Online Appendix

"(Pro)-Social Learning and Selective Disclosure"

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A Heterogeneous Payoffs

We now extend the analysis to allow for heterogeneous payoffs, by introducing an idiosyncratic component to utility. We also relax the restriction to equilibria in which P2 is assumed to fully disclose. Besides adding realism this will serve to show that, under general conditions on the form of this heterogeneity, disclosure is still polarized and positively biased, and that all equilibria are *necessarily* EE's in which P2 strictly prefers to disclose.

Let the payoff to agent t from receiving signal x now be $x\epsilon_t$, where each ϵ_t is drawn from a distribution H, independently from x. Without loss of generality, we assume that $\mathbb{E}(\epsilon) = 1$ and H has full support on $[0, \infty)$, with a density h that is everywhere positive. We further assume that the realization of their own ϵ_t is observable to an agent prior to their consumption decision –e.g., it represents the intensity of their need for such a product– whereas the value $x\epsilon_t$ (or, equivalently, x itself) is revealed only when consumption occurs. Thus ϵ_t guides the experimentation decision a_t, but when $a_t = 1$ the relevant information for the disclosure decision d_t remains x itself, since ϵ_t is irrelevant to any successor. Formally, consumption rules now map both from beliefs and shocks, i.e. $a_t : [0, 1] \times [0, \infty) \to \{0, 1\}$, while disclosure rules remain as before.

The expected values, from P1's perspective, of subsequent players' consumptions are now:

$$u(r \mid q) \equiv \int_{c/r}^{\infty} (q\epsilon - c)\, dH(\epsilon),$$

$$\Lambda(r \mid q) \equiv \mathbb{E}_{\epsilon,z}(u\,(r^z\epsilon \mid q^z\epsilon)) = \int_0^1 \int_{\frac{c}{r^z}}^{\infty} (q^z\epsilon - c)\, dH(\epsilon) dF_q(z).$$

We start with some basic properties of u and Λ.

Lemma A.1. *1. Both maps $r \mapsto u(r \mid q)$ and $\Lambda(r \mid q)$ are strictly maximized at q.*

2. $\Lambda(r \mid q) \geq (>) u(r \mid q)$ for all $r \leq (<) q$.

Proof. Direct calculation verifies that $\frac{\partial u}{\partial r} = -\frac{c^2}{r^2}\left(\frac{q}{r} - 1\right) h\left(\frac{c}{r}\right)$, which is equal to zero if and only if $q = r$. Since $\Lambda(r \mid q) = \mathbb{E}_{\epsilon,z}(u\left(r^z \epsilon \mid q^z \epsilon\right))$, point 1 is verified. To verify point 2, note that

$$
\begin{aligned}
\Lambda(r \mid q) - u(r \mid q) &= \int_0^1 \int_{\frac{c}{r^z}}^{\infty} \left(q^z \epsilon - c\right) dH(\epsilon) dF_q(z) - u(r \mid q) \\
&= \int_0^{\hat{x}} \int_{\frac{c}{r^z}}^{\infty} \left(q^z \epsilon - c\right) dH(\epsilon) dF_q(z) \\
&+ \underbrace{\int_{\hat{x}}^1 \int_{\frac{c}{r^z}}^{\infty} \left(q^z \epsilon - c\right) dH(\epsilon) dF_q(z)}_{\geq u(r|q)} - u(r \mid q) \geq 0,
\end{aligned}
$$

where the last inequality holds because $z \mapsto q^z \epsilon - c$ is positive on the range $[c/r^z, \infty)$ by the MLRP, since by assumption $r \leq q$. $\square$

Note that since $V_2(r \mid q) = u(r \mid q)$, Lemma A.1 implies that full disclosure by P2 is a *strictly* dominant strategy. In Section 2, P2 was indifferent over posterior beliefs that induce the same action by P3. Now, greater accuracy leads to a strictly lower chance of erroneous consumption choices by P3 due to idiosyncratic shocks.

As before, this allows us to simplify player 1's value function,

$$
V_1(r \mid q) = u(r \mid q) + \Big(\alpha C(r) \Lambda(r \mid q) + (1 - \alpha + \alpha\left(1 - C(r)\right) u(r \mid q)\Big), \tag{A.1}
$$

where

$$
C(r) \equiv \int_{c/r}^{\infty} dH(\epsilon)
$$

is the probability of consumption given a prior belief r, prior to the realization of ϵ.

A.0.1 Selected Disclosure

In order to draw comparison to the results in Sections 3 and 4, we first adapt the definition of experimentation equilibria in the most natural manner. Now, let

$$X_E(\sigma) = \{x \in N_1(d) \mid a_2(p^{\emptyset}, \epsilon) > a_2(p^x, \epsilon) \quad \forall \epsilon \in [0, \infty)\}$$

denote the experimentation set for an equilibrium σ. First, we recover the result of polarized disclosure.

Lemma A.2. *(Polarized disclosure) Fix $r \in (0, 1)$. Then,* $\lim_{q \to 0,1} [V_1(q \mid q) - V_1(r \mid q)] > 0$.

Proof. For the lower limit, note that

$$u(r \mid 0) = \int_{c/r}^{\infty} -c \, dH(\epsilon) < 0, \quad \Lambda(r \mid 0) = \int_0^1 \int_{c/r^z}^{\infty} -c \, dH(\epsilon) dF_q(z) < 0,$$

whereas $u(0 \mid 0) = \Lambda(0 \mid 0) = 0$. Thus, by the expression for $V_1(r \mid q)$ given in (A.1), $V_1(r \mid q) < 0 = V_1(q \mid q)$. For the upper limit, note that

$$\Lambda(r \mid 1) = \int_0^1 \int_{\frac{c}{r^z}}^{\infty} (\epsilon - c) \, dH(\epsilon) dF_q(z),$$

which is strictly increasing in r by the MLRP, since the integrand is strictly positive. Similarly, $r \mapsto u(r \mid 1)$ is strictly increasing. Finally, $r \mapsto C(r)$ is also strictly increasing, and thus so is $r \mapsto V_1(r \mid 1)$. Therefore, the claim is verified. $\square$

Next, we demonstrate that for any prior $p \in (0, 1)$, any posterior $q \geqslant p$ (i.e. any signal $x \geqslant \hat{x}$) is disclosed by P1. Note that whereas in the baseline model (Lemma A.2) it was dominant for all posteriors $q \geq c$ to be disclosed, here this is no longer necessarily the case.

Lemma A.3. *If $p^x \geqslant p$, then $d_1(x) = 1$ is a strictly dominant strategy.*

Proof. Suppose not, so that there exists an $x > \hat{x}$ such that $d_1(x) = 0$. Take the largest such x and let $q = p^x$. By construction, to satisfy the equilibrium belief condition (3) it must be that $p^\emptyset < q$. But then by Lemma A.1, $u(p^\emptyset \mid q) < u(q \mid q)$ and $\Lambda(p^\emptyset \mid q) < \Lambda(q \mid q)$, while we also have $C(p^\emptyset) < C(q)$ as $r \mapsto C(r)$ is strictly increasing. Combining, we have that $V_1(p^\emptyset \mid q) < V_1(q \mid q)$. □

Finally, we prove that non-disclosure of signals that convey marginally bad news (namely, such that the posterior p^x is just below the prior p) is optimal. This result has no direct analog in the baseline model, insofar as non-disclosure now occurs at signals the revelation of which would have induced consumption with strictly positive probability ($c < p^x < p$).

Lemma A.4. *(Positive selection) Let $\tilde{V}_1(q) \equiv V_1(r \mid q)$. Then $\tilde{V}_1'(q) > \frac{\partial V_1}{\partial q}|_{r=q}$.*

Proof. Since ${\tilde{V}_1}'(q) = \partial V_1(r \mid q)/\partial r|_{r=q} + \partial V_1(r \mid q)/\partial q|_{r=q}$, the claim is equivalent to proving that $\partial V_1(r \mid q)/\partial r|_{p=q} > 0$. But

$$\begin{aligned}\frac{\partial V_1|(r \mid q)}{\partial r}|_{r=q} = \underbrace{\frac{\partial u}{\partial r}_{r=q}}_{=0} &+ \frac{c}{q^2} h\left(\frac{c}{q}\right)\Big[\Lambda(q \mid q) + (1 - C(q)) u(q \mid q)\Big] \\ &+ C\left(\frac{c}{q}\right)\Big[\underbrace{\frac{\partial \Lambda}{\partial r}|_{r=q}}_{=0} + (1 - C(q)) \underbrace{\frac{\partial u}{\partial r}|_{r=q}}_{=0} - C'(q) u(q \mid q)\Big] \\ = \Lambda(q \mid q) - C\left(\frac{c}{q}\right) u(q \mid q) \frac{c}{q^2} h\left(\frac{c}{q}\right) > 0,\end{aligned}$$

where the last inequality holds because $C(q \mid q) < 1$ and $\Lambda(q \mid q) > u(q \mid q)$. □

In particular, for $x = \hat{x} - \varepsilon$ where ε is small, non-disclosure is optimal. Combining Lemmas A.3 and A.4 with a continuity argument yields that non-disclosure takes place in (at least) some interval $[\hat{x} - \varepsilon, \hat{x})$, and thus disclosure is positively biased. Furthermore, we have:

Lemma A.5. *Any equilibrium is an EE.*

Proof. To see that all equilibria admit a non-empty experimentation region, note that Lemmas A.2 and A.3 imply that in any equilibrium σ, for each p there exists a minimal posterior

$\underline{q}(p) < p$ that is concealed. Continuity of $r \mapsto V_1(r \mid q)$ then ensures the existence of a $\delta > 0$ such that posteriors in the interval $[\underline{q}(p), \underline{q}(p) + \delta)$ are concealed. But for δ sufficiently small, $\underline{q}(p) + \delta < p$, and so $[\underline{q}(p), \underline{q}(p) + \delta) \subset X_E(\sigma)$. $\square$

B Optimal Feedback – Persuasion

We now turn to the benchmark wherein P1 can commit to an arbitrary messaging rule prior to receiving their private signal x (Kamenica and Gentzkow, 2011). Formally, P1 chooses an *information structure*, consisting of a message space $\mathcal{S}$ along with a collection of conditional probabilities $(\pi(\cdot \mid x))_{x \in X}$, where $\pi(s \mid x)$ denotes the likelihood of P1 sending the message s given that they received signal x. Let $\mathcal{M} = X \cup \{\emptyset\}$ denote the (rich) message space that naturally associates messages with outcomes, as well as a privileged message $\emptyset$ that denotes no signal reported. We may take $\mathcal{S} = \mathcal{M}$. Since communication is no longer constrained to be verifiable, we can set $\alpha = 1$ without loss of generality. Contrasting this case with that of hard-evidence disclosure will thus shed light on how ex-post IC constraints shape optimal feedback. Recently developed techniques in the persuasion literature allow us to completely characterize the solution (Dworczak and Martini, 2019). Denote $V_1(q \mid q)$ by $V_1(q)$ for simplicity. The following result is illustrated in Figure 2.

Theorem B.1. *There exist $\underline{q}^*(p) < c < \bar{q}^*(p)$ such that the solution to the persuasion problem takes the following form: reveal x if either $p^x < \underline{q}^*(p)$ or $p^x \geqslant \bar{q}^*(p)$, and pool all x such that $p^x \in [\underline{q}^*(p), \bar{q}^*(p))$. Furthermore, $\underline{q}^*(p), \bar{q}^*(p)$ solve*

$$\mathbb{E}_p(q \mid q \in [\underline{q}^*(p), \bar{q}^*(p))) \equiv \frac{\int_{\underline{q}^*(p)}^{\bar{q}^*(p)} q \, dG_p(q)}{\int_{\underline{q}^*(p)}^{\bar{q}^*(p)} dG_p(q)} = c, \tag{B.1}$$

and

$$\frac{V_1(\bar{q}^*(p))}{V_1(c)} = \frac{\bar{q}^*(p) - \underline{q}^*(p)}{c - \underline{q}^*(p)}. \tag{B.2}$$

Proof. Since $q \mapsto V_1(r \mid q)$ is affine, standard arguments imply that the problem faced by

P1 under commitment is to solve

$$v^*(p) = \max_{H \in \Delta([0,1])} \int_0^1 V_1(q)\, dH(q), \tag{B.3}$$

subject to the constraint that H is a mean-preserving contraction of G_p (Kamenica and Gentzkow, 2011). First, we prove that $V_1(q)$ is convex on $[c, 1]$. To see this, note that Lemma 1 implies that $V_1(q \mid q) = \sup_{r \in [0,1]} V_1(r \mid q)$ for $q \in [c, 1]$, and that $q \mapsto V_1(r \mid q)$ is affine. The convexity of $V_1(q)$ then follows from standard results in convex duality (Rockafellar, 1997, Theorem 13.2).

We may now apply (Dworczak and Martini, 2019, Theorem 1). In particular, consider the function ψ defined by

$$\psi(q) = \begin{cases} V_1(q) & \text{if} \quad p^x \geqslant \bar{q}^*(p) \\ V_1(c)\left(\frac{q - \underline{q}^*(p)}{c - \underline{q}^*(p)}\right) & \text{if} \quad p^x \in [\underline{q}^*(p), \bar{q}^*(p)) \\ V_1(q) & \text{if} \quad p^x < \underline{q}^*(p), \end{cases}$$

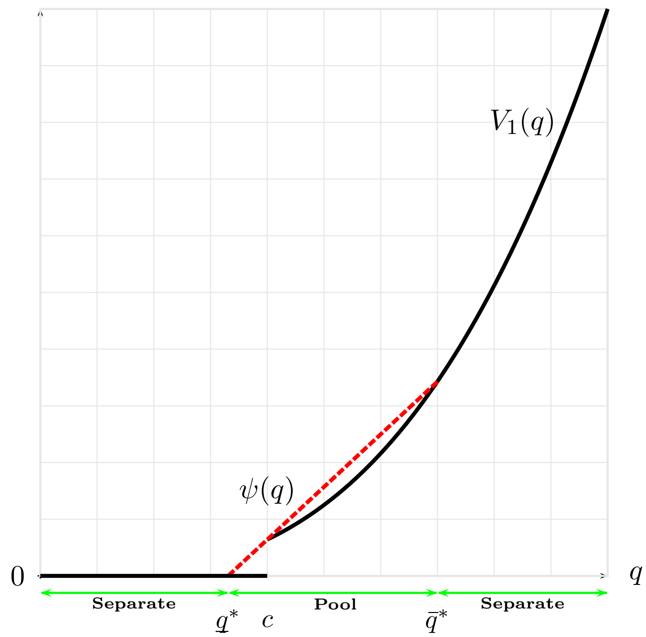
and the distribution $H_p : [0,1] \to [0,1]$ defined by

$$H_p(q) = \begin{cases} G_p(q) & \text{if} \quad p^x \geqslant \bar{q}^*(p) \\ G_p(c) + \mathbb{I}_{q \geq c}[G_p(\bar{q}^*(p)) - G_p(\underline{q}^*(p))] & \text{if} \quad p^x \in [\underline{q}^*(p), \bar{q}^*(p)) \\ G_p(q) & \text{if} \quad p^x < \underline{q}^*(p), \end{cases}$$

which reveals q when either $q \geq \bar{q}^*(p)$ or $q \leq \underline{q}^*(p)$ and pools otherwise. It is readily verified that ψ and H together satisfy conditions 3.1-3.3 of (Dworczak and Martini, 2019, Theorem 1), and thus constitute a solution to the commitment problem. Finally, note that since $q \mapsto G_p(q)$ is continuous and strictly increasing, so too are $\underline{q}^*(p), \bar{q}^*(p)$. $\square$

Communication under persuasion is also both polarized (pooling takes place on an interior interval) and positively selected (the average belief conditional on pooling is c, which

Figure 2: Persuasion Solution



Disclosure under commitment. Value function $V_1(q) \equiv V_1(q \mid q)$: solid black lines. $\underline{q}^*, \bar{q}^*$ are determined by both $\mathbb{E}(q \mid q \in [\underline{q}^*, \bar{q}^*)) = c$ and lying on a straight-line segment $\psi(q)$ (dotted red) intersecting $V_1(q)$ at $\underline{q}^*, \bar{q}^*$ and c.

is less than the prior p). In contrast to the disclosure benchmark, however, this pooling interval remains even when the prior p is close to c. Crucially, under persuasion, P1 can "pool down" by pooling posteriors above c with those below c, while still averaging to c (equation (B.1)). This allows them to maintain a positive-measure pooling interval as the prior p converges to either c or 1. In contrast, under disclosure such pooling down cannot occur, since P1 finds it ex-post optimal to disclose (separate) at all posteriors above c. This heavily constrains their ability to not disclose at posteriors below c. The logic presented here highlights the role of ex-post optimality (equation (4)) that disclosure rules must satisfy in shaping optimal feedback.

Corollary B.1. *Both $\underline{q}^*(p)$ and $\bar{q}^*(p)$ are strictly decreasing in p. Furthermore, $lim_{p \to c,1} \underline{q}^*(p) < c < \lim_{p \to c,1} \bar{q}^*(p)$.*

Proof. Note that the constraint (B.2) is independent of p, whereas a simple application of the posterior monotonicity property (Proposition 4 in Smith et al. (2021)) implies that for fixed $\underline{q}, \bar{q}, \mathbb{E}_p(q \mid q \in [\underline{q}, \bar{q}))$ is strictly increasing in p. Thus, to keep $\mathbb{E}_p(q \mid q \in [\underline{q}, \bar{q}))$ fixed, we must lower both $\underline{q}$ and $\bar{q}$. The final part of the corollary follows by noting that $V(q)$ is strictly increasing and convex for $q \geq c$ and strictly positive at c, and thus for all $p \in [c, 1]$ the line segment intersecting the three points $(\underline{q}^*(p), 0), (c, V_1(c)), (\bar{q}^*(p)$ and $V_1(\bar{q}^*(p)))$ can only exist if $\underline{q}^*(p) \neq \bar{q}^*(p)$, while the constraint that $\mathbb{E}_p(q \mid q \in [\underline{q}^*(p), \bar{q}^*(p))) = c$ further implies that $\underline{q}^*(p) < c < \bar{q}^*(p)$. □

Finally, notice that the persuasion outcome — which did not assume information to be verifiable – can be implemented via commitment to the verifiable disclosure rule

$$d(x) = \begin{cases} 1 & \text{if} \quad p^x \geqslant \bar{q}^*(p) \\ \emptyset & \text{if} \quad p^x \in [\underline{q}^*(p), \bar{q}^*(p)) \\ 1 & \text{if} \quad p^x < \underline{q}^*(p). \end{cases}$$

This is due to the simple structure of optimal persuasion; it is not only monotone partitional (Dworczak and Martini, 2019), but includes only one pooling region (see Figure 2). Thus, the pooling region can be interpreted as non-disclosure and the separating regions as disclosure, satisfying the verifiability assumption. In this sense, the benefit of persuasion over (ex-post) verifiable disclosure comes directly from which posteriors (signals) are credibly concealed, rather than the communication language itself.

C Biased Feedback – Cheap Talk

We now consider a natural variant on our baseline model, by relaxing the requirement of hard evidence disclosure and instead permitting arbitrary message reporting (cheap talk). Such a variant is important for several reasons. First, in many applied settings, it might not only be feasible but strategically optimal for consumers to misreport their experiences. The hard-evidence baseline abstracts from this possibility, thus providing a useful benchmark;

even when fake reviews are impossible, might there be scope for strategic disclosure? In this section, we explore the extent of strategic information transmission when lying is both feasible and costless. Second, by studying an alternative, well-established form of equilibrium information transmission, we make clear the features of strategic disclosure that are invariant to the information-sharing technology available to agents.

Specifically, we endow each agent with a rich messaging space $\mathcal{M} = [0,1] \times \{\emptyset\}$ that allows not only for full separation but also for agents to send a privileged message that pools with non-arriving consumers, so that messaging rules (previously, disclosure rules) are now mappings $d_t : X \times [0,1] \to \mathcal{M}$.[21] Again, full transparency is dominant for P2, so we focus on P1's messaging strategy. Let $r^*(m)$ denote P2's equilibrium belief upon observing message m. Then the IC constraint (4) is replaced with the condition

$$d(x) \in \underset{m \in \text{supp}(d)}{\arg\min}\ V_1(r^*(m) \mid p^x). \tag{C.1}$$

We focus on the case where $\alpha = 1$. Combining various insights learned through the baseline analysis, we summize that all equilibria must admit a partitional structure. The proof of Theorem C.1 is constructive. First, we identify a lower-bound on the degree of experimentation possible; there exists a $\underline{q}_{min}$ such that $V_1(c \mid \underline{q}_{min})$, thus any type lower prefers to terminate experimentation, regardless of the continuation belief r. Each equilibrium is then determined by its associated $\underline{q} \in [\underline{q}_{min}, c]$, that is the lowest type whose message induces experimentation. See Figure 3 for a graphical illustration of this construction.

Theorem C.1. *All equilibria are partitional. That is, for all $r \in [0,1]$ induced in equilibrium, the set of q in which r is induced forms an interval in $[0,1]$. Furthermore, there must be at most finitely many such intervals on $[c,1]$.*

Proof. We proceed with a series of lemmas.

[21] We focus on pure-strategy equilibria for simplicity, noting the usual implementation via uniform randomization in cheap-talk games

Lemma C.1. *All equilibria are partitional. Furthermore, there must be at most countably infinitely many such intervals on* $[c, 1]$.

Proof. Lemma 1 tells us that $r \mapsto V_1(r \mid q)$ is maximized at $r = q$, and that $q \mapsto V_1(r \mid q)$ is strictly increasing, so that $\arg\max_{r\in[0,1]} V_1(r \mid q)$ is strictly increasing in q, which proves the first claim. To prove the second one, we argue that there can be no interval in $[c, 1]$ on which separation can occur. Suppose there were, and take the lowest such interval $[q_1, q_2], q_1 \leq q_2$. If $q_2 < 1$, then we claim that types $q \in (q_2 + \varepsilon]$ have an incentive to pool with q_2. For since this was the lowest separating interval, it must be that types $q \in (q_2 + \varepsilon]$ induce a belief $\hat{q} = q_2 + \delta, \delta > 0$. By Lemma A.1, $V_1(\hat{q}_2 \mid q_2 + \varepsilon) < V_1(q_2 \mid q_2 + \varepsilon) \approx V_1(q_2 \mid q_2) + \varepsilon V_1'(q_2 \mid q_2)$ for small enough $\varepsilon > 0$. If $q_2 = 1$, then we claim that $q \in (q_1 - \varepsilon]$ have an incentive to pool with q_1 by analogous reasoning. □

Lemma C.2. *There exists* $\underline{q}_{min} < c$ *such that full revelation is weakly dominant for all types* $q \in [0, \underline{q}_{min})$.

Proof. $\underline{q}_{min}$ is the unique root of $q \mapsto V_1(c \mid q)$ on $[0, c]$, which is well-defined since the map is continuous, strictly increasing with $V_1(c \mid 0) < 0 < V_1(c \mid c)$. □

It is thus without loss to associate an equilibrium with a lowest type $\underline{q} > 0$ that forms part of a pooling interval that itself induces experimentation. More specifically, combining with Lemma C.1, an equilibrium can be described by a (possibly infinite) sequence ($\underline{q} \equiv q_0 < q_1 < q_2 < \ldots$ such that types in $[q_i, q_{i+1})$ pool and $\hat{\underline{q}} \equiv \mathbb{E}(q \mid q \in [\underline{q}, q_1)) \geq c$. More generally, we denote $\hat{q}_{i+1} = \mathbb{E}(q \mid q \in [q_i, q_{i+1}))$.

We next prove that the two first intervals $[\underline{q}, q_1), [q_1, q_2)$ cannot be "too small" as types just below $\underline{q}$ would then profitably deviate by pooling with $[q_1, q_2)$ to induce $\hat{q}_1$.

Lemma C.3. *For all* $\underline{q} \in [\underline{q}_{min}, c]$ *there exists* $\hat{q}_{2,min} > c$ *such that in any equilibrium,* $\hat{q}_2 \geq \hat{q}_{2,min}$.

Proof. If not, then for any $\varepsilon > 0$ there exists an equilibrium with $\hat{q}_2 \leq c + \varepsilon$. But since by definition $\hat{\underline{q}} \geq c$, it must be that $\underline{q} > \underline{q}_{min}$ for sufficiently small ε, and by the sandwich theorem, $V_1(\hat{\underline{q}} \mid \underline{q}) > 0$, violating the IC constraint at $\underline{q}$. □

Lemma C.4. *All equilibrium partitions essentially admit at most finitely many intervals covering* $[c, 1]$.

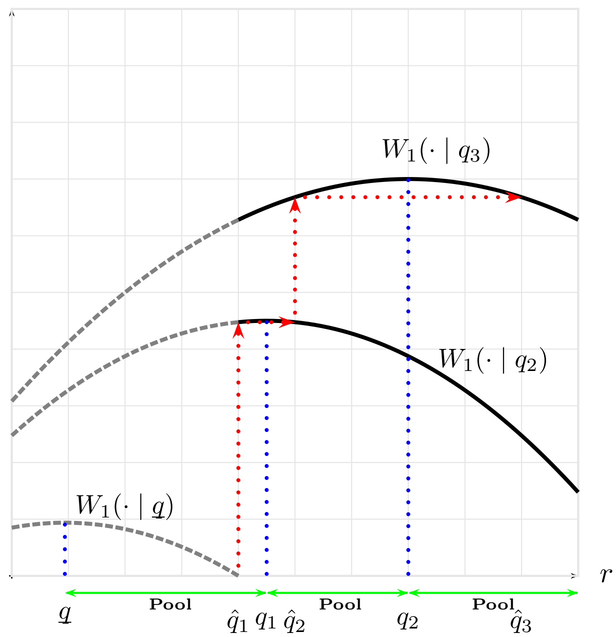
Proof. We proceed constructively, via the following algorithm:

1. Fix a $\underline{q} \geq \underline{q}_{min}$. Compute $\hat{q}_{max} \equiv \mathbb{E}_p(q \mid q \in [\underline{q}, 1])$.
   (a) If $V_1(\hat{q}_{max} \mid \underline{q}) > 0$, then $N^*(\underline{q}) = 0$ and $\underline{q}$ cannot be implemented in equilibrium.
   (b) If not, then there exists a unique $q_1 > c$ such that $V_1(\hat{q}_1 \mid \underline{q}) = 0$, where $\hat{q}_1 \equiv \mathbb{E}(q \mid q \in [\underline{q}, q_1])$. (Such a value exists by continuity and strict monotonicity of $r \mapsto V_1(r \mid q)$ on $[q, 1]$, the Intermediate Value Theorem and because $V_1(c \mid \underline{q}) > V_1(c \mid \underline{q}_{min}) = 0$ by Lemma C.2).
2. Compute $V_1(\hat{q}_1 \mid q_1)$.
   (a) If $V_1(1 \mid q_1) \geq V_1(\hat{q}_1 \mid q_1)$, then $N^*(\underline{q}) = 1$.
   (b) If not, then there exists a unique $q_2 > q_1$ such that $V_1(\hat{q}_2 \mid q_1) = V_1(\hat{q}_1 \mid q_1)$, where $\hat{q}_2$ is analogously defined, and q_2 exists by the same reasoning as q_1.
3. Repeat from step 2.

Finally, we argue that this algorithm terminates in finitely many steps. Suppose not. Then for all $\varepsilon > 0$, there exists an equilibrium and an interval $[q_i, q_{i+1}) \subset [c, 1]$ such that $q_{i+1} - q_i \leq \varepsilon$. Without loss, assume equality, and further assume that $[q_i, q_{i+1})$ is the lowest such interval (this is possible due to Lemma C.3). Let $\hat{q}_{i+1} = \mathbb{E}(q \mid q \in [q_i, q_{i+1}))$. Then there exists $\delta(\varepsilon) < \varepsilon$ such that $\hat{q}_{i+1} - q_i = \delta(\varepsilon)$. The Mean Value Theorem implies that

$$V_1(\hat{q}_i \mid q_i) - V_1(q_i \mid q_i) = V_1'(\varphi_1 \mid q_i)(\hat{q}_1 - q_i),$$

Figure 3: Cheap-Talk Equilibria: Construction



An equilibrium with three pooling intervals covering $[c, 1]$. Relaxed value function $W_1(r \mid q)$. For $r \geq c$, $W_1(\cdot \mid q) = V_1(\cdot \mid q)$ (solid black lines). For $r < c$, $V_1(\cdot \mid q) = 0$.

for some $\varphi_i \in (\hat{q}_i, q_i)$. But since $r \mapsto V_1(r \mid q)$ has a global maximum at q, we know that

$$V_1(q_i + \delta(\varepsilon) \mid q_i) - V_1(q_i \mid q_i) \approx \frac{\partial^2 V_1}{\partial r^2}(q_i \mid q_i)\delta(\varepsilon)^2.$$

Combining these terms implies that $q_i - \hat{q}_i = \kappa\delta(\varepsilon)$, for some $\kappa > 0$, and so $\hat{q}_{i+1} - \hat{q}_i = (\hat{q}_{i+1} - q_i) + (q_i - \hat{q}_i) = \kappa_i\delta(\varepsilon)$, for some $\kappa_i > 0$. Now, since $\varepsilon > 0$, there exists a finite $I > 0$ such that $q_{i-I} \leq \underline{q}$ (if not, then Lemma C.3 is violated) and thus a simple inductive argument implies that $\hat{q}_1 - \hat{\underline{q}} = \kappa_{i-I}\delta(\varepsilon)$, for some $\kappa_{i-I} > 0$. Taking ε (and thus $\delta(\varepsilon) < \varepsilon$) sufficiently small violates Lemma C.3. $\square$

$\square$