Supplementary Appendix for “Information Acquisition and Product Differentiation Perception”

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A Remaining Proof of Lemma 4 in Biglaiser et al. (2023)

We start with the buyer's payoff. The buyer expects that sellers set the symmetric mixed pricing strategy by $\hat{F} \equiv F(\cdot|\hat{\gamma}) \in \Delta([\underline{\hat{p}}, \overline{\hat{p}}])$. Hence, the buyer's gross expected payoff of choosing information precision γ given sellers' strategy $\hat{F}$ is

$$\begin{aligned}
& \frac{1}{2}\Big\{\Pr(p_L - p_R < t\gamma \big| \hat{F}) \times \mathbb{E}\left[v + t\frac{1+\gamma}{2} - p_L \Big| p_L - p_R < t\gamma, \hat{F}\right] \\
& \qquad + \Pr(p_L - p_R > t\gamma \big| \hat{F}) \times \mathbb{E}\left[v + t\frac{1-\gamma}{2} - p_R \Big| p_L - p_R > t\gamma, \hat{F}\right]\Big\} \\
& + \frac{1}{2}\Big\{\Pr(p_R - p_L < t\gamma \big| \hat{F}) \times \mathbb{E}\left[v + t\frac{1+\gamma}{2} - p_R \Big| p_R - p_L < t\gamma, \hat{F}\right] \\
& \qquad + \Pr(p_R - p_L > t\gamma \big| \hat{F}) \times \mathbb{E}\left[v + t\frac{1-\gamma}{2} - p_L \Big| p_R - p_L > t\gamma, \hat{F}\right]\Big\} \\
= \;& v + \frac{t}{2} - \Pr(p_L - p_R < t\gamma \big| \hat{F}) \times \mathbb{E}\left[p_L - \frac{t\gamma}{2} \Big| p_L - p_R < t\gamma, \hat{F}\right] \\
& - \Pr(p_L - p_R > t\gamma \big| \hat{F}) \times \mathbb{E}\left[p_R + \frac{t\gamma}{2} \Big| p_L - p_R > t\gamma, \hat{F}\right]
\end{aligned}$$

and the buyer's value of information acquisition $b(\gamma, \hat{\gamma})$ is

$$\begin{aligned} b(\gamma,\hat{\gamma}) &= \frac{t}{2} - \Pr(p_L - p_R < t\gamma \big| \hat{F}) \times \mathbb{E}\left[p_L - \frac{t\gamma}{2}\Big| p_L - p_R < t\gamma, \hat{F}\right] \\ &\quad - \Pr(p_L - p_R > t\gamma \big| \hat{F}) \times \mathbb{E}\left[p_R + \frac{t\gamma}{2}\Big| p_L - p_R > t\gamma, \hat{F}\right] \\ &= \frac{t}{2} - \underbrace{\int_{\underline{\hat{p}}}^{\hat{\overline{p}}} \int_{\underline{\hat{p}}}^{\min\{p'+t\gamma, \overline{p}\}} (p - \frac{t\gamma}{2}) d\hat{F}(p) d\hat{F}(p')}_{\text{area L: obedience}} \\ &\quad - \underbrace{\int_{\underline{\hat{p}}}^{\hat{\overline{p}}} \int_{p'+t\gamma}^{\hat{\overline{p}}} (p' + \frac{t\gamma}{2}) d\hat{F}(p) d\hat{F}(p')}_{\text{area R: dis-obedience}} \end{aligned} \tag{1}$$

When $\gamma < 2\hat{\gamma}$, the obedience part of equation (1) is

$$\begin{aligned} & \int_{\underline{\hat{p}}}^{\hat{\overline{p}}} \int_{\underline{\hat{p}}}^{\min\{p'+t\gamma,\overline{p}\}} (p - \frac{t\gamma}{2}) d\hat{F}(p) d\hat{F}(p') \\ = & \int_{\underline{\hat{p}}}^{\hat{\overline{p}}} \int_{\max\{p-t\gamma,\underline{p}\}}^{\hat{\overline{p}}} (p - \frac{t\gamma}{2}) d\hat{F}(p') d\hat{F}(p) \\ = & \int_{\underline{\hat{p}}}^{\underline{\hat{p}}+t\gamma} \int_{\underline{\hat{p}}}^{\hat{\overline{p}}} (p - \frac{t\gamma}{2}) d\hat{F}(p') d\hat{F}(p) + \int_{\underline{\hat{p}}+t\gamma}^{\hat{\overline{p}}} \int_{p-t\gamma}^{\hat{\overline{p}}} (p - \frac{t\gamma}{2}) d\hat{F}(p') d\hat{F}(p) \\ = & \int_{\underline{\hat{p}}}^{\underline{\hat{p}}+t\gamma} (p - \frac{t\gamma}{2}) d\hat{F}(p) + \int_{\underline{\hat{p}}+t\gamma}^{\hat{\overline{p}}} (p - \frac{t\gamma}{2}) \left[1 - \hat{F}(p - t\gamma)\right] d\hat{F}(p) \end{aligned} \tag{2}$$

and the disobedience part of equation (1) is

$$\begin{aligned} \int_{\underline{\hat{p}}}^{\hat{\overline{p}}} \int_{p'+t\gamma}^{\hat{\overline{p}}} (p' + \frac{t\gamma}{2}) d\hat{F}(p) d\hat{F}(p') &= \int_{\underline{\hat{p}}}^{\hat{\overline{p}}-t\gamma} \int_{p'+t\gamma}^{\hat{\overline{p}}} (p' + \frac{t\gamma}{2}) d\hat{F}(p) d\hat{F}(p') \\ &= \int_{\underline{\hat{p}}}^{\hat{\overline{p}}-t\gamma} (p' + \frac{t\gamma}{2}) \left[1 - \hat{F}(p' + t\gamma)\right] d\hat{F}(p') \end{aligned} \tag{3}$$

Plugging equations (2) and (3) into the right hand side of equation (1) yields

$$\begin{aligned} b(\gamma,\hat{\gamma}) &= \frac{t}{2} - \int_{\underline{\hat{p}}}^{\underline{\hat{p}}+t\gamma} (p - \frac{t\gamma}{2}) d\hat{F}(p) - \int_{\underline{\hat{p}}+t\gamma}^{\hat{\overline{p}}} (p - \frac{t\gamma}{2}) \left[1 - \hat{F}(p - t\gamma)\right] d\hat{F}(p) \\ &\quad - \int_{\underline{\hat{p}}}^{\hat{\overline{p}}-t\gamma} (p' + \frac{t\gamma}{2}) \left[1 - \hat{F}(p' + t\gamma)\right] d\hat{F}(p'). \end{aligned} \tag{4}$$

The following shows the details of computations and simplifications of the partial derivatives separately: Step 1 demonstrates $b_1(\gamma, \gamma) = \beta_1 t$, and Step 2 shows $b_2(\gamma, \gamma) = -\beta_2 t + \beta_0$.

Step 1. We show that $b_1(\gamma,\gamma)=\beta_1 t$ with $\beta_1>0$. Using the expression of $b(\gamma,\hat{\gamma})$ in (4), the partial derivative of $b(\gamma,\hat{\gamma})$ with respect to the first argument γ is

$$\begin{aligned}
b_1(\gamma,\hat{\gamma}) &= \frac{t}{2}\hat{F}(\underline{\hat{p}}+t\gamma)-t(\underline{\hat{p}}+\frac{t\gamma}{2})\hat{f}(\underline{\hat{p}}+t\gamma) \\
&\quad +t(\underline{\hat{p}}+\frac{t\gamma}{2})\hat{f}(\underline{\hat{p}}+t\gamma)-\int_{\underline{\hat{p}}+t\gamma}^{\hat{\bar{p}}}\left[t(p-\frac{t\gamma}{2})\hat{f}(p-t\gamma)-\frac{t}{2}(1-\hat{F}(p-t\gamma))\right]d\hat{F}(p) \\
&\quad +\int_{\underline{\hat{p}}}^{\hat{\bar{p}}-t\gamma}\left[t(p+\frac{t\gamma}{2})\hat{f}(p+t\gamma)-\frac{t}{2}(1-\hat{F}(p+t\gamma))\right]d\hat{F}(p) \\
&= \frac{t}{2}\hat{F}(\underline{\hat{p}}+t\gamma)-\int_{\underline{\hat{p}}+t\gamma}^{\hat{\bar{p}}}\left[t(p-\frac{t\gamma}{2})\hat{f}(p-t\gamma)-\frac{t}{2}(1-\hat{F}(p-t\gamma))\right]d\hat{F}(p) \\
&\quad +\int_{\underline{\hat{p}}}^{\hat{\bar{p}}-t\gamma}\left[t(p+\frac{t\gamma}{2})\hat{f}(p+t\gamma)-\frac{t}{2}(1-\hat{F}(p+t\gamma))\right]d\hat{F}(p) \\
&= \frac{t}{2}\hat{F}(\underline{\hat{p}}+t\gamma)-\int_{\underline{\hat{p}}}^{\hat{\bar{p}}-t\gamma}\left[t(p+\frac{t\gamma}{2})\hat{f}(p)-\frac{t}{2}(1-\hat{F}(p))\right]d\hat{F}(p) \\
&\quad +\int_{\underline{\hat{p}}+t\gamma}^{\hat{\bar{p}}}\left[t(p-\frac{t\gamma}{2})\hat{f}(p)-\frac{t}{2}(1-\hat{F}(p))\right]d\hat{F}(p) \qquad (5)
\end{aligned}$$

where the first equality is by partial derivatives and the last equality comes from changes in the limits of integration.

Then we plug the equilibrium $\hat{F}(p)$ into equation (5). By the continuity of $b_1(\gamma,\hat{\gamma})$ in γ, it's without loss of generality to consider $\gamma\geq\hat{\gamma}$. For $\gamma\geq\hat{\gamma}$, with the equilibrium $\hat{F}(p)$ and $\hat{f}(p)$ the partial derivative $b_1(\gamma,\hat{\gamma})$ in (5) becomes

$$\begin{aligned}
&\frac{t}{2}\left[2-\frac{t\hat{\gamma}(1+\sqrt{2})}{(\sqrt{2}-1)t\hat{\gamma}+t\gamma}\right]-\int_{\underline{\hat{p}}}^{\hat{\bar{p}}-t\gamma}\left[t(p+\frac{t\gamma}{2})\frac{t\hat{\gamma}(1+\sqrt{2})}{(p+t\hat{\gamma})^2}-\frac{t}{2}\frac{t\hat{\gamma}(1+\sqrt{2})}{p+t\hat{\gamma}}\right]\frac{t\hat{\gamma}(1+\sqrt{2})}{(p+t\hat{\gamma})^2}dp \\
&+\int_{\underline{\hat{p}}+t\gamma}^{\hat{\bar{p}}}\left[t(p-\frac{t\gamma}{2})\frac{t\hat{\gamma}(1+\sqrt{2})}{(p-t\hat{\gamma})^2}-\frac{t}{2}(\frac{t\hat{\gamma}(1+\sqrt{2})}{p-t\hat{\gamma}}-1)\right]\frac{t\hat{\gamma}(1+\sqrt{2})}{(p-t\hat{\gamma})^2}dp \\
=\;&\frac{t}{2}\left[2\sqrt{2}-1+\frac{(3+2\sqrt{2})t\gamma}{(\sqrt{2}-1)t\hat{\gamma}+t\gamma}\right]-\frac{t(\hat{\bar{p}}-t\hat{\gamma})^2}{2}\int_{\underline{\hat{p}}}^{\hat{\bar{p}}-t\gamma}\frac{p+t\gamma-t\hat{\gamma}}{(p+t\hat{\gamma})^4}dp \\
&+\frac{t(\hat{\bar{p}}-t\hat{\gamma})}{2}\int_{\underline{\hat{p}}+t\gamma}^{\hat{\bar{p}}}\left[\frac{(\hat{\bar{p}}-t\hat{\gamma})(p-t\gamma+t\hat{\gamma})}{(p-t\hat{\gamma})^2}+1\right]\frac{1}{(p-t\hat{\gamma})^2}dp \\
=\;&\frac{t}{2}\left[2\sqrt{2}-1+\frac{(3+2\sqrt{2})t\gamma}{(\sqrt{2}-1)t\hat{\gamma}+t\gamma}\right]-\frac{t(\hat{\bar{p}}-t\hat{\gamma})^2}{2}\int_{\underline{\hat{p}}}^{\hat{\bar{p}}-t\gamma}\frac{p+t\gamma-t\hat{\gamma}}{(p+t\hat{\gamma})^4}dp \\
&+\frac{t(\hat{\bar{p}}-t\hat{\gamma})^2}{2}\int_{\underline{\hat{p}}+t\gamma}^{\hat{\bar{p}}}\frac{p-t\gamma+t\hat{\gamma}}{(p-t\hat{\gamma})^4}dp+\frac{t(\hat{\bar{p}}-t\hat{\gamma})}{2}\int_{\underline{\hat{p}}+t\gamma}^{\hat{\bar{p}}}\frac{1}{(p-t\hat{\gamma})^2}dp
\end{aligned}$$

Further simplification yields

$$
\begin{aligned}
&\frac{t}{2}\left[2\sqrt{2}-1+\frac{(3+2\sqrt{2})t\gamma}{(\sqrt{2}-1)t\hat{\gamma}+t\gamma}\right]\\
&-\frac{t(\bar{\hat{p}}-t\hat{\gamma})^2}{2}\left[-\frac{1}{2}(\frac{1}{(\bar{\hat{p}}-t\gamma+t\hat{\gamma})^2}-\frac{1}{(\underline{\hat{p}}+t\hat{\gamma})^2})-\frac{t\gamma-2t\hat{\gamma}}{3}(\frac{1}{(\bar{\hat{p}}-t\gamma+t\hat{\gamma})^3}-\frac{1}{(\underline{\hat{p}}+t\hat{\gamma})^3})\right]\\
&+\frac{t(\bar{\hat{p}}-t\hat{\gamma})^2}{2}\left[-\frac{1}{2}(\frac{1}{(\bar{\hat{p}}-t\hat{\gamma})^2}-\frac{1}{(\underline{\hat{p}}+t\gamma-t\hat{\gamma})^2})+\frac{t\gamma-2t\hat{\gamma}}{3}(\frac{1}{(\bar{\hat{p}}-t\hat{\gamma})^3}-\frac{1}{(\underline{\hat{p}}+t\gamma-t\hat{\gamma})^3})\right]\\
&+\frac{t(\bar{\hat{p}}-t\hat{\gamma})}{2}\left[-(\frac{1}{\bar{\hat{p}}-t\hat{\gamma}}-\frac{1}{\underline{\hat{p}}+t\gamma-t\hat{\gamma}})\right]
\end{aligned}
$$

Again reorganizing the above expression, we have

$$
\begin{aligned}
&b_1(\gamma,\hat{\gamma})\\
=\;&\frac{t}{2}\left[2\sqrt{2}-1+\frac{(3+2\sqrt{2})t\gamma}{(\sqrt{2}-1)t\hat{\gamma}+t\gamma}\right]+\frac{t}{2}\left[\frac{1}{2}(\frac{(\bar{\hat{p}}-t\hat{\gamma})^2}{(\bar{\hat{p}}-t\gamma+t\hat{\gamma})^2}-1)+\frac{t\gamma-2t\hat{\gamma}}{3}(\frac{(\bar{\hat{p}}-t\hat{\gamma})^2}{(\bar{\hat{p}}-t\gamma+t\hat{\gamma})^3}-\frac{1}{\underline{\hat{p}}+t\hat{\gamma}})\right]\\
&+\frac{t}{2}\left[\frac{1}{2}(\frac{(\bar{\hat{p}}-t\hat{\gamma})^2}{(\underline{\hat{p}}+t\gamma-t\hat{\gamma})^2}-1)+\frac{t\gamma-2t\hat{\gamma}}{3}(\frac{1}{\bar{\hat{p}}-t\hat{\gamma}}-\frac{(\bar{\hat{p}}-t\hat{\gamma})^2}{(\underline{\hat{p}}+t\gamma-t\hat{\gamma})^3})\right]+\frac{t}{2}(\frac{\bar{\hat{p}}-t\hat{\gamma}}{\underline{\hat{p}}+t\gamma-t\hat{\gamma}}-1)
\end{aligned}
$$

Hence, evaluating $b_1(\gamma,\hat{\gamma})$ at $\hat{\gamma}=\gamma\ \forall\gamma\in(0,1)$ yields

$$
\begin{aligned}
&b_1(\gamma,\gamma)\\
=\;&\frac{t}{2}\left[2\sqrt{2}-1+\frac{(3+2\sqrt{2})t\gamma}{\sqrt{2}t\gamma}\right]+\frac{t}{2}\left[\frac{1}{2}(\frac{(1+\sqrt{2})^2(t\gamma)^2}{(2+\sqrt{2})^2(t\gamma)^2}-1)-\frac{t\gamma}{3}(\frac{(1+\sqrt{2})^2(t\gamma)^2}{(2+\sqrt{2})^3(t\gamma)^3}-\frac{1}{(1+\sqrt{2})t\gamma})\right]\\
&+\frac{t}{2}\left[\frac{1}{2}(\frac{(1+\sqrt{2})^2(t\gamma)^2}{2(t\gamma)^2}-1)-\frac{t\gamma}{3}(\frac{1}{(1+\sqrt{2})t\gamma}-\frac{(1+\sqrt{2})^2(t\gamma)^2}{(\sqrt{2}t\gamma)^3})\right]+\frac{t}{2}(\frac{(1+\sqrt{2})t\gamma}{\sqrt{2}t\gamma}-1)\\
=\;&\frac{t}{2}\left[2\sqrt{2}-1+\frac{(3+2\sqrt{2})}{\sqrt{2}}\right]+\frac{t}{2}\left[\frac{1}{2}(\frac{(1+\sqrt{2})^2}{(2+\sqrt{2})^2}-1)-\frac{1}{3}(\frac{(1+\sqrt{2})^2}{(2+\sqrt{2})^3}-\frac{1}{1+\sqrt{2}})\right]\\
&+\frac{t}{2}\left[\frac{1}{2}(\frac{(1+\sqrt{2})^2}{2}-1)-\frac{1}{3}(\frac{1}{1+\sqrt{2}}-\frac{(1+\sqrt{2})^2}{(\sqrt{2})^3})\right]+\frac{t}{2}(\frac{1+\sqrt{2}}{\sqrt{2}}-1)\\
=\;&\frac{t}{2}\left[\frac{9\sqrt{2}}{2}+1+\frac{(1+\sqrt{2})^2}{6\sqrt{2}}(1-\frac{1}{(1+\sqrt{2})^3})\right]\\
=\;&\beta_1 t
\end{aligned}
$$

where $\beta_1\equiv\frac{1}{2}\left[\frac{9\sqrt{2}}{2}+1+\frac{(1+\sqrt{2})^2}{6\sqrt{2}}(1-\frac{1}{(1+\sqrt{2})^3})\right]\approx 4.00102>0$.

Step 2. We show that $b_2(\gamma,\gamma) = -\beta_2 t + \beta_0$ with $\beta_0 > 0$ and $\beta_2 > 0$. Using the expression of $b(\gamma,\hat{\gamma})$ in (4), the partial derivative of $b(\gamma,\hat{\gamma})$ with respect to the second argument $\hat{\gamma}$ is

$$
\begin{aligned}
& b_2(\gamma,\hat{\gamma}) \\
= \; & -(\underline{\hat{p}} + \frac{t\gamma}{2})\hat{f}(\underline{\hat{p}} + t\gamma)\sqrt{2}t + (\underline{\hat{p}} - \frac{t\gamma}{2})\hat{f}(\underline{\hat{p}})\sqrt{2}t - \int_{\underline{\hat{p}}}^{\underline{\hat{p}}+t\gamma} (p - \frac{t\gamma}{2})\hat{f}'(p)dp \\
& -(\hat{\overline{p}} - \frac{t\gamma}{2})\left[1 - \hat{F}(\hat{\overline{p}} - t\gamma)\right]\hat{f}(\hat{\overline{p}})(2+\sqrt{2})t \\
& +(\underline{\hat{p}} + \frac{t\gamma}{2})\left[1 - \hat{F}(\underline{\hat{p}})\right]\hat{f}(\underline{\hat{p}} + t\gamma)\sqrt{2}t \\
& -\int_{\underline{\hat{p}}+t\gamma}^{\hat{\overline{p}}} (p - \frac{t\gamma}{2})\left[(1 - \hat{F}(p - t\gamma))\hat{f}'(p) - \hat{f}(p - t\gamma)\hat{f}(p)\right]dp \\
& -(\hat{\overline{p}} - \frac{t\gamma}{2})(1 - \hat{F}(\hat{\overline{p}}))\hat{f}(\hat{\overline{p}} - t\gamma)(2+\sqrt{2})t \\
& +(\underline{\hat{p}} + \frac{t\gamma}{2})\left[1 - \hat{F}(\underline{\hat{p}} + t\gamma)\right]\hat{f}(\underline{\hat{p}})\sqrt{2}t \\
& -\int_{\underline{\hat{p}}}^{\hat{\overline{p}}-t\gamma} (p' + \frac{t\gamma}{2})\left[(1 - \hat{F}(p' + t\gamma))\hat{f}'(p') - \hat{f}(p' + t\gamma)\hat{f}(p')\right]dp' \\
= \; & (\underline{\hat{p}} - \frac{t\gamma}{2})\hat{f}(\underline{\hat{p}})\sqrt{2}t - \int_{\underline{\hat{p}}}^{\underline{\hat{p}}+t\gamma} (p - \frac{t\gamma}{2})\hat{f}'(p)dp \\
& -(\hat{\overline{p}} - \frac{t\gamma}{2})\left[1 - \hat{F}(\hat{\overline{p}} - t\gamma)\right]\hat{f}(\hat{\overline{p}})(2+\sqrt{2})t \\
& -\int_{\underline{\hat{p}}+t\gamma}^{\hat{\overline{p}}} (p - \frac{t\gamma}{2})\left[(1 - \hat{F}(p - t\gamma))\hat{f}'(p) - \hat{f}(p - t\gamma)\hat{f}(p)\right]dp \\
& +(\underline{\hat{p}} + \frac{t\gamma}{2})\left[1 - \hat{F}(\underline{\hat{p}} + t\gamma)\right]\hat{f}(\underline{\hat{p}})\sqrt{2}t \\
& -\int_{\underline{\hat{p}}}^{\hat{\overline{p}}-t\gamma} (p + \frac{t\gamma}{2})\left[(1 - \hat{F}(p + t\gamma))\hat{f}'(p) - \hat{f}(p + t\gamma)\hat{f}(p)\right]dp
\end{aligned}
$$

We evaluate the previous expression at $\hat{\gamma} = \gamma$ and get

$$
\begin{aligned}
b_2(\gamma,\gamma) \; = \; & (\underline{p} - \frac{t\gamma}{2})f(\underline{p})\sqrt{2}t - \int_{\underline{p}}^{\underline{p}+t\gamma} (p - \frac{t\gamma}{2})f'(p)dp \\
& -(\overline{p} - \frac{t\gamma}{2})\left[1 - F(\overline{p} - t\gamma)\right]f(\overline{p})(2+\sqrt{2})t \\
& -\int_{\underline{p}+t\gamma}^{\overline{p}} (p - \frac{t\gamma}{2})\left[(1 - F(p - t\gamma))f'(p) - f(p - t\gamma)f(p)\right]dp \\
& +(\underline{p} + \frac{t\gamma}{2})\left[1 - F(\underline{p} + t\gamma)\right]f(\underline{p})\sqrt{2}t \\
& -\int_{\underline{p}}^{\overline{p}-t\gamma} (p + \frac{t\gamma}{2})\left[(1 - F(p + t\gamma))f'(p) - f(p + t\gamma)f(p)\right]dp
\end{aligned} \qquad (6)
$$

We simplify the right-hand side (RHS) of (6) as follows. First, the three items without integrals in the RHS of (6) are

$$
\begin{aligned}
& (\underline{p}-\frac{t\gamma}{2})f(\underline{p})\sqrt{2}t-(\overline{p}-\frac{t\gamma}{2})\left[1-F(\overline{p}-t\gamma)\right]f(\overline{p})(2+\sqrt{2})t \\
& +(\underline{p}+\frac{t\gamma}{2})\left[1-F(\underline{p}+t\gamma)\right]f(\underline{p})\sqrt{2}t \\
= \; & (\sqrt{2}-\frac{1}{2})t\gamma\frac{t\gamma(1+\sqrt{2})}{(\underline{p}+t\gamma)^2}\sqrt{2}t \\
& -(2+\sqrt{2}-\frac{1}{2})t\gamma\frac{t\gamma(1+\sqrt{2})}{\overline{p}}\frac{t\gamma(1+\sqrt{2})}{(\overline{p}-t\gamma)^2}(2+\sqrt{2})t \\
& +(\sqrt{2}+\frac{1}{2})t\gamma\frac{t\gamma(1+\sqrt{2})}{\underline{p}+2t\gamma}\frac{t\gamma(1+\sqrt{2})}{(\underline{p}+t\gamma)^2}\sqrt{2}t \\
= \; & (\sqrt{2}-\frac{1}{2})\frac{1}{1+\sqrt{2}}\sqrt{2}t-(\frac{3}{2}+\sqrt{2})t+(\sqrt{2}+\frac{1}{2})\frac{1}{(2+\sqrt{2})}\sqrt{2}t \\
= \; & -\frac{1+2\sqrt{2}}{1+\sqrt{2}}t
\end{aligned}
$$

The remaining items with integrals in the RHS of (6) are

$$
\begin{aligned}
& -\int_{\underline{p}}^{\underline{p}+t\gamma}(p-\frac{t\gamma}{2})f'(p)dp \\
& -\int_{\underline{p}+t\gamma}^{\overline{p}}(p-\frac{t\gamma}{2})\left[(1-F(p-t\gamma))f'(p)-f(p-t\gamma)f(p)\right]dp \\
& -\int_{\underline{p}}^{\overline{p}-t\gamma}(p+\frac{t\gamma}{2})\left[(1-F(p+t\gamma))f'(p)-f(p+t\gamma)f(p)\right]dp \\
= \; & -\int_{\underline{p}}^{\underline{p}+t\gamma}(p-\frac{t\gamma}{2})f'(p)dp \\
& -\int_{\underline{p}}^{\overline{p}-t\gamma}(p+\frac{t\gamma}{2})\left[(1-F(p))f'(p+t\gamma)-f(p)f(p+t\gamma)\right]dp \\
& -\int_{\underline{p}}^{\overline{p}-t\gamma}(p+\frac{t\gamma}{2})\left[(1-F(p+t\gamma))f'(p)-f(p+t\gamma)f(p)\right]dp \\
= \; & -\int_{\underline{p}}^{\underline{p}+t\gamma}(p-\frac{t\gamma}{2})f'(p)dp \\
& -\int_{\underline{p}}^{\overline{p}-t\gamma}(p+\frac{t\gamma}{2})\left[(1-F(p))f'(p+t\gamma)+(1-F(p+t\gamma))f'(p)-2f(p)f(p+t\gamma)\right]dp \\
= \; & -\int_{\underline{p}}^{\underline{p}+t\gamma}2pf'(p)-(p+\frac{t\gamma}{2})f'(p)F(p+t\gamma)dp \\
& -\int_{\underline{p}}^{\overline{p}-t\gamma}(p+\frac{t\gamma}{2})\left[(1-F(p))f'(p+t\gamma)-2f(p)f(p+t\gamma)\right]dp
\end{aligned} \tag{7}
$$

where the first equality is from the change in the limits of integration and the last two are simply reorganizing. Plugging the equilibrium $F(p)$ and $f(p)$ into (7), the sum of items with integrals in the RHS of (6) becomes

$$
\begin{aligned}
& -\int_{\underline{p}}^{\underline{p}+t\gamma}(p+\frac{t\gamma}{2})\left[\frac{t\gamma(1+\sqrt{2})}{p+t\gamma}\frac{t\gamma(1+\sqrt{2})(-2)}{p^3}-2\frac{t\gamma(1+\sqrt{2})}{(p+t\gamma)^2}\frac{t\gamma(1+\sqrt{2})}{p^2}\right]dp \\
& -\int_{\underline{p}}^{\overline{p}-t\gamma}\left[2p\frac{t\gamma(1+\sqrt{2})(-2)}{(p+t\gamma)^3}-(p+\frac{t\gamma}{2})(2-\frac{t\gamma(1+\sqrt{2})}{p})\frac{t\gamma(1+\sqrt{2})(-2)}{(p+t\gamma)^3}\right]dp \\
= \; & 2(t\gamma(1+\sqrt{2}))^2\int_{\underline{p}}^{\overline{p}-t\gamma}(p+\frac{t\gamma}{2})\left[\frac{1}{(p+t\gamma)p^3}+\frac{1}{p^2(p+t\gamma)^2}\right]dp \\
& +2t\gamma(1+\sqrt{2})\int_{\underline{p}}^{\overline{p}-t\gamma}\left[\frac{2p}{(p+t\gamma)^3}-\frac{2p+t\gamma-t\gamma(1+\sqrt{2})-\frac{t\gamma t\gamma(1+\sqrt{2})}{2p}}{(p+t\gamma)^3}\right]dp \\
= \; & 2(t\gamma(1+\sqrt{2}))^2\int_{\underline{p}}^{\overline{p}-t\gamma}(p+\frac{t\gamma}{2})\left[\frac{1}{(p+t\gamma)p^3}+\frac{1}{p^2(p+t\gamma)^2}\right]dp \\
& +2t\gamma(1+\sqrt{2})\int_{\underline{p}}^{\overline{p}-t\gamma}\frac{\sqrt{2}t\gamma+\frac{t\gamma t\gamma(1+\sqrt{2})}{2p}}{(p+t\gamma)^3}dp \\
= \; & t\gamma(t\gamma(1+\sqrt{2}))^2\int_{\underline{p}}^{\overline{p}-t\gamma}\left[\frac{1}{(p+t\gamma)p^3}+\frac{1}{p^2(p+t\gamma)^2}\right]dp \\
& +2(t\gamma(1+\sqrt{2}))^2\int_{\underline{p}}^{\overline{p}-t\gamma}\left[\frac{1}{(p+t\gamma)p^2}+\frac{1}{p(p+t\gamma)^2}\right]dp \\
& +2\sqrt{2}(1+\sqrt{2})(t\gamma)^2\int_{\underline{p}}^{\overline{p}-t\gamma}\frac{1}{(p+t\gamma)^3}dp \\
& +(1+\sqrt{2})^2(t\gamma)^3\int_{\underline{p}}^{\overline{p}-t\gamma}\frac{1}{p(p+t\gamma)^3}dp
\end{aligned} \tag{8}
$$

In what follows, we simplify (8) item by item to show that it is a positive constant, denoted by β_0.

- $t\gamma(t\gamma(1+\sqrt{2}))^2\int_{\underline{p}}^{\overline{p}-t\gamma}\left[\frac{1}{(p+t\gamma)p^3}+\frac{1}{p^2(p+t\gamma)^2}\right]dp$ is a constant as

$$
\begin{aligned}
& \int_{\underline{p}}^{\overline{p}-t\gamma}\frac{1}{(p+t\gamma)p^3}dp \\
= \; & \frac{1}{(t\gamma)^3}\ln\frac{\overline{p}-t\gamma}{\underline{p}}+\frac{1}{(t\gamma)^2}(\frac{1}{\overline{p}-t\gamma}-\frac{1}{\underline{p}})-\frac{1}{2t\gamma}(\frac{1}{(\overline{p}-t\gamma)^2}-\frac{1}{\underline{p}^2})-\frac{1}{(t\gamma)^3}\ln\frac{\overline{p}}{\underline{p}+t\gamma} \\
= \; & \frac{1}{(t\gamma)^3}\ln\frac{1+\sqrt{2}}{\sqrt{2}}+\frac{1}{(t\gamma)^3}(\frac{1}{1+\sqrt{2}}-\frac{1}{\sqrt{2}})-\frac{1}{2(t\gamma)^3}(\frac{1}{(1+\sqrt{2})^2}-\frac{1}{2})-\frac{1}{(t\gamma)^3}\ln\frac{2+\sqrt{2}}{1+\sqrt{2}} \\
= \; & \frac{1}{(t\gamma)^3}\left[\ln\frac{1+\sqrt{2}}{\sqrt{2}}+(\frac{1}{1+\sqrt{2}}-\frac{1}{\sqrt{2}})-\frac{1}{2}(\frac{1}{(1+\sqrt{2})^2}-\frac{1}{2})-\ln\frac{2+\sqrt{2}}{1+\sqrt{2}}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\underline{p}}^{\overline{p}-t\gamma} \frac{1}{p^2(p+t\gamma)^2} dp \\
= &-2\frac{1}{(t\gamma)^3}\ln\frac{\overline{p}-t\gamma}{\underline{p}} - \frac{1}{(t\gamma)^2}(\frac{1}{\overline{p}-t\gamma} - \frac{1}{\underline{p}}) + 2\frac{1}{(t\gamma)^3}\ln\frac{\overline{p}}{\underline{p}+t\gamma} - \frac{1}{(t\gamma)^2}(\frac{1}{\overline{p}} - \frac{1}{\underline{p}+t\gamma}) \\
= &-2\frac{1}{(t\gamma)^3}\ln\frac{1+\sqrt{2}}{\sqrt{2}} - \frac{1}{(t\gamma)^3}(\frac{1}{1+\sqrt{2}} - \frac{1}{\sqrt{2}}) + 2\frac{1}{(t\gamma)^3}\ln\frac{2+\sqrt{2}}{1+\sqrt{2}} - \frac{1}{(t\gamma)^3}(\frac{1}{2+\sqrt{2}} - \frac{1}{1+\sqrt{2}}) \\
= &\frac{1}{(t\gamma)^3}\left[-2\ln\frac{1+\sqrt{2}}{\sqrt{2}} - (\frac{1}{1+\sqrt{2}} - \frac{1}{\sqrt{2}}) + 2\ln\frac{2+\sqrt{2}}{1+\sqrt{2}} - (\frac{1}{2+\sqrt{2}} - \frac{1}{1+\sqrt{2}})\right]
\end{aligned}
$$

This item equals

$$
\begin{aligned}
&(1+\sqrt{2})^2\left[-\frac{1}{2}(\frac{1}{(1+\sqrt{2})^2} - \frac{1}{2}) - \ln\frac{1+\sqrt{2}}{\sqrt{2}} + \ln\sqrt{2} - \frac{1-\sqrt{2}}{2+\sqrt{2}}\right] \\
= &-\frac{1}{2} + \frac{1}{4}(1+\sqrt{2})^2 + (1+\sqrt{2})^2\ln\frac{2}{1+\sqrt{2}} + \frac{1}{\sqrt{2}}
\end{aligned}
$$

- $2(t\gamma(1+\sqrt{2}))^2\int_{\underline{p}}^{\overline{p}-t\gamma}\left[\frac{1}{(p+t\gamma)p^2} + \frac{1}{p(p+t\gamma)^2}\right]dp$ is a constant as

$$
\begin{aligned}
&\int_{\underline{p}}^{\overline{p}-t\gamma} \frac{1}{(p+t\gamma)p^2} dp \\
= &\frac{1}{(t\gamma)^2}\ln\frac{\overline{p}}{\underline{p}+t\gamma} - \frac{1}{(t\gamma)^2}\ln\frac{\overline{p}-t\gamma}{\underline{p}} - \frac{1}{t\gamma}(\frac{1}{\overline{p}-t\gamma} - \frac{1}{\underline{p}}) \\
= &\frac{1}{(t\gamma)^2}\ln\frac{2+\sqrt{2}}{1+\sqrt{2}} - \frac{1}{(t\gamma)^2}\ln\frac{1+\sqrt{2}}{\sqrt{2}} - \frac{1}{(t\gamma)^2}(\frac{1}{1+\sqrt{2}} - \frac{1}{\sqrt{2}}) \\
= &\frac{1}{(t\gamma)^2}\left[\ln\frac{2+\sqrt{2}}{1+\sqrt{2}} - \ln\frac{1+\sqrt{2}}{\sqrt{2}} - (\frac{1}{1+\sqrt{2}} - \frac{1}{\sqrt{2}})\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\underline{p}}^{\overline{p}-t\gamma} \frac{1}{p(p+t\gamma)^2} dp \\
= &\frac{1}{(t\gamma)^2}\ln\frac{\overline{p}-t\gamma}{\underline{p}} - \frac{1}{(t\gamma)^2}\ln\frac{\overline{p}}{\underline{p}+t\gamma} + \frac{1}{t\gamma}(\frac{1}{\overline{p}} - \frac{1}{\underline{p}+t\gamma}) \\
= &\frac{1}{(t\gamma)^2}\ln\frac{1+\sqrt{2}}{\sqrt{2}} - \frac{1}{(t\gamma)^2}\ln\frac{2+\sqrt{2}}{1+\sqrt{2}} + \frac{1}{(t\gamma)^2}(\frac{1}{2+\sqrt{2}} - \frac{1}{1+\sqrt{2}}) \\
= &\frac{1}{(t\gamma)^2}\left[\ln\frac{1+\sqrt{2}}{\sqrt{2}} - \ln\frac{2+\sqrt{2}}{1+\sqrt{2}} + (\frac{1}{2+\sqrt{2}} - \frac{1}{1+\sqrt{2}})\right]
\end{aligned}
$$

This item equals $2(1+\sqrt{2})^2\left[\frac{1}{2+\sqrt{2}}-\frac{2}{1+\sqrt{2}}+\frac{1}{\sqrt{2}}\right]=2$

- $2\sqrt{2}(1+\sqrt{2})(t\gamma)^2\int_{\underline{p}}^{\overline{p}-t\gamma}\frac{1}{(p+t\gamma)^3}dp$ is a constant as

$$\int_{\underline{p}}^{\overline{p}-t\gamma}\frac{1}{(p+t\gamma)^3}dp=-\frac{1}{2}(\frac{1}{\overline{p}^2}-\frac{1}{(\underline{p}+t\gamma)^2})=-\frac{1}{2(t\gamma)^2}(\frac{1}{(2+\sqrt{2})^2}-\frac{1}{(1+\sqrt{2})^2})$$

This item equals $\sqrt{2}(1+\sqrt{2})(\frac{1}{(1+\sqrt{2})^2}-\frac{1}{(2+\sqrt{2})^2})=\frac{1}{\sqrt{2}(1+\sqrt{2})}$

- $(1+\sqrt{2})^2(t\gamma)^3\int_{\underline{p}}^{\overline{p}-t\gamma}\frac{1}{p(p+t\gamma)^3}dp$ is a constant as

$$\begin{aligned}
&\int_{\underline{p}}^{\overline{p}-t\gamma}\frac{1}{p(p+t\gamma)^3}dp\\
=&\frac{1}{(t\gamma)^3}\ln\frac{\overline{p}-t\gamma}{\underline{p}}-\frac{1}{(t\gamma)^3}\ln\frac{\overline{p}}{\underline{p}+t\gamma}+\frac{1}{(t\gamma)^2}(\frac{1}{\overline{p}}-\frac{1}{\underline{p}+t\gamma})-\frac{1}{2t\gamma}(\frac{1}{\overline{p}^2}-\frac{1}{(\underline{p}+t\gamma)^2})\\
=&\frac{1}{(t\gamma)^3}\ln\frac{1+\sqrt{2}}{\sqrt{2}}-\frac{1}{(t\gamma)^3}\ln\frac{2+\sqrt{2}}{1+\sqrt{2}}+\frac{1}{(t\gamma)^3}(\frac{1}{2+\sqrt{2}}-\frac{1}{1+\sqrt{2}})-\frac{1}{2(t\gamma)^3}(\frac{1}{(2+\sqrt{2})^2}-\frac{1}{(1+\sqrt{2})^2})\\
=&\frac{1}{(t\gamma)^3}\left[\ln\frac{1+\sqrt{2}}{\sqrt{2}}-\ln\frac{2+\sqrt{2}}{1+\sqrt{2}}+(\frac{1}{2+\sqrt{2}}-\frac{1}{1+\sqrt{2}})-\frac{1}{2}(\frac{1}{(2+\sqrt{2})^2}-\frac{1}{(1+\sqrt{2})^2})\right]
\end{aligned}$$

This item equals

$$\begin{aligned}
&(1+\sqrt{2})^2\left[\ln\frac{1+\sqrt{2}}{\sqrt{2}}-\ln\frac{2+\sqrt{2}}{1+\sqrt{2}}+(\frac{1}{2+\sqrt{2}}-\frac{1}{1+\sqrt{2}})-\frac{1}{2}(\frac{1}{(2+\sqrt{2})^2}-\frac{1}{(1+\sqrt{2})^2})\right]\\
=&(1+\sqrt{2})^2\left[\ln\frac{1+\sqrt{2}}{2}+\frac{1-\sqrt{2}}{\sqrt{2}(1+\sqrt{2})}+\frac{1}{4(1+\sqrt{2})^2}\right]\\
=&(1+\sqrt{2})^2\ln\frac{1+\sqrt{2}}{2}-\frac{1}{\sqrt{2}}+\frac{1}{4}
\end{aligned}$$

By summarizing the above four items, we get the sum of items with integrals in the RHS of (6)

$$\begin{aligned}
\beta_0 &= -\frac{1}{2}+\frac{1}{4}(1+\sqrt{2})^2+(1+\sqrt{2})^2\ln\frac{2}{1+\sqrt{2}}+\frac{1}{\sqrt{2}}+2+\frac{1}{\sqrt{2}(1+\sqrt{2})}+(1+\sqrt{2})^2\ln\frac{1+\sqrt{2}}{2}-\frac{1}{\sqrt{2}}+\frac{1}{4}\\
&= \frac{1}{4}(1+\sqrt{2})^2+\frac{1}{\sqrt{2}(1+\sqrt{2})}+\frac{7}{4}=\frac{7}{2}
\end{aligned}$$

Hence, with $\gamma=\hat{\gamma}$

$$b_2(\gamma,\gamma)=-\beta_2 t+\beta_0$$

where $\beta_2=\frac{1+2\sqrt{2}}{1+\sqrt{2}}$ and $\beta_0=\frac{7}{2}$.

References

Biglaiser, G., J. Gu, and F. Li (2023). Information acquisition and product differentiation perception. Technical report, University of North Carolina.