{k'}))h(v_{k'} \mid v_{k'-1})\Big) \\
&= h(x \mid v_0).
\end{aligned}$$

For $\mu_k^{\mathbf{a}} = h(\cdot \mid v_0)$, (SA.2) holds by Lemma 2.

So suppose $k \in \{2, \ldots, n\}$. Let $r^* \in \{0 \ldots, k-1\}$ be the number of products such that $a_{k'} = \mathtt{h}$ for $k' < k$ and $a_{k''} \neq \mathtt{g}$ for $k' < k'' < k$. Then

$$\begin{aligned}
\mu_k^{\mathbf{a}}(x) &= \sum_{\mathbf{v}: v_k = x} \mu^{\mathbf{a}}(\mathbf{v}) \\
&= \sum_{\mathbf{v}: v_k = x} \prod_{k'} \Big(\zeta(a_{k'}) g_i(v_{k'}) + (1 - \zeta(a_{k'})) h(v_{k'} \mid v_{k'-1}) \Big) \\
&= \sum_{v_1, \ldots, v_{k-1}} \prod_{k' < k} \Big(\zeta(a_{k'}) g_i(v_{k'}) + (1 - \zeta(a_{k'})) h(v_{k'} \mid v_{k'-1}) \Big) h(x \mid v_{k-1}) \\
&= \sum_{v_{k-r^*-1}, \ldots, v_{k-1}} e(v_{k-r^*-1}) \left(\prod_{k'=k-r^*}^{k-1} h(v_{k'} \mid v_{k'-1}) \right) h(x \mid v_{k-1}),
\end{aligned}$$

where $e \in \{g_i, h(\cdot \mid v_0)\}$.

We show by induction that

$$\sum_{v_{k-r-1}, \ldots, v_{k-1}} e(v_{k-r-1}) \left(\prod_{k'=k-r}^{k-1} h(v_{k'} \mid v_{k'-1}) \right) h(\cdot \mid v_{k-1}) \in \Delta X$$

is equal to

$$\left(\frac{t}{1-\lambda} \right)^{r+1} e + \frac{1 - \left(\frac{t}{1-\lambda} \right)^{r+1}}{1 - \frac{t}{1-\lambda}} \left(\frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i \right) \in \Delta X$$

for all $r \in \mathbb{N}$. If $r = 0$, then

$$\begin{aligned}
\sum_{v_{k-1}} e(v_{k-1}) h(\cdot \mid v_{k-1}) &= \sum_{v_{k-1}} e(v_{k-1}) \frac{t}{1-\lambda} \delta^{v_{k-1}} + \frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i \\
&= \frac{t}{1-\lambda} e + \frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i.
\end{aligned}$$

Suppose equality holds for a given $r \geq 0$. Then equality holds for $r+1$:

$$\begin{aligned}
& \sum_{v_{k-r-2}, \ldots, v_{k-1}} e(v_{k-r-2}) \left(\prod_{k'=k-r-1}^{k-1} h(v_{k'} \mid v_{k'-1}) \right) h(\cdot \mid v_{k-1}) \\
=\ & \sum_{v_{k-r-2}, \ldots, v_{k-1}} e(v_{k-r-2}) \left(\prod_{k'=k-r-1}^{k-1} h(v_{k'} \mid v_{k'-1}) \right) \frac{t}{1-\lambda} \delta^{v_{k-1}} + \frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i \\
=\ & \frac{t}{1-\lambda} \sum_{v_{k-r-2}, \ldots, v_{k-2}} e(v_{k-r-2}) \left(\prod_{k'=k-r-1}^{k-2} h(v_{k'} \mid v_{k'-1}) \right) h(\cdot \mid v_{k-2}) + \frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i \\
=\ & \frac{t}{1-\lambda} \left(\left(\frac{t}{1-\lambda} \right)^{r+1} e + \frac{1 - \left(\frac{t}{1-\lambda} \right)^{r+1}}{1 - \frac{t}{1-\lambda}} \left(\frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i \right) \right) + \frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i \\
=\ & \left(\frac{t}{1-\lambda} \right)^{r+2} e + \frac{1 - \left(\frac{t}{1-\lambda} \right)^{r+2}}{1 - \frac{t}{1-\lambda}} \left(\frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i \right).
\end{aligned}$$

We have shown that

$$\mu_k^{\mathbf{a}} = \left(\frac{t}{1-\lambda}\right)^{r^*+1} e + \frac{1-\left(\frac{t}{1-\lambda}\right)^{r^*+1}}{1-\frac{t}{1-\lambda}} \left(\frac{1-t}{1-\lambda} f - \frac{\lambda}{1-\lambda} g_i\right).$$

So

$$\begin{aligned}
& \max_p p \sum_{x \geq p} \mu_k^{\mathbf{a}}(x) \\
= \quad & \max_p p \sum_{x \geq p} \left(\left(\frac{t}{1-\lambda}\right)^{r^*+1} e(x) + \frac{1-\left(\frac{t}{1-\lambda}\right)^{r^*+1}}{1-\frac{t}{1-\lambda}} \left(\frac{1-t}{1-\lambda} f(x) - \frac{\lambda}{1-\lambda} g_i(x)\right)\right) \\
\leq \quad & \left(\frac{t}{1-\lambda}\right)^{r^*+1} \max_p p \sum_{x \geq p} e(x) + \frac{1-\left(\frac{t}{1-\lambda}\right)^{r^*+1}}{1-\frac{t}{1-\lambda}} \max_p p \sum_{x \geq p} \left(\frac{1-t}{1-\lambda} f(x) - \frac{\lambda}{1-\lambda} g_i(x)\right) \\
\leq \quad & \left(\frac{t}{1-\lambda}\right)^{r^*+1} x_i + \frac{1-\left(\frac{t}{1-\lambda}\right)^{r^*+1}}{1-\frac{t}{1-\lambda}} \max_p p \sum_{x \geq p} \left(\frac{1-t}{1-\lambda} f(x) - \frac{\lambda}{1-\lambda} g_i(x)\right).
\end{aligned}$$

For any $p \in \{x_1, \ldots, x_{i-1}\}$:

$$p \sum_{x \geq p} \left(\frac{1-t}{1-\lambda} f(x) - \frac{\lambda}{1-\lambda} g_i(x)\right) \leq \frac{1-t}{1-\lambda} p - \frac{\lambda}{1-\lambda} p,$$

implying

$$\begin{aligned}
p \sum_{x \geq p} \mu_k^{\mathbf{a}}(x) \quad & \leq \quad \left(\frac{t}{1-\lambda}\right)^{r^*+1} x_i + \frac{1-\left(\frac{t}{1-\lambda}\right)^{r^*+1}}{1-\frac{t}{1-\lambda}} \frac{1-t-\lambda}{1-\lambda} p \\
& = \quad \left(\frac{t}{1-\lambda}\right)^{r^*+1} x_i + \left(1-\left(\frac{t}{1-\lambda}\right)^{r^*+1}\right) p \\
& \leq \quad x_i.
\end{aligned}$$

For any $p \in \{x_i, \ldots, x_m\}$:

$$p \sum_{x \geq p} \left(\frac{1-t}{1-\lambda} f(x) - \frac{\lambda}{1-\lambda} g_i(x)\right) \leq \frac{1-t}{1-\lambda} \pi_0 - \frac{\lambda}{1-\lambda} x_i,$$

implying

$$\begin{aligned}
p \sum_{x \geq p} \mu_k^{\mathbf{a}}(x) \quad & \leq \quad \left(\frac{t}{1-\lambda}\right)^{r^*+1} x_i + \frac{1-\left(\frac{t}{1-\lambda}\right)^{r^*+1}}{1-\frac{t}{1-\lambda}} \frac{1-t-\lambda}{1-\lambda} x_i \\
& = \quad x_i.
\end{aligned}$$

Thus, (SA.2) holds.

By (SA.1) and (SA.2), there exists an optimal strategy ρ for the seller with the following property: for every market $\mu^{\mathbf{a}}$ such that $a_k = \mathsf{g}$ for some product k, offer such a product at price p^*.

Lastly, observe that the only market $\mu^{\mathbf{a}}$ with $a_k \neq \mathsf{g}$ for all $k \in \{1, \dots, n\}$ has

$$\tau(\mu^{\mathbf{a}}) = (1-\lambda)^n.$$

Let π_n be the surplus of the seller, and u_n the consumer surplus, under this market segmentation and such an optimal strategy. Then

$$\lim_{n\to\infty} \pi_n = \lim_{n\to\infty} (1-(1-\lambda)^n) x_i = x_i,$$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1-(1-\lambda)^n) \sum_{x \geq p} g_i(x)(x-p^*) = c_i.$$

□

SA.2 Non-Identical Distributions

Our model in Section I assumes that a consumer's valuation for any product $k \in \{1, \dots, n\}$ is drawn from the same distribution f. For the characterization of feasible surplus pairs in Theorem 1, f mattered only through π_0, the maximum producer surplus without market segmentation. In this section, we extend our characterization to a more general setting in which the valuations for different products may be drawn from different distributions.

To work with an infinite sequence of distributions, we impose a lower bound on the probabilities. Fix some $\varepsilon \in (0, 1/m)$. Let F be the subset of distributions $e \in \Delta X$ such that

$$e(x) \geq \varepsilon, \quad \forall x \in X. \tag{SA.3}$$

Throughout in this section, we hold fixed an arbitrary sequence $(f_l)_{l\in\mathbb{N}}$ of distributions in F. For any given number of products $n \in \mathbb{N}$, replace the definition of the aggregate market $\bar{\mu}$ in (1) by

$$\bar{\mu}(\mathbf{v}) = \prod_k f_k(v_k), \quad \forall \mathbf{v} \in X^n,$$

that is, the valuations for products $k = 1, \dots, n$ are independently drawn from $f_1, \dots, f_n$.

The maximum producer surplus without market segmentation now depends on the number of products; denote it by

$$\pi'_{0,n} := \max\left\{\max_p p \sum_{x \geq p} f_k(x),\, k = 1,\dots,n\right\}.$$

Define furthermore

$$\pi'_0 := \sup_{n \in \mathbb{N}} \pi'_{0,n},$$

and suppose that there is some $x' \in X$ such that $\pi'_0 = x'$.

We can now state our generalization of Theorem 1. The only change concerns the maximum producer surplus without market segmentation.

Theorem SA.1. *For every $n \in \mathbb{N}$, the set S_n of feasible surplus pairs is contained in*

$$S'_n := \left\{(\pi,u) \in \mathbb{R}^2 \mid \pi \in [\pi'_{0,n}, x_m], u \in [0, \overline{u}(\pi)]\right\}.$$

Moreover, for every

$$(\pi,u) \in S' := \left\{(\pi,u) \in \mathbb{R}^2 \mid \pi \in [\pi'_0, x_m], u \in [0, \overline{u}(\pi)]\right\},$$

there exists a sequence $\Big((\pi_n,u_n)\Big)_{n \in \mathbb{N}}$ such that $(\pi_n,u_n) \in S_n$ and $(\pi_n,u_n) \underset{n\to\infty}{\longrightarrow} (\pi,u)$.

Fix any of the distributions g_i used in the proof of Theorem 1. By (SA.3), there exists for every $f' \in F$ a distribution $h' \in \Delta X$ with $\varepsilon g_i + (1-\varepsilon)h' = f'$. The proof of Theorem SA.1 is then analogous to the proof of Theorem 1, and therefore omitted. In particular, to attain a point $(\pi,u) \in S'$ with $\pi = x_i \in \{x', \dots, x_m\}$, decompose each distribution f_k into g_i and a residual $h_k \in \Delta X$, with the same weights ε and $1-\varepsilon$ for each product k.[1] As in the original model, the seller prefers to offer a product for which the consumers' valuations are distributed according to g_i rather than the residual. In a second step, construct a market segmentation τ by independently drawing g_i or h_k for each product $k = 1,\dots,n$, again with the same weights ε and $1-\varepsilon$ for each product. For large n, it is then again almost certain that the valuations for at least one product are distributed according to g_i.

[1] The weight ε plays the role of the weight λ in Lemma 2.

SA.3 Continuous Valuations

In this section, given any topological space Y, $\mathcal{B}(Y)$ denotes the Borel σ-algebra, and ΔY denotes the set of Borel probability measures on Y. We endow ΔY with the weak* topology.[2]

Let now $X = [0,1]$, and let f be an atomless probability measure in ΔX with full support. The rest of the model is analogous to the model in Section I.

The aggregate market $\bar{\mu} \in \Delta X^n$ is defined by

$$\bar{\mu}(B_1 \times \cdots \times B_n) = \prod_k f(B_k), \quad \forall B_1 \times \cdots \times B_n \in \prod_k \mathcal{B}(X).$$

A market segmentation is a distribution $\tau \in \Delta\Delta X^n$ of markets $\mu \in \Delta X^n$ that averages to $\bar{\mu}$:

$$\int \mu(B_1 \times \cdots \times B_n)\tau(\mathrm{d}\mu) = \prod_k f(B_k), \quad \forall B_1 \times \cdots \times B_n \in \prod_k \mathcal{B}(X). \tag{SA.4}$$

We use the notation μ_k for the k^{th}-marginal of market μ.[3]

A strategy of the seller is a mapping $\rho : \Delta X^n \times \mathcal{B}(\{1,\ldots,n\} \times X) \to [0,1]$ such that $\rho(\mu,\cdot) \in \Delta(\{1,\ldots,n\} \times X)$ for all $\mu \in \Delta X^n$ and $\mu \mapsto \rho(\mu,\{k\} \times B)$ is measurable for all $\{k\} \times B \in \mathcal{B}(\{1,\ldots,n\} \times X)$. Thus, a strategy selects, potentially randomly, a product $k \in \{1,\ldots,n\}$ to be offered and a price $p \in X$ to be charged for any market $\mu \in \Delta X^n$.

The producer surplus under market segmentation τ and strategy ρ is

$$\Pi_\tau(\rho) := \int\int p\mu_k([p,1])\rho(\mu,\mathrm{d}(k,p))\tau(\mathrm{d}\mu),$$

and the consumer surplus is

$$U_\tau(\rho) := \int\int\int_p^1 (x-p)\mu_k(\mathrm{d}x)\rho(\mu,\mathrm{d}(k,p))\tau(\mathrm{d}\mu).$$

We repeatedly use that the consumer surplus can be written as

$$U_\tau(\rho) = \int\int\int_p^1 \mu_k([x,1])\mathrm{d}x\rho(\mu,\mathrm{d}(k,p))\tau(\mathrm{d}\mu),$$

[2] All probability measures on product spaces in this subsection are uniquely defined by its values on the products of the Borel σ-algebras (see Aliprantis and Border, 2006, Thms. 4.44 and 10.10). We write "for all products of Borel sets" rather than "for all Borel sets of the product space" where convenient.

[3] Thus, $\mu_k \in \Delta X$, with $\mu_k(B) = \int_{\mathbf{v}:v_k\in B}\mu(\mathrm{d}\mathbf{v})$ for all $B \in \mathcal{B}(X)$.

using a well-known formula for expected values.[4] Strategy ρ^* is optimal for the seller under market segmentation τ if $\rho^* \in \arg\max_\rho \Pi_\tau(\rho)$.

A combination of producer and consumer surplus (π, u) is feasible if there exist a market segmentation τ and an optimal strategy ρ such that $(\pi, u) = (\Pi_\tau(\rho), U_\tau(\rho))$. For given n, the set of feasible surplus pairs is denoted by S_n.

For any $\pi \in (0,1]$, define $g_\pi \in \Delta X$ by

$$g_\pi([0,x]) := \begin{cases} 0 & \text{if } x \in [0,\pi), \\ 1 - \frac{\pi}{x} & \text{if } x \in [\pi, 1), \\ 1 & \text{if } x = 1. \end{cases} \tag{SA.5}$$

Thus, g_π assigns zero probability to $[0,\pi)$, and for any given $x \in [\pi, 1]$, the set $[x,1]$ has probability π/x. The probability measure g_π is the analog to g_i in the original model. Note that

$$\int_p^1 p g_\pi(\mathrm{d}v) = p g_\pi([p,1]) = \begin{cases} p & \text{if } p \in [0,\pi), \\ \pi & \text{if } p \in [\pi, 1], \end{cases} \tag{SA.6}$$

analogous to property (2) of the distributions $\{g_i\}_{i=1}^m$ in the original model.

Lastly, define

$$\overline{u}(\pi) := \int_\pi^1 g_\pi([x,1])\mathrm{d}x = \int_\pi^1 \pi/x \mathrm{d}x = -\pi \ln \pi, \quad \forall \pi \in (0,1],$$

$$\pi_0 := \max_p p f([p,1]),$$

and the set

$$S := \Big\{(\pi, u) \in \mathbb{R}^2 \mid \pi \in [\pi_0, 1], u \in [0, \overline{u}(\pi)]\Big\}.$$

We now show that Theorem 1, as stated in Section II, extends to this model. Our proof of Theorem 1 uses three lemmas. The first lemma is the analog to Lemma 1.

Lemma SA.1. *For every $n \in \mathbb{N}$, $(\pi, u) \in S_n$ implies $u \leq \overline{u}(\pi)$.*

Proof. Let τ be any market segmentation, and let ρ be any strategy that is optimal given τ such that $\Pi_\tau(\rho) = \pi$ and $U_\tau(\rho) = u$. Define the probability measure $h \in \Delta X$ by

$$h(B) := \int \int \mu_k(B) \rho(\mu, \mathrm{d}(k,p)) \tau(\mathrm{d}\mu), \quad \forall B \in \mathcal{B}(X).$$

[4] See formula (21.9) in Billingsley (1995).

By the optimality of ρ, we have for any $q \in [\pi, 1]$

$$\begin{aligned}\int\int q\mu_k([q,1])\rho(\mu, \mathrm{d}(k,p))\tau(\mathrm{d}\mu) &\leq \int\int p\mu_k([q,1])\rho(\mu, \mathrm{d}(k,p))\tau(\mathrm{d}\mu) \\ &= \Pi_\tau(\rho) = \pi = qg_\pi([q,1]).\end{aligned}$$

Dividing through by q, we see that g_π first-order stochastically dominates h. Hence,

$$\begin{aligned}u &= \int\int\int_p^1 \mu_k([x,1])\mathrm{d}x\rho(\mu, \mathrm{d}(k,p))\tau(\mathrm{d}\mu) \\ &= \int\int\int \mu_k([x,1])\mathrm{d}x\rho(\mu, \mathrm{d}(k,p))\tau(\mathrm{d}\mu) - \int\int\int_0^p \mu_k([x,1])\mathrm{d}x\rho(\mu, \mathrm{d}(k,p))\tau(\mathrm{d}\mu) \\ &\leq \int\int\int \mu_k([x,1])\mathrm{d}x\rho(\mu, \mathrm{d}(k,p))\tau(\mathrm{d}\mu) - \int\int\int_0^p \mu_k([p,1])\mathrm{d}x\rho(\mu, \mathrm{d}(k,p))\tau(\mathrm{d}\mu) \\ &= \int\int\int \mu_k([x,1])\mathrm{d}x\rho(\mu, \mathrm{d}(k,p))\tau(\mathrm{d}\mu) - \pi \\ &= \int\int\int \mu_k([x,1])\rho(\mu, \mathrm{d}(k,p))\tau(\mathrm{d}\mu)\mathrm{d}x - \pi \\ &= \int h([x,1])\mathrm{d}x - \pi \\ &\leq \int g_\pi([x,1])\mathrm{d}x - \pi \\ &= \overline{u}(\pi),\end{aligned}$$

where we used Fubini's Theorem for the fifth row. □

The next lemma is similar to Lemma 3.

Lemma SA.2. *Let $e, h \in \Delta X$ and $\lambda \in (0,1)$ such that*

$$\lambda e(B) + (1-\lambda)h(B) = f(B) \quad \forall B \in \mathcal{B}(X), \tag{SA.7}$$

$$\max_p pe([p,1]) \geq \max_p ph([p,1]). \tag{SA.8}$$

Let $p^* \in \arg\max_p pe([p,1])$. There exists a sequence $\big((\pi_n, u_n)\big)_{n\in\mathbb{N}}$ such that $(\pi_n, u_n) \in S_n$ and

$$(\pi_n, u_n) \underset{n\to\infty}{\longrightarrow} \left(p^* e([p^*,1]), \int_{p^*}^1 e([x,1])\mathrm{d}x\right). \tag{SA.9}$$

Proof. Fix some $n \in \mathbb{N}$. We present a market segmentation τ supported on 2^n markets. As in the proof of Lemma 3, τ is constructed by independently drawing e or h with probability λ and $1-\lambda$, respectively, for each product $k = 1, \ldots, n$.

Formally, define for every tuple $\mathbf{a} = (a_1, \ldots, a_n) \in \{e, h\}^n$ a market $\mu^{\mathbf{a}}$ with

$$\mu^{\mathbf{a}}(B_1 \times \cdots \times B_n) := \prod_k a_k(B_k), \quad \forall B_1 \times \cdots \times B_n \in \prod_k \mathcal{B}(X).$$

Using the notation $\zeta(a_k) := \lambda$ if $a_k = e$ and $\zeta(a_k) := 1 - \lambda$ if $a_k = h$, set

$$\tau(\mu^{\mathbf{a}}) := \prod_k \zeta(a_k), \quad \forall \mathbf{a} \in \{e, h\}^n.$$

Then τ is a market segmentation:

$$\sum_{\mathbf{a}} \tau(\mu^{\mathbf{a}})\mu^{\mathbf{a}}(B_1 \times \cdots \times B_n) = \prod_k f(B_k) = \bar{\mu}(B_1 \times \cdots \times B_n), \quad \forall B_1 \times \cdots \times B_n \in \prod_k \mathcal{B}(X).$$

For every market $\mu^{\mathbf{a}}$, $a_k = e$ implies $\mu^{\mathbf{a}}_k = e$, and $a_k = h$ implies $\mu^{\mathbf{a}}_k = h$.

Next, we describe a strategy ρ as follows:

- For every market $\mu^{\mathbf{a}}$ in the support of τ such that $a_k = e$ for some product k, offer any such product at price p^*.
- For the unique market $\mu^{\mathbf{a}}$ in the support of τ such that $a_k = h$ for all products k, offer product $k = 1$ at some fixed price $p' \in \arg\max_x ph([x, 1])$.
- For every market outside of the support of τ, offer product $k = 1$ at price p'.

We have not specified how ρ selects among products k with $a_k = e$, but this indeterminacy will not matter. Note furthermore that V is compact and metrizable, which implies that ΔX^n is metrizable (Aliprantis and Border, 2006, Thm. 15.11). Hence, any finite subset of ΔX^n is a Borel set. Because the support of τ is finite, this implies that $\mu \mapsto \rho(\mu, \{k\} \times B)$ is measurable for all $k \in \{1, \ldots, n\}$ and all $B \in \mathcal{B}(X)$, as required by the definition of a strategy. By (SA.7), ρ is optimal.

Lastly, observe that the only market $\mu^{\mathbf{a}}$ with $a_k = h$ for all $k \in \{1, \ldots, n\}$ has

$$\tau(\mu^{\mathbf{a}}) = (1 - \lambda)^n.$$

Let π_n be the surplus of the seller, and u_n the consumer surplus, under this market segmentation and such an optimal strategy. Then

$$\lim_{n\to\infty} \pi_n = \lim_{n\to\infty} (1 - (1 - \lambda)^n)p^* e([p^*, 1]) = p^* e([p^*, 1]),$$
$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (1 - (1 - \lambda)^n) \int_{p^*}^1 e([x, 1])\mathrm{d}x = \int_{p^*}^1 e([x, 1])\mathrm{d}x.$$

$\square$

Lemma SA.2 is not directly useful because the probability measures g_π have an atom at $x=1$ whereas f is atomless; thus, f cannot be split into some g_π and some h as in (SA.7). We will approximate the respective g_π by atomless probability measures. When doing so, we must furthermore make sure that the seller sets the right price. To this end, we now state a third lemma.

Fix some $\pi \in (0,1)$, some $p \in [\pi,1]$, and some $\epsilon \in [0,1]$. For $N \in \mathbb{N}$, let

$$\pi = x_0^N < x_1^N < \ldots < x_N^N = 1$$

be a collection of points in $[\pi,1]$ of equal distance. For each $i=1,\ldots,N$, define

$$\alpha_{\pi,p}^{\epsilon,N}(i) := \frac{(1-\epsilon)g_\pi([x_{i-1}^N, x_i^N]) + \epsilon\delta^p([x_{i-1}^N, x_i^N])}{f([x_{i-1}^N, x_i^N])},$$

where $\delta^p \in \Delta X$ denotes the Dirac measure centered on p. Let

$$\lambda_{\pi,p}^N := \Big(\max_{\substack{\epsilon\in[0,1] \\ i\in\{1,\ldots,N\}}} a_{\pi,p}^{\epsilon,N}(i) \Big)^{-1} \tag{SA.10}$$

and note that $\lambda_{\pi,p}^N \in (0,1)$ because g_π and δ^p are probability measures and assign probability one to $[\pi,1]$ whereas f assigns probability strictly less than one to $[\pi,1]$. For each $i=1,\ldots,N$, let furthermore

$$\beta_{\pi,p}^{\epsilon,N}(i) = \frac{1}{1-\lambda_{\pi,p}^N} - \frac{\lambda_{\pi,p}^N}{1-\lambda_{\pi,p}^N}\alpha_{\pi,p}^{\epsilon,N}(i).$$

We now introduce two probability measures in ΔX: the probability measure $e_{\pi,p}^{\epsilon,N}$, which has support $[\pi,1]$ and is given by

$$e_{\pi,p}^{\epsilon,N}([x,x_i^N]) := \alpha_{\pi,p}^{\epsilon,N}(i) f([x,x_i^N]), \quad \forall x \in [x_{i-1}^N, x_i^N], \forall i=1,\ldots,N, \tag{SA.11}$$

and the probability measure $h_{\pi,p}^{\epsilon,N}$ given by

$$h_{\pi,p}^{\epsilon,N}([0,x_i]) := \frac{1}{1-\lambda_{\pi,p}^N} f([0,x]), \quad \forall x \in [0,\pi),$$
$$h_{\pi,p}^{\epsilon,N}([x,x_i]) := \beta_{\pi,p}^{\epsilon,N}(i) f([x,x_i^N]), \quad \forall x \in [x_{i-1}^N, x_i^N], \forall i=1,\ldots,N.$$

Then, f is a mixture of $e_{\pi,p}^{\epsilon,N}$ and $h_{\pi,p}^{\epsilon,N}$:

$$\lambda_{\pi,p}^N e_{\pi,p}^{\epsilon,N}(B) + (1-\lambda_{\pi,p}^N) h_{\pi,p}^{\epsilon,N}(B) = f(B) \quad \forall B \in \mathcal{B}(X). \tag{SA.12}$$

Lemma SA.3. *Let $\pi \in (0,1)$ and $p \in [\pi,1]$. For every $\epsilon \in (0,1]$, let $(p^{\epsilon,N})_{N\in\mathbb{N}}$ be a sequence of prices such that*

$$p^{\epsilon,N} \in \arg\max_x x e^{\epsilon,N}_{\pi,p}([x,1]), \quad \forall N \in \mathbb{N}.$$

Then,

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} p^{\epsilon,N} e^{\epsilon,N}_{\pi,p}([p^{\epsilon,N},1]) = \pi, \tag{SA.13}$$

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} \int_{p^{\epsilon,N}}^1 e^{\epsilon,N}_{\pi,p}([x,1])\mathrm{d}x = \int_p^1 g_\pi([x,1])\mathrm{d}x. \tag{SA.14}$$

If furthermore $\pi > \pi_0$, then

$$\lim_{\epsilon\to 0}\max_x x h^{\epsilon,N}_{\pi,p}([x,1]) < \pi_0, \quad \forall N \in \mathbb{N}. \tag{SA.15}$$

Proof. Let $\hat{e}^\epsilon_{\pi,p} \in \Delta X$ be the probability measure given by

$$\hat{e}^\epsilon_{\pi,p}(B) := (1-\epsilon)g_\pi(B) + \epsilon\delta^p(B), \quad \forall B \in \mathcal{B}(X). \tag{SA.16}$$

To prove (SA.13), we first show

$$\lim_{N\to\infty} p^{\epsilon,N} = p, \tag{SA.17}$$

$$\lim_{N\to\infty} e^{\epsilon,N}_{\pi,p}([p^{\epsilon,N},1]) = \hat{e}^\epsilon_{\pi,p}([p,1]). \tag{SA.18}$$

Fix some $\epsilon \in (0,1]$. For all $x \in [x^N_{i-1}, x^N_i]$ and all $i = 1,\dots,N-1$,

$$e^{\epsilon,N}_{\pi,p}([x,1]) = \frac{\hat{e}^\epsilon_{\pi,p}([x^N_{i-1},x^N_i])}{f([x^N_{i-1},x^N_i])} f([x,x^N_i]) + \hat{e}^\epsilon_{\pi,p}([x^N_i,1]).$$

Hence,

$$\lim_{N\to\infty} e^{\epsilon,N}_{\pi,p}([x,1]) = \hat{e}^\epsilon_{\pi,p}([x,1]), \quad \forall x \in [\pi,1). \tag{SA.19}$$

It follows that if $p<1$, then $\lim_{N\to\infty} p^{\epsilon,N} = p$ because $x \mapsto x\hat{e}^\epsilon_{\pi,p}([x,1])$ is uniquely maximized at $x = p$ by (SA.6). If $p = 1$, then $\lim_{N\to\infty} p^{\epsilon,N} = p$ because

$$\lim_{N\to\infty} x^N_{N-1} e^{\epsilon,N}_{\pi,p}([x^N_{N-1},1]) = \lim_{N\to\infty} x^N_{N-1}\hat{e}^\epsilon_{\pi,p}([x^N_{N-1},1]) = \hat{e}^\epsilon_{\pi,p}(\{1\}).$$

This shows (SA.17).

If $p^{\epsilon,N} \in [x^N_{i-1}, x^N_i]$, then

$$e^{\epsilon,N}_{\pi,p}([p^{\epsilon,N},1]) \le e^{\epsilon,N}_{\pi,p}([x^N_{i-1},1]) = \hat{e}^\epsilon_{\pi,p}([x^N_{i-1},1]) \le \hat{e}^\epsilon_{\pi,p}\left(\left[p^{\epsilon,N} - \frac{1-\pi}{N}, 1\right]\right),$$

where the last inequality holds because $[x_{i-1}^N, x_i^N]$ has length $(1-\pi)/N$. Hence,

$$\limsup_{N\to\infty} e_{\pi,p}^{\epsilon,N}([p^{\epsilon,N},1]) \leq \limsup_{N\to\infty} \hat{e}_{\pi,p}^{\epsilon}\left(\left[p^{\epsilon,N} - \frac{1-\pi}{N}, 1\right]\right) \leq \hat{e}_{\pi,p}^{\epsilon}([p,1])$$

because $x \mapsto \hat{e}_{\pi,p}^{\epsilon}([x,1])$ is upper semicontinuous. On the other hand,

$$\liminf_{N\to\infty} e_{\pi,p}^{\epsilon,N}([p^{\epsilon,N},1]) \geq \hat{e}_{\pi,p}^{\epsilon}([p,1])$$

because otherwise

$$\liminf_{N\to\infty} p^{\epsilon,N} e_{\pi,p}^{\epsilon,N}([p^{\epsilon,N},1]) = p \liminf_{N\to\infty} e_{\pi,p}^{\epsilon,N}([p^{\epsilon,N},1]) < p\hat{e}_{\pi,p}^{\epsilon}([p,1]),$$

which contradicts the optimality of $p^{\epsilon,N}$. This shows (SA.18). Together, (SA.17) and (SA.18) imply

$$\lim_{N\to\infty} p^{\epsilon,N} e_{\pi,p}^{\epsilon,N}([p^{\epsilon,N},1]) = p\hat{e}_{\pi,p}^{\epsilon}([p,1]).$$

Letting ϵ go to zero concludes the proof of (SA.13):

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} p^{\epsilon,N} e_{\pi,p}^{\epsilon,N}([p^{\epsilon,N},1]) = p g_{\pi}([p,1]) = \pi.$$

Next, we show (SA.14). For given $\epsilon \in (0,1]$, the Dominated Convergence Theorem implies

$$\begin{aligned}
\lim_{N\to\infty}\int_{p^{\epsilon,N}}^{1} e_{\pi,p}^{\epsilon,N}([x,1])\mathrm{d}x &= \int \lim_{N\to\infty} \mathbf{1}_{[p^{\epsilon,N},1]}(x) e_{\pi,p}^{\epsilon,N}([x,1])\mathrm{d}x \\
&= \int \mathbf{1}_{[p,1]}(v)\hat{e}_{\pi,p}^{\epsilon}([x,1])\mathrm{d}x \\
&= \int_{p}^{1} \hat{e}_{\pi,p}^{\epsilon}([x,1])\mathrm{d}x
\end{aligned}$$

where the second equality holds by (SA.19) and (SA.17). Letting ϵ go to zero yields (SA.14).

Finally, we prove (SA.15). Fix $N \in \mathbb{N}$. For any $\epsilon \in (0,1]$, we have

$$h_{\pi,p}^{\epsilon,N}([x,1]) = 1 - \frac{1}{1-\lambda_{\pi,p}^{N}} f([0,x]) < f([x,1]), \quad \forall x \in [0,\pi).$$

Hence

$$\lim_{\epsilon\to 0}\max_{x\in[0,\pi]} x h_{\pi,p}^{\epsilon,N}([x,1]) \leq \max_{x\in[0,\pi]} x f([x,1]) \leq \pi_0.$$

On the other hand, if $x \in [x_{i-1}^N, x_i^N]$, $i = 1, \ldots, N-1$, then

$$h_{\pi,p}^{\epsilon,N}([x,1]) = \frac{1}{1-\lambda_{\pi,p}^N} f([x,1]) - \frac{\lambda_{\pi,p}^N}{1-\lambda_{\pi,p}^N} \left(\frac{\hat{e}_{\pi,p}^{\epsilon}([x_{i-1}^N, x_i^N])}{f([x_{i-1}^N, x_i^N])} f([x, x_i^N]) + \hat{e}_{\pi,p}^{\epsilon}([x_i^N, 1]) \right).$$

By the Maximum Theorem,

$$\lim_{\epsilon \to 0} \max_{x \in [x_{i-1}^N, x_i^N]} x h_{\pi,p}^{\epsilon,N}([x,1]) = \max_{x \in [x_{i-1}^N, x_i^N]} x h_{\pi,p}^{0,N}([x,1]), \quad \forall i = 1, \ldots, N-1.$$

Analogously,

$$\lim_{\epsilon \to 0} \max_{x \in [x_{N-1}^N, 1]} x h_{\pi,p}^{\epsilon,N}([x,1]) = \max_{x \in [x_{N-1}^N, 1]} x h_{\pi,p}^{0,N}([x,1]).$$

It remains to show that

$$\max_{x \in [\pi,1]} x h_{\pi,p}^{0,N}([x,1]) < \pi_0. \tag{SA.20}$$

Note that if $x \in \{x_0^N, \ldots, x_{N-1}^N\}$, then

$$x e_{\pi,p}^{0,N}([x,1]) = x g_\pi([x,1]) = \pi > \pi_0 \geq x f([x,1])$$

and thus

$$e_{\pi,p}^{0,N}([x,1]) > f([x,1]).$$

Because the ratio

$$\frac{e_{\pi,p}^{0,N}([x, x_i^N])}{f([x, x_i^N])} = \frac{g_\pi([x_{i-1}^N, x_i^N])}{f([x_{i-1}^N, x_i^N])}$$

is the same for all $x \in (x_{i-1}^N, x_i^N)$, $i = 1, \ldots, N$, it follows that

$$e_{\pi,p}^{0,N}([x,1]) > f([x,1]), \quad \forall x \in [\pi, 1].$$

Using (SA.12), we conclude

$$h_{\pi,p}^{0,N}([x,1]) < f([x,1]), \quad \forall x \in [\pi, 1],$$

and thus

$$x h_{\pi,p}^{0,N}([x,1]) < x f([x,1]) \leq \pi_0, \quad \forall x \in [\pi, 1],$$

which implies (SA.20). $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. The first part of the theorem holds by Lemma SA.1. To prove the second part, let $(\pi,u)\in S$. Because the function $p\mapsto\int_p^1 g_\pi([x,1])\mathrm{d}x$ is continuous, there exists a price $p\in[\pi,1]$ such that

$$u=\int_p^1 g_\pi([x,1])\mathrm{d}x.$$

The function $\pi\mapsto\overline{u}(\pi)=-\pi\ln\pi$ is continuous. To prove the theorem, we may therefore assume $1>\pi>\pi_0$, because if for any such π and any $u_\pi\in[0,\overline{u}(\pi)]$ there exists a sequence $\big((\pi_n,u_n)\big)_{n\in\mathbb{N}}$ converging to (π,u_π), then there also exists a sequence converging to (π,u) for $\pi\in\{\pi_0,1\}$ and any $u\in[0,\overline{u}(\pi)]$.

We now apply Lemma SA.3. For every $\epsilon\in(0,1]$, let $(p^{\epsilon,N})_{N\in\mathbb{N}}$ be a sequence of prices such that

$$p^{\epsilon,N}\in\arg\max_x x e_{\pi,p}^{\epsilon,N}([x,1]),\quad\forall N\in\mathbb{N}.$$

Then by Lemma SA.3,

$$\begin{aligned}\lim_{\epsilon\to0}\lim_{N\to\infty}p^{\epsilon,N}e_{\pi,p}^{\epsilon,N}([p^{\epsilon,N},1])&=\pi,\\ \lim_{\epsilon\to0}\lim_{N\to\infty}\int_{p^{\epsilon,N}}^1 e_{\pi,p}^{\epsilon,N}([x,1])\mathrm{d}x&=\int_p^1 g_\pi([x,1])\mathrm{d}x.\end{aligned}$$

To prove the theorem, it therefore suffices to show that for any ϵ below some cutoff and for any N above some cutoff, there exists a sequence $\big((\pi_n,u_n)\big)_{n\in\mathbb{N}}$ such that $(\pi_n,u_n)\in S_n$ and

$$(\pi_n,u_n)\underset{n\to\infty}{\longrightarrow}\left(p^{\epsilon,N}e_{\pi,p}^{\epsilon,N}([p^{\epsilon,N},1]),\int_{p^{\epsilon,N}}^1 e_{\pi,p}^{\epsilon,N}([x,1])\mathrm{d}x\right).$$

Invoking Lemma SA.3, let ϵ be small enough and N big enough such that

$$p^{\epsilon,N}e_{\pi,p}^{\epsilon,N}([p^{\epsilon,N},1])\geq\pi_0\geq\max_x x h_{\pi,p}^{\epsilon,N}([x,1]).$$

Then, such a sequence $\big((\pi_n,u_n)\big)_{n\in\mathbb{N}}$ exists by Lemma SA.2. $\square$

SA.4 Seller Offers Two Products

In this section, we consider the setting of Section SA.3, with f being the Lebesgue measure on $X=[0,1]$, and assume that the seller can offer two products in each market.

Let $\mathcal{K}$ be the set of all subsets of $\{1,\ldots,n\}$ that have two elements.[5] A *strategy* of the seller is a mapping $\rho : \Delta X^n \times \mathcal{B}(\mathcal{K} \times X^n) \to [0,1]$ such that $\rho(\mu,\cdot) \in \Delta(\mathcal{K} \times X^n)$ for all $\mu \in \Delta X^n$ and $\mu \mapsto \rho(\mu, K \times B)$ is measurable for all $K \times B \in \mathcal{B}(\mathcal{K} \times X^n)$. Thus, a strategy selects, for any given market $\mu \in \Delta X^n$ and potentially randomly, a set $K \in \mathcal{K}$ of two products to be offered and prices for all products $k \in \{1,\ldots,n\}$.[6] We denote a vector of prices in X^n by $\mathbf{p} := (p_1,\ldots,p_n)$.

A *selection* of the consumers is a measurable mapping $\sigma : X^n \times \mathcal{K} \times X^n \times \{1,\ldots,n\} \to [0,1]$, where $\sigma_{\mathbf{v},K,\mathbf{p}}(k)$ denotes the probability that the consumer buys product k at $(\mathbf{v}, K, \mathbf{p})$, where $\sigma_{\mathbf{v},K,\mathbf{p}}(k) = 0$ for all products $k \notin K$ and

$$\sum_{k \in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k) \leq 1.$$

Thus, a selection selects, potentially randomly, for any vector of valuations $\mathbf{v} \in X^n$, any set of offered products K, and any vector of prices $\mathbf{p} \in X^n$, a product to be purchased, if any.

The *producer surplus* under market segmentation τ, strategy ρ, and selection σ is

$$\Pi_\tau(\rho,\sigma) := \int\int\int \sum_{k \in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k) p_k \mu(\mathrm{d}\mathbf{v}) \rho(\mu, \mathrm{d}(K,\mathbf{p})) \tau(\mathrm{d}\mu),$$

and the *consumer surplus* is

$$U_\tau(\rho,\sigma) := \int\int\int \sum_{k \in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k)(v_k - p_k) \mu(\mathrm{d}\mathbf{v}) \rho(\mu, \mathrm{d}(K,\mathbf{p})) \tau(\mathrm{d}\mu).$$

A strategy ρ^* is *optimal* for the seller under market segmentation τ and selection σ if $\rho^* \in \arg\max_\rho \Pi_\tau(\rho,\sigma)$. A selection σ^* is *optimal* for the consumers under market segmentation τ and strategy σ if $\sigma^* \in \arg\max_\sigma U_\tau(\rho,\sigma)$.

The producer surplus without market segmentation is again denoted by π_0. Without market segmentation, the seller optimally offers any two products; the valuations for the two products are independently drawn from the uniform distribution on $[0,1]$; a consumer optimally buys that product for which the difference between valuation and price is greatest,

[5]At some points, we make the dependence of $\mathcal{K}$ on n clear in the notation and write $\mathcal{K}_n$ instead of $\mathcal{K}$.

[6]Although the seller offers only two products, we assume that she chooses prices for all products in $\{1,\ldots,n\}$. This is to simplify the notation.

provided the difference is positive.[7] By Pavlov (2011, Example 1), it is optimal for the seller to set the same price for both products in this case, so that

$$\pi_0 = \max_p p(1-p^2) = \frac{2}{3}\frac{1}{\sqrt{3}}. \tag{SA.21}$$

A combination of producer and consumer surplus (π, u) is *feasible* if there exist a market segmentation τ, an optimal strategy ρ, and an optimal selection ρ such that $(\pi, u) = (\Pi_\tau(\rho,\sigma), U_\tau(\rho,\sigma))$. For given n, the set of feasible surplus pairs is again denoted by S_n.

The other notation is as in Section SA.3. In particular, we use again the set S, the probability measures g_π, and the function $\overline{u}$ defined there.

We now show that Theorem 1, as stated in Section II, extends to this model.

SA.4.1 Proof of the First Part of Theorem 1

We wish to show that for every $n \in \mathbb{N}$, the set S_n of feasible surplus pairs is a subset of

$$S = \Big\{(\pi,u) \in \mathbb{R}^2 \mid \pi \in [\pi_0, 1], u \in [0, \overline{u}(\pi)]\Big\}.$$

Let $n \in \mathbb{N}$ and $(\pi_n, u_n) \in S_n$. By the same arguments as in the original model, it holds that $\pi_n \in [\pi_0, 1]$ and $u_n \geq 0$. The following lemma, the analog to Lemma 1, thus concludes the proof of the first part of Theorem 1.

Lemma SA.4. *For every $n \in \mathbb{N}$, $(\pi, u) \in S_n$ implies $u \leq \overline{u}(\pi)$.*

Proof. Let τ be any market segmentation, let ρ be any strategy, and let σ be any selection such that ρ and σ are optimal and $\Pi_\tau(\rho,\sigma) = \pi$ and $U_\tau(\rho,\sigma) = u$.

We first show that

$$\int\int\int \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k) \mathbf{1}_{v_k \geq q}(\mathbf{v})\mu(\mathrm{d}\mathbf{v})\rho(\mu, \mathrm{d}(K,\mathbf{p}))\tau(\mathrm{d}\mu) \leq g_\pi([q,1]), \quad \forall q \in [\pi, 1]. \tag{SA.22}$$

[7] A monopolist's problem of choosing prices for $l > 1$ products of which the buyer buys at most one is known in the literature as the "Bayesian unit-demand pricing problem" (see Chawla, Hartline, and Kleinberg, 2007). Solutions have been obtained only for special cases (see the literature overviews in Cai and Daskalakis (2015) and Chen, Diakonikolas, Paparas, Sun, and Yannakakis (2018)).

By contradiction, suppose that there exists $q \in [\pi, 1]$ such that

$$\int\int\int \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k)\mathbf{1}_{v_k\geq q}(\mathbf{v})\mu(\mathrm{d}\mathbf{v})\rho(\mu, \mathrm{d}(K,\mathbf{p}))\tau(\mathrm{d}\mu) > g_\pi([q,1]).$$

Let $\mathbf{p}'$ be the vector of prices where $p'_k = q$ for every product $k \in \{1, \ldots, n\}$. We have

$$\begin{aligned} & \int\int\int \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k)\mathbf{1}_{v_k\geq q}(\mathbf{v})\mu(\mathrm{d}\mathbf{v})\rho(\mu, \mathrm{d}(K,\mathbf{p}))\tau(\mathrm{d}\mu) \\ \leq \quad & \int\int\int \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}'}(k)\mathbf{1}_{v_k\geq q}(\mathbf{v})\mu(\mathrm{d}\mathbf{v})\rho(\mu, \mathrm{d}(K,\mathbf{p}))\tau(\mathrm{d}\mu) \end{aligned}$$

since by the optimality of σ,

$$\sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}'}(k)\mathbf{1}_{v_k\geq q}(\mathbf{v}) = \mathbf{1}_{\exists k\in K: v_k\geq q}(\mathbf{v}) \geq \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k)\mathbf{1}_{v_k\geq q}(\mathbf{v}),$$
$$\forall K \in \mathcal{K}, \forall \mathbf{p} \in X^n, \forall \mathbf{v} \in X^n.$$

Hence,

$$\begin{aligned} & q\int\int\int \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}'}(k)\mathbf{1}_{v_k\geq q}(\mathbf{v})\mu(\mathrm{d}\mathbf{v})\rho(\mu, \mathrm{d}(K,\mathbf{p}))\tau(\mathrm{d}\mu) \\ \geq \quad & q\int\int\int \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k)\mathbf{1}_{v_k\geq q}(\mathbf{v})\mu(\mathrm{d}\mathbf{v})\rho(\mu, \mathrm{d}(K,\mathbf{p}))\tau(\mathrm{d}\mu) \\ > \quad & q g_\pi([q,1]) \\ = \quad & \pi. \end{aligned}$$

But then the strategy ρ' that differs from ρ in that it chooses the price vector $\mathbf{p}'$ for every market μ results in a strictly higher producer surplus than ρ, contradicting the optimality of ρ. Hence, (SA.22) holds.

Define now the probability measure $h \in \Delta X$ by

$$h([x,1]) = \int\int\int \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k)\mathbf{1}_{v_k\geq x}(\mathbf{v})\mu(\mathrm{d}\mathbf{v})\rho(\mu, \mathrm{d}(K,\mathbf{p}))\tau(\mathrm{d}\mu), \quad \forall x \in (0,1]$$

and $h([0,1])=1$. By (SA.22), g_π first-order stochastically dominates h. Hence,

$$\begin{aligned}
u &= \int\int\int \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k)(v_k-p_k)\mu(\mathrm{d}\mathbf{v})\rho(\mu,\mathrm{d}(K,\mathbf{p}))\tau(\mathrm{d}\mu) \\
&= \int\int\int \sum_{k\in K} \sigma_{\mathbf{v},K,\mathbf{p}}(k)v_k\mu(\mathrm{d}\mathbf{v})\rho(\mu,\mathrm{d}(K,\mathbf{p}))\tau(\mathrm{d}\mu)-\pi \\
&= \int xh(\mathrm{d}x)-\pi \\
&= \int h([x,1])\mathrm{d}x-\pi \\
&\leq \int g_\pi([x,1])\mathrm{d}x-\pi \\
&= \overline{u}(\pi).
\end{aligned}$$

□

SA.4.2 Proof of the Second Part of Theorem 1: Preliminaries

To prove the second part of Theorem 1, we first state auxiliary results.

Let ξ be a probability measure in Δ^n, and let $K\in\mathcal{K}$. We denote by ξ_K the probability measure in ΔX^2 defined by

$$\xi_K\Big(\prod_{k\in K} B_k\Big) := \xi\Big(\prod_{k=1}^n B_k\Big)$$

where $\prod_{k=1}^n B_k\in\prod_{k=1}^n\mathcal{B}(X)$ such that $B_k=X$ for every $k\notin K$. This extends the notion of a marginal to two-dimensional marginals. Moreover, we say that a probability measure $\xi\in\Delta X^n$ is *invariant to permutation* if

$$\xi\Big(\prod_{k=1}^n B_k\Big)=\xi\Big(\prod_{k=1}^n B_{\psi(k)}\Big)$$

for every $\prod_{k=1}^n B_k\in\prod_{k=1}^n\mathcal{B}(X)$ and every permutation ψ of $\{1,\ldots,n\}$.

Lemma SA.5. *a) Let $\zeta,\phi\in\Delta X^2$ and $\lambda\in(0,1)$ such that, for any $K\in\mathcal{K}$,*

$$\lambda\zeta\Big(\prod_{k\in K} B_k\Big)+(1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big)=\prod_{k\in K} f(B_k),\quad \forall\prod_{k\in K} B_k\in\prod_{k\in K}\mathcal{B}(X).$$

For any $n\geq 3$, there exist $\xi^n\in\Delta X^{n-2}$ and $\gamma\in\Delta X^n$ such that

$$\begin{aligned}
\prod_{k=1}^n f(B_k) &= (1-(1-\lambda)^{n\div 2})\sum_{K\in\mathcal{K}}\frac{1}{|\mathcal{K}|}\zeta\Big(\prod_{k\in K} B_k\Big)\xi^n\Big(\prod_{k\notin K} B_k\Big) \\
&\quad +(1-\lambda)^{n\div 2}\gamma\Big(\prod_{k=1}^n B_k\Big),\quad \forall\prod_{k=1}^n B_k\in\prod_{k=1}^n\mathcal{B}(X).
\end{aligned}\tag{SA.23}$$

Moreover, if ζ is invariant to permutation then there exist such ξ^n, γ with ξ^n being invariant to permutation.

b) *For every $n \geq 4$, let $K_n \in \mathcal{K}_n$. Let ξ^n be as in part a) and invariant to permutation. Then,*

$$\lim_{n\to\infty} \xi^n_{K_n}\Big(B' \times B''\Big) = f(B')f(B''), \quad \forall B' \times B'' \in \mathcal{B}(X) \times \mathcal{B}(X).$$

Proof. Part a): We show (SA.23) first for $n = 3$ and $n = 4$. Afterwards, we will be able to show (SA.23) for every $n > 4$ by induction.

Let $n = 3$, and fix any $\prod_{k=1}^n B_k \in \prod_{k=1}^n \mathcal{B}(X)$. Then,

$$\begin{aligned}
\prod_{k=1}^n f(B_k) &= \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\left(\lambda\zeta\Big(\prod_{k\in K} B_k\Big) + (1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big)\right) f\big(\prod_{k\notin K} B_k\big) \\
&= \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\lambda\zeta\Big(\prod_{k\in K} B_k\Big) f\big(\prod_{k\notin K} B_k\big) + (1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big) f\big(\prod_{k\notin K} B_k\big) \\
&= (1-(1-\lambda)) \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\zeta\Big(\prod_{k\in K} B_k\Big) f\big(\prod_{k\notin K} B_k\big) + (1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big) f\big(\prod_{k\notin K} B_k\big).
\end{aligned}$$

Thus, (SA.23) holds for $n = 3$.

Let now $n = 4$, and fix any $\prod_{k=1}^n B_k \in \prod_{k=1}^n \mathcal{B}(X)$. Then,

$$\begin{aligned}
\prod_{k=1}^n f(B_k) &= \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\left(\lambda\zeta\Big(\prod_{k\in K} B_k\Big) + (1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big)\right)\left(\lambda\zeta\Big(\prod_{k\notin K} B_k\Big) + (1-\lambda)\phi\Big(\prod_{k\notin K} B_k\Big)\right) \\
&= \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\left(\lambda\zeta\Big(\prod_{k\in K} B_k\Big)\left(\frac{1}{2}\lambda\zeta\Big(\prod_{k\notin K} B_k\Big) + (1-\lambda)\phi\Big(\prod_{k\notin K} B_k\Big)\right)\right. \\
&\qquad \left. + \lambda\zeta\Big(\prod_{k\notin K} B_k\Big)\left(\frac{1}{2}\lambda\zeta\Big(\prod_{k\in K} B_k\Big) + (1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big)\right)\right) \\
&\qquad + (1-\lambda)^2 \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\phi\Big(\prod_{k\in K} B_k\Big)\phi\Big(\prod_{k\notin K} B_k\Big) \\
&= \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\lambda\zeta\Big(\prod_{k\in K} B_k\Big)\left(\lambda\zeta\Big(\prod_{k\notin K} B_k\Big) + 2(1-\lambda)\phi\Big(\prod_{k\notin K} B_k\Big)\right) \\
&\qquad + (1-\lambda)^2 \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\phi\Big(\prod_{k\in K} B_k\Big)\phi\Big(\prod_{k\notin K} B_k\Big) \\
&= (1-(1-\lambda)^2) \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\zeta\Big(\prod_{k\in K} B_k\Big)\left(\frac{\lambda^2}{1-(1-\lambda)^2}\zeta\Big(\prod_{k\notin K} B_k\Big) + \frac{2\lambda(1-\lambda)}{1-(1-\lambda)^2}\phi\Big(\prod_{k\notin K} B_k\Big)\right) \\
&\qquad + (1-\lambda)^2 \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\phi\Big(\prod_{k\in K} B_k\Big)\phi\Big(\prod_{k\notin K} B_k\Big).
\end{aligned}$$

Thus, (SA.23) holds for $n=4$.

Now suppose that (SA.23) holds for an arbitrary given $n \geq 4$. We show that (SA.23) then holds for $n+1$. Fix any $\prod_{k=1}^{n+1} B_k \in \prod_{k=1}^{n+1} \mathcal{B}(X)$. We have

$$\begin{aligned}
\prod_{k=1}^{n+1} f(B_k) &= \sum_{K \in \mathcal{K}_{n+1}} \frac{1}{|\mathcal{K}_{n+1}|} \left(\lambda\zeta\Big(\prod_{k\in K} B_k\Big) + (1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big)\right) \prod_{k\notin K} f(B_k) \\
&= \sum_{K \in \mathcal{K}_{n+1}} \frac{1}{|\mathcal{K}_{n+1}|} \left(\lambda\zeta\Big(\prod_{k\in K} B_k\Big) + (1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big)\right) \\
&\quad \cdot \left((1-(1-\lambda)^{(n-1)\div 2}) \sum_{H \in \mathcal{K}_{n+1}\setminus\{K\}} \frac{1}{|\mathcal{K}_{n-1}|} \zeta\Big(\prod_{k\in H} B_k\Big) \xi^{n-1}\Big(\prod_{k\notin H\cup K} B_k\Big) \right. \\
&\qquad \left. + (1-\lambda)^{(n-1)\div 2} \gamma\Big(\prod_{k\notin K} B_k\Big)\right) \\
&= \sum_{K \in \mathcal{K}_{n+1}} \frac{1}{|\mathcal{K}_{n+1}|} \lambda\zeta\Big(\prod_{k\in K} B_k\Big) \\
&\quad \cdot \left(\frac{1}{2}(1-(1-\lambda)^{(n-1)\div 2}) \sum_{H \in \mathcal{K}_{n+1}\setminus\{K\}} \frac{1}{|\mathcal{K}_{n-1}|} \zeta\Big(\prod_{k\in H} B_k\Big) \xi^{n-1}\Big(\prod_{k\notin H\cup K} B_k\Big) \right. \\
&\qquad \left. + (1-\lambda)^{(n-1)\div 2} \gamma\Big(\prod_{k\notin K} B_k\Big)\right) \\
&\quad + \sum_{K \in \mathcal{K}_{n+1}} \frac{1}{|\mathcal{K}_{n+1}|} (1-(1-\lambda)^{(n-1)\div 2}) \sum_{H \in \mathcal{K}_{n+1}\setminus\{K\}} \frac{1}{|\mathcal{K}_{n-1}|} \zeta\Big(\prod_{k\in H} B_k\Big) \xi^{n-1}\Big(\prod_{k\notin H\cup K} B_k\Big) \\
&\qquad \cdot \left(\frac{1}{2}\lambda\zeta\Big(\prod_{k\in K} B_k\Big) + (1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big)\right) \\
&\quad + \sum_{K \in \mathcal{K}_{n+1}} \frac{1}{|\mathcal{K}_{n+1}|} (1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big)(1-\lambda)^{(n-1)\div 2} \gamma\Big(\prod_{k\notin K} B_k\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{K}_{n+1}} \frac{1}{|\mathcal{K}_{n+1}|}\zeta\Big(\prod_{k\in K} B_k\Big) \\
&\quad \cdot \Bigg((1-(1-\lambda)^{(n-1)\div 2}) \sum_{H\in\mathcal{K}_{n+1}\setminus\{K\}} \frac{1}{|\mathcal{K}_{n-1}|}\xi^{n-1}\Big(\prod_{k\notin H\cup K} B_k\Big)\Bigg(\lambda\zeta\Big(\prod_{k\in H} B_k\Big)+(1-\lambda)\phi\Big(\prod_{k\in H} B_k\Big)\Bigg) \\
&\qquad + \lambda(1-\lambda)^{(n-1)\div 2}\gamma\Big(\prod_{k\notin K} B_k\Big)\Bigg) \\
&\quad + \sum_{K\in\mathcal{K}_{n+1}} \frac{1}{|\mathcal{K}_{n+1}|}(1-\lambda)\phi\Big(\prod_{k\in K} B_k\Big)(1-\lambda)^{(n-1)\div 2}\gamma\Big(\prod_{k\notin K} B_k\Big) \\
&= (1-(1-\lambda)^{(n+1)\div 2}) \sum_{K\in\mathcal{K}_{n+1}} \frac{1}{|\mathcal{K}_{n+1}|}\zeta\Big(\prod_{k\in K} B_k\Big) \\
&\quad \cdot \Bigg(\frac{1-(1-\lambda)^{(n-1)\div 2}}{1-(1-\lambda)^{(n+1)\div 2}} \sum_{H\in\mathcal{K}_{n+1}\setminus\{K\}} \frac{1}{|\mathcal{K}_{n-1}|}\xi^{n-1}\Big(\prod_{k\notin H\cup K} B_k\Big)\prod_{k\in H} f(B_k) \\
&\qquad + \frac{\lambda(1-\lambda)^{(n-1)\div 2}}{1-(1-\lambda)^{(n+1)\div 2}}\gamma\Big(\prod_{k\notin K} B_k\Big)\Bigg) \\
&\quad + (1-\lambda)^{(n+1)\div 2} \sum_{K\in\mathcal{K}_{n+1}} \frac{1}{|\mathcal{K}_{n+1}|}\phi\Big(\prod_{k\in K} B_k\Big)\gamma\Big(\prod_{k\notin K} B_k\Big).
\end{aligned}
$$

Thus, (SA.23) holds for $n+1$.

Now suppose that ζ is invariant to permutation and ξ^n, γ satisfy (SA.23). For $n \geq 3$ and $K \in \mathcal{K}_n$, let $\Psi([n])$ be the set of all permutations of $\{1,\dots,n\}$, and let $\Psi([n]\setminus K)$ be the set of all permutations of $\{1,\dots,n\}\setminus K$. Then,

$$
\begin{aligned}
\prod_{k=1}^{n} f(B_k) &= \sum_{\psi\in\Psi([n])} \frac{1}{n!}\prod_{k=1}^{n} f(B_{\psi(k)}) \\
&= (1-(1-\lambda)^{n\div 2}) \sum_{\psi\in\Psi([n])} \frac{1}{n!} \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\zeta\Big(\prod_{k\in K} B_{\psi(k)}\Big)\xi^n\Big(\prod_{k\notin K} B_{\psi(k)}\Big) \\
&\quad + (1-\lambda)^{n\div 2} \sum_{\psi\in\Psi([n])} \frac{1}{n!}\gamma\Big(\prod_{k=1}^{n} B_{\psi(k)}\Big) \\
&= (1-(1-\lambda)^{n\div 2}) \sum_{K\in\mathcal{K}} \frac{1}{|\mathcal{K}|}\zeta\Big(\prod_{k\in K} B_k\Big)\Bigg(\sum_{\psi\in\Psi([n]\setminus K)} \frac{1}{(n-2)!}\xi^n\Big(\prod_{k\notin K} B_{\psi(k)}\Big)\Bigg) \\
&\quad + (1-\lambda)^{n\div 2} \sum_{\psi\in\Psi([n])} \frac{1}{n!}\gamma\Big(\prod_{k=1}^{n} B_{\psi(k)}\Big),
\end{aligned}
$$

where the second equality holds because ζ is invariant to permutation. This completes the proof of part a) because

$$\prod_{k\notin K}\mathcal{B}(X)\ni\prod_{k\notin K}B_k\mapsto\sum_{\psi\in\Psi([n]\setminus K)}\frac{1}{(n-2)!}\xi^n\Big(\prod_{k\notin K}B_{\psi(k)}\Big)$$

is invariant to permutation.

Part b): For every $n\geq 4$, let $\{k'(n),k''(n)\}\in\mathcal{K}_n$ and, moreover, $\prod_{k=1}^n B_k\in\prod_{k=1}^n\mathcal{B}(X)$ with $B_{k'(n)}=B'$, $B_{k''(n)}=B''$, and $B_k=X$ for every $k\notin\{k'(n),k''(n)\}$. Then,

$$\begin{aligned}
f(B_{k'})f(B_{k''}) &= \prod_{k=1}^n f(B_k)\\
&= (1-(1-\lambda)^{n\div 2})\sum_{K\in\mathcal{K}_n}\frac{1}{|\mathcal{K}_n|}\zeta\Big(\prod_{k\in K}B_k\Big)\xi^n\Big(\prod_{k\notin K}B_k\Big)+(1-\lambda)^{n\div 2}\gamma\Big(\prod_{k=1}^n B_k\Big)\\
&= (1-(1-\lambda)^{n\div 2})\sum_{K\in\mathcal{K}_n:K\cap\{k'(n),k''(n)\}=\emptyset}\frac{1}{|\mathcal{K}_n|}\zeta\Big(\prod_{k\in K}B_k\Big)\xi^n\Big(\prod_{k\notin K}B_k\Big)\\
&\quad +(1-(1-\lambda)^{n\div 2})\sum_{K\in\mathcal{K}_n:K\cap\{k'(n),k''(n)\}\neq\emptyset}\frac{1}{|\mathcal{K}_n|}\zeta\Big(\prod_{k\in K}B_k\Big)\xi^n\Big(\prod_{k\notin K}B_k\Big)\\
&\quad +(1-\lambda)^{n\div 2}\gamma\Big(\prod_{k=1}^n B_k\Big)\\
&= (1-(1-\lambda)^{n\div 2})\sum_{K\in\mathcal{K}_n:K\cap\{k'(n),k''(n)\}=\emptyset}\frac{1}{|\mathcal{K}_n|}\xi^n\Big(\prod_{k\notin K}B_k\Big)\\
&\quad +(1-(1-\lambda)^{n\div 2})\sum_{K\in\mathcal{K}_n:K\cap\{k'(n),k''(n)\}\neq\emptyset}\frac{1}{|\mathcal{K}_n|}\zeta\Big(\prod_{k\in K}B_k\Big)\xi^n\Big(\prod_{k\notin K}B_k\Big)\\
&\quad +(1-\lambda)^{n\div 2}\gamma\Big(\prod_{k=1}^n B_k\Big)\\
&=: a_n.
\end{aligned}$$

Thus,

$$\begin{aligned}
f(B_{k'})f(B_{k''}) &= \lim_{n\to\infty}a_n\\
&= \lim_{n\to\infty}\sum_{K\in\mathcal{K}_n:K\cap\{k'(n),k''(n)\}=\emptyset}\frac{1}{|\mathcal{K}_n|}\xi^n\Big(\prod_{k\notin K}B_k\Big)\\
&= \lim_{n\to\infty}\sum_{K\in\mathcal{K}_n:K\cap\{k'(n),k''(n)\}=\emptyset}\frac{1}{|\mathcal{K}_n|}\xi^n\Big(B'\times B''\times\underbrace{X\times\cdots\times X}_{n-4\text{ times}}\Big)\\
&= \lim_{n\to\infty}\xi^n\Big(B'\times B''\times X\times\cdots\times X\Big)\\
&= \lim_{n\to\infty}\xi^n_{\{1,2\}}\Big(B'\times B''\Big)\\
&= \lim_{n\to\infty}\xi^n_{\{k'(n),k''(n)\}}\Big(B'\times B''\Big),
\end{aligned}$$

where the third line and the last line hold because ξ^n is invariant to permutation. $\square$

For $(p_l, p_k) \in [0,1]^2$, define

$$\begin{aligned}
c^\pi(p_l,p_k) &:= \frac{1}{2}p_l\left(g_\pi\Big([p_l,1]\Big)p_k + \int_{p_l}^1 g_\pi([v_l,1])\mathrm{d}v_k\right) + \frac{1}{2}p_k\int_{p_k}^1\Big(1+g_\pi\Big([0,v_k-p_k+p_l]\Big)\Big)\,\mathrm{d}v_k \\
&= \frac{1}{2}p_l\left(g_\pi\Big([p_l,1]\Big)p_k + \int_{p_l}^1 (v_l-p_l)g_\pi(\mathrm{d}v_l)\right) + \frac{1}{2}p_k\int_{p_k}^1\Big(1+g_\pi\Big([0,v_k-p_k+p_l]\Big)\Big)\,\mathrm{d}v_k \\
&= \frac{1}{2}p_l\int_{p_l}^1 (v_l-p_l+p_k)g_\pi(\mathrm{d}v_l) + \frac{1}{2}p_k\int_{p_k}^1\Big(1+g_\pi\Big([0,v_k-p_k+p_l]\Big)\Big)\,\mathrm{d}v_k \\
&= \frac{1}{2}p_l\int_{p_l}^1\int_{v_k:v_l-p_l\geq v_k-p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) + \frac{1}{2}p_k\int_{p_k}^1\left(1+\int_{v_l:v_l-p_l\leq v_k-p_k}\mathrm{d}g_\pi(\mathrm{d}v_l)\right)\mathrm{d}v_k.
\end{aligned}$$

Since the function c^π is continuous, it has a maximum on $[0,1]^2$. We show in Lemma SA.7 below that

$$\max_{(p_l,p_k)\in[0,1]^2} c^\pi(p_l,p_k) < \pi, \quad \forall \pi \in [\pi_0, 1].$$

We need the following lemma, which provides an upper bound on the maximum value.

Lemma SA.6. *It holds that*

$$\max_{(p_l,p_k)\in[0,1]^2} c^\pi(p_l,p_k) \leq \max_{p_l\in[\pi,1],p_k\in[0,p_l]} p_k(1-p_k) + \frac{\pi}{2}\left(1+2p_k-p_l-\frac{p_k}{p_l}\right).$$

Proof. We show first that

$$\max_{(p_l,p_k)\in[0,1]^2} c^\pi(p_l,p_k) = \max_{(p_l,p_k)\in[0,1]^2:p_l\geq p_k} c^\pi(p_l,p_k). \tag{SA.24}$$

Let $p_l < p_k$. Then,

$$c^\pi(p_k,p_k) = \frac{1}{2}p_k\int_{p_k}^1\int_{v_k:v_l\geq v_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) + \frac{1}{2}p_k\int_{p_k}^1\left(1+\int_{v_l:v_l\leq v_k}\mathrm{d}g_\pi(\mathrm{d}v_l)\right)\mathrm{d}v_k.$$

Hence, $2\Big(c^\pi(p_k,p_k)-c^\pi(p_l,p_k)\Big)$ is equal to

$$\begin{aligned}
& p_k\int_{p_k}^1\int_{v_k:v_l\geq v_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l)+p_k\int_{p_k}^1\int_{v_l:v_l\leq v_k}\mathrm{d}g_\pi(\mathrm{d}v_l)\mathrm{d}v_k \\
& \quad -p_l\int_{p_l}^1\int_{v_k:v_l-p_l+p_k\geq v_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l)-p_k\int_{p_k}^1\int_{v_l:v_l\leq v_k-p_k+p_l}\mathrm{d}g_\pi(\mathrm{d}v_l)\mathrm{d}v_k \\
= \; & p_k\int_{p_k}^1\int_{v_k:v_l\geq v_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l)+p_k\int_{p_k}^1\int_{v_l:v_k-p_k+p_l\leq v_l\leq v_k} g_\pi(\mathrm{d}v_l)\mathrm{d}v_k \\
& \quad -p_l\int_{p_l}^1\int_{v_k:v_l-p_l+p_k\geq v_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
= \; & p_k\int_{p_k}^1\int_{v_k:v_k\leq p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l)+p_k\int_{p_k}^1\int_{v_k:p_k\leq v_k\leq v_l}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
& \quad +p_k\int_{p_k}^1\int_{v_l:v_k-p_k+p_l\leq v_l\leq v_k} g_\pi(\mathrm{d}v_l)\mathrm{d}v_k \\
& \quad -p_l\int_{p_l}^1\int_{v_k:v_k\leq p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l)-p_l\int_{p_l}^1\int_{v_k:p_k\leq v_k\leq v_l-p_l+p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
= \; & \min\{\pi,p_k\}v_k+p_k\int_{p_k}^1\int_{v_k:p_k\leq v_k\leq v_l}\mathrm{d}v_k g_\pi(\mathrm{d}v_l)+p_k\int_{p_k}^1\int_{v_l:v_k-p_k+p_l\leq v_l\leq v_k} g_\pi(\mathrm{d}v_l)\mathrm{d}v_k \\
& \quad -\min\{\pi,p_l\}v_k-p_l\int_{p_l}^1\int_{v_k:p_k\leq v_k\leq v_l-p_l+p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
\geq \; & p_k\int_{p_k}^1\int_{v_k:p_k\leq v_k\leq v_l}\mathrm{d}v_k g_\pi(\mathrm{d}v_l)+p_k\int_{p_k}^1\int_{v_l:v_k-p_k+p_l\leq v_l\leq v_k} g_\pi(\mathrm{d}v_l)\mathrm{d}v_k \\
& \quad -p_l\int_{p_l}^1\int_{v_k:p_k\leq v_k\leq v_l-p_l+p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
= \; & p_k\int_{p_k}^1\int_{v_k:p_k\leq v_k\leq v_l}\mathrm{d}v_k g_\pi(\mathrm{d}v_l)+p_k\int_{p_l}^1\int_{v_k:v_l\leq v_k\leq v_l-p_l+p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
& \quad -p_l\int_{p_l}^1\int_{v_k:p_k\leq v_k\leq v_l-p_l+p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
= \; & p_k\int_{p_l}^1\int_{v_k:p_k\leq v_k\leq v_l}\mathrm{d}v_k g_\pi(\mathrm{d}v_l)+p_k\int_{p_l}^1\int_{v_k:v_l\leq v_k\leq v_l-p_l+p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
& \quad -p_l\int_{p_l}^1\int_{v_k:p_k\leq v_k\leq v_l-p_l+p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
= \; & (p_k-p_l)\int_{p_l}^1\int_{v_k:p_k\leq v_k\leq v_l-p_l+p_k}\mathrm{d}v_k g_\pi(\mathrm{d}v_l) \\
\geq \; & 0.
\end{aligned}$$

This shows (SA.24).

Let now $p_l \geq p_k$. Then,

$$\begin{aligned}
c^{\pi}(p_l, p_k) &= \frac{1}{2}p_l \int_{p_l}^{1} \int_{v_k : v_k \leq v_l - p_l + p_k} \mathrm{d}v_k g_{\pi}(\mathrm{d}v_l) + \frac{1}{2}p_k \int_{p_k}^{1} \left(1 + \int_{v_l : v_l \leq v_k - p_k + p_l} \mathrm{d}g_{\pi}(\mathrm{d}v_l)\right) \mathrm{d}v_k \\
&= \frac{1}{2}p_l \int_{p_l}^{1} \int_{v_k : v_k \leq p_k} \mathrm{d}v_k g_{\pi}(\mathrm{d}v_l) + \frac{1}{2}p_l \int_{p_l}^{1} \int_{v_k : p_k \leq v_k \leq v_l - p_l + p_k} \mathrm{d}v_k g_{\pi}(\mathrm{d}v_l) \\
&\quad + \frac{1}{2}p_k \int_{p_k}^{1} \left(1 + \int_{v_l : v_l \leq p_l} \mathrm{d}g_{\pi}(\mathrm{d}v_l)\right) \mathrm{d}v_k \\
&\quad + \frac{1}{2}p_k \int_{p_k}^{1} \int_{v_l : p_l \leq v_l \leq v_k - p_k + p_l} \mathrm{d}g_{\pi}(\mathrm{d}v_l) \mathrm{d}v_k \\
&\leq \frac{1}{2}p_l \int_{p_l}^{1} \int_{v_k : v_k \leq p_k} \mathrm{d}v_k g_{\pi}(\mathrm{d}v_l) + \frac{1}{2}p_l \int_{p_l}^{1} \int_{v_k : p_k \leq v_k \leq 1 - p_l + p_k} \mathrm{d}v_k g_{\pi}(\mathrm{d}v_l) \\
&\quad + \frac{1}{2}p_k \int_{p_k}^{1} \left(1 + \int_{v_l : v_l \leq p_l} \mathrm{d}g_{\pi}(\mathrm{d}v_l)\right) \mathrm{d}v_k \\
&\quad + \frac{1}{2}p_k \int_{1 - p_l + p_k}^{1} \int_{v_l : p_l \leq v_l \leq 1} \mathrm{d}g_{\pi}(\mathrm{d}v_l) \mathrm{d}v_k \\
&=: a(p_l, p_k),
\end{aligned}$$

where the inequality holds by $p_l \geq p_k$. There are two cases.

Case 1: $p_l \leq \pi$. Then,

$$a(p_l, p_k) = \frac{1}{2}p_l p_k + \frac{1}{2}p_l(1 - p_l) + \frac{1}{2}p_k(1 - p_k) + \frac{1}{2}p_k(p_l - p_k),$$

so that

$$\begin{aligned}
\frac{\partial a(p_l, p_k)}{\partial p_l} &= \frac{1}{2}p_k + \frac{1}{2}(1 - 2p_l) + \frac{1}{2}p_k = \frac{1}{2} + p_k - p_l, \\
\frac{\partial a(p_l, p_k)}{\partial p_k} &= \frac{1}{2}p_l + \frac{1}{2}(1 - 2p_k) + \frac{1}{2}(p_l - 2p_k) = \frac{1}{2} - 2p_k + p_l.
\end{aligned}$$

It follows that if

$$a(p_l^*, p_k^*) = \max_{p_l \in [0, \pi], p_k \in [0, p_l]} a(p_l, p_k),$$

then $p_k^* = p_l^*$ or $p_k^* = 1/4 + p_l^*/2$. With both solutions for p_k, we obtain

$$\frac{\partial a(p_l, p_k)}{\partial p_l} > 0, \quad \forall p_l \in [0, 1].$$

Hence,

$$a(p_l^*, p_k^*) = \max_{p_l \in [0, \pi], p_k \in [0, p_l]} a(p_l, p_k)$$

implies $p_l^* = \pi$. We can therefore drop Case 1 and concentrate on the following Case 2.

Case 2: $p_l \geq \pi$. Then,

$$
\begin{aligned}
a(p_l,p_k) &= \frac{1}{2}p_l\frac{\pi}{p_l}p_k + \frac{1}{2}p_l\frac{\pi}{p_l}(1-p_l) + \frac{1}{2}p_k(1-p_k)\left(1+1-\frac{\pi}{p_l}\right) + \frac{1}{2}p_k(p_l-p_k)\frac{\pi}{p_l} \\
&= \frac{\pi}{2}p_k + \frac{\pi}{2}(1-p_l) + p_k(1-p_k)\left(1-\frac{\pi}{2p_l}\right) + p_k(p_l-p_k)\frac{\pi}{2p_l} \\
&= p_k(1-p_k) + \frac{\pi}{2}\left(p_k+1-p_l-\frac{p_k(1-p_k)}{p_l}+\frac{p_k(p_l-p_k)}{p_l}\right) \\
&= p_k(1-p_k) + \frac{\pi}{2}\left(1+2p_k-p_l-\frac{p_k}{p_l}\right).
\end{aligned}
$$

□

We use Lemma SA.6 to prove the following lemma. We note that in the proof, we use Wolfram Mathematica to solve

$$\max_{x\in[0,1]} x(1-x)+\frac{1}{3}\frac{1}{\sqrt{3}}\left(2(x-\sqrt{x})-1\right);$$

according to Wolfram Mathematica, the maximum is equal to -0.0185221.[8]

Lemma SA.7. *For every $\pi \in [\pi_0,1]$, it holds that*

$$\max_{(p_l,p_k)\in[0,1]^2} c^\pi(p_l,p_k) < \pi.$$

Proof. By Lemma SA.6,

$$
\begin{aligned}
\max_{(p_l,p_k)\in[0,1]^2} c^\pi(p_l,p_k) - \pi &\leq \max_{p_l\in[\pi,1],p_k\in[0,p_l]} p_k(1-p_k) + \frac{\pi}{2}\left(1+2p_k-p_l-\frac{p_k}{p_l}\right) - \pi \\
&= \max_{p_l\in[\pi,1],p_k\in[0,p_l]} p_k(1-p_k) + \frac{\pi}{2}\left(2p_k-p_l-\frac{p_k}{p_l}-1\right) \\
&\leq \max_{p_l\in(0,1],p_k\in[0,p_l]} p_k(1-p_k) + \frac{\pi}{2}\left(2p_k-p_l-\frac{p_k}{p_l}-1\right).
\end{aligned}
$$

Define

$$m(p_l,p_k) := p_k(1-p_k) + \frac{\pi}{2}\left(2p_k-p_l-\frac{p_k}{p_l}-1\right).$$

[8]The code is: Maximize$\left[\left\{\frac{2(x-x^{0.5})-1}{3\ 3^{0.5}}+(1-x)x, x\geq 0, x\leq 1\right\},\{x\}\right]$. The output is: $\{-0.0185221,\{x\to 0.564362\}\}$.

Then,

$$\frac{\partial m(p_l,p_k)}{\partial p_l} = \frac{\pi}{2}\left(-1+\frac{p_k}{p_l^2}\right),$$
$$\frac{\partial m(p_l,p_k)}{\partial p_k} = 1-2p_k+\frac{\pi}{2}\left(2-\frac{1}{p_l}\right).$$

It follows that if

$$m(p_l^*,p_k^*) = \max_{p_l\in(0,1],p_k\in[0,p_l]} m(p_l,p_k)$$

with $p_l^*=1$, then

$$\left.\frac{\partial m(p_l,p_k)}{\partial p_k}\right|_{(1,p_k^*)} = 0 \iff p_k^* = \frac{1}{2}+\frac{\pi}{4},$$

since

$$\left.\frac{\partial m(p_l,p_k)}{\partial p_k}\right|_{(1,0)} = 1+\frac{\pi}{2} > 0,$$
$$\left.\frac{\partial m(p_l,p_k)}{\partial p_k}\right|_{(1,1)} = -1+\frac{\pi}{2} < 0.$$

But

$$\left.\frac{\partial m(p_l,p_k)}{\partial p_l}\right|_{\left(1,\frac{1}{2}+\frac{\pi}{4}\right)} < 1.$$

Hence,

$$m(p_l^*,p_k^*) = \max_{p_l\in(0,1],p_k\in[0,p_l]} m(p_l,p_k)$$

implies $p_l^*\in(0,1)$. It follows that

$$\left.\frac{\partial m(p_l,p_k)}{\partial p_l}\right|_{(p_l,p_k)=(p_l^*,p_k^*)} = 0 \iff p_l^* = \sqrt{p_k^*}\in(0,1).$$

We conclude that

$$\begin{aligned}
\max_{(p_l,p_k)\in[0,1]^2} c^\pi(p_l,p_k)-\pi &\le \max_{p_k\in(0,1):p_k\le\sqrt{p_k}} p_k(1-p_k)+\frac{\pi}{2}\left(2p_k-\sqrt{p_k}-\frac{p_k}{\sqrt{p_k}}-1\right)\\
&\le \max_{p_k\in(0,1)} p_k(1-p_k)+\frac{\pi}{2}\Big(2p_k-\sqrt{p_k}-\frac{p_k}{\sqrt{p_k}}-1\Big)\\
&= \max_{p_k\in[0,1]} p_k(1-p_k)+\frac{\pi}{2}\Big(2(p_k-\sqrt{p_k})-1\Big).
\end{aligned}$$

Now, at

$$\pi=\pi_0=\frac{2}{3}\frac{1}{\sqrt{3}},$$

we have

$$\max_{p_k \in [0,1]} p_k(1-p_k) + \frac{\pi}{2}\Big(2(p_k - \sqrt{p_k}) - 1\Big) \approx -0.0185221 < 0$$

as mentioned in the text above the lemma. Because $2(p_k - \sqrt{p_k}) - 1 < 0$ for every $p_k \in [0,1]$,

$$\max_{p_k \in [0,1]} p_k(1-p_k) + \frac{\pi}{2}\Big(2(p_k - \sqrt{p_k}) - 1\Big)$$

is strictly decreasing in π. Thus,

$$\max_{(p_l,p_k) \in [0,1]^2} c^{\pi}(p_l, p_k) - \pi \leq \max_{p_k \in [0,1]} p_k(1-p_k) + \frac{\pi}{2}\Big(2(p_k - \sqrt{p_k}) - 1\Big) < 0, \quad \forall \pi \in [\pi_0, 1].$$

$\square$

SA.4.3 Proof of the Second Part of Theorem 1

Let $(\pi, u) \in S$. We wish to show that there exists a sequence $\big((\pi_n, u_n)\big)_{n \in \mathbb{N}}$ such that $(\pi_n, u_n) \in S_n$ and $(\pi_n, u_n) \underset{n \to \infty}{\longrightarrow} (\pi, u)$.

Because the function $p \mapsto \int_p^1 g_\pi([x,1])\mathrm{d}x$ is continuous, there exists a price $p' \in [\pi, 1]$ such that

$$u = \int_{p'}^1 g_\pi([x,1])\mathrm{d}x.$$

Throughout in this subsection, we hold this price p' fixed.

SA.4.3.1 Construction of Market Segmentations

Fix some $\epsilon \in (0,1]$ and some $N \in \mathbb{N}$. Define the probability measure $f^N \in \Delta X$ by

$$f^N(B) := \frac{f(B \cap [0,1/N])}{f([0,1/N])} = N f(B \cap [0,1/N]), \quad \forall B \in \mathcal{B}(X).$$

Moreover, define the probability measure $\zeta^{\epsilon,N} \in \Delta X^2$ by

$$\zeta^{\epsilon,N}(B_1 \times B_2) := \frac{1}{2} e^{\epsilon,N}_{\pi,p'}(B_1) f^N(B_2) + \frac{1}{2} f^N(B_1) e^{\epsilon,N}_{\pi,p'}(B_2), \quad \forall B_1 \times B_2 \in \mathcal{B}(X) \times \mathcal{B}(X),$$

where $e^{\epsilon,N}_{\pi,p'} \in \Delta X$ was defined in (SA.11). Let

$$\hat{\lambda}_N := \min\left\{\lambda^N_{\pi,p'}, \frac{1}{N}\right\} \in (0,1),$$

where $\lambda^N_{\pi,p'}$ was defined in (SA.10). For every $B \in \mathcal{B}(X)$, we then have

$$\hat{\lambda}_N f^N(B) \leq f(B) \quad \text{and} \quad \hat{\lambda}_N e^{\epsilon,N}_{\pi,p'} \leq f(B);$$

hence,

$$\hat{\lambda}_N \zeta^{\epsilon,N}(B_1 \times B_2) \leq f(B_1)f(B_2), \quad \forall B_1 \times B_2 \in \mathcal{B}(X) \times \mathcal{B}(X).$$

It follows that

$$\mathcal{B}(X) \times \mathcal{B}(X) \ni B_1 \times B_2 \mapsto \phi^{\epsilon,N}(B_1 \times B_2) := \frac{f(B_1)f(B_2) - \hat{\lambda}_N \zeta^{\epsilon,N}(B_1 \times B_2)}{1 - \hat{\lambda}_N}$$

is a probability measure in ΔX^2 and

$$\begin{gathered} \hat{\lambda}^N \zeta^{\epsilon,N}(B_1 \times B_2) + (1 - \hat{\lambda}^N)\phi^{\epsilon,N}(B_1 \times B_2) = f(B_1)f(B_2), \\ \forall B_1 \times B_2 \in \mathcal{B}(X) \times \mathcal{B}(X). \end{gathered} \tag{SA.25}$$

Invoking (SA.25), we apply part a) of Lemma SA.5: for any $n \geq 3$, there exist $\gamma \in \Delta^n$ and $\xi^n \in \Delta X^{n-2}$ such that

$$\begin{aligned} \prod_{k=1}^n f(B_k) = {} & (1 - (1 - \hat{\lambda}_N)^{n \div 2}) \sum_{K \in \mathcal{K}} \frac{1}{|\mathcal{K}|} \zeta^{\epsilon,N}\Big(\prod_{k \in K} B_k\Big) \xi^{\epsilon,N,n}\Big(\prod_{k \notin K} B_k\Big) \\ & + (1 - \hat{\lambda}_N)^{n \div 2} \gamma\Big(\prod_{k=1}^n B_k\Big), \quad \forall \prod_{k=1}^n B_k \in \prod_{k=1}^n \mathcal{B}(X). \end{aligned} \tag{SA.26}$$

Moreover, since $\zeta^{\epsilon,N}$ is invariant to permutation, we can assume by part a) of Lemma SA.5 that $\xi^{\epsilon,N,n}$ is invariant to permutation.

Note that (SA.26) defines a market segmentation under which market γ is drawn with probability $(1 - \hat{\lambda}_N)^{n \div 2}$ and, with probability $(1 - (1 - \hat{\lambda}_N)^{n \div 2})$, a market $\mu^{\epsilon,N,n,K} \in \Delta X^n$ defined by

$$\mu^{\epsilon,N,n,K}\Big(\prod_{k \in K} B_k\Big) := \zeta^{\epsilon,N}\Big(\prod_{k \in K} B_k\Big) \xi^{\epsilon,N,n}\Big(\prod_{k \notin K} B_k\Big), \quad \forall \prod_{k=1}^n B_k \in \prod_{k=1}^n \mathcal{B}(X)$$

is drawn uniformly at random over $K \in \mathcal{K}$. We denote this market segmentation by $\tau^{\epsilon,N,n}$.

SA.4.3.2 Producer Surplus and Consumer Surplus

If ρ and σ are a strategy for the seller and a selection for the consumers, respectively, then the producer surplus under market segmentation $\tau^{\epsilon,N,n}$, $\Pi_{\tau^{\epsilon,N,n}}(\rho,\sigma)$, is equal to

$$(1-\hat{\lambda}_N)^{n\div 2}\int\int\sum_{k\in K'}\sigma_{\mathbf{v},K',\mathbf{p}}(k)p_k\gamma(\mathrm{d}\mathbf{v})\rho(\gamma,\mathrm{d}(K',\mathbf{p}))$$
$$+(1-(1-\hat{\lambda}_N)^{n\div 2})\sum_{K\in\mathcal{K}}\frac{1}{|\mathcal{K}|}\int\int\sum_{k\in K'}\sigma_{\mathbf{v},K',\mathbf{p}}(k)p_k\mu^{\epsilon,N,n,K}(\mathrm{d}\mathbf{v})\rho(\mu^{\epsilon,N,n,K},\mathrm{d}(K',\mathbf{p})),$$

and the consumer surplus $U_{\tau^{\epsilon,N,n}}(\rho,\sigma)$ is equal to

$$(1-\hat{\lambda}_N)^{n\div 2}\int\int\sum_{k\in K'}\sigma_{\mathbf{v},K',\mathbf{p}}(k)(v_k-p_k)\gamma(\mathrm{d}\mathbf{v})\rho(\gamma,\mathrm{d}(K,\mathbf{p}))$$
$$+(1-(1-\hat{\lambda}_N)^{n\div 2})\sum_{K\in\mathcal{K}}\frac{1}{|\mathcal{K}|}\int\int\sum_{k\in K'}\sigma_{\mathbf{v},K',\mathbf{p}}(k)(v_k-p_k)\mu^{\epsilon,N,n,K}(\mathrm{d}\mathbf{v})\rho(\mu^{\epsilon,N,n,K},\mathrm{d}(K',\mathbf{p})).$$

To simplify the notation, we denote the contribution to producer surplus from market $\mu^{\epsilon,N,n,K}$ by

$$c_{\Pi}^{\epsilon,N,n,K}(\rho,\sigma):=\int\int\sum_{k\in K'}\sigma_{\mathbf{v},K',\mathbf{p}}(k)p_k\mu^{\epsilon,N,n,K}(\mathrm{d}\mathbf{v})\rho(\mu^{\epsilon,N,n,K},\mathrm{d}(K',\mathbf{p}))$$

and the contribution to consumer surplus by

$$c_{U}^{\epsilon,N,n,K}(\rho,\sigma):=\int\int\sum_{k\in K'}\sigma_{\mathbf{v},K',\mathbf{p}}(k)(v_k-p_k)\mu^{\epsilon,N,n,K}(\mathrm{d}\mathbf{v})\rho(\mu^{\epsilon,N,n,K},\mathrm{d}(K',\mathbf{p})).$$

For $\{k',k''\}\in\mathcal{K}_n$ and $(p_{k'},p_{k''})\in[0,1]^2$ define also

$$\begin{aligned}\tilde{c}_{\Pi}^{\epsilon,N,n,K}(\{k',k''\},p_{k'},p_{k''}) &:= \mu_{\{k',k''\}}^{\epsilon,N,n,K}\Big(\{v_{k'}\geq p_{k'},v_{k'}-p_{k'}\geq v_{k''}-p_{k''}\}\Big)p_{k'}\\ &\quad+\mu_{\{k',k''\}}^{\epsilon,N,n,K}\Big(\{v_{k''}\geq p_{k''},v_{k''}-p_{k''}>v_{k'}-p_{k'}\}\Big)p_{k''},\\ \tilde{c}_{U}^{\epsilon,N,n,K}(\{k',k''\},p_{k'},p_{k''}) &:= \int\Bigg(\mathbf{1}_{v_{k'}\geq p'_k,v_{k'}-p_{k'}\geq v_{k''}-p_{k''}}(v_{k'},v_{k''})(v_{k'}-p_{k'})\\ &\quad+\mathbf{1}_{v_{k''}\geq p_{k''},v_{k''}-p_{k''}>v_{k'}-p_{k'}}(v_{k'},v_{k''})(v_{k''}-p_{k''})\Bigg)\mu^{\epsilon,N,n,K}(\mathrm{d}\mathbf{v}),\\ M^{\epsilon,N,n,K} &:= \arg\max_{\{k',k''\}\in\mathcal{K}_n,(p_{k'},p_{k''})\in[0,1]^2}\tilde{c}_{\Pi}^{\epsilon,N,n,K}(\{k',k''\},p_{k'},p_{k''}).\end{aligned}$$

We describe an optimal selection for the consumers: given any set of offered products $\{k',k''\}\in\mathcal{K}_n$ and corresponding prices $(p_{k'},p_{k''})\in[0,1]^2$, buy product k' if and only if $v_{k'}\geq p_{k'}$ and $v_{k'}-p_{k'}\geq v_{k''}-p_{k''}$; buy product k'' if and only if $v_{k''}\geq p_{k''}$ and $v_{k''}-p_{k''}>v_{k'}-p_{k'}$, where $p_{k'} \geq p_{k''}$.[9] Denote this selection by σ^*. We also describe an optimal strategy for the seller, restricted to market $\mu^{\epsilon,N,n,K}$, given σ^*: choose $(\{k',k''\},(p_{k'},p_{k''})) \in M^{\epsilon,N,n,K}$ so as to maximize $\tilde{c}_U^{\epsilon,N,n,K}(\{k',k''\},p_{k'},p_{k''})$. We leave the strategy unspecified for markets $\mu \notin \{\mu^{\epsilon,N,n,K}, K \in \mathcal{K}_n\}$, as these will be unimportant, and denote it by ρ^*. Then, for every $K \in \mathcal{K}_n$,

$$c_\Pi^{\epsilon,N,n,K}(\rho^*,\sigma^*) = \max_{\{k',k''\}\in\mathcal{K}_n,(p_{k'},p_{k''})\in[0,1]^2} \tilde{c}_\Pi^{\epsilon,N,n,K}(\{k',k''\},p_{k'},p_{k''}) =: \hat{c}_\Pi^{\epsilon,N,n,K}, \tag{SA.27}$$

$$c_U^{\epsilon,N,n,K}(\rho^*,\sigma^*) = \max_{(\{k',k''\},(p_{k'},p_{k''}))\in M^{\epsilon,N,n,K}} \tilde{c}_U^{\epsilon,N,n,K}(\{k',k''\},p_{k'},p_{k''}) =: \hat{c}_U^{\epsilon,N,n,K}. \tag{SA.28}$$

We can now prove the second part of Theorem 1 by showing that for every $K = \{k',k''\}, k',k'' \in \mathbb{N}, k' \neq k''$,

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\lim_{n\to\infty} \hat{c}_\Pi^{\epsilon,N,n,K} = \pi, \tag{SA.29}$$

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\lim_{n\to\infty} \hat{c}_U^{\epsilon,N,n,K} = u. \tag{SA.30}$$

SA.4.3.3 Limits

We will see that the limits (SA.29) and (SA.30) hold if, for sufficiently small ϵ, we can choose N sufficiently large such that as n grows without bound, the seller eventually offers in market $\mu^{\epsilon,N,n,K}$ the two products in K. The limits (SA.29) and (SA.30) will then follow from Lemma SA.3 in Section SA.3.

The following lemma concerns the case that the seller offers in market $\mu^{\epsilon,N,n,K}$ the two products in K.

Lemma SA.8. *Let $k',k'' \in \mathbb{N}, k' \neq k''$. Let $n \geq \max\{k',k''\}$. Then,*

$$\lim_{\epsilon\to 0}\lim_{N\to\infty} \max_{(p_{k'},p_{k''})\in[0,1]^2} \tilde{c}_\Pi^{\epsilon,N,n,\{k',k''\}}(\{k',k''\},p_{k'},p_{k''}) = \pi \tag{SA.31}$$

For every $\epsilon \in (0,1]$, let furthermore $\left((p_{k'}^{\epsilon,N},p_{k''}^{\epsilon,N})\right)_{N\in\mathbb{N}}$ be a sequence of prices such that

$$(p_{k'}^{\epsilon,N},p_{k''}^{\epsilon,N}) \in \arg\max_{(p_{k'},p_{k''})\in[0,1]^2} \tilde{c}_\Pi^{\epsilon,N,n,\{k',k''\}}(\{k',k''\},p_{k'},p_{k''}), \quad \forall N \in \mathbb{N}.$$

[9] Thus, the consumer breaks ties in favor of the seller under this selection. With this tie-breaking rule, $\tilde{c}_\Pi^{\epsilon,N,n,K}$ is upper semicontinuous, which implies that $M^{\epsilon,N,n,K}$ is nonempty and compact (see Aliprantis and Border, 2006, Thm. 2.43). We note that by the Dominated Convergence Theorem, $\tilde{c}_U^{\epsilon,N,n,K}$ is continuous and therefore has a maximum on $M^{\epsilon,N,n,K}$; this is used in (SA.28).

Then,

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\tilde{c}_U^{\epsilon,N,n,K}(\{k',k''\},p_{k'}^{\epsilon,N},p_{k''}^{\epsilon,N}) = \int_p^1 g_\pi([x,1])\mathrm{d}x. \tag{SA.32}$$

Proof. We first prove (SA.31). We consider here the case that $p_{k'} \geq p_{k''}$; the other case is analogous.

If $p_{k'} \leq 1/N$, then

$$\tilde{c}_\Pi^{\epsilon,N,n,\{k',k''\}}(\{k',k''\},p_{k'},p_{k''}) \leq \frac{1}{N}.$$

If $p_{k''} \leq 1/N \leq p_{k'}$, then

$$\begin{aligned}
\tilde{c}_\Pi^{\epsilon,N,n,\{k',k''\}}(\{k',k''\},p_{k'},p_{k''}) &= \zeta^{\epsilon,N}\Big(\{v_{k'} \geq p_{k'}, v_{k'}-p_{k'} \geq v_{k''}-p_{k''}\}\Big)p_{k'} \\
&\quad + \zeta^{\epsilon,N}\Big(\{v_{k''} \geq p_{k''}, v_{k''}-p_{k''} > v_{k'}-p_{k'}\}\Big)p_{k''} \\
&\leq \zeta^{\epsilon,N}\Big(\{v_{k'} \geq p_{k'}\}\Big)p_{k'} + \frac{1}{N} \\
&= \frac{1}{2}e_{\pi,p'}^{\epsilon,N}([p_{k'},1])p_{k'} + \frac{1}{N} \\
&\leq \max_{p_{k'}} \frac{1}{2}e_{\pi,p'}^{\epsilon,N}([p_{k'},1])p_{k'} + \frac{1}{N}.
\end{aligned}$$

By Lemma SA.3, it follows that

$$\begin{aligned}
& \lim_{\epsilon\to 0}\lim_{N\to\infty}\max_{p_{k'},p_{k''}\leq p_{k'}} \tilde{c}_\Pi^{\epsilon,N,n,\{k',k''\}}(\{k',k''\},p_{k'},p_{k''}) \\
\leq\; & \lim_{\epsilon\to 0}\lim_{N\to\infty}\max_{p_{k'}} \frac{1}{2}e_{\pi,p'}^{\epsilon,N}([p_{k'},1])p_{k'} + \frac{1}{N} \\
=\; & \frac{1}{2}\pi.
\end{aligned}$$

If finally $p_{k''} > 1/N$, then

$$\begin{aligned}
\tilde{c}_\Pi^{\epsilon,N,n,\{k',k''\}}(\{k',k''\},p_{k'},p_{k''}) &= \zeta^{\epsilon,N}\Big(\{v_{k'} \geq p_{k'}, v_{k'}-p_{k'} \geq v_{k''}-p_{k''}\}\Big)p_{k'} \\
&\quad + \zeta^{\epsilon,N}\Big(\{v_{k''} \geq p_{k''}, v_{k''}-p_{k''} > v_{k'}-p_{k'}\}\Big)p_{k''} \\
&= \frac{1}{2}e_{\pi,p'}^{\epsilon,N}([p_{k'},1])p_{k'} + \frac{1}{2}e_{\pi,p'}^{\epsilon,N}([p_{k''},1])p_{k''} \\
&\leq \max_x x e_{\pi,p'}^{\epsilon,N}([x,1]).
\end{aligned}$$

By Lemma SA.3 again, it follows that

$$\begin{aligned}
& \lim_{\epsilon\to 0}\lim_{N\to\infty}\max_{p_{k'},p_{k''}\le p'_k} \tilde{c}_{\Pi}^{\epsilon,N,n,\{k',k''\}}(\{k',k''\},p_{k'},p_{k''}) \\
\le\ & \lim_{\epsilon\to 0}\lim_{N\to\infty}\max_{x\in[\pi,1]} x e^{\epsilon,N}_{\pi,p'}([x,1]) \\
=\ & \pi.
\end{aligned}$$

This shows (SA.31) and, by (SA.14), also (SA.32). □

The next lemma concerns the case that the seller offers in market $\mu^{\epsilon,N,n,K}$ none of the two products in K. Since $\zeta^{\epsilon,N}$ and $\xi^{\epsilon,N,n}$ are invariant to permutation, $\tilde{c}_{\Pi}^{\epsilon,N,n,K}(\{l',l''\},p_{l'},p_{l''})$ is the same for every $\{l',l''\}\in\mathcal{K}_n$ with $\{l',l''\}\cap K=\emptyset$.

Lemma SA.9. *Let $k',k''\in\mathbb{N}, k'\neq k''$. Let $l',l''\in\mathbb{N}, l'\neq l''$ with $\{l',l''\}\cap\{k',k''\}=\emptyset$. Let $\epsilon\in(0,1]$ and $N\in\mathbb{N}$. Then,*

$$\limsup_{n\to\infty}\max_{(p_{l'},p_{l''})\in[0,1]^2} \tilde{c}_{\Pi}^{\epsilon,N,n,\{k',k''\}}(\{l',l''\},p_{l'},p_{l''}) < \pi. \tag{SA.33}$$

Proof. Note that

$$\begin{aligned}
\tilde{c}_{\Pi}^{\epsilon,N,n,\{k',k''\}}(\{l',l''\},p_{l'},p_{l''}) &\le \mu^{\epsilon,N,n,\{k',k''\}}_{\{l',l''\}}\Big(\{v_{l'}\ge p_{l'}, v_{l'}-p_{l'}\ge v_{l''}-p_{l''}\}\Big)p_{l'} \\
&\quad + \mu^{\epsilon,N,n,\{k',k''\}}_{\{l',l''\}}\Big(\{v_{l''}\ge p_{l''}, v_{l''}-p_{l''}\ge v_{l'}-p_{l'}\}\Big)p_{l''} \\
&= \xi^{\epsilon,N,n}_{\{l',l''\}}\Big(\{v_{l'}\ge p_{l'}, v_{l'}-p_{l'}\ge v_{l''}-p_{l''}\}\Big)p_{l'} \\
&\quad + \xi^{\epsilon,N,n}_{\{l',l''\}}\Big(\{v_{l''}\ge p_{l''}, v_{l''}-p_{l''}\ge v_{l'}-p_{l'}\}\Big)p_{l''} \\
&=: a^n(p_{l'},p_{l''}).
\end{aligned}$$

Since $(p_{l'},p_{l''})\mapsto a_n$ is upper semicontinuous, there exists $(p^n_{l'},p^n_{l''})\in[0,1]^2$ such that

$$a^n(p^n_{l'},p^n_{l''}) = \max_{(p_{l'},p_{l''})\in[0,1]^2} a^n(p_{l'},p_{l''}).$$

Since $[0,1]^2$ is sequentially compact, there exists a converging subsequence $\Big((p^{n_t}_{l'},p^{n_t}_{l''})\Big)_{t\in\mathbb{N}}$ with

$$\lim_{t\to\infty} a^{n_t}(p^{n_t}_{l'},p^{n_t}_{l''}) = \limsup_{n\to\infty}\max_{(p_{l'},p_{l''})\in[0,1]^2} a^n(p_{l'},p_{l''}).$$

Denoting

$$\lim_{t\to\infty}(p^{n_t}_{l'},p^{n_t}_{l''}) =: (p^*_{l'},p^*_{l''}),$$

it follows that for every $y>0$, there exists $t'\in\mathbb{N}$ such that for every $t>t'$,

$$\begin{aligned}a^{n_t}(p_{l'}^{n_t},p_{l''}^{n_t}) &\le \xi^{\epsilon,N,n_t}_{\{l',l''\}}\Big(\{v_{l'}\ge p^*_{l'}-y, v_{l'}-v_{l''}\ge p^*_{l'}-p^*_{l''}-y\}\Big)(p^*_{l'}+y)\\ &\quad +\xi^{\epsilon,N,n_t}_{\{l',l''\}}\Big(\{v_{l''}\ge p^*_{l'}-y, v_{l''}-v_{l'}\ge p^*_{l''}-p^*_{l'}-y\}\Big)(p^*_{l''}+y).\end{aligned}$$

Hence, for every $y>0$,

$$\begin{aligned}\lim_{t\to\infty} a^{n_t}(p_{l'}^{n_t},p_{l''}^{n_t}) &\le \lim_{n\to\infty}\xi^{\epsilon,N,n}_{\{l',l''\}}\Big(\{v_{l'}\ge p^*_{l'}-y, v_{l'}-v_{l''}\ge p^*_{l'}-p^*_{l''}-y\}\Big)(p^*_{l'}+y)\\ &\quad +\xi^{\epsilon,N,n}_{\{l',l''\}}\Big(\{v_{l''}\ge p^*_{l'}-y, v_{l''}-v_{l'}\ge p^*_{l''}-p^*_{l'}-y\}\Big)(p^*_{l''}+y).\end{aligned}$$

By part b) of Lemma SA.5, since f is the Lebesgue measure on $[0,1]$,

$$\begin{aligned}&\lim_{n\to\infty}\xi^{\epsilon,N,n}_{\{l',l''\}}\Big(\{v_{l'}\ge p^*_{l'}-y, v_{l'}-v_{l''}\ge p^*_{l'}-p^*_{l''}-y\}\Big)(p^*_{l'}+y)\\ &\quad +\xi^{\epsilon,N,n}_{\{l',l''\}}\Big(\{v_{l''}\ge p^*_{l'}-y, v_{l''}-v_{l'}\ge p^*_{l''}-p^*_{l'}-y\}\Big)(p^*_{l''}+y)\\ =\;& (p^*_{l'}+y)\int_{p^*_{l'}-y}^1 v_{l'}-(p^*_{l'}-p^*_{l''}-y)\mathrm{d}v_{l'}+(p^*_{l''}+y)\int_{p^*_{l''}-y}^1 v_{l''}-(p^*_{l''}-p^*_{l'}-y)\mathrm{d}v_{l''}.\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{t\to\infty} a^{n_t}(p_{l'}^{n_t},p_{l''}^{n_t})\\ \le\;& \lim_{y\to 0}(p^*_{l'}+y)\int_{p^*_{l'}-y}^1 v_{l'}-(p^*_{l'}-p^*_{l''}-y)\mathrm{d}v_{l'}+(p^*_{l''}+y)\int_{p^*_{l''}-y}^1 v_{l''}-(p^*_{l''}-p^*_{l'}-y)\mathrm{d}v_{l''}\\ =\;& p^*_{l'}\int_{p^*_{l'}}^1 v_{l'}-(p^*_{l'}-p^*_{l''})\mathrm{d}v_{l'}+p^*_{l''}\int_{p^*_{l''}}^1 v_{l''}-(p^*_{l''}-p^*_{l'})\mathrm{d}v_{l''}\\ \le\;& \max_{(p_{l'},p_{l''})\in[0,1]^2} p_{l'}\int_{p_{l'}}^1 v_{l'}-(p_{l'}-p_{l''})\mathrm{d}v_{l'}+p_{l''}\int_{p_{l''}}^1 v_{l''}-(p_{l''}-p_{l'})\mathrm{d}v_{l''}\\ =\;& \pi_0\\ <\;& \pi.\end{aligned}$$

This shows (SA.33). $\square$

The next lemma concerns the case that the seller offers in market $\mu^{\epsilon,N,n,K}$ one of the two products in K and one product that does not belong to K. Since $\zeta^{\epsilon,N}$ and $\xi^{\epsilon,N,n}$ are invariant to permutation, $\tilde{c}_{\Pi}^{\epsilon,N,n,K}(\{l',l''\},p_{l'},p_{l''})$ has the same value for every $\{l',l''\}\in\mathcal{K}$ with $|\{l',l''\}\cap K|=1$.

Lemma SA.10. *Let $k', k'' \in \mathbb{N}, k' \neq k''$. Let $l', l'' \in \mathbb{N}, l' \neq l''$ with $|\{l', l''\} \cap \{k', k''\}| = 1$. Then*

$$\limsup_{\epsilon \to 0} \limsup_{N \to \infty} \limsup_{n \to \infty} \max_{(p_{l'}, p_{l''}) \in [0,1]^2} \tilde{c}_{\Pi}^{\epsilon, N, n, \{k', k''\}}(\{l', l''\}, p_{l'}, p_{l''}) < \pi. \tag{SA.34}$$

Proof. As in the proof of the previous lemma, note that

$$\begin{aligned}\tilde{c}_{\Pi}^{\epsilon,N,n,\{k',k''\}}(\{l',l''\},p_{l'},p_{l''}) \leq\ & \mu_{\{l',l''\}}^{\epsilon,N,n,\{k',k''\}}\Big(\{v_{l'} \geq p_{l'}, v_{l'} - p_{l'} \geq v_{l''} - p_{l''}\}\Big)p_{l'} \\ &+ \mu_{\{l',l''\}}^{\epsilon,N,n,\{k',k''\}}\Big(\{v_{l''} \geq p_{l''}, v_{l''} - p_{l''} \geq v_{l'} - p_{l'}\}\Big)p_{l''}.\end{aligned}$$

Without loss of generality, suppose that product l' is the product contained in $\{k', k''\}$. Furthermore, let $p_{l'} \geq 1/N$; as in the proof of Lemma SA.8, this will be the case for sufficiently large N when $\tilde{c}_{\Pi}^{\epsilon,N,n,\{k',k''\}}(\{l',l''\},p_{l'},p_{l''})$ is maximized. Then,

$$\begin{aligned} & \mu_{\{l',l''\}}^{\epsilon,N,n,\{k',k''\}}\Big(\{v_{l'} \geq p_{l'}, v_{l'} - p_{l'} \geq v_{l''} - p_{l''}\}\Big)p_{l'} \\ & \quad + \mu_{\{l',l''\}}^{\epsilon,N,n,\{k',k''\}}\Big(\{v_{l''} \geq p_{l''}, v_{l''} - p_{l''} \geq v_{l'} - p_{l'}\}\Big)p_{l''} \\ = \quad & \frac{1}{2}p_{l'}\int_{p_{l'}}^{1} \xi_{l''}^{\epsilon,N,n}\Big([0, v_{l'} - p_{l'} + p_{l''}]\Big)e_{\pi,p'}^{\epsilon,N}(\mathrm{d}v_{l'}) \\ & \quad + \frac{1}{2}p_{l''}\int_{p_{l''}}^{1}\Big(1 + e_{\pi,p'}^{\epsilon,N}\Big([0, v_{l''} - p_{l''} + p_{l'}]\Big)\Big)\xi_{l''}^{\epsilon,N,n}(\mathrm{d}v_{l''}) \\ =: \quad & a^{\epsilon,N,n}(p_{l'}, p_{l''}).\end{aligned}$$

Since $(p_{l'}, p_{l''}) \mapsto a^{\epsilon,N,n}(p_{l'}, p_{l''})$ is upper semicontinuous, there exists $(p_{l'}^{\epsilon,N,n}, p_{l''}^{\epsilon,N,n}) \in [0,1]^2$ such that

$$a^{\epsilon,N,n}(p_{l'}^{\epsilon,N,n}, p_{l''}^{\epsilon,N,n}) = \max_{(p_{l'},p_{l''}) \in [0,1]^2} a^{\epsilon,N,n}(p_{l'}, p_{l''}).$$

Since $[0,1]^2$ is sequentially compact, there exists a converging subsequence

$$\Big((p_{l'}^{\epsilon_r,N_s,n_t}, p_{l''}^{\epsilon_r,N_s,n_t})\Big)_{r\in\mathbb{N},s\in\mathbb{N},t\in\mathbb{N}}$$

with

$$\lim_{r\to\infty}\lim_{s\to\infty}\lim_{t\to\infty} a^{\epsilon_r,N_s,n_t}(p_{l'}^{\epsilon_r,N_s,n_t}, p_{l''}^{\epsilon_r,N_s,n_t}) = \limsup_{\epsilon \to 0} \limsup_{N \to \infty} \limsup_{n \to \infty} \max_{(p_{l'},p_{l''}) \in [0,1]^2} a^{\epsilon,N,n}(p_{l'}, p_{l''}).$$

Denoting

$$\lim_{r\to\infty}\lim_{s\to\infty}\lim_{t\to\infty}(p_{l'}^{\epsilon_r,N_s,n_t}, p_{l''}^{\epsilon_r,N_s,n_t}) =: (p_{l'}^{*}, p_{l''}^{*}),$$

it follows that for every $y>0$, there exists $r'\in\mathbb{N}$ such that for every $r>r'$,

$$\begin{aligned}
&\lim_{s\to\infty}\lim_{t\to\infty} a^{\epsilon_r,N_s,n_t}(p_{l'}^{\epsilon_r,N_s,n_t},p_{l''}^{\epsilon_r,N_s,n_t})\\
\leq\ &\lim_{s\to\infty}\lim_{t\to\infty}\frac{1}{2}(p_{l'}^*+y)\int_{p_{l'}^*-y}^1 \xi_{l''}^{\epsilon_r,N_s,n_t}\Big([0,v_{l'}-p_{l'}^*+p_{l''}^*+y]\Big)e_{\pi,p'}^{\epsilon_r,N_s}(\mathrm{d}v_{l'})\\
&\quad+\frac{1}{2}(p_{l''}^*+y)\int_{p_{l''}^*-y}^1\Big(1+e_{\pi,p'}^{\epsilon_r,N_s}\Big([0,v_{l''}-p_{l''}^*+p_{l'}^*+y]\Big)\Big)\xi_{l''}^{\epsilon_r,N_s,n_t}(\mathrm{d}v_{l''}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\lim_{r\to\infty}\lim_{s\to\infty}\lim_{t\to\infty} a^{\epsilon_r,N_s,n_t}(p_{l'}^{\epsilon_r,N_s,n_t},p_{l''}^{\epsilon_r,N_s,n_t})\\
\leq\ &\lim_{r\to\infty}\lim_{s\to\infty}\lim_{t\to\infty}\frac{1}{2}(p_{l'}^*+y)\int_{p_{l'}^*-y}^1 \xi_{l''}^{\epsilon_r,N_s,n_t}\Big([0,v_{l'}-p_{l'}^*+p_{l''}^*+y]\Big)e_{\pi,p'}^{\epsilon_r,N_s}(\mathrm{d}v_{l'})\\
&\quad+\frac{1}{2}(p_{l''}^*+y)\int_{p_{l''}^*-y}^1\Big(1+e_{\pi,p'}^{\epsilon_r,N_s}\Big([0,v_{l''}-p_{l''}^*+p_{l'}^*+y]\Big)\Big)\xi_{l''}^{\epsilon_r,N_s,n_t}(\mathrm{d}v_{l''}),\quad\forall y>0.
\end{aligned}\tag{SA.35}$$

By part b) of Lemma SA.5, since f is the Lebesgue measure on $[0,1]$,

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{2}(p_{l'}^*+y)\int_{p_{l'}^*-y}^1 \xi_{l''}^{\epsilon_r,N_s,n_t}\Big([0,v_{l'}-p_{l'}^*+p_{l''}^*+y]\Big)e_{\pi,p'}^{\epsilon_r,N_s}(\mathrm{d}v_{l'})\\
&\quad+\frac{1}{2}(p_{l''}^*+y)\int_{p_{l''}^*-y}^1\Big(1+e_{\pi,p'}^{\epsilon_r,N_s}\Big([0,v_{l''}-p_{l''}^*+p_{l'}^*+y]\Big)\Big)\xi_{l''}^{\epsilon_r,N_s,n_t}(\mathrm{d}v_{l''})\\
=\ &\frac{1}{2}(p_{l'}^*+y)\int_{p_{l'}^*-y}^1 f\Big([0,v_{l'}-p_{l'}^*+p_{l''}^*+y]\Big)e_{\pi,p'}^{\epsilon_r,N_s}(\mathrm{d}v_{l'})\\
&\quad+\frac{1}{2}(p_{l''}^*+y)\int_{p_{l''}^*-y}^1\Big(1+e_{\pi,p'}^{\epsilon_r,N_s}\Big([0,v_{l''}-p_{l''}^*+p_{l'}^*+y]\Big)\Big)f(\mathrm{d}v_{l''})\\
=\ &\frac{1}{2}(p_{l'}^*+y)\int_{p_{l'}^*-y}^1 (v_{l'}-p_{l'}^*+p_{l''}^*+y)e_{\pi,p'}^{\epsilon_r,N_s}(\mathrm{d}v_{l'})\\
&\quad+\frac{1}{2}(p_{l''}^*+y)\int_{p_{l''}^*-y}^1\Big(1+e_{\pi,p'}^{\epsilon_r,N_s}\Big([0,v_{l''}-p_{l''}^*+p_{l'}^*+y]\Big)\Big)\mathrm{d}v_{l''}\\
=\ &\frac{1}{2}(p_{l'}^*+y)\left(e_{\pi,p'}^{\epsilon_r,N_s}\Big([p_{l'}^*-y,1]\Big)p_{l''}^*+\int_{p_{l'}^*-y}^1 e_{\pi,p'}^{\epsilon_r,N_s}([v_{l'},1])\mathrm{d}v_{l''}\right)\\
&\quad+\frac{1}{2}(p_{l''}^*+y)\int_{p_{l''}^*-y}^1\Big(1+e_{\pi,p'}^{\epsilon_r,N_s}\Big([0,v_{l''}-p_{l''}^*+p_{l'}^*+y]\Big)\Big)\mathrm{d}v_{l''}.
\end{aligned}$$

By (SA.19) and (SA.16),

$$\lim_{r\to\infty}\lim_{s\to\infty} e_{\pi,p'}^{\epsilon_r,N_s}=g_\pi.$$

Because $x\mapsto g_\pi[x,1]$ is continuous, the Portmanteau Theorem therefore implies

$$\lim_{r\to\infty}\lim_{s\to\infty} e_{\pi,p'}^{\epsilon_r,N_s}\Big([p_{l'}^*-y,1]\Big)=g_\pi\Big([p_{l'}^*-y,1]\Big).$$

Since $v_{l'} \mapsto e^{\epsilon_r, N_s}_{\pi,p'}([v_{l'},1])$ and $v_{l''} \mapsto e^{\epsilon_r, N_s}_{\pi,p'}\Big([0, v_{l''} - p^*_{l''} + p^*_{l'} + y]\Big)$ are monotone and therefore have at most countably many discontinuity points, the Dominated Convergence Theorem thus implies

$$\begin{aligned}
&\lim_{r\to\infty}\lim_{s\to\infty}\frac{1}{2}(p^*_{l'}+y)\left(e^{\epsilon_r,N_s}_{\pi,p'}\Big([p^*_{l'}-y,1]\Big)p^*_{l''}+\int_{p^*_{l'}-y}^{1} e^{\epsilon_r,N_s}_{\pi,p'}([v_{l'},1])\mathrm{d}v_{l''}\right)\\
&\quad+\frac{1}{2}(p^*_{l''}+y)\int_{p^*_{l''}-y}^{1}\Big(1+e^{\epsilon_r,N_s}_{\pi,p'}\Big([0,v_{l''}-p^*_{l''}+p^*_{l'}+y]\Big)\Big)\,\mathrm{d}v_{l''}\\
=\;&\frac{1}{2}(p^*_{l'}+y)\left(g_\pi\Big([p^*_{l'}-y,1]\Big)p^*_{l''}+\int_{p^*_{l'}-y}^{1} g_\pi([v_{l'},1])\mathrm{d}v_{l''}\right)\\
&\quad+\frac{1}{2}(p^*_{l''}+y)\int_{p^*_{l''}-y}^{1}\Big(1+g_\pi\Big([0,v_{l''}-p^*_{l''}+p^*_{l'}+y]\Big)\Big)\,\mathrm{d}v_{l''}.
\end{aligned}$$

Because $x \mapsto g_\pi\Big([x,1]\Big)$ is continuous, and $x \mapsto g_\pi\Big([0,x]\Big)$ is continuous except at $x=1$,

$$\begin{aligned}
&\lim_{y\to 0}\frac{1}{2}(p^*_{l'}+y)\left(g_\pi\Big([p^*_{l'}-y,1]\Big)p^*_{l''}+\int_{p^*_{l'}-y}^{1} g_\pi([v_{l'},1])\mathrm{d}v_{l''}\right)\\
&\quad+\frac{1}{2}(p^*_{l''}+y)\int_{p^*_{l''}-y}^{1}\Big(1+g_\pi\Big([0,v_{l''}-p^*_{l''}+p^*_{l'}+y]\Big)\Big)\,\mathrm{d}v_{l''}\\
=\;&\frac{1}{2}p^*_{l'}\left(g_\pi\Big([p^*_{l'},1]\Big)p^*_{l''}+\int_{p^*_{l'}}^{1} g_\pi([v_{l'},1])\mathrm{d}v_{l''}\right)\\
&\quad+\frac{1}{2}p^*_{l''}\int_{p^*_{l''}}^{1}\Big(1+g_\pi\Big([0,v_{l''}-p^*_{l''}+p^*_{l'}]\Big)\Big)\,\mathrm{d}v_{l''}.
\end{aligned}$$

By (SA.35), (SA.34) thus holds if

$$\frac{1}{2}p^*_{l'}\left(g_\pi\Big([p^*_{l'},1]\Big)p^*_{l''}+\int_{p^*_{l'}}^{1} g_\pi([v_{l'},1])\mathrm{d}v_{l''}\right)+\frac{1}{2}p^*_{l''}\int_{p^*_{l''}}^{1}\Big(1+g_\pi\Big([0,v_{l''}-p^*_{l''}+p^*_{l'}]\Big)\Big)\,\mathrm{d}v_{l''}<\pi.$$

This inequality does hold, as we stated in Lemma SA.7. □

Taken together, Lemmas SA.8, SA.9, and SA.10 show the limit (SA.29) regarding the producer surplus and the limit (SA.30) regarding the consumer surplus. This concludes the proof of the second part of Theorem 1.

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