Online Appendix: Negative Weights are No Concern in Design-Based Specifications

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Appendix A: Proof of Convex Ex-Ante Weights in Section I

Consider the numerator of $\beta = E[\psi_i \beta_i]/E[\psi_i]$. We have $E[\psi_i \beta_i] = E[E[\psi_i \mid w_i, \beta_i]\beta_i]$ and

$$E[\psi_i \mid w_i, \beta_i] = E[\tilde{x}_i^2 \mid w_i, \beta] + E[\tilde{x}_i \mid w_i, \beta_i] w_i' \lambda = Var(x_i \mid w_i, \beta_i).$$

Similarly, for the denominator, $E[\psi_i] = E[E[\psi_i \mid w_i, \beta_i]] = E[Var(x_i \mid w_i, \beta_i)]$.

Q.E.D.

Appendix B: Proof of Proposition 1

We assume that all the relevant moments exist and the conditions for Fubini's theorem hold. Consider first the numerator of $\beta = E[\tilde{z}_i y_i]/E[\tilde{z}_i x_i]$. Following the same steps as in footnote 7, $E[\tilde{z}_i y_i(0)] = 0$ under either Assumption $1'$ or $2'$. Thus:

$$\begin{aligned} E[\tilde{z}_i y_i] &= E[\tilde{z}_i y_i(0)] + E\left[\tilde{z}_i \int_0^{x_i} \frac{\partial}{\partial x} y_i(x) dx\right] \\ &= E\left[\int_0^{\infty} \psi_i(x) \beta_i(x) dx\right] \end{aligned}$$

for $\psi_i(x) \equiv \tilde{z}_i \mathbf{1}[x_i \geq x]$. Similarly, for the denominator, $E[\tilde{z}_i x_i] = E\left[\int_0^{\infty} \psi_i(x) dx\right]$. The ex-post $\psi_i(x)$ weights can clearly be negative, since $E[\tilde{z}_i] = 0$.

Under Assumption $2'$:

$$\begin{aligned} E\left[\int_0^{\infty} \psi_i(x) \beta_i(x) dx\right] &= E\left[\int_0^{\infty} E\left[\psi_i(x) \beta_i(x) \mid y_i(\cdot), w_i\right] dx\right] \\ &= E\left[\int_0^{\infty} \phi_i(x) \beta_i(x) dx\right] \end{aligned}$$

for $\phi_i(x) \equiv E[\psi_i(x) \mid y_i(\cdot), w_i] = Cov(\tilde{z}_i, \mathbf{1}[x_i \geq x] \mid y_i(\cdot), w_i)$. Moreover, under Assumption 3, the ex-ante $\phi_i(x)$ weights are non-negative:

$$Cov(\tilde{z}_i, \mathbf{1}[x_i \geq x] \mid y_i(\cdot), w_i) = Cov(\tilde{z}_i, Pr(x_i \geq x \mid z_i, y_i(\cdot), w_i) \mid y_i(\cdot), w_i) \geq 0,$$

since both $\tilde{z}_i$ and $Pr(x_i \geq x \mid z_i, y_i(\cdot), w_i)$ are non-decreasing in z_i, conditional on $(y_i(\cdot), w_i)$.

Q.E.D.

Appendix C: Formula Treatments and Instruments

Borusyak and Hull (2023), henceforth BH, study formula treatments and instruments of the form $z_i = f_i(g, s)$ for known $f_i(\cdot)$, observed shocks $g = (g_1, \ldots, g_K)$ that potentially vary at a different "level" k, and other observed data s of arbitrary structure. They assume the shocks are exogenous in the sense of $g \perp\!\!\!\perp y(\cdot) \mid v$, where $y(\cdot)$ is the set of potential outcomes and v is some set of observed variables that includes s. They further assume the shock "assignment process"—i.e., the conditional distribution of g given v—is known or consistently estimable. Borusyak, Hull and Jaravel (2022), henceforth BHJ, study the class of shift-share formulas $z_i = \sum_k s_{ik} g_k$ under the identifying assumption closer to Assumption $2'$: that $E[g_k \mid y(\cdot), v] = q_k' \xi$ for some observed q_k collected in v along with

$s = \{s_{ik}\}$. This assumption weakens the full independence condition in BH while also only specifying the mean of shocks, rather than the full assignment process.

BH show that OVB is avoided in their setting when the controls in w_i are functions of v and linearly span $\mu_i = E[f_i(g,s) \mid v]$: the expected instrument over draws of the shocks. Similarly, BHJ show that OVB is avoided with shift-share z_i when $\sum_k s_{ik} q_k$ is controlled for, which follows because w_i spans $\mu_i = E[\sum_k s_{ik} E[g_k \mid y(\cdot), v] \mid v] = \sum_k s_{ik} q_k' \xi$. While both BH and BHJ focus on constant-effects models, where OVB is the only concern, they also discuss identification of convex-weighted averages of heterogeneous effects.

We show Assumption 2′ is satisfied in both frameworks. Suppose w_i is a function of v and linearly spans μ_i: i.e., $\mu_i = w_i'\lambda$. Then, under either the BH or BHJ assumptions:

$$\begin{aligned} E[z_i \mid y_i(\cdot), w_i] &= E[E[f_i(g,s) \mid v, y_i(\cdot)] \mid y_i(\cdot), w_i] \\ &= E[E[f_i(g,s) \mid v] \mid y_i(\cdot), w_i] \\ &= E[\mu_i \mid y_i(\cdot), w_i] \\ &= w_i'\lambda. \end{aligned}$$

We also show how convexity of ex-ante weights can follow with formula instruments in a way similar (but not identical) to imposing Assumption 3. Specifically, we verify $\phi_i(x) \geq 0$ for a shift-share case where treatment is generated by the general causal model $x_i = h(g, \eta_i)$ where h is monotone in g. Suppose the instrument $z_i = \sum_k s_{ik} g_k$ is constructed from the correct shocks g_k but the shares $s_{ik} \geq 0$ do not correctly capture the dependence of x_i on g, which can be nonlinear or linear with different shares. Thus, the first-stage is non-causal. Assume that, conditional on $(y_i(\cdot), \eta_i, s_i, w_i)$ for $s_i = \{s_{ik}\}$, the g_k are drawn independently and, without loss, with zero mean (such that $\mu_i = 0$, making controls unnecessary). Then, for any x and almost-surely:

$$Cov\left(\sum_k s_{ik} g_k,\ \mathbf{1}\left[h(g,\eta_i) \geq x\right] \mid y_i(\cdot), \eta_i, s_i, w_i\right) \geq 0$$

by Lemma 2 of BH, since both $\sum_k s_{ik} g_k$ and $\mathbf{1}\left[h(g,\eta_i) \geq x\right]$ are non-decreasing functions of the mutually independent g_k, conditionally on $(y_i(\cdot), \eta_i, s_i, w_i)$. Hence:

$$\phi_i(x) = E\left[Cov\left(\sum_k s_{ik} g_k, \mathbf{1}\left[h(g,\eta_i) \geq x\right] \mid y_i(\cdot), \eta_i, s_i, w_i\right) \mid y_i(\cdot), w_i\right] \geq 0.$$

We finally note that both BHJ and BH do not require random or exchangeable samples, allowing each unit to have its own data-generating process. Our results apply in that case if moments are replaced with their full-sample versions: i.e., replacing $E[\tilde{z}_i y_i]$ with $\frac{1}{N}\sum_{i=1}^{N} E[\tilde{z}_i y_i]$ where N is the number of observations.

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REFERENCES

Borusyak, Kirill, and Peter Hull. 2023. "Non-Random Exposure to Exogenous Shocks." *Econometrica*, 91(6): 2155–2185.

Borusyak, Kirill, Peter Hull, and Xavier Jaravel. 2022. "Quasi-Experimental Shift-Share Research Designs." *Review of Economic Studies*, 89(1): 181–213.