Supplemental Appendix for "Optimal Trade Mechanisms with Adverse Selection and Inferential Naivety," Takeshi Murooka and Takuro Yamashita

A Optimal Menu Mechanism and Its Properties under Power Distribution

This section investigates properties of the optimal menu mechanism under Power distribution. We first show that the followig two-option menu is optimal under Power distribution.

Proposition 8. Let F be a power distribution, $F(v) = v^\gamma$, with $\gamma > 0$ and $\alpha\mu < 1$. Then, the optimal menu mechanism is $\{(q_m, p_m)\}_{m=0}^M$ with $M = 2$, $(q_0, p_0) = (0, 0)$, $p_1 > \alpha\mu$, and $(q_2, p_2) = (1, \alpha\mu)$.

Proof. Suppose that there is an optimal menu mechanism with $M \geq 3$, which we denote by $\{(q_m, p_m)\}_{m=0}^M$. It suffices to construct another optimal menu mechanism with $M = 2$.

It is without loss to assume that p_m is decreasing in m, $p_M = \alpha\mu < \ldots < p_1 \leq 1$, $q_M = 1$, and that there exists a sequence of threshold seller types $0 = v_{M+1} < \ldots < v_1 \leq v_0 = 1$ such that any seller type $v \in (v_{m+1}, v_m)$ chooses option (q_m, p_m) in the associated equilibrium. More specifically, we have $q_1(p_1 - v_1) \geq 0$, v_2 is the seller type who is indifferent between (q_1, p_1) and (q_2, p_2), and so on. Without loss, we can assume that the rational buyer's individual rationality given (q_2, p_2) is binding: $(\alpha E[v|v \in (v_3, v_2)] - p_2)q_2 = 0$.[31]

Consider an alternative mechanism, which is exactly the same as the original one, except that, instead of (q_1, p_1) and (q_2, p_2), it offers (q_2', p_2') (together with $\{(q_m, p_m)\}_{m=3}^M$) where

[31] If $(\alpha E[v|v \in (v_3, v_2)] - p_2)q_2 > 0$, then we can be strictly better off by modifying the mechanism as follows. Fix $\varepsilon > 0$ small, and consider an alternative mechanism $\{(\hat{q}_m, \hat{p}_m)\}_{m=0}^M$, with $\hat{q}_2\hat{p}_2 = q_2p_2 - \varepsilon$, $\hat{q}_2 = q_2 - \frac{\varepsilon}{v_3}$, and the rest is the same as $\{(q_m, p_m)\}_{m=0}^M$. The same thresholds obtain for the seller types, except that threshold v_2 becomes $\hat{v}_2 = \frac{q_2p_2 - q_1p_1 - \varepsilon}{q_2 - q_1 - \frac{\varepsilon}{v_3}}$. Note that all the constraints on the buyer side continue to be satisfied. The welfare change with small ε is approximately $(v_2 - v_3)v_2f(v_2) - \int_{v_3}^{v_2} vdF > 0$ as $vf(v)$ is increasing for a power distribution.

the same v_3 is indifferent between (q_2', p_2') and (q_3, p_3), and the rational buyer's individual rationality holds with equality, that is, $\alpha E[v|v \in (v_3, p_2')] = p_2'$. In what follows, we focus on the case where $p_2' \leq 1$; the other case is similar.

The expected trade surplus in the new mechanism is higher than in the original mechanism by at least Δ, where:

$$\begin{aligned}\Delta &= q_2' \int_{v_3}^{p_2'} xdF - q_2 \int_{v_3}^{v_2} xdF - q_1 \int_{v_2}^{v_1} xdF \\ &\propto \int_{v_3}^{p_2'} xdF \frac{p_2 - v_3}{p_2' - v_3} - \int_{v_3}^{v_2} xdF - \int_{v_2}^{v_1} xdF \frac{p_2 - v_2}{v_1 - v_2}.\end{aligned}$$

Note that the new mechanism has exactly one less option than in the original mechanism. Therefore, if $\Delta \geq 0$, then we complete the proof by induction.

From here on, the notation is modified as follows: We use v instead of v_3, βv instead of v_2, and δv instead of p_2'. Note that $\beta \geq 1$ is a free parameter, while δ is determined as a function of F and α:

$$\delta v = \frac{\alpha \int_v^{\delta v} xdF}{F(\delta v) - F(v)}.$$

By construction, we have $\delta \geq \beta \geq 1$.

We now show that $\Delta \geq 0$. First, observe that $v_1 \leq \delta \beta v$. Thus:

$$\Delta \propto \int_v^{\delta v} xdF \frac{p_2 - v}{\delta v - v} - \int_v^{\beta v} xdF - \int_{\beta v}^{v_1} xdF \frac{p_2 - \beta v}{v_1 - \beta v}.$$

Because $xf(x)$ is increasing, this is at least:

$$\begin{aligned}&\int_v^{\delta v} xdF \frac{p_2 - v}{\delta v - v} - \int_v^{\beta v} xdF - \int_{\beta v}^{\delta \beta v} xdF \frac{p_2 - \beta v}{\delta \beta v - \beta v} \\ \propto\ & \beta \int_v^{\delta v} xdF(p_2 - v) - \beta v \int_v^{\beta v} xdF(\delta - 1) - \int_{\beta v}^{\delta \beta v} xdF(p_2 - \beta v) \\ \equiv\ & \Delta'.\end{aligned}$$

Note that

$$\begin{aligned}p_2 &= \frac{\alpha \int_v^{\beta v} xdF}{F(\beta v) - F(v)} \\ &= \frac{(F(\delta v) - F(v)) \int_v^{\beta v} xdF}{(F(\beta v) - F(v)) \int_v^{\delta v} xdF} \delta v,\end{aligned}$$

and thus,

$$\begin{aligned}
p_2 - v &= \frac{\delta(F(\delta v)-F(v))\int_v^{\beta v} xdF - (F(\beta v)-F(v))\int_v^{\delta v} xdF}{(F(\beta v)-F(v))\int_v^{\delta v} xdF} v,\\
p_2 - \beta v &= \frac{\delta(F(\delta v)-F(v))\int_v^{\beta v} xdF - \beta(F(\beta v)-F(v))\int_v^{\delta v} xdF}{(F(\beta v)-F(v))\int_v^{\delta v} xdF} v.
\end{aligned}$$

Hence:

$$\begin{aligned}
\Delta' &\propto \beta \int_v^{\delta v} xdF \left(\delta(F(\delta v)-F(v))\int_v^{\beta v} xdF - (F(\beta v)-F(v))\int_v^{\delta v} xdF\right)\\
&\quad -\beta \int_v^{\beta v} xdF(\delta-1)(F(\beta v)-F(v))\int_v^{\delta v} xdF\\
&\quad -\int_{\beta v}^{\delta\beta v} xdF \left(\delta(F(\delta v)-F(v))\int_v^{\beta v} xdF - \beta(F(\beta v)-F(v))\int_v^{\delta v} xdF\right)\\
&= \beta \int_v^{\delta v} xdF \left(\delta(F(\delta v)-F(\beta v))\int_v^{\beta v} xdF - (F(\beta v)-F(v))\int_{\beta v}^{\delta v} xdF\right)\\
&\quad -\int_{\beta v}^{\delta\beta v} xdF \left(\delta(F(\delta v)-F(v))\int_v^{\beta v} xdF - \beta(F(\beta v)-F(v))\int_v^{\delta v} xdF\right).
\end{aligned}$$

Because $F(\delta v)F(\beta v) = F(\delta\beta v)F(v)$ and $\int_a^b xdF = c(bF(b)-aF(a))$ for any $a < b$ for some constant $c > 0$, we finally have:

$$\begin{aligned}
\Delta' &\propto F(v)\left(\delta(F(\delta v)-F(\beta v))\int_v^{\beta v} xdF - (F(\beta v)-F(v))\int_{\beta v}^{\delta v} xdF\right)\\
&\quad -F(\beta v)\left(\delta(F(\delta v)-F(v))\int_v^{\beta v} xdF - \beta(F(\beta v)-F(v))\int_v^{\delta v} xdF\right)\\
&= \delta F(v)(F(\delta v)-F(\beta v))(\beta v F(\beta v)-vF(v)) - F(v)(F(\beta v)-F(v))(\delta v F(\delta v)-\beta v F(\beta v))\\
&\quad -\delta F(\beta v)(F(\delta v)-F(v))(\beta v F(\beta v)-vF(v)) - \beta F(\beta v)(F(\beta v)-F(v))(\delta v F(\delta v)-vF(v))\\
&= 0.
\end{aligned}$$

□

To investigate the naive type's payoff under the optimal mechanism with incentive-feasible cross-subsidization in Example 2 (ii), we focus on the case in which rational type's IR is not binding; when rational type's IR is binding, there is no incentive-feasible cross subsidization from the rational type to the naive type. In this case, $(q_1, p_1) = (1, 1)$ and $(q_2, p_2) = (1, \alpha\mu)$.

The threshold type v^* is obtained from the seller's indifference condition: $(1-\psi)(\alpha\mu - v^*) = \psi(1-v^*) \;\Leftrightarrow\; v^* = \frac{(1-\psi)\alpha\mu-\psi}{1-2\psi}$. Note that $v^* < \alpha\mu$ if and only if $\alpha\mu < 1$ and $\psi < \frac{1}{2}$. Note also that $\frac{dv^*}{d\psi} = -\frac{1-\alpha\mu}{(1-2\psi)^2} < 0$.

Under this mechanism, naive type's actual expected payoff is $\int_0^{v^*}(\alpha v - \alpha\mu)dF = \int_0^{v^*}(\alpha\gamma v^\gamma - \alpha\mu\gamma v^{\gamma-1})dv = [\alpha\frac{\gamma}{1+\gamma}v^{1+\gamma} - \alpha\mu v^\gamma]_0^{v^*} = -\alpha\mu(1-v^*)(v^*)^\gamma < 0$. Also, rational type's expected payoff is $\int_{v^*}^1(\alpha v - 1)dF = \int_{v^*}^1(\alpha\gamma v^\gamma - \gamma v^{\gamma-1})dv = [\alpha\frac{\gamma}{1+\gamma}v^{1+\gamma} - v^\gamma]_{v^*}^1 = \alpha\mu - 1 + (v^*)^\gamma - \alpha\mu(v^*)^{1+\gamma}$.

The expected buyer surplus is:

$$\begin{aligned} & -(1-\psi)\alpha\mu(1-v^*)(v^*)^\gamma + \psi\left[\alpha\mu - 1 + (v^*)^\gamma - \alpha\mu(v^*)^{1+\gamma}\right] \\ = & -(1-2\psi)\alpha\mu(1-v^*)(v^*)^\gamma - \psi(1-\alpha\mu)[1-(v^*)^\gamma] < 0. \end{aligned}$$

Although the ex-ante expected buyer surplus is always non-positive and maximized at $\psi = 1$ as in Proposition 1, it can be non-monotonic. For example, at $\psi = 0$, the ex-ante expected buyer surplus is decreasing in ψ if α and γ are sufficiently small.

As discussed in Example 2.2, naive type's actual expected payoff with the maximal cross-subsidization is:

$$-\alpha\mu(1-v^*)(v^*)^\gamma + \frac{\psi}{1-\psi}\left[\alpha\mu - 1 + (v^*)^\gamma - \alpha\mu(v^*)^{1+\gamma}\right].$$

Its derivative with respect to ψ is:

$$
\begin{aligned}
&\frac{1}{(1-\psi)^2}\left[\alpha\mu - 1 + (v^*)^\gamma - \alpha\mu(v^*)^{1+\gamma}\right] \\
&- \frac{1-\alpha\mu}{(1-2\psi)^2}\left[-\alpha\mu\gamma(v^*)^{\gamma-1} + \alpha\mu(1+\gamma)(v^*)^\gamma + \frac{\psi}{1-\psi}\gamma(v^*)^{\gamma-1} - \frac{\psi}{1-\psi}\alpha\mu(1+\gamma)(v^*)^\gamma\right] \\
=&\frac{1}{(1-\psi)^2}\left[\alpha\mu - 1 + (v^*)^\gamma - \alpha\mu(v^*)^{1+\gamma}\right] \\
&- \frac{1-\alpha\mu}{(1-2\psi)^2}(v^*)^{\gamma-1}\underbrace{\left[-\alpha\mu\gamma + \frac{\psi}{1-\psi}\gamma - \frac{1-2\psi}{1-\psi}\alpha\mu(1+\gamma)\frac{(1-\psi)\alpha\mu-\psi}{1-2\psi}\right]}_{=\frac{1}{1-\psi}(\alpha-1)\gamma[(1-\psi)\alpha\mu-\psi]} \\
=&\frac{1}{(1-\psi)^2}\left[\alpha\mu - 1 + (v^*)^\gamma - \alpha\mu(v^*)^{1+\gamma}\right] - \frac{1-\alpha\mu}{(1-2\psi)}(v^*)^{\gamma-1}\frac{1}{1-\psi}(\alpha-1)\gamma v^* \\
=&\frac{1}{(1-\psi)^2}\left[\alpha\mu - 1 + (v^*)^\gamma - \alpha\mu(v^*)^{1+\gamma} - \underbrace{\frac{1-\psi}{1-2\psi}(1-\alpha\mu)}_{=1-v^*}(\alpha-1)\gamma(v^*)^\gamma\right] \\
=&\frac{1}{(1-\psi)^2}\left\{\alpha\mu - 1 + [1-(\alpha-1)\gamma - \gamma v^*](v^*)^\gamma\right\}.
\end{aligned}
$$

Because $[1-(\alpha-1)\gamma-\gamma v](v)^\gamma$ subject to $v \in (0, \alpha\mu)$ is maximized at $v = 1-\alpha\mu$, the derivative is at most

$$
\begin{aligned}
&\frac{1}{(1-\psi)^2}\left\{\alpha\mu - 1 + \underbrace{[1-(\alpha-1)\gamma-\gamma(1-\alpha\mu)]}_{=1-\alpha\mu}(1-\alpha\mu)^\gamma\right\} \\
=& -\frac{1}{(1-\psi)^2}(1-\alpha\mu)\left[1-(1-\alpha\mu)^\gamma\right] < 0.
\end{aligned}
$$

Hence, the naive type's actual expected payoff with the maximal cross-subsidization is decreasing in ψ, implying that the payoff of the naive type is strictly lower than the one-price mechanism.

We next turn to Example 3. The expected trade surplus is:

$$
\begin{aligned}
&(1-\psi)\int_0^{v^*}(\alpha-1)vdF + \psi\int_{v^*}^1(\alpha-1)vdF \\
=&(1-\psi)(\alpha-1)[\mu v^{1+\gamma}]_0^{v^*} + \psi(\alpha-1)[\mu v^{1+\gamma}]_{v^*}^1 \\
=&(\alpha-1)\mu\left[(1-2\psi)(v^*)^{1+\gamma}+\psi\right].
\end{aligned}
$$

Note that its derivative at $\psi = 0$ is positive if $\gamma < \frac{1}{5}$ and $\alpha < \frac{5}{4}$.

B Interim Approach

This section analyzes the environment in which the seller and buyer *simultaneously* participate in the mechanism and report their messages to the mechanism; in other words, the buyer's constraints are imposed at the *interim* stage, rather than after observing the seller's choice. In what follows, we focus on the environment under the binary distribution and discuss how the results in the main text qualitatively hold.

Recall the binary distribution in which $v \in \{0, 1\}$ with $\Pr(v = 1) = \mu$. Let $q_r(v)$ denote the probability of trading between the seller with value v and the rational type of the buyer, and $t_r(v)$ denote the expected payment.[32] Let $q_n(v), t_n(v)$ be the analogous expressions for the naive type of the buyer. Let:

$$q(v) = \psi q_r(v) + (1 - \psi) q_n(v),$$
$$t(v) = \psi t_r(v) + (1 - \psi) t_n(v).$$

For the seller, as in the main text, we can focus only on the incentive compatibility constraint on the value-0 seller:

$$t(0) \geq t(1),$$

and the individual rationality constraint of the value-1 seller:

$$t(1) \geq q(1).$$

For the buyer, we now consider the (anticipated) interim participation constraints:[33]

$$\alpha \mu q_r(1) \geq \mu t_r(1) + (1 - \mu) t_r(0),$$
$$\alpha \mu (\mu q_n(1) + (1 - \mu) q_n(0)) \geq \mu t_n(1) + (1 - \mu) t_n(0).$$

In what follows, we focus on the case with:

$$(1 - \psi) \alpha \mu (1 - \mu) \leq \psi (1 - \alpha \mu), \tag{4}$$

[32] Thus, $t_r(v)$ equals the probability of trade times the price, in case no payment is charged without trading.

[33] We ignore the incentive compatibility constraints of the buyer, as they are not binding.

though the conclusion is qualitatively similar with the opposite condition as we discuss in Footnote 34 below.

Consider the trade surplus maximization problem:

$$
\begin{aligned}
\max \quad & \mu(\alpha-1)q(1) \\
\text{sub. to} \quad & t(0) \geq t(1) \geq q(1), \\
& \alpha\mu q_r(1) - \mu t_r(1) - (1-\mu)t_r(0) \geq 0, \\
& \alpha\mu(\mu q_n(1) + (1-\mu)q_n(0)) - \mu t_n(1) - (1-\mu)t_n(0) \geq 0.
\end{aligned}
$$

It is optimal to set $t(0) = t(1) = q(1)$ and $q_n(0) = 1$. Also, $q_r(0)$ does not matter, so let us set $q_r(0) = 0$. By combining the buyer's interim participation constraints:

$$
\begin{aligned}
& \psi\alpha\mu q_r(1) + (1-\psi)[\alpha\mu(\mu q_n(1) + (1-\mu)q_n(0))] \\
& \geq \psi(\mu t_r(1) + (1-\mu)t_r(0)) + (1-\psi)(\mu t_n(1) + (1-\mu)t_n(0)) \\
\Leftrightarrow \quad & \psi\alpha\mu q_r(1) + (1-\psi)\alpha\mu^2 q_n(1) + (1-\psi)\alpha\mu(1-\mu) \\
& \geq \mu t(1) + (1-\mu)t(0) = q(1) = \psi q_r(1) + (1-\psi)q_n(1).
\end{aligned}
$$

Hence, the following is a relaxed problem of the original problem:

$$
\begin{aligned}
\max \quad & \mu(\alpha-1)q(1) \\
\text{sub. to} \quad & \psi\alpha\mu q_r(1) + (1-\psi)\alpha\mu^2 q_n(1) + (1-\psi)\alpha\mu(1-\mu) \\
& \geq \psi q_r(1) + (1-\psi)q_n(1).
\end{aligned}
$$

Because $\mu < 1$, fixed any $q(1)$, it is optimal to set $q_r(1)$ as high as possible (and $q_n(1)$ as low as possible). Under the assumption (4), we have $q_n(1) = 0$ and $q_r(1)$ is given by:

$$
q_r(1) = \frac{(1-\psi)\alpha\mu(1-\mu)}{\psi(1-\alpha\mu)},
$$

which is between 0 and 1 (because $\alpha\mu < 1$).[34]

We show that this relaxed solution is feasible with appropriate $t_r(\cdot), t_n(\cdot)$ (or equivalently, appropriate trade prices), and hence constitutes a solution to the original problem. Indeed,

[34] When the assumption (4) is violated, we have $q_r(1) = 1$ and $q_n(1) \in (0, 1)$.

by setting:

$$t_r(1) = \alpha q_r(1), \qquad t_n(1) = -\frac{\alpha(\alpha-1)\mu(1-\mu)}{1-\alpha\mu},$$
$$t_r(0) = 0, \qquad t_n(0) = \frac{\alpha\mu(1-\mu)}{1-\alpha\mu},$$

we can satisfy all the constraints.

The "interim" solution here has a *double-separation* feature qualitatively similar to our main case, in the sense that the low-value seller trades (only) with the naive buyer, and the high-value seller trades (only) with the rational buyer. The basic logic is similar. First, matched with the naive buyer with a relatively high price, the low-value seller's truth-telling constraint is maintained; the naive buyer is happy to trade because he (wrongly) believes that the expected valuation given trading is $\alpha\mu$. Then, the high-value seller can be matched with the rational buyer in order to generate trade surplus.

However, the trade probability is slightly different from the main case, and this is because the interim participation constraints are less stringent than the ones in the main case. Relatedly, the possibility of cross-subsidization in this interim model makes the solution satisfy all the relevant constraints with equality, while the rational buyer sometimes earns information rent in the main case.

Finally, even in this interim formulation, it is impossible to achieve any trade without the naive buyer's loss. Indeed, the naive buyer's *actual* expected payoff in this case is:

$$\mu(\alpha \cdot 0 - t_n(1)) + (1-\mu)(0 - t_n(0)) < 0.$$

C Counterexample of the Optimality of Menu Mechanisms

This section provides a counterexample with three types showing that the optimal mechanism may not be a menu mechanism in general.

Because the seller of any type v may potentially trade with both types of the buyer, in what follows, we denote a generic allocation by $((q_n, p_n), (q_r, p_r))$ (instead of (q, p) as before), where (q_n, p_n) represents the allocation if the buyer reports the naive type, and (q_r, p_r) represents the allocation if the buyer reports the rational type. A mechanism is a menu mechanism if every allocation possible in the mechanism has the form either $((q_n, p_n), (0, 0))$ or $((0, 0), (q_r, p_r))$.

Example 5. $v \in \{0, v^*, 1\}$, where F is such that $v^* = \alpha\mu$ (or slightly below $\alpha\mu$). For simplicity, assume ψ is small enough so that the optimal two-option menu mechanism comprises $((1, \alpha\mu), (0, 0))$ and $((0, 0), (1, 1))$.

Now consider its variation by adding the third option: $((\varepsilon, \alpha\mu), (1, 1))$ for some $\varepsilon > 0$. First, it is easy to see that the naive buyer's IC and IR are satisfied (under truth-telling). Similarly, the rational buyer's IC is satisfied, and his IR is satisfied if and only if $\alpha v^* - 1 \geq 0$ (equivalently, $\alpha^2\mu \geq 1$). For the seller, $v = 0$ chooses $((1, \alpha\mu), (0, 0))$ and $v = 1$ chooses $((0, 0), (1, 1))$ as before, while $v = v^*$ is indifferent between $((\varepsilon, \alpha\mu), (1, 1))$ and $((0, 0), (1, 1))$. Thus, letting this type choose $((\varepsilon, \alpha\mu), (1, 1))$ (if one prefers, we can imagine v^* slightly below $\alpha\mu$), the total surplus strictly increases.

In relation to Theorem 2, in this counterexample, notice that the higher-value seller trades with the buyer through multiple prices, leading to the optimality of a non-menu mechanism. Theorem 2 excludes such a possibility.