ONLINE APPENDIX

A Generalized Model of Advertised Sales

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Appendix B - Main Proofs

Proof of Lemma 1. First, note that $u_i < u^m$ is always strictly dominated by $u_i = u^m$ for any $\eta_i \in \{0, 1\}$. Increasing u_i to u^m would i) raise firm i's profits per consumer as $\pi_i'(u) > 0$ for $u_i < u^m$, and yet ii) never reduce the number of consumers that it trades with. Second, $u_i > u^m$ is strictly dominated by $u_i = u^m$ when $\eta_i = 0$. Reducing u_i to u^m would i) strictly increase firm i's profits per-consumer as $\pi_i'(u) < 0$ for $u_i > u^m$, but ii) never reduce the number of consumers that it trades with, since non-advertised offers are unobserved to consumers and consumers can only visit one firm. Third, for any tie-break probability, $x_i(T) \in [0, 1]$, setting $u_i = u^m$ and $\eta_i = 1$ with positive probability is strictly dominated by setting $u_i = u^m$ and $\eta_i = 0$. Given $u_i = u^m$, moving from advertising to not advertising would i) strictly reduce firm i's advertising costs, $A_i > 0$, and ii) never reduce the number of consumers that it trades with since $x_i(T)$ is independent of advertising decisions via Assumption X. $\square$

Proof of Lemma 2. First, any sales equilibrium must have $k^* \geq 2$ because there can be no sales equilibrium with $k^* = 1$. If so, firm i would win the shoppers with probability one whenever advertising as then $u_i > u^m$ and $u_j = u^m$ $\forall j \neq i$. Hence, in such instances, i's strategy cannot be defined as it would always want to relocate its probability mass closer to u^m. Second, given this, one can then adapt standard arguments (e.g. Baye et al. (1992)), to show that for at least two firms i and j, u must be a point of increase of $F_i(u)$ and $F_j(u)$ at any $u \in (u^m, \bar{u}]$. Third, by adapting standard arguments (e.g. Narasimhan (1988), Baye et al. (1992), Arnold et al. (2011)) firms cannot use point masses on any $u > u^m$. Fourth, any firm with $\alpha_i > 0$ must have $\alpha_i \in (0, 1)$ in equilibrium. To see this, suppose $\alpha_i = 1$ for some i and note from above that at least two firms must randomize just above u^m. If so, the expected profits from advertising just above u^m must equal $\theta_j \pi_j^m - A_j$ for at least one such firm $j \neq i$ as there can be no mass points at $u > u^m$. However, firm j could earn $\theta_j \pi_j^m > \theta_j \pi_j^m - A_j$ from not advertising; a contradiction. Finally, suppose $n = 2$. As a consequence of previous arguments, in any sales equilibrium both firms must share a common advertised utility support, $(u^m, \bar{u}]$, with no gaps. $\square$

Proof of Lemma 3. Assume the opposite and consider the following exhaustive cases. First, consider a potential tie involving at least one advertising firm and at least one non-advertising firm. If so, any advertising firms in T must set $u > u^m$, and any non-advertising firms in T must set u^m in equilibrium; a contradiction. Second, consider a potential tie involving only advertising firms. For such a tie to arise, at least two firms must put positive probability mass on some utility level, $u > u^m$. However, such mass points cannot exist in equilibrium via Lemma 2. Third, consider a potential tie involving only non-advertising firms, but where $|T| < n$. If so, the firms in T must set u^m, and any remaining firm, $j \notin T$, must set $u_j > u^m$ in equilibrium, a contradiction. $\square$

Proof of Lemma 4. Firm i's expected profits from advertising just above u^m must equal $\pi_i^m[\theta_i + (1-\theta)\Pi_{j \neq i}(1-\alpha_j)] - A_i$, where for a cost of A_i it can win the shoppers outright with the probability that its rivals set u^m and do not advertise, $\Pi_{j \neq i}(1-\alpha_j)$. If firm i uses sales, we know from the text that its expected profits from advertising an offer just above u^m must equal its expected profits from not advertising, (1). Hence, by equating these two expressions one can solve for

$$\Pi_{j \neq i}(1-\alpha_j) = \frac{A_i}{(1-x_i^*)(1-\theta)\pi_i^m}.$$

The expression in (2) can then be derived by plugging this back in to (1). $\square$

Proof of Lemma 5. Suppose firm i uses sales in equilibrium and $\bar{u} > u^m$. i) For this to be optimal, it must be that $\bar{u} \leq \tilde{u}_i$. Suppose not. Then from the derivation of (4), we know $\pi_i(\bar{u})(1-\theta_{-i}) - A_i < \theta_i \pi_i^m$ such that firm i would strictly prefer to deviate from $u_i = \bar{u}$. ii) To derive (5), note that (1) expresses $\bar{\Pi}_i$ for a given x_i^*, and that i must expect to earn $\bar{\Pi}_i$ for $u_i = u^m$ and for all $u_i \in (u^m, \bar{u}]$. If i set $u_i = \bar{u}$ it would attract the shoppers with probability one because there are no mass points on $u \in (u^m, \bar{u}]$. Hence, it must be that $\bar{\Pi}_i = (1-\theta_{-i})\pi_i(\bar{u}) - A_i$. Solving this implies $x_i^* = \chi_i(\bar{u})$. $\square$

Proof of Lemma 6. First, given $x_1^* + x_2^* = 1$ and $x_i^* = \chi_i(\bar{u})$, it must be that $\chi_1(\bar{u}) + \chi_2(\bar{u}) = 1$. $\chi_1(\bar{u}) + \chi_2(\bar{u})$ is defined on $\bar{u} \in (u^m, \min\{\tilde{u}_1, \tilde{u}_2\})$ and is strictly decreasing. Hence, we know the solution for $\bar{u}$ will be unique, if it exists. Second, the expression for α_i can be calculated using the expression from the proof of Lemma 4, $\Pi_{j\neq i}(1-\alpha_j) = \frac{A_i}{(1-x_i^*)(1-\theta)\pi_i^m}$, and so the unique expression (7) follows for $n = 2$. Third, to derive $F_i(u)$, we require firm i's equilibrium profits, $\bar{\Pi}_i$, to equal its expected profits for all $u_i \in (u^m, \bar{u}]$, $\pi_i(u)[\theta_i + (1-\theta)F_j(u)] - A_i$. Using (2) and rearranging for $F_j(u)$ implies the unique expression (8). □

Proof of Proposition 1. Part a). If a sales equilibrium exists, Lemmas 1-6 have characterized its unique properties. We now demonstrate that this sales equilibrium exists and that no other equilibrium can exist when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1-\theta$.

First, we show that no other equilibrium can exist. The only other candidate is a non-sales equilibrium where $\alpha_1 = \alpha_2 = 0$ and $u_1 = u_2 = u^m$. For this to be an equilibrium, we require that no firm i can profitably deviate to advertising a utility slightly above u^m to attract all the shoppers. For a given x_i^*, this requires $\pi_i^m[\theta_i + x_i^*(1-\theta)] \geq \pi_i^m[\theta_i + (1-\theta)] - A_i$ or $\frac{A_i}{\pi_i^m} \geq (1-\theta)(1-x_i^*)$. The same condition for j yields $\frac{A_j}{\pi_j^m} \geq (1-\theta)x_i^*$, and so for both to hold we need $1 - \frac{A_i}{(1-\theta)\pi_i^m} \leq x_i^* \leq \frac{A_j}{(1-\theta)\pi_j^m}$. However, no such x_i^* can exist when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1-\theta$.

Second, we demonstrate the unique sales equilibrium exists. For this, it is sufficient to show that $\chi_1(\bar{u}) + \chi_2(\bar{u}) = 1$ implies a solution $\bar{u} \in (u^m, \min\{\tilde{u}_1, \tilde{u}_2\})$. This follows as $\chi_1(\bar{u}) + \chi_2(\bar{u})$ is i) strictly decreasing in $\bar{u} \in (u^m, \min\{\tilde{u}_1, \tilde{u}_2\})$, ii) below 1 for $\bar{u}$ sufficiently close to $\min\{\tilde{u}_1, \tilde{u}_2\}$ and iii) above 1 for $\bar{u}$ sufficiently close to u^m when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1-\theta$. It then follows that $x_i^* = \chi_i(\bar{u}) \in (0,1)$ for $i = \{1,2\}$. One can then verify that $\alpha_i^* = 1 - \frac{A_j}{x_i^*(1-\theta)\pi_j^m} \in (0,1)$, $F_i(\cdot)$ is increasing over $(u^m, \bar{u}]$, and $F_i(\bar{u}) = 1$ for both firms.

Part b). As demonstrated in Part a), a sales equilibrium only exists when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} < 1-\theta$. However, we now demonstrate that a non-sales equilibrium exists when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} \geq 1-\theta$. From above, a non-sales equilibrium requires $1 - \frac{A_i}{(1-\theta)\pi_i^m} \leq x_i^* \leq \frac{A_j}{(1-\theta)\pi_j^m}$ for each i, or equivalently, $x_i^* = 1 - x_j^* \in [\chi_i(u^m), 1-\chi_j(u^m)]$. This interval is non-empty when $\frac{A_1}{\pi_1^m} + \frac{A_2}{\pi_2^m} \geq 1-\theta$. □

Proof of Lemma 7. First, let $\tilde{u}_i > \bar{u}$. To show why $\alpha_i > 0$ in equilibrium, suppose not, with $\alpha_i = 0$. From our restrictions, firm i would then have $x_i^* = 0$. Thus, by the definition of $\tilde{u}_i$, i would be indifferent between never advertising, and advertising $\tilde{u}_i$ provided it attracted all the shoppers. Given $\tilde{u}_i > \bar{u}$, i must then strictly prefer to deviate to set

$\eta_i = 1$ with $u_i = \bar{u}$ where it could win the shoppers with probability one; a contradiction. Second, let $\tilde{u}_i \leq \bar{u}$. To show why $\alpha_i = 0$ in equilibrium, suppose not, with $\alpha_i > 0$. From our restrictions, firm i would then have $x_i^* > 0$. Thus, using the definition of $\tilde{u}_i$, i would be unwilling to advertise over the whole required support $u \in (u^m, \bar{u}]$, and would strictly prefer to deviate to $\alpha_i = 0$. Finally, statements i) and ii) in the Lemma then follow immediately given our two settings where $u^m < \tilde{u}_i = \tilde{u}$ for all i, or $u^m < \tilde{u}_n < ... < \tilde{u}_1$. $\square$

Proof of Proposition 2. In line with the sketch of the proof under the proposition, we proceed by proving a number of claims.

Claim 1: In any sales equilibrium under our restrictions, a) the equilibrium tie-break probabilities, x^*, and upper bound, $\bar{u}$, are uniquely (implicitly) defined by (10) and (11), and b) these solutions must satisfy (9) for k^* to be consistent with equilibrium.

Proof of 1a: We know from (5), that any advertising firm, $i \leq k^*$, must have $x_i^* = \chi_i(\bar{u})$. From Lemma 7, an advertising firm must have $\tilde{u}_i > \bar{u}$ such that $x_i^* = \chi_i(\bar{u}) > 0$ as required. In addition, from our restrictions, $x_i^* = 0$ for all non-advertising firms, $i > k^*$. Hence, (10) applies. As $\sum_{i=1}^n x_i^*$ must sum to one, it then also follows that $\bar{u}$ is implicitly defined by (11). Note $\sum_{i=1}^{k^*} \chi_i(\bar{u})$ is strictly decreasing on $\bar{u} \in (u^m, \tilde{u}_{k^*})$. Hence, the solution for $\bar{u}$ will be unique.

Proof of 1b: First, suppose $k^* = n$. Then from Lemma 7, we require the solution to (11) to lie within $\bar{u} \in (u^m, \tilde{u}_n)$. Thus, we require $\sum_{i=1}^n \chi_i(u^m) > 1$ and $\sum_{i=1}^n \chi_i(\tilde{u}_n) < 1$ as consistent with (9). Note that $\bar{u} \in (u^m, \tilde{u}_n)$ also guarantees a unique interior value for $x_i^* \in (0,1)$ $\forall i \leq k^*$. Second, suppose $k^* \in [2, n)$. Then from Lemma 7, we require the solution to (11) to lie within $\bar{u} \in [\tilde{u}_{k^*+1}, \tilde{u}_{k^*})$. Thus, we require $\sum_{i=1}^{k^*} \chi_i(\tilde{u}_{k^*+1}) \geq 1$ and $\sum_{i=1}^{k^*} \chi_i(\tilde{u}_{k^*}) < 1$ as consistent with (9). Note that $\bar{u} \in [\tilde{u}_{k^*+1}, \tilde{u}_{k^*})$ also guarantees a unique interior value for $x_i^* \in (0,1)$ $\forall i \leq k^*$ under our restrictions.

Claim 2: Whenever a sales equilibrium exists under our restrictions, $k^* \in [2, n]$ is uniquely defined by (9) provided $1 < \sum_{i=1}^n \chi_i(u^m)$.

Proof: Using Claim 1, it is useful to summarize and re-notate the following results. First, for any $k^* \in [2, n]$, $\sum_{i=1}^{k^*} \chi_i(\bar{u})$ is strictly decreasing on $\bar{u} \in (u^m, \tilde{u}_{k^*})$. Second, using (9), if $k^* = n$, then we require $\underline{I}_n \equiv \sum_{i=1}^n \chi_i(\tilde{u}_n) < 1 < \sum_{i=1}^n \chi_i(u^m) \equiv \bar{I}_n$. Third, if $k^* = k \in (2, n]$, then we require $\underline{I}_k \equiv \sum_{i=1}^{k^*} \chi_i(\tilde{u}_{k^*}) < 1 \leq \sum_{i=1}^{k^*} \chi_i(\tilde{u}_{k^*+1}) \equiv \bar{I}_k$. Hence, for k^* to be uniquely defined, there must exist exactly one value of k^* for which either $1 \in (\underline{I}_n, \bar{I}_n)$ or $1 \in (\underline{I}_k, \bar{I}_k]$. Provided $\sum_{i=1}^n \chi_i(u^m) \equiv \bar{I}_n > 1$, this then follows because i) $\underline{I}_{z+1} = \bar{I}_z$ for any $z \in (2, n]$ (as $\sum_{i=1}^{z+1} \chi_i(\tilde{u}_{z+1}) = \sum_{i=1}^{z} \chi_i(\tilde{u}_{z+1})$ given $\chi_{z+1}(\tilde{u}_{z+1}) = 0$ from (4)), and ii) $\underline{I}_2 \equiv \sum_{i=1}^2 \chi_i(\tilde{u}_2) < 1$ (as $\sum_{i=1}^2 \chi_i(\tilde{u}_2) = \chi_1(\tilde{u}_2) \in (0,1)$).

Claim 3: Whenever a sales equilibrium exists under our restrictions, the firms' advertising probabilities and offer distributions are uniquely defined. Firms $i > k^*$ have $\alpha_i = 0$ and $u_i = u^m$, and firms $i \leq k^*$ have:

$$\alpha_i = 1 - \frac{\left[\Pi_{j=1}^{k^*}\gamma_j(u^m)\right]^{\frac{1}{k^*-1}}}{\gamma_i(u^m)} \in (0,1)$$

$$F_i(u) = \frac{\left[\Pi_{j=1}^{k^*}\gamma_j(u)\right]^{\frac{1}{k^*-1}}}{\gamma_i(u)}$$

$$\textit{where} \qquad \gamma_i(u) = \frac{\pi_i(\bar{u})(1-\theta_{-i}) - \theta_i\pi_i(u)}{(1-\theta)\pi_i(u)}$$

In addition, $\forall i$, each firm i's equilibrium profits remain equal to (2).

Proof: The behavior of firms $i > k^*$ follows immediately from Lemma 1. To derive α_i, first recall the expression from the proof of Lemma 4, $\Pi_{j\neq i}(1-\alpha_j) = \frac{A_i}{(1-x_i^*)(1-\theta)\pi_i^m}$. As $\alpha_i = 0$ for all $i > k^*$, this also equals $\Pi_{j\neq i\in K^*}(1-\alpha_j)$. After plugging in $x_i^* = \chi_i(\bar{u})$, $\Pi_{j\neq i\in K^*}(1-\alpha_j) = \gamma_i(u^m)$, where $\gamma_i(u)$ is given above. By then multiplying this equation across the k^* firms, we get $\Pi_{i=1}^{k^*}[\Pi_{j\neq i\in K^*}(1-\alpha_j)] \equiv \Pi_{i=1}^{k^*}(1-\alpha_i)^{k^*-1} = \Pi_{i=1}^{k^*}\gamma_i(u^m)$, such that $\Pi_{i=1}^{k^*}(1-\alpha_i) = \left[\Pi_{i=1}^{k^*}\gamma_i(u^m)\right]^{\frac{1}{(k^*-1)}}$. Then, by returning to $\Pi_{j\neq i\in K^*}(1-\alpha_j) = \gamma_i(u^m)$ and multiplying both sides by $1-\alpha_i$ we get $\Pi_{j=1}^{k^*}(1-\alpha_j) = (1-\alpha_i)\gamma_i(u^m)$, which after substitution provides our expression for α_i. Similar steps can be then used to derive the expression for the unique utility distribution, $F_i(u)$. One can verify that $\alpha_i \in (0,1)$ and $F_i(\bar{u}) = 1$ $\forall i \leq k^*$ as required given $\bar{u} \in (\tilde{u}_{k^*+1}, \tilde{u}_{k^*}]$. Finally, to verify each firm's equilibrium profits, remember that each firm must earn (1) for a given set of tie-break probabilities. After substituting out for $\Pi_{j\neq i}(1-\alpha_j)$ from above, this equals (2). Note that (2) applies not only to firms that use sales, but also to those that do not because they have $x_i^* = 0$ under our assumptions such that $\bar{\Pi}_i = \theta_i\pi_i^m$ as consistent with them pricing only to their non-shoppers. □

Proof of Corollary 1. i) Let $A \to 0$. Using (3) and past results, $\sum_{i=1}^{k^*}\chi_i(\tilde{u}_{k^*}) = \sum_{i=1}^{k^*-1}\chi_i(\tilde{u}_{k^*}) \to (k^*-1)$ for any $k^* \in [2,n]$. Hence, the conditions in (9) can only be satisfied when $k^* = 2$. ii) Let $A \to \frac{(n-1)(1-\theta)}{\sum_{i=1}^{n}\frac{1}{\pi_i^m}}$. Using (3), $\sum_{i=1}^{n}\chi_i(u^m) = n - \frac{nA}{(1-\theta)\sum_{i=1}^{n}\pi_i^m} \to 1$ such that the solution to $\bar{u}$ in (11) converges to $u^m < \tilde{u}_n$ from above. Hence, it must be that $\bar{u} \in (u^m, \tilde{u}_n)$ as only consistent with $k^* = n$. □

Proof of Corollary 2. From above, firms with a higher $\tilde{u}_i$ are more likely to use sales. Hence, we require $\frac{\partial \tilde{u}_i}{\partial \rho_i} > 0$. Rewrite (4) as $(1-\theta_{-i})\pi(\tilde{u}_i, \rho_i) - A_i = \theta_i \pi(u^m, \rho_i)$. Then note that $\frac{\partial \tilde{u}_i}{\partial \rho_i} = \frac{\theta_i \pi_\rho(u^m,\rho_i) - (1-\theta_{-i})\pi_\rho(\tilde{u}_i,\rho_i)}{(1-\theta_{-i})\pi_u(\tilde{u}_i,\rho_i)}$. As $\pi_u(\tilde{u}_i, \rho_i) < 0$ given $\tilde{u}_i > u^m$, then $\frac{\partial \tilde{u}_i}{\partial \rho_i}$ is positive whenever $\frac{1-\theta_{-i}}{\theta_i} > \frac{\pi_\rho(u^m,\rho_i)}{\pi_\rho(\tilde{u}_i,\rho_i)}$. This is satisfied when $\theta_i = (\theta/n)\ \forall i$ and $\pi_{\rho u} \geq 0$ for $u > u^m$ because i) $\frac{1-\theta_{-i}}{\theta_i} = \frac{n-(n-1)\theta}{\theta} > 1$ given $\theta \in (0,1)$, and ii) $\frac{\pi_\rho(u^m,\rho_i)}{\pi_\rho(\tilde{u}_i,\rho_i)} \leq 1$ given $\tilde{u}_i > u^m$. □

Proof of Proposition 3. i) Given $\bar{\Pi}_i = \pi(\bar{u}(\cdot))(1 - \sum_{j\neq i}\theta_j(\cdot)) - A - \tau e_i$, firm i's first-order condition wrt e_i can be expressed by (12) when evaluated at symmetry with $\theta_j(\cdot) = \theta(\cdot)/n\ \forall j$. ii) For the comparative statics, we first re-write the FOC in terms of model primitives by using (11) to derive $\frac{\partial \bar{u}(\cdot)}{\partial e_i}$. When evaluated at symmetry, this equals $\frac{[\pi^m + \pi(\bar{u}(\cdot))(n-1)]}{\pi'(\bar{u}(\cdot))[n-(n-1)\theta(\cdot)]}\left(\frac{\partial \theta_i(\cdot)}{\partial e_i} - (n-1)\frac{\partial \theta_j(\cdot)}{\partial e_i}\right)$ where $\pi(\bar{u}(\cdot)) = \frac{\theta(\cdot)\pi^m + \frac{An^2}{(n-1)}}{n-(n-1)\theta(\cdot)}$. By substituting these in and rearranging, one can rewrite the FOC as: $\frac{\partial \theta_i(\cdot)}{\partial e_i}(\pi^m + An) + \frac{\partial \theta_j(\cdot)}{\partial e_i}[\pi^m(1-\theta(\cdot))(n-1) - An] - \tau[n - \theta(\cdot)(n-1)] = 0$. We now denote the LHS of this equation as $H(\cdot)$ and apply the implicit function theorem. At any symmetric equilibrium, the associated second-order condition must be negative, such that $\frac{\partial H(\cdot)}{\partial e_i} \equiv \frac{\partial^2 \bar{\Pi}_i}{\partial e_i^2} < 0$. Hence, it follows that $\frac{\partial e}{\partial A} \gtreqless 0$ if $\frac{\partial H(\cdot)}{\partial A} = n\big(\frac{\partial \theta_i(\cdot)}{\partial e_i} - \frac{\partial \theta_j(\cdot)}{\partial e_i}\big) \gtreqless 0$. Hence, given our assumptions about the form of $\theta_i(\cdot)$, the statics follow as $\frac{\partial H(\cdot)}{\partial A} > 0$ under own loyalty-increasing actions, but $\frac{\partial H(\cdot)}{\partial A} < 0$ under own loyalty-decreasing actions. □

Proof of Proposition 4. Let $\pi_i(u) = \pi(u)$, $A_i = A$ and $\theta_j = \theta - \theta_i$. From (6), $\frac{\partial \bar{u}}{\partial \theta_i} = 0$ after we impose symmetry ex post with $\theta_i = \theta_j = \theta/2$. By using this with the derivative of (5), we gain $\frac{\partial x_i^*}{\partial \theta_i} = -\frac{A[\pi^m - \pi(\bar{u})]}{[\pi(\bar{u})(1-(\theta/2)) - (\theta/2)\pi^m]^2} < 0$. These two results also help us find the remaining derivatives. Using (2) or $\bar{\Pi}_i = (1-\theta_j)\pi_i(\bar{u}) - A_i$ gives $\frac{\partial \bar{\Pi}_i}{\partial \theta_i} = \pi(\bar{u}) > 0$ and $\frac{\partial \bar{\Pi}_j}{\partial \theta_i} = -\pi(\bar{u}) < 0$, and using (7) gives $\frac{\partial \alpha_i}{\partial \theta_i} = -\frac{[\pi^m - \pi(\bar{u})]}{(1-\theta)\pi^m} < 0$, and $\frac{\partial \alpha_j}{\partial \theta_i} = \frac{\pi^m - \pi(\bar{u})}{(1-\theta)\pi^m} > 0$. Further, from (8), $\frac{\partial F_i}{\partial \theta_i} = \frac{\pi(u) - \pi(\bar{u})}{(1-\theta)\pi(u)} > 0$ and $\frac{\partial F_j}{\partial \theta_i} = -\frac{\pi(u) - \pi(\bar{u})}{(1-\theta)\pi(u)} < 0$ for all relevant u, such that $E(u_i)$ decreases and $E(u_j)$ increases. □

Proof of Proposition 5. Given $\pi_i(u) = \pi(u)$ and $\theta_i = \theta/2$, note from (5) and (6) that $A_i + A_j = \pi(\bar{u})(1 - \frac{\theta}{2}) - \frac{\theta}{2}\pi^m = \frac{A_j}{x_i}$, such that $x_i^* = \frac{A_j}{A_i + A_j}$. For the profit results, substitute x_i^* into (2) to give $\bar{\Pi}_i = \frac{\theta}{2}\pi^m + A_j$. For the remaining results, substitute x_i^* into (7) to give $\alpha_i = 1 - \frac{A_i + A_j}{(1-\theta)\pi^m}$, and into (8) to obtain $F_i(u) = \frac{(\theta/2)[\pi^m - \pi(u)] + [A_i + A_j]}{(1-\theta)\pi(u)}$. An increase in A_i then decreases α_i and α_j, and increases $F_i(u)$ and $F_j(u)$ for all relevant u. □

Proof of Proposition 6. Given $A_i = A$ and $\theta_i = \theta/2$, note from (6) that $\frac{\partial \bar{u}}{\partial \rho_i}|_{\rho_i=\rho_j=\rho} = \frac{(1-(\theta/2))\pi_\rho(\bar{u},\rho)-(\theta/2)\pi_\rho(u^m,\rho)}{-(2-\theta)\pi_u(\bar{u},\rho)}$. This is positive as both the denominator and numerator are positive given $\theta \in (0,1)$, $\pi_{\rho u}(\cdot) \geq 0$ and $\bar{u} > u^m$. Then, using (5) and the above, $\frac{\partial x_i^*}{\partial \rho_i} = \frac{A[(2-\theta)\pi_\rho(\bar{u},\rho)-\theta\pi_\rho(u^m,\rho)]}{[(2-\theta)\pi(\bar{u},\rho)-\theta\pi(u^m,\rho)]^2}$, which has the same sign as $\frac{\partial \bar{u}}{\partial \rho_i}|_{\rho_i=\rho_j=\rho}$. Note $\bar{\Pi}_i = (1-\frac{\theta}{2})\pi(\bar{u},\rho_i)-A$. At the point of symmetry, it then follows that $\frac{\partial \bar{\Pi}_i}{\partial \rho_i} = (1-\frac{\theta}{2})\left(\pi_\rho(\bar{u},\rho) + \frac{\partial \bar{u}}{\partial \rho_i}\pi_u(\bar{u},\rho)\right)$ which equals $\frac{1}{2}[(1-(\theta/2))\pi_\rho(\bar{u},\rho)+(\theta/2)\pi_\rho(u^m,\rho)] > 0$. Similarly, note $\bar{\Pi}_j = (1-\frac{\theta}{2})\pi(\bar{u},\rho_j)-A$. Then $\frac{\partial \bar{\Pi}_j}{\partial \rho_i} = \frac{1}{2}\theta\pi_\rho(u^m,\rho)$ which has the opposite sign of $\frac{\partial \bar{u}}{\partial \rho_i}|_{\rho_i=\rho_j=\rho}$. Using (7), one can then prove $\frac{\partial \alpha_i}{\partial \rho_i}$ has the same sign as $\frac{\partial \bar{u}}{\partial \rho_i}|_{\rho_i=\rho_j=\rho}$. Using (8) one can show that $\frac{\partial F_i(u)}{\partial \rho_i}$ has the opposite sign to $\frac{\partial \bar{u}}{\partial \rho_i}|_{\rho_i=\rho_j=\rho}$ for all relevant u. □

Appendix C - Supplementary Equilibrium Details

Sections C1 and C2 provide extra information about the equilibrium when i) advertising costs tend to zero, and ii) the single visit assumption is relaxed.

C1. Equilibrium when Advertising Costs Tend to Zero

To ease exposition and to best connect to the existing literature, we illustrate the case of near-zero advertising costs for the duopoly equilibrium. Suppose the firms are asymmetric, but $A_1 = A_2 = A \to 0$. The equilibrium depends upon $\tilde{u}_1 \gtreqless \tilde{u}_2$. Without loss of generality, suppose $\tilde{u}_i < \tilde{u}_j$ such that $\pi_i(u)(1-\theta_j)-A-\theta_i\pi_i^m < \pi_j(u)(1-\theta_i)-A-\theta_j\pi_j^m$ at $u \in (u^m, \tilde{u}_i]$. Using (5) and (6), for $\bar{u}$ to exist within $(u^m, \tilde{u}_i]$ and for x_i^* and x_j^* to be well defined, it must be that $\bar{u} \to \tilde{u}_i$ such that $x_i^* \to 0$ and $x_j^* \to 1$. Given this, we know $\lim_{A\to 0} \bar{\Pi}_i = \theta_i\pi_i^m$ and $\lim_{A\to 0} \bar{\Pi}_j = \lim_{A\to 0}(1-\theta_i)\pi_j(\bar{u}) = (1-\theta_i)\pi_j\left(\pi_i^{-1}\left(\frac{\theta_i\pi_i^m}{1-\theta_j}\right)\right) > \theta_j\pi_j^m$. Further, from (8), we know $\lim_{A\to 0} F_i(u) = \lim_{A\to 0} \frac{\bar{\Pi}_j-\theta_j\pi_j(u)}{(1-\theta)\pi_j(u)}$ and $\lim_{A\to 0} F_j(u) = \lim_{A\to 0} \frac{\bar{\Pi}_i-\theta_i\pi_i(u)}{(1-\theta)\pi_i(u)}$. Finally, from (7), $\alpha_j \to 1$, while firm i advertises with probability $\lim_{A\to 0} \alpha_i = 1-\frac{\bar{\Pi}_j-\theta_j\pi_j^m}{(1-\theta)\pi_j^m} \in (0,1)$. This limit equilibrium converges to the equilibrium of a model that allows for $A = 0$ explicitly without our tie-break rule. There, both firms advertise with probability one and use equivalent utility distributions except that firm i advertises u^m with a probability mass equivalent to $\lim_{A\to 0}(1-\alpha_i)$.

To show how this connects to much of the past literature which has considered various asymmetries in non-shopper shares, product values and/or costs under unit demand and the restriction, $A_i = A_j = 0$, consider the following example. Suppose consumers have unit demands. From above, the equilibrium then depends upon $\tilde{u}_1 \gtreqless \tilde{u}_2$, or $(1-\theta_1)(V_1 - c_1) - (1-\theta_2)(V_2 - c_2) \lesseqgtr 0$. For instance, when this is negative, $x_1^* \to 0$ and $x_2^* \to 1$, such that $\bar{\Pi}_1 \to \theta_1(V_1 - c_1)$, and $\bar{\Pi}_2 \to (1-\theta_1)[(V_2 - c_2) - \bar{u}]$, where $\bar{u} \to \left(\frac{(1-\theta)(V_1-c_1)}{1-\theta_2}\right)$. By then denoting $\Delta V = V_1 - V_2$, and noting that $F_1(u_2) = Pr(u_1 \leq u_2) = 1 - F_1(p_2 + \Delta V)$ and $F_2(u_1) = 1 - F_2(p_1 - \Delta V)$, it follows that $F_1(p) = 1 - \left[\frac{\bar{\Pi}_2 - \theta_2(p-\Delta V - c_2)}{(1-\theta)(p-\Delta V - c_2)}\right] = 1 + \frac{\theta_2}{1-\theta} - \frac{(1-\theta_1)(\theta_1(V_1-c_1)+(1-\theta_2)(c_1-c_2-\Delta V))}{(1-\theta_2)(1-\theta)(p-\Delta V-c_2)}$ on $[V_1 - \bar{u}, V_1)$ and $F_2(p) = 1 - \left[\frac{\bar{\Pi}_1 - \theta_1(p+\Delta V - c_1)}{(1-\theta)(p+\Delta V - c_1)}\right] = 1 - \left[\frac{\theta_1(V_2-p)}{(1-\theta)(p+\Delta V-c_1)}\right]$ on $[V_2 - \bar{u}, V_2)$, where $\alpha_2 \to 1$ but where firm 1 refrains from advertising with probability $1 - \alpha_1 = 1 - F_1(V_1) \in (0,1)$.

C2: Relaxing the Single Visit Assumption

Here, we explain how the model can be generalized to allow the shoppers to sequentially visit multiple firms. We focus on duopoly - similar (more lengthy) arguments can also be made for $n > 2$ firms. Suppose the cost of visiting any first firm is $s(1)$ and the cost of visiting any second firm is $s(2)$. The main model implicitly assumes $s(1) = 0$ and $s(2) = \infty$. However, we now use some arguments related to the Diamond paradox (Diamond, 1971) to show that our equilibrium remains under sequential visits for any $s(2) > 0$ provided that i) the costs of any first visit are not too large, $s(1) \in [0, u^m)$, and ii) shoppers can only purchase from a single firm. The latter 'one-stop shopping' assumption is frequently assumed in consumer search models and the wider literature on price discrimination.

First, suppose $s(1) \in [0, u^m)$ but maintain $s(2) = \infty$. Beyond $s(1) = 0$, this now permits cases where $s(1) \in (0, u^m)$ provided $u^m > 0$ as consistent with downward-sloping demand and linear prices. In this case, shoppers will still be willing to make a first visit and the equilibrium will remain unchanged.

Second, suppose $s(1) \in [0, u^m)$ but allow for any $s(2) > 0$ subject to a persistent 'one-stop shopping' assumption such that shoppers cannot buy from more than one firm. By assumption, the behavior of the non-shoppers will remain unchanged. Therefore, to demonstrate that our equilibrium remains robust, we need to show that shoppers will endogenously refrain from making a second visit. Initially suppose that the firms keep playing their original equilibrium strategies and that a given shopper receives $h \in \{0, 1, 2\}$ adverts. Given $s(2) > 0$ and one-stop shopping, the gains from any second visit will always be strictly negative for all h. In particular, if $h = 0$, then any second visit would be sub-optimal as both firms will offer u^m. Alternatively, if $h \geq 1$, then a shopper will first visit the firm with the highest advertised utility, $u^* > u^m$, and any second visit will be be sub-optimal as it will necessarily offer $u < u^*$. Now suppose that the firms can deviate from their original equilibrium strategies. To see that the logic still holds, note that only the behavior of any non-advertising firms is relevant and that such firms are unable to influence any second visit decisions due to their inability to communicate or commit to any $u < u^m$. Hence, firms' advertising and utility incentives remain unchanged and the original equilibrium still applies.

References

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