Online Appendix for "Cournot Fire Sales"

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This online appendix provides technical details and extensions.

A. CONDITIONS FOR UNIFIED MODEL

To capture both canonical frameworks in our unified model, we impose conditions on the liquidity shock θ_1 that guarantee $c_1 = \theta_1$, i.e. a corner solution for intertemporal substitution between $t = 1$ and $t = 2$. Here, we explicitly study the intertemporal optimization at $t = 1$ with extra consumption e_1 such that $c_1 = \theta_1 + e_1$ and derive necessary conditions for $e_1 = 0$.

In the good state, the liquidity shock is $\overline{\theta}$ and intertemporal optimization at $t = 1$ solves

$$\max_{e_1} \left\{ u\big(\overline{\theta} + e_1\big) + \beta u\big(\ell - \overline{\theta} - e_1 + Rk\big) \right\}.$$

For for $e_1 = 0$ we need

$$u'\big(\overline{\theta}\big) \leq \beta u'\big(\ell - \overline{\theta} + Rk\big)$$

which implicitly defines a unique lower bound on $\overline{\theta}$.

In the bad state, a lucky agent decides between extra consumption or extra asset purchases but, since $p \leq 1$, leaves no liquidity for $t = 2$ so the optimization is

$$\max_{e_1} \left\{ u(\theta_L + e_1) + \beta u\left(R\left(k + \frac{\ell + \delta_H k - \theta_L - e_1}{p} \right) \right) \right\}.$$

For for $e_1 = 0$ we need

$$u'(\theta_L) < \beta \frac{R}{p} u'\left(R\left(k + \frac{\ell + \delta_H k - \theta_L}{p} \right) \right)$$

which implicitly defines a unique lower bound on θ_L.

For an unlucky agent in the case of partial liquidation, the intertemporal optimization is

$$\max_{e_1} \left\{ u(\theta_H + e_1) + \beta u\left(R\left(k - \frac{\theta_H + e_1 - (\ell + \delta_L k)}{p} \right) \right) \right\}.$$

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For for $e_1 = 0$ we need

$$u'(\theta_H) < \beta \frac{R}{p} u'\left(R\left(k - \frac{\theta_H - (\ell + \delta_L k)}{p}\right)\right)$$

which implicitly defines a unique lower bound on θ_H. However, note that the lower bound for θ_L is higher than that for θ_H so the assumption $\theta_H > \theta_L$ guarantees that θ_H is above its lower bound.

B. Strategic interim behavior

In the main analysis, we suppose that agents strategically choose portfolios at $t = 0$ (understanding that their portfolios will affect future prices), but in later periods agents act as price takers. In this section, we extend the previous analysis to allow agents to also act strategically when assets trade.

To incorporate strategic behavior in the interim period, we suppose that buyers choose a value of funds f with which they purchase assets, and sellers choose a quantity of assets s to sell. The price is determined given the funds supplied to purchase assets and the quantity of assets supplied, as in the canonical strategic market game of Shapley and Shubik (1977), which converges to Walras in the limit (Dubey and Geanakoplos, 2003).

B1. Strategic interim behavior in the productivity shock model

Consider the productivity shock model and now suppose that firms act strategically in the market for capital at $t = 1$. We show that low-productivity firms still find it optimal to sell the least amount of capital necessary to repay their debt. Taking into account their effect on price means that the amount they sell is the solution to a fixed point condition, but their sales are still an increasing function of the capital they hold. We also show that, under mild conditions, high-productivity firms still find it optimal to use all their funds, just as in the non-strategic case. Allowing for strategic behavior at $t = 1$ therefore does not change the fact that additional investment in capital at $t = 0$ drives up the price paid as a buyer and down the price received as a seller. The potential for overcorrection of the externality is therefore unchanged.

Firms with low productivity shocks choose an amount s of capital to sell. Firms with high productivity shocks choose an amount f of funds to purchase capital. Market clearing with N low types, N high types and $2N$ households requires

$$\sum_{i \in L} s_i = \sum_{j \in H} \frac{f_j}{p} + 2N\left(\frac{a}{p} - 1\right),$$

which implies a price of capital given by

$$p(s,f) = \frac{2Na + \sum_{j\in H} f_j}{2N + \sum_{i\in L} s_i}. \tag{B1}$$

SELLERS. — Seller i chooses s_i, taking as given other sellers' choices s_{-i} and buyers choices f, to solve the problem

$$\begin{aligned} &\max_{s_i} \{k_i - s_i + A_L k_i + p s_i - r b_i\} \\ &\text{s.t.} \quad p s_i \geq r b_i - A_L k_i \\ &\qquad\; s_i \leq k_i \\ &\qquad\; p = p(s_i, s_{-i}, f) \end{aligned}$$

For the seller constraint $p s_i \geq r b_i - A_L k_i$ to be binding, i.e. for them not wanting to sell more than necessary to repay debt, we need the price elasticity with respect to s_i to satisfy:

$$-\frac{\partial p}{\partial s_i}\frac{s_i}{p} > 1 - \frac{1}{p} \tag{B2}$$

For the case $p < 1$ that we are interested in, this condition is satisfied by a positive price elasticity, which we naturally have from the price function (B1):

$$-\frac{\partial p}{\partial s_i}\frac{s_i}{p} = \frac{s_i}{2N + \sum_{j\in L} s}$$

A seller acting strategically at $t = 1$ therefore finds it optimal to sell the least amount of capital possible to repay their debt. Since they take into account their effect on the price, their optimal sales are given by a fixed point condition

$$p(s_i, s_{-i}, f)\, s_i = r b_i - A_L k_i.$$

Solving for s_i and substituting in $b_i = k_i - n$, we obtain

$$s_i = \frac{\left(2N + \sum_{j\in L\setminus i} s_j\right)\left((r - A_L)\, k_i - rn\right)}{2Na + \sum_{j\in H} f_j - \left((r - A_L)\, k_i - rn\right)},$$

which is increasing in k_i. Since the optimal level of sales with strategic behavior at $t = 1$ is increasing in the level of capital chosen at $t = 0$, the comparative statics underlying the results in the main text remain unchanged.

BUYERS. — Buyer i chooses f_i, taking as given other buyers' choices f_{-i} and sellers' choices s, to solve the problem

$$\begin{aligned}\max_{f_i} & \left\{k_i + \frac{f_i}{p} + A_H k_i - f_i - rb_i\right\} \\ \text{s.t.} \quad & f_i \leq A_H k_i - rb_i \\ & p = p(s, f_i, f_{-i})\end{aligned}$$

For the buyer constraint $f_i \leq A_H k_i - rb_i$ to be binding, i.e. for them to use all their funds, we the price elasticity with respect to f_i to satisfy:

$$\frac{\partial p}{\partial f_i}\frac{f_i}{p} < 1 - p \tag{B3}$$

From the price function (B1) we have an elasticity with respect to f_i given by

$$\frac{\partial p}{\partial f_i}\frac{f_i}{p} = \frac{f_i}{2Na + \sum_{j\in H} f_j}.$$

Substituting in this elasticity, using (B1), and the equilibrium conditions $f_i = f$ and $s_i = s$ for all i, condition (B3) becomes

$$p < \frac{2Na + (N-1)f}{2Na + Nf}. \tag{B4}$$

Given that we are interested in the case $p < 1$, this is a weak condition that holds for sufficiently low p. A buyer acting strategically at $t = 1$ then finds it optimal to use all their funds to buy capital, exactly as in the case without strategic interaction at $t = 1$.

Figure B1 illustrates that condition (B4) is satisfied for almost all parameter combinations shown in Figure 1 in the main text. The figure shows the difference between the equilibrium price from the main text and the threshold from condition (B4) which is negative if the condition is satisfied.

B2. Strategic interim behavior in the liquidity shock model

We now consider the liquidity shock model and suppose that banks act strategically in the asset market at $t = 1$. We show that for $N > 1$, banks with liquidity shocks still find it optimal to sell all their assets. We also show that, under mild conditions, banks without liquidity shocks still find it optimal to use all their funds, just as in the non-strategic case. In sum, allowing for strategic behavior at $t = 1$ has no effect on the choices at $t = 1$ and therefore no effect on the optimization at $t = 0$. We derive the conditions in the more general setting with outside liquidity $N\phi \geq 0$ (see Appendix D).

Banks with liquidity shocks choose an amount s of assets to sell. Banks without liq-

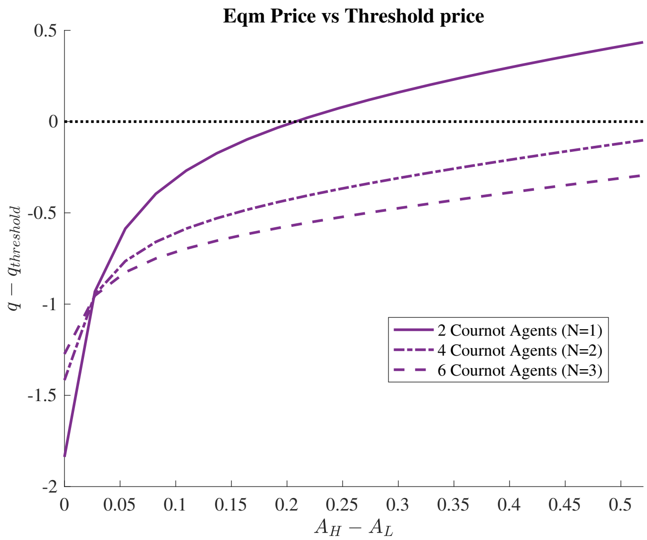

FIGURE B1. CONDITION FOR STRATEGIC INTERIM BEHAVIOR IN PRODUCTIVITY SHOCK MODEL.

Note: The figure shows the difference between the equilibrium price in the productivity shock model from the main text and the threshold from condition (B4) such that non-strategic interim behavior is w.l.o.g. for different levels of idiosyncratic productivity risk, $A_H - A_L$. With relative risk aversion 1, $\alpha = 0.85$, $a = 0.93$, $n = 1$, $N \in \{1,2,3\}$, $\mathbb{E}[A] = 1.05$, $r = 1.02$, and $\underline{A} = 0.99$ (see Section II.C).

uidity shocks choose an amount f of funds to purchase assets. Market clearing implies an asset price given by

$$p(s,f) = \frac{N\phi + \sum_{j \in H} f_j}{\sum_{i \in L} s_i}. \tag{B5}$$

SELLERS. — Seller i chooses s_i, taking as given other sellers' choices s_{-i} and buyers choices f, to solve the problem

$$\begin{aligned} &\max_{s_i} u(\ell_i + p s_i) \\ &\text{s.t.} \quad s_i \leq 1 - \ell_i \\ &\qquad\quad p = p(s_i, s_{-i}, f) \end{aligned}$$

For the seller constraint $s_i \leq 1 - \ell_i$ to be binding, i.e. for them to sell all their assets, we need the price elasticity with respect to s_i to satisfy:

$$-\frac{\partial p}{\partial s_i}\frac{s_i}{p} < 1$$

This is satisfied by the price function (B5) for $N > 1$:

$$-\frac{\partial p}{\partial s_i}\frac{s_i}{p} = \frac{s_i}{\sum_{j \in L} s_j}$$

A seller acting strategically at $t = 1$ therefore finds it optimal to sell all their assets, exactly as in the case without strategic interaction at $t = 1$.

BUYERS. — Buyer i chooses f_i, taking as given other buyers' choices f_{-i} and sellers' choices s, to solve the problem

$$\begin{aligned}\max_{f_i} u\left(\ell_i - f_i + R\frac{f_i}{p} + R(1-\ell_i)\right)& \\ \text{s.t.} \quad f_i \leq \ell_i & \\ p = p(s, f_i, f_{-i})&\end{aligned}$$

For the buyer constraint $f_i \leq \ell_i$ to be binding, i.e. for them to use all their funds, we need the price elasticity with respect to f_i to satisfy:

$$\frac{\partial p}{\partial f_i}\frac{f_i}{p} < 1 - \frac{p}{R} \tag{B6}$$

From the price function (B5) we have an elasticity with respect to f_i given by

$$\frac{\partial p}{\partial f_i}\frac{f_i}{p} = \frac{f_i}{N\phi + \sum_{j \in H} f_j}$$

Substituting in this elasticity, using (B5), and the equilibrium conditions $f_i = f$ and $s_i = s$ for all i, condition (B6) becomes

$$p < \frac{N\phi + (N-1)f}{N\phi + Nf} R \tag{B7}$$

Given that we are interested in the case $p < 1$ and have $R > 1$, this condition holds for sufficiently low p (i.e. high α) and/or large R. A buyer acting strategically at $t = 1$ then finds it optimal to use all their funds to buy assets, exactly as in the case without strategic interaction at $t = 1$. Figure B2 illustrates that condition (B7) is satisfied for the relevant case of high α where Cournot leads to severe underprovision of liquidity (see Figure 3 in the main text). The figure shows the difference between the equilibrium price and the threshold from condition (B4), which is negative if the condition is satisfied. Note that for more outside liquidity (higher ϕ), the region where strategic interim behavior is irrelevant increases considerably.

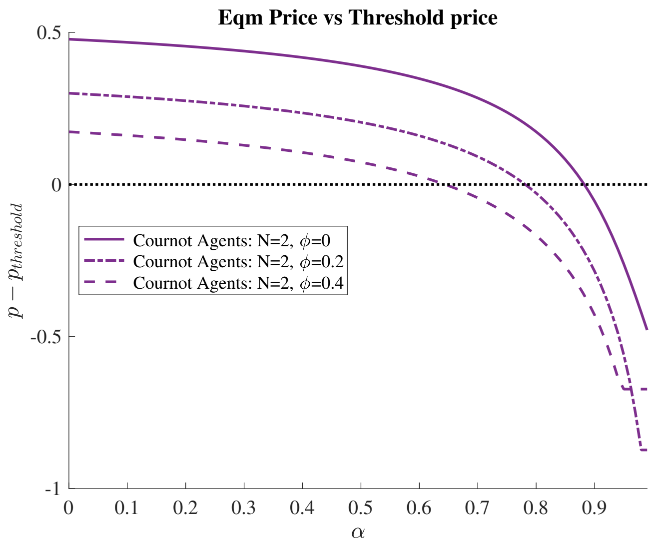

FIGURE B2. CONDITION FOR STRATEGIC INTERIM BEHAVIOR IN LIQUIDITY SHOCK MODEL.

Note: The figure shows the difference between the equilibrium price in the liquidity shock model from the main text and the threshold from condition (B7) such that non-strategic interim behavior is w.l.o.g. for different values of the probability of the good state, α. With log utility, $\beta = 0.96$, $R = 1.03/\beta$, and $N = 2$ (see Section III.C).

C. A SINGLE MODEL OF BANKS AND REAL INVESTMENT

In the main text, we present two separate models, one with a choice of investment level funded with debt and one with a choice of portfolio allocation between liquid and illiquid assets. In this appendix, we show how the two models can be viewed as representing two parts of an economy with financial intermediation. The liquidity trade-off model then represents the main decision of banks in allocating households savings between liquid assets and illiquid loans to firms; the leverage trade-off model then represents the main decision of equity constrained firms borrowing from banks to invest in productive assets.

The economy consists of households, banks, and firms, active in a real sector with productivity shocks and a financial sector with liquidity shocks. Households have funds to invest but are inefficient at operating capital and face uncertain consumption needs, while firms are efficient at operating capital but have small initial endowments. Banks, which are mutually owned by households, sit squarely between the two agents, taking deposits from households and providing loans to firms. There are three periods, $t = 0, 1', 1'', 2$. Liquidity shocks determining banks' solvency occur at $t = 1'$, while productivity shocks determining firms' output occur at $t = 1''$. The shocks can lead to fire sales of financial assets and real assets in the financial and real sector, respectively. Period $t = 2$ functions as "the future" or a continuation value for production in the economy.

At $t = 0$ banks have access to liquid and illiquid investment technologies. With the liquid technology, one unit of capital invested at $t = 0$ produces one unit of consumption good ("output") either at $t = 1'$ or $t = 1''$. With the illiquid technology, one unit of capital invested at $t = 0$ produces $R > 1$ units of output at $t = 1''$ but nothing at $t = 1'$. Illiquid investments can be traded at $t = 1'$ at any endogenous price p.

In each period, firms have access to a linear production technology using capital. Production at $t = 1''$ is risky: capital k invested at $t = 0$ produces Ak units of output at $t = 1''$, where A is uncertain with $\mathbb{E}[A] > R$ (i.e. the interest rate r from the productivity shock model equals the project return R from the liquidity shock model). To simplify, production at $t = 2$ is risk-free, with every unit of period-1 capital producing one unit of output at $t = 2$. Firms have small endowments of capital at $t = 0$, denoted by n, and have utility function $v(c)$ over consumption in period 2.

In each period, households have access to a production technology that takes capital k and yields $F(k) = a\log(1+k)$ units of consumption goods in the next period. We suppose that $a < 1$ so that households are never the efficient users of capital. Each period contains a new generation of households endowed with one unit of capital. Importantly, however, households born at $t = 0$ are subject to liquidity shocks à la Diamond and Dybvig (1983): they will either consume at $t = 1''$ (late types), or they will receive a liquidity shock and be forced to consume at $t = 1'$ (early types). Households receive utility $u(c)$ over early consumption and $\beta u(c)$ over late consumption, with $\beta \leq 1$ and $\beta R > 1$. To simplify the analysis, we suppose that households born at $t = 1''$ are not subject to liquidity shocks.

Banks pool resources from many households in order to offer deposit contracts that provide liquidity in the sense of Diamond and Dybvig (1983). Banks serve a restricted economic area as in Allen and Gale (2004), able to take deposits only from a set of households with correlated liquidity needs (i.e. banks cannot serve the entire population of households and completely diversify away liquidity shocks). To simplify, we assume that the households in each area have perfectly correlated liquidity needs. As a result, a bank whose consumers are early types will be forced to liquidate its assets in the interim period. Thus, we can say that the bank is itself subject to liquidity shocks.

At $t = 0$, firms can borrow from banks by issuing non-contingent debt due at $t = 1''$. Firms cannot default, and therefore bank loans to firms are identical to investments in the illiquid technology. Thus, firms can borrow at a gross interest rate R. Firms cannot borrow new funds at $t = 1''$ but must repay debt using proceeds from production or from selling capital at an endogenous price p.

The economy therefore features the following financial frictions. In the financial sector, banks and households cannot insure against liquidity risk, and the price of illiquid assets is determined by cash-in-the-market pricing (i.e. bank capital is slow moving and so demand for assets must come from banks who do not receive liquidity shocks). In the real sector, firms cannot insure against productivity shocks (i.e. they are restricted to borrow using non-contingent debt), and firms are subject to borrowing constraints at $t = 1''$ (they cannot borrow to repay/roll over debts).

We assume that, relative to the firm sector, household endowments are sufficiently large and liquidity demands sufficiently small so that in equilibrium banks' demands for illiquid investments exceed firms' demands for borrowing. As a result, the flow of funds in the economy in equilibrium can be described as follows. At $t = 0$, households deposit all capital with banks. Banks allocate a fraction ℓ of capital to liquid investments and a fraction $1-\ell$ to illiquid investments, a portion of which are loans to firms at an interest rate R. Firms borrow b units from banks, allowing them to invest $k=n+b$ in capital for risky projects. (Our relative size assumption means that in equilibrium $b<1-\ell$.) At $t=1''$, firms repay debts, perhaps by selling capital to households at price p in order to do so, and remaining capital is invested in projects to produce at $t=2$.

Given these assumptions, we can solve for equilibrium in this economy by considering the financial and real sectors separately. First, we can consider the equilibrium provision of liquidity by banks at $t=0$ and analyze how internalizing price impacts in the market for illiquid investments affects the price p and the level of liquidity ℓ. Second, we can consider the equilibrium borrowing decision of firms at $t=0$ and analyze how internalizing price impacts in the market for capital affects the price p, the level of investment k, and borrowing b. Because all bank loans are risk-free, outcomes in the real sector (firm production and fire sales in capital) do not affect behavior or outcomes in the financial sector (provision of liquidity and fire sales in illiquid assets), and vice versa. As a result, we can also analyze pecuniary externalities in financial and real markets separately, and a Social Planner attempting to correct each externality can consider them separately without considering interactions between real and financial markets. The results therefore correspond to those in the main text.

D. Liquidity shock model with outside buyers

We now consider the case of outside buyers in the liquidity shock model of Section III. Specifically, we assume that there are N outside buyers with $\phi \geq 0$ in cash to buy assets at $t=1$. This collapses to the model in the main text for $\phi=0$. With outside liquidity, the equilibrium price (19) becomes

$$p=\frac{N\phi+\sum_{j\in\text{buy}}\ell_j.}{\sum_{i\in\text{sell}}(1-\ell_i)}$$

The first-order condition (20) of Walrasian equilibrium remains unchanged and still implies $p<1$ for $\alpha>0$ and $p=1$ for $\alpha=0$.[1] However, due to the additional outside liquidity ϕ, the equilibrium inside liquidity ℓ will be lower. For example, in the case $\alpha=0$, the equilibrium $p=1$ implies that we can solve for ℓ in closed form and it is decreasing in ϕ:

$$\ell=\frac{1-\phi}{2}$$

The Social Planner first-order condition (21) is affected by ϕ through the price effect

$$\frac{dp}{d\ell}=\frac{\phi+1}{(1-\ell)^2}.$$

[1]Note that payoffs are in terms of the individual choice ℓ_i and the equilibrium price so their formulas are as in the main text.

Combined with the effect of ϕ on p, the additional outside liquidity ϕ will therefore also lower the efficient level of liquidity ℓ. Consider again the case $\alpha = 0$ where the Social Planner implements the standard risk sharing $u'(c_L) = \beta R u'(c_H)$ of Diamond and Dybvig (1983). Since outside liquidity substitutes for inside liquidity, the same risk sharing can be achieved with lower ℓ.

Finally, we turn to the Cournot first-order condition (22). Notably, the outside liquidity ϕ appears only in the price impact as perceived by an L type who sells assets, changing the price impacts in (23) as follows:

$$\frac{dp}{d\ell_L} = \frac{1}{N} \frac{\phi + \ell}{(1-\ell)^2}$$

This price impact, which in the first-order condition weighs the benefit of holding extra liquidity, is increasing in ϕ. The presence of outside liquidity therefore biases downward the inside liquidity ℓ in the Cournot equilibrium as well. The Cournot equilibrium can still lead to lower liquidity than the Walrasian equilibrium if the seller price impact is sufficiently low relative to the buyer price impact. We still have that the ratio of the two satisfies

$$\frac{dp_L/d\ell_i}{dp_H/d\ell_i} = \frac{\phi + \ell}{1-\ell} = p.$$

Since $\phi > 0$ bounds the price (and therefore the ratio of price impacts) away from zero, higher outside liquidity attenuates the underprovision of liquidity in the Cournot equilibrium.

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