Appendix (For Online Publication)

Nonrivalry and the Economics of Data

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A Proposition 1 (the Optimal Allocation)

The social planner problem is :

$$\max_{\{L_{pt},x_t,\tilde{x}_t\}} \int_0^\infty e^{-\tilde{\rho}t} L_0 \left(\log c_t - \frac{\kappa}{2}\frac{1}{N_t}x_t^2 - \frac{\tilde{\kappa}}{2}\tilde{x}_t^2\right)\, dt, \quad \tilde{\rho} \equiv \rho - g_L$$

s.t.

$$c_t = Y_t/L_t$$

$$Y_t = N_t^{\frac{1}{\sigma-1}} \left(\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}$$

$$\dot{N}_t = \frac{1}{\chi}(L_t - L_{pt})$$

$$L_t = L_0 e^{g_L t}$$

Next, define Hamiltonian with state variable N_t, control variables $\{L_{pt}, x_t, \tilde{x}_t\}$ and co-state variable μ_t:

$$H(L_{pt}, x_t, \tilde{x}_t, N_t, \mu_t) = \log Y_t/L_t - \frac{\kappa}{2}\frac{1}{N_t}x_t^2 - \frac{\tilde{\kappa}}{2}\tilde{x}_t^2 + \mu_t \frac{1}{\chi}(L_t - L_{pt})$$

The FOC are:

$$\begin{cases} \dfrac{\partial H}{\partial L_{pt}} = 0 \\ \dfrac{\partial H}{\partial x_t} = 0 \\ \dfrac{\partial H}{\partial \tilde{x}_t} = 0 \\ \tilde{\rho} = \dfrac{\partial H/\partial N_t}{\mu_t} + \dfrac{\dot{\mu}_t}{\mu_t} \end{cases}$$

Start with $\frac{\partial H}{\partial L_{pt}} = 0$, which implies

$$\frac{1}{Y_t}\frac{\partial Y_t}{\partial L_{pt}} = \frac{\mu_t}{\chi}$$

and therefore

$$L_{pt} = \frac{\chi}{1-\eta} \cdot \frac{1}{\mu_t}.$$

Next, consider $\frac{\partial H}{\partial x_t} = 0$:

$$\frac{1}{Y_t}\frac{\partial Y_t}{\partial x_t} = \frac{\kappa}{N_t}x_t.$$

Computing the marginal product and substituting gives

$$x_t = \frac{\alpha}{\kappa}\frac{\eta}{1-\eta}\frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t}.$$

The next FOC is similar, $\frac{\partial H}{\partial \tilde{x}_t} = 0$:

$$\frac{1}{Y_t}\frac{\partial Y_t}{\partial \tilde{x}_t} = \tilde{\kappa}\tilde{x}_t.$$

Computing the marginal product and substituting gives

$$\tilde{x}_t = \frac{1-\alpha}{\tilde{\kappa}}\frac{\eta}{1-\eta}\frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t}.$$

Finally, the last FOC is the arbitrage-like equation, $\tilde{\rho} = \frac{\partial H/\partial N_t}{\mu_t} + \frac{\dot{\mu}_t}{\mu_t}$. The derivative of the Hamiltonian $\frac{\partial H}{\partial N_t}$ is:

$$\begin{aligned}\frac{\partial H}{\partial N_t} &= \frac{1}{Y_t}\left(\frac{1}{\sigma-1}\frac{Y_t}{N_t} - \frac{\alpha x_t}{N_t^2}\frac{\eta}{1-\eta}\frac{Y_t}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t}\right)\\ &= \frac{1}{N_t}\left(\frac{1}{\sigma-1} - \frac{\alpha x_t}{N_t}\frac{\eta}{1-\eta}\frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t}\right).\end{aligned}$$

Substituting this into the FOC and rearranging to solve for μ_t gives

$$\begin{aligned}\mu_t &= \frac{\partial H/\partial N_t}{\tilde{\rho} - \dot{\mu}_t/\mu_t}\\ &= \frac{\frac{1}{N_t}\left(\frac{1}{\sigma-1} - \frac{\alpha x_t}{N_t}\frac{\eta}{1-\eta}\frac{1}{\frac{\alpha x_t}{N_t}+(1-\alpha)\tilde{x}_t}\right)}{\tilde{\rho} - \dot{\mu}_t/\mu_t}.\end{aligned}$$

A.1 Solving the Optimal Allocation

The 4 FOCs are:

$$L_{pt} = \frac{\chi}{1-\eta}\cdot\frac{1}{\mu_t} \tag{A}$$

$$x_t = \frac{\alpha}{\kappa} \frac{\eta}{1-\eta} \frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t} \tag{B}$$

$$\tilde{x}_t = \frac{1-\alpha}{\tilde{\kappa}} \frac{\eta}{1-\eta} \frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t} \tag{C}$$

$$\mu_t = \frac{\frac{1}{N_t}\left(\frac{1}{\sigma-1} - \frac{\alpha x_t}{N_t}\frac{\eta}{1-\eta}\frac{1}{\frac{\alpha x_t}{N_t}+(1-\alpha)\tilde{x}_t}\right)}{\tilde{\rho} - \dot{\mu}_t/\mu_t} \tag{D}$$

Solving for $\tilde{x}_t$ and x_t (eqs. 24 and 25).

Divide (B) by (C):

$$x_t = \frac{\alpha}{1-\alpha}\left(\frac{\tilde{\kappa}}{\kappa}\right)\tilde{x}_t$$

Replace in (C):

$$\begin{aligned}\tilde{x}_t &= \frac{1-\alpha}{\tilde{\kappa}} \frac{\eta}{1-\eta}\left(\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t\right)^{-1} \\ &= \frac{1-\alpha}{\tilde{\kappa}} \frac{\eta}{1-\eta}\left(\frac{\alpha\frac{\alpha}{(1-\alpha)}\left(\frac{\tilde{\kappa}}{\kappa}\right)\tilde{x}_t}{N_t} + (1-\alpha)\tilde{x}_t\right)^{-1}\end{aligned}$$

Solve for $\tilde{x}_t$:

$$\tilde{x}_t^2 = \left(\frac{\eta}{1-\eta}\frac{1}{\tilde{\kappa}}\right)\left[\frac{(1-\alpha)}{\frac{\alpha\frac{\alpha}{(1-\alpha)}\left(\frac{\tilde{\kappa}}{\kappa}\right)}{N_t} + (1-\alpha)}\right]$$

and $\lim_{N_t\to\infty}\left[\frac{(1-\alpha)}{\frac{\alpha\frac{\alpha}{(1-\alpha)}\left(\frac{\tilde{\kappa}}{\kappa}\right)}{N_t}+(1-\alpha)}\right] = 1$. Hence the optimal values for $\tilde{x}_t$ and x_t as N_t grows large are:

$$\tilde{x}_t^{sp} = \left(\frac{\eta}{1-\eta}\frac{1}{\tilde{\kappa}}\right)^{\frac{1}{2}} \tag{24}$$

$$x_t^{sp} = \frac{\alpha}{(1-\alpha)}\left(\frac{\tilde{\kappa}}{\kappa}\right)\left(\frac{\eta}{1-\eta}\frac{1}{\tilde{\kappa}}\right)^{\frac{1}{2}}. \tag{25}$$

Solving for L_{it}^{sp} (eq. 26).

From (A), along a balanced growth path, $g_\mu = g_{L_p} = g_L$. Substituting this into (D) along a balanced growth path gives

$$\mu_t^{sp} N_t^{sp} \to \frac{1}{\rho}\frac{1}{\sigma-1}$$

since the last term in the numerator of (D) goes to zero as N_t gets large.

Using this fact and combining (A) and (D) gives

$$L_i^{sp} = \left(\frac{L_{pt}}{N_t}\right)^{sp} = \frac{\rho\chi(\sigma-1)}{1-\eta} =: \nu_{sp} \tag{26}$$

Solving for N_t^{sp} (eq. 27).

From the definition of the innovation process, $\frac{\dot{N}_t}{N_t} = \frac{1}{\chi}(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t})$ and therefore

$$\frac{L_t}{N_t} = \frac{\dot{N}_t}{N_t}\chi + \frac{L_{pt}}{N_t}$$

Along a balanced growth path, $\frac{\dot{N}_t}{N_t} = g_N = g_{Lp}$ and $g_{L_p} = g_L$ therefore $\frac{\dot{N}_t}{N_t} = g_L$ and we get:

$$\frac{L_t}{N_t} = g_L\chi + \frac{L_{pt}}{N_t}$$

From above we know that when N_t grows large, then $\frac{L_{pt}}{N_t} = \nu_{sp}$ and therefore

$$N_t^{sp} = \frac{L_t}{\chi g_L + \nu_{sp}} =: \psi_{sp} L_t \tag{27}$$

Solving for L_{pt}^{sp} (eq. 28).

Use $L_{pt} = \frac{\rho\chi(\sigma-1)}{1-\eta}N_t$ and equation (27) directly above to get:

$$\begin{aligned} L_{pt}^{sp} &= \frac{\rho\chi\left(\sigma-1\right)}{1-\eta}N_t \\ &= \frac{\rho\chi\left(\sigma-1\right)}{1-\eta}\psi_{sp}L_t \\ &= \nu_{sp}\psi_{sp}L_t \end{aligned} \tag{28}$$

Solving for Y_t^{sp} (eq. 29).

Aggregate production is given by $Y_t = N_t^{\frac{1}{\sigma-1}}\left(\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t\right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}$. Use the optimal expression for N_t^{sp}, L_{pt}^{sp}, $\tilde{x}_{sp}$ and x_{sp} to get:

$$\begin{aligned}
Y_t &= N_t^{\frac{1}{\sigma-1}} \left(\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t \right)^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}. \\
&= (\psi_{sp} L_t)^{\frac{1}{\sigma-1}} \left(\frac{\alpha}{N_t} \frac{\alpha}{1-\alpha} \left(\frac{\tilde{\kappa}}{\kappa} \right) \tilde{x}_{sp} + (1-\alpha)\tilde{x}_{sp} \right)^{\frac{\eta}{1-\eta}} (\nu_{sp}\psi_{sp} L_t)^{\frac{1}{1-\eta}} \\
&= [v_{sp}]^{\frac{1}{1-\eta}} \left(\frac{\alpha}{N_t} \frac{\alpha}{1-\alpha} \left(\frac{\tilde{\kappa}}{\kappa} \right) \tilde{x}_{sp} + (1-\alpha)\tilde{x}_{sp} \right)^{\frac{\eta}{1-\eta}} (\psi_{sp} L_t)^{\frac{1}{1-\eta}+\frac{1}{\sigma-1}}
\end{aligned}$$

Because $\frac{\alpha}{N_t}\frac{\alpha}{1-\alpha}\left(\frac{\tilde{\kappa}}{\kappa}\right)\tilde{x}_{sp} \to 0$ as N_t grows, then:

$$Y_t = \left[v_{sp} (1-\alpha)^\eta \tilde{x}_{sp}^\eta \right]^{\frac{1}{1-\eta}} (\psi_{sp} L_t)^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}} \tag{29}$$

Solving for c_t^{sp} (eq. 30).

By definition, consumption per capita is:

$$\begin{aligned}
c_t &= \frac{Y_t}{L_t} \\
&= \frac{\left[v_{sp} (1-\alpha)^\eta \tilde{x}_{sp}^\eta \right]^{\frac{1}{1-\eta}} (\psi_{sp} L_t)^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}}}{L_t} \\
&= \left[v_{sp} (1-\alpha)^\eta \tilde{x}_{sp}^\eta \right]^{\frac{1}{1-\eta}} (\psi_{sp})^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}} L_t^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}}
\end{aligned} \tag{30}$$

Solving for g_c^{sp} (eq. 31).

Using the definition of consumption per capita growth and equation 30 yields:

$$\begin{aligned}
g_c^{sp} &= g_{Y/L}^{sp} \\
&= \left(\frac{1}{\sigma-1} + \frac{\eta}{1-\eta} \right) g_L
\end{aligned} \tag{31}$$

Solving for D_{it}^{sp} (eq. 32).

From equation 19, $Y_t = N_t^{\frac{1}{\sigma-1}} D_{it}^\eta L_{pt}$. Plugging the previous results yields:

$$\begin{aligned}
D_{it}^{sp} &= \left(\frac{Y_t}{N_t^{\frac{1}{\sigma-1}} L_{pt}} \right)^{\frac{1}{\eta}} \\
&= \left(\frac{\left[v_{sp} (1-\alpha)^\eta \tilde{x}_{sp}^\eta \right]^{\frac{1}{1-\eta}} (\psi_{sp} L_t)^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}}}{(\psi_{sp} L_t)^{\frac{1}{\sigma-1}} \nu_{sp}\psi_{sp} L_t} \right)^{\frac{1}{\eta}}
\end{aligned}$$

$$= \left([(1-\alpha)\tilde{x}_{sp}]^{\frac{\eta}{1-\eta}} (v_{sp}\psi_{sp}L_t)^{\frac{\eta}{1-\eta}} \right)^{\frac{1}{\eta}}$$
$$= [(1-\alpha)\tilde{x}_{sp}v_{sp}\psi_{sp}L_t]^{\frac{1}{1-\eta}} \tag{32}$$

Solving for D_t^{sp} (eq. 33).

By definition:

$$\begin{aligned} D_t^{sp} &= N_t^{sp} D_{it}^{sp} \\ &= (\psi_{sp}L_t)\left([(1-\alpha)\tilde{x}_{sp}v_{sp}\psi_{sp}L_t]^{\frac{1}{1-\eta}} \right) \\ &= [(1-\alpha)\tilde{x}_{sp}v_{sp}]^{\frac{1}{1-\eta}} (\psi_{sp}L_t)^{\frac{1}{1-\eta}+1} \end{aligned} \tag{33}$$

Solving for Y_{it}^{sp} (eq. 34).

Using equation 18 and the results for L_{pt} and D_{it}:

$$\begin{aligned} Y_{it}^{sp} &= D_{it}^{\eta} \frac{L_{pt}}{N_t} \\ &= [(1-\alpha)\tilde{x}_{sp}v_{sp}\psi_{sp}L_t]^{\frac{\eta}{1-\eta}} (\nu_{sp}) \\ &= \left[(1-\alpha)^{\eta} \tilde{x_{sp}^{\eta}} v_{sp} \right]^{\frac{1}{1-\eta}} (\psi_{sp}L_t)^{\frac{\eta}{1-\eta}} \end{aligned} \tag{34}$$

Solving for U_0^{sp} (eq. 35).

By definition $U_0 = \int_0^\infty e^{-(\rho-g_L)t} L_0 u(c_t, x_t, \tilde{x}_t)dt$. Then:

$$\begin{aligned} U_0 &= \int_0^\infty e^{-(\rho-g_L)t} L_0 u(c_t, x_t, \tilde{x}_t)dt \\ &= L_0 \int_0^\infty e^{-(\tilde{\rho})t} \Big[\log(\left[v_{sp}(1-\alpha)^{\eta}\tilde{x}_{sp}^{\eta} \right]^{\frac{1}{1-\eta}} (\psi_{sp})^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}} L_t^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}}) ... \\ &... - \frac{\kappa}{2N_t} x_{sp}^2 - \frac{\tilde{\kappa}}{2}\tilde{x}_{sp}^2 \Big] \end{aligned}$$

Next, assume $\frac{\kappa}{2N_t}x_{sp}^2 \simeq 0$ and use $c_0 = [v_{sp}(1-\alpha)^{\eta}\tilde{x}_{sp}^{\eta}]^{\frac{1}{1-\eta}} (\psi_{sp})^{\frac{1}{\sigma-1}+\frac{1}{1-\eta}} L_0^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}}$. Then:

$$\begin{aligned} U_0 &= L_0 \int_0^\infty e^{-\tilde{\rho}t} \left[\log c_0 + \left(\frac{1}{\sigma-1} + \frac{\eta}{1-\eta} \right) g_L t \right] dt + ... \\ &- L_0 \frac{\tilde{\kappa}}{2}\tilde{x}_{sp}^2 \int_0^\infty e^{-\tilde{\rho}t} dt \\ &= L_0 \frac{1}{\tilde{\rho}} \left(\log c_0 - \frac{\tilde{\kappa}}{2}\tilde{x}_{sp}^2 \right) + L_0 \left(\frac{1}{\sigma-1} + \frac{\eta}{1-\eta} \right) g_L \int_0^\infty e^{-\tilde{\rho}t} t dt \end{aligned}$$

$$
\begin{aligned}
&= L_0 \frac{1}{\tilde{\rho}} \left(\log c_0 - \frac{\tilde{\kappa}}{2} \tilde{x}_{sp}^2 \right) + L_0 \left(\frac{1}{\sigma - 1} + \frac{\eta}{1 - \eta} \right) g_L \frac{1}{\tilde{\rho}^2} \\
&= L_0 \frac{1}{\tilde{\rho}} \left(\log c_0 - \frac{\tilde{\kappa}}{2} \tilde{x}_{sp}^2 + \frac{g_c}{\tilde{\rho}} \right) \qquad (35)
\end{aligned}
$$

B Competitive Equilibrium When Firms Own Data

B.1 Household Problem

The problem is defined by:

$$
\begin{aligned}
U_0 = &\max_{\{c_{it}\}} \int_0^\infty e^{-(\tilde{\rho})} L_0 u(c_t, x_{it}, \tilde{x}_{it}) dt \\
s.t.\ c_t &= \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}} \\
\dot{a}_t &= (r_t - g_L) a_t + w_t - \int_0^{N_t} p_{it} c_{it} di
\end{aligned}
$$

Define the current value Hamiltonian with state variable a_t, control variable c_{it} and co-state variable μ_t:

$$
H(a_t, c_{it}, \mu_t) = u(\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}, x_{it}, \tilde{x}_{it}) + \mu_t \left[(r_t - g_L) a_t + w_t - \int_0^{N_t} p_{it} c_{it} di \right]
$$

The FOCs are:

$$
\begin{cases}
\dfrac{\partial H}{\partial c_{it}} = 0 \\
\dfrac{\partial H}{\partial a_t} = \tilde{\rho} \mu_t - \dot{\mu}_t
\end{cases}
$$

First, start with $\frac{\partial H}{\partial c_{it}} = 0$:

$$
\frac{1}{\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}} \frac{\sigma}{\sigma - 1} \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{1}{\sigma-1}} \frac{\sigma - 1}{\sigma} c_{it}^{\frac{-1}{\sigma}} - \mu_t p_{it} = 0
$$

$$
\Rightarrow c_t^{\frac{1-\sigma}{\sigma}} c_{it}^{\frac{-1}{\sigma}} - \mu_t p_{it} = 0
$$

$$\Rightarrow \; c_t^{\sigma-1} c_{it} = (\mu_t p_{it})^{-\sigma}$$

$$\Rightarrow \; \left(c_t^{\sigma-1} c_{it}\right)^{\frac{\sigma-1}{\sigma}} = (\mu_t p_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_{it}^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} (p_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} \int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_t^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{\sigma-1}{\sigma} + \frac{(1-\sigma)^2}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; \mu_t = \frac{1}{c_t \left(\int_0^{N_t} p_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}}$$

Next, use the fact that the price of c_t is normalized to 1, i.e., $P_t = \left(\int_0^{N_t} p_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} = 1$. Next, plug the expression for μ_t in the FOC and this gives equation (39):

$$c_{it} = c_t p_{it}^{-\sigma}$$

Next, compute the FOC $\frac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t$:

$$\mu_t (r_t - g_L) = \tilde{\rho}\mu_t - \dot{\mu}_t$$

$$\Rightarrow \; (r_t - g_L) = \tilde{\rho} - \frac{\dot{\mu}_t}{\mu_t}$$

But $\mu_t = c_t^{-1}$ then $\frac{\dot{\mu}_t}{\mu_t} = -g_c$. Thus:

$$\Rightarrow \; (r_t - g_L) = \tilde{\rho} + g_c$$

B.2 Firm Problem

The firm problem is:

$$
\begin{aligned}
r_t V_{it} = & \max_{\{L_{it}, D_{bit}, x_{it}, \tilde{x}_{it}\}} \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} - p_{bt} D_{bit} + p_{sit} \tilde{x}_{it} Y_{it} - \delta(\tilde{x_{it}}) V_{it} + \dot{V}_{it} \\
& s.t.\ Y_{it} = D_{it}^{\eta} L_{it} \\
& D_{it} = \alpha x_{it} Y_{it} + (1-\alpha) D_{bit} \\
& p_{sit} = \lambda_{DI} N_t^{-\frac{1}{\epsilon}} \left(\frac{B_t}{\tilde{x}_{it} Y_{it}}\right)^{\frac{1}{\epsilon}} \\
& x_{it} \in [0;1] \\
& \tilde{x}_{it} \in [0;1]
\end{aligned}
$$

taking as given λ_{DI}, B_t, N_t, p_{bt} and Y_t. To solve this problem, write the Lagrangean:

$$
\begin{aligned}
\mathbb{L} = (Y_t)^{\frac{1}{\sigma}} Y_{it}^{1-\frac{1}{\sigma}} - w_t L_{it} - p_{bt} D_{bit} + p_{sit} \tilde{x}_{it} Y_{it} - \delta(\tilde{x_{it}}) V_{it} + \dot{V}_{it} \ldots \\
+ \mu_{x0}(x_{it}) + \mu_{\tilde{x}0}(\tilde{x}_{it}) + \mu_{\tilde{x}1}(1-\tilde{x}_{it}) + \mu_{x1}(1-x_{it})
\end{aligned}
$$

Simplify using the constraints:

$$
\begin{aligned}
\mathbb{L} = (Y_t)^{\frac{1}{\sigma}} Y_{it}^{1-\frac{1}{\sigma}} - w_t L_{it} - p_{bt} D_{bit} + \lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (\tilde{x}_{it} Y_{it})^{1-\frac{1}{\epsilon}} - \delta(\tilde{x_{it}}) V_{it} + \dot{V}_{it} \ldots \\
+ \mu_{x0}(x_{it}) + \mu_{\tilde{x}0}(\tilde{x}_{it}) + \mu_{\tilde{x}1}(1-\tilde{x}_{it}) + \mu_{x1}(1-x_{it})
\end{aligned}
$$

Now take the FOCs.

1) Start with the FOC w.r.t. to L_{it}:

$$
\begin{aligned}
\frac{\partial \mathbb{L}}{\partial L_{it}} &= 0 \\
\Leftrightarrow\ (1-\frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} \frac{\partial Y_{it}}{\partial L_{it}} - w_t + (1-\frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (\tilde{x}_{it})^{1-\frac{1}{\epsilon}} \frac{\partial Y_{it}}{\partial L_{it}} &= 0 \\
\frac{\partial Y_{it}}{\partial L_{it}} \left[(1-\frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} \right] &= w_t
\end{aligned}
$$

And using $Y_{it} = D_{it}^{\eta} L_{it}$ and assuming D_{it} depends on L_{it}, then by implicit derivation:

$$\frac{\partial Y_{it}}{\partial L_{it}} = \eta D_{it}^{\eta-1} L_{it} \frac{\partial D_{it}}{\partial L_{it}} + D_{it}^{\eta}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \eta \frac{Y_{it}}{D_{it}} \frac{\partial D_{it}}{\partial L_{it}} + \frac{Y_{it}}{L_{it}}$$

Next, compute $\frac{\partial D_{it}}{\partial L_{it}}$ using $D_{it} = \alpha x_{it} Y_{it} + (1-\alpha) D_{bit}$:

$$\frac{\partial D_{it}}{\partial L_{it}} = \alpha x_{it} \frac{\partial Y_{it}}{\partial L_{it}}$$

Substituting above:

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \eta \frac{Y_{it}}{D_{it}} \alpha x_{it} \frac{\partial Y_{it}}{\partial L_{it}} + \frac{Y_{it}}{L_{it}}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \frac{\frac{Y_{it}}{L_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}}$$

The FOC for L_{it} is then:

$$\frac{\frac{Y_{it}}{L_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}} \left[(1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} + (1 - \frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} \right] = w_t$$

2) Compute the FOC w.r.t. to D_{bit}:

$$\frac{\partial \mathbb{L}}{\partial D_{bit}} = 0$$

$$\Leftrightarrow \left(1 - \frac{1}{\sigma}\right) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} \frac{\partial Y_{it}}{\partial D_{bit}} + \left(1 - \frac{1}{\epsilon}\right) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (\tilde{x}_{it})^{1-\frac{1}{\epsilon}} \frac{\partial Y_{it}}{\partial D_{bit}} - p_{bt} = 0$$

$$\frac{\partial Y_{it}}{\partial D_{bit}} \left[\left(1 - \frac{1}{\sigma}\right) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} + \left(1 - \frac{1}{\epsilon}\right) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} \right] = p_{bt}$$

And using $Y_{it} = D_{it}^{\eta} L_{it}$ we have:

$$\frac{\partial Y_{it}}{\partial D_{bit}} = \eta \frac{Y_{it}}{D_{it}} \frac{\partial D_{it}}{\partial D_{bit}}$$

and using $D_{it} = \alpha x_{it} Y_{it} + (1-\alpha) D_{bit}$:

$$\frac{\partial D_{it}}{\partial D_{bit}} = \alpha x_{it} \frac{\partial Y_{it}}{\partial D_{bit}} + (1-\alpha)$$

Substituting above:

$$\frac{\partial Y_{it}}{\partial D_{bit}} = \eta \frac{Y_{it}}{D_{it}} \left(\alpha x_{it} \frac{\partial Y_{it}}{\partial D_{bit}} + (1-\alpha) \right)$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial D_{bit}} = \frac{(1-\alpha)\eta \frac{Y_{it}}{D_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}}$$

Then the FOC for D_{bit} is

$$\frac{(1-\alpha)\eta \frac{Y_{it}}{D_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}} \left[(1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} + (1 - \frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} \right] = p_{bt}$$

3) Compute the FOC w.r.t. to x_{it}:

$$\frac{\partial \mathbb{L}}{\partial x_{it}} = 0$$

$$\Leftrightarrow (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} \frac{\partial Y_{it}}{\partial x_{it}} + (1 - \frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (\tilde{x}_{it})^{1-\frac{1}{\epsilon}} \frac{\partial Y_{it}}{\partial x_{it}} + \mu_{x0} - \mu_{x1} = 0$$

$$\frac{\partial Y_{it}}{\partial x_{it}} \left[(1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} + (1 - \frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} \right] = -\mu_{x0} + \mu_{x1}$$

Now to compute $\frac{\partial Y_{it}}{\partial x_{it}}$ use $Y_{it} = D_{it}^{\eta} L_{it}$:

$$\frac{\partial Y_{it}}{\partial x_{it}} = \eta \frac{Y_{it}}{D_{it}} \frac{\partial D_{it}}{\partial x_{it}}$$

and using $D_{it} = \alpha x_{it} Y_{it} + (1-\alpha) D_{bit}$ and implicit derivation:

$$\frac{\partial D_{it}}{\partial x_{it}} = \alpha x_{it} \frac{\partial Y_{it}}{\partial x_{it}} + Y_{it} \alpha$$

Thus:

$$\Rightarrow \frac{\partial Y_{it}}{\partial x_{it}} = \eta \frac{Y_{it}}{D_{it}} \left[\alpha x_{it} \frac{\partial Y_{it}}{\partial x_{it}} + Y_{it}\alpha \right]$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial x_{it}} = \frac{\eta \frac{Y_{it}}{D_{it}} Y_{it}\alpha}{1 - \alpha x_{it}\eta \frac{Y_{it}}{D_{it}}}$$

Then the FOC w.r.t. to x_{it} is:

$$\frac{\eta \frac{Y_{it}}{D_{it}} Y_{it}\alpha}{1 - \alpha x_{it}\eta \frac{Y_{it}}{D_{it}}} \left[(1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} + (1 - \frac{1}{\epsilon}) Y_{it}^{-\frac{1}{\epsilon}} \lambda_{DI} N_t^{-\frac{1}{\epsilon}} B_t^{\frac{1}{\epsilon}} \tilde{x}_{it}^{1-\frac{1}{\epsilon}} \right] = -\mu_{x0} + \mu_{x1}$$

Now, note that the LHS is > 0, then:

$$\mu_{x1} > \mu_{x0} \geq 0$$

$$\Rightarrow \mu_{x1} > 0$$

$$\Rightarrow x_{it} = 1 \tag{A.1}$$

4) Compute the FOC w.r.t. to $\tilde{x}_{it}$:

$$\frac{\partial \mathbb{L}}{\partial \tilde{x}_{it}} = 0$$

$$\Leftrightarrow \lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (Y_{it})^{1-\frac{1}{\epsilon}} \tilde{x}_{it}^{-\frac{1}{\epsilon}} \left(1 - \frac{1}{\epsilon}\right) - \delta'(\tilde{x_{it}}) V_{it} + \mu_{\tilde{x}0} - \mu_{\tilde{x}1} = 0$$

$$\lambda_{DI} N_t^{-\frac{1}{\epsilon}} (B_t)^{\frac{1}{\epsilon}} (Y_{it})^{1-\frac{1}{\epsilon}} \tilde{x}_{it}^{-\frac{1}{\epsilon}} \left(1 - \frac{1}{\epsilon}\right) - \delta'(\tilde{x_{it}}) V_{it} = -\mu_{\tilde{x}0} + \mu_{\tilde{x}1}$$

Or also:

$$p_{sit} Y_{it} \left(1 - \frac{1}{\epsilon}\right) - \delta'(\tilde{x_{it}}) V_{it} = -\mu_{\tilde{x}0} + \mu_{\tilde{x}1}$$

Finally, the 4 FOCs of the firm problem are:

$$\begin{cases} \frac{\frac{Y_{it}}{L_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=w_t \quad (A) \\ \frac{(1-\alpha)\eta\frac{Y_{it}}{D_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=p_{bt} \quad (B) \\ x_{it}=1 \quad (C) \\ p_{sit}Y_{it}\left(1-\frac{1}{\epsilon}\right)-\delta'(\tilde{x_{it}})V_{it}+\mu_{\tilde{x}0}-\mu_{\tilde{x}1}=0 \quad (D) \end{cases}$$

5) Solution in terms of equilibrium aggregates

Divide (A) by (B) :

$$\left[\frac{\frac{Y_{it}}{L_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}\right]\left[\frac{(1-\alpha)\eta\frac{Y_{it}}{D_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha x_{it}}\right]^{-1}=\frac{w_t}{p_{bt}}$$
$$\Rightarrow \frac{D_{it}}{(1-\alpha)\eta L_{it}}=\frac{w_t}{p_{bt}}$$

Hence:

$$\Rightarrow D_{it}=\frac{w_t}{p_{bt}}(1-\alpha)\eta L_{it} \tag{A.2}$$

Next, substitute in (A) and use (C):

$$\frac{\frac{Y_{it}}{L_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha}\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=w_t$$

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=w_t\frac{L_{it}}{Y_{it}}\left(1-\eta\frac{Y_{it}}{D_{it}}\alpha\right)$$

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=w_t\frac{L_{it}}{Y_{it}}-w_t\eta\frac{L_{it}}{D_{it}}\alpha$$

But from above $D_{it}=\frac{w_t}{p_{bt}}(1-\alpha)\eta L_{it}$, therefore substitute in RHS:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}\right]=w_t\frac{L_{it}}{Y_{it}}-\alpha(\frac{p_{bt}}{1-\alpha})$$

Then:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}+\alpha(\frac{p_{bt}}{1-\alpha})\right]\frac{Y_{it}}{L_{it}}=w_t \tag{A.3}$$

Next, we need to compute $\frac{Y_{it}}{L_{it}}$ using: $Y_{it}=D_{it}^{\eta}L_{it}$:

$$\frac{Y_{it}}{L_{it}}=D_{it}^{\eta}=\left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta}L_{it}^{\eta} \tag{A.4}$$

B.3 Data Intermediary Problem

The problem faced by the data intermediary is :

$$\begin{aligned}&\max_{p_{bt},D_{sit}}\ p_{bt}\int_0^{N_t}D_{bit}di-\int_0^{N_t}p_{sit}D_{sit}di\\ &s.t. D_{bit}\leq B_t=\left(N_t^{-\frac{1}{\epsilon}}\int_0^{N_t}(D_{sit})^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}\\ &p_{bt}\leq p_{bt}^{*}\end{aligned}$$

where p_{sit}, i.e. the purchase price of data is taken as given.

B.3.1 The downward-sloping demand curve: data intermediary cost minimization

To compute the demand curve of the data intermediary, we solve the following cost minimization problem:

$$\begin{aligned}&\min_{D_{sit}}\int_0^{N_t}p_{sit}D_{sit}di\\ &s.t.\ D_{bit}\leq B_t=\left(N_t^{-\frac{1}{\epsilon}}\int_0^{N_t}(D_{sit})^{\frac{\epsilon-1}{\epsilon}}di\right)^{\frac{\epsilon}{\epsilon-1}}\end{aligned}$$

The Lagrangean is given by:

$$\mathbb{L} = \int_0^{N_t} p_{sit} D_{sit} di + \lambda_{DI} \left[D_{bit} - \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} (D_{sit})^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \right]$$

By symmetry:

$$\mathbb{L} = \int_0^{N_t} p_{sit} D_{sit} di + \lambda_{DI} \left[D_{bit} - \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} (D_{sit})^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \right]$$

Taking FOC yields:

$$\frac{\partial \mathbb{L}}{\partial D_{sit}} = 0$$

$$\Leftrightarrow \; p_{sit} - \lambda_{DI} \frac{\epsilon}{\epsilon - 1} \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} (D_{sit})^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}-1} \frac{\epsilon - 1}{\epsilon} D_{sit}{}^{\frac{\epsilon-1}{\epsilon}-1} = 0$$

$$p_{sit} - \lambda_{DI} B_t^{\frac{1}{\epsilon}} D_{sit}{}^{\frac{\epsilon-1}{\epsilon}-1} N_t^{-\frac{1}{\epsilon}} = 0$$

$$\lambda_{DI} \left(\frac{B_t}{D_{sit}} \right)^{\frac{1}{\epsilon}} N_t^{-\frac{1}{\epsilon}} = p_{sit}$$

We get the following demand curve:

$$p_{sit} = \lambda_{DI} N_t^{-\frac{1}{\epsilon}} \left(\frac{B_t}{D_{sit}} \right)^{\frac{1}{\epsilon}}$$

which is equation (44), which is used as a constraint in the firm problem. Next, by symmetry

$$B_t = \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} (D_{sit})^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} = N_t^{\frac{1}{-\epsilon}\frac{\epsilon}{\epsilon-1}+\frac{\epsilon}{\epsilon-1}} D_{sit} = N_t D_{sit} \tag{A.5}$$

Substituting in equation (44)

$$p_{sit} = \lambda_{DI} N_t^{-\frac{1}{\epsilon}} \left(\frac{N_t D_{sit}}{D_{sit}} \right)^{\frac{1}{\epsilon}}$$

$$\Rightarrow p_{sit} = \lambda_{DI}$$

B.3.2 The zero profit condition

The objective function is increasing in p_{bt}. It is clear that $p_{bt} = p_{bt}^*$, which is the price given by the zero profit condition.Use the zero profit condition:

$$\Pi = 0$$
$$p_{bt}\int_0^{N_t} D_{bit}di - \int_0^{N_t} p_{sit}D_{sit}di = 0$$
$$p_{bt}\int_0^{N_t} B_t di - \int_0^{N_t} p_{sit}D_{sit}di = 0$$
$$p_{bt}B_tN_t - p_{sit}D_{sit}N_t = 0$$

Thus we get:

$$p_{bt} = \frac{p_{sit}D_{sit}}{B_t}$$

Next, from the demand curve, we have $\lambda_{DI}\left(\frac{B_t}{D_{sit}}\right)^{\frac{1}{\epsilon}} N_t^{-\frac{1}{\epsilon}} = p_{sit}$ and solving for B_t we get $B_t = \left(\frac{p_{sit}}{\lambda_{DI}}\right)^{\epsilon} N_tD_{sit} = N_tD_{sit}$ where the last equation comes from the fact that $p_{sit} = \lambda_{DI}$. Finally:

$$p_{bt} = \frac{p_{sit}D_{sit}}{N_tD_{sit}}$$

$$\Rightarrow p_{bt} = \frac{p_{sit}}{N_t} \tag{A.6}$$

B.4 Free Entry and the Creation of New Varieties

The free entry condition is given by:

$$\chi w_t = V_{it} + \frac{\int_0^{N_t}\delta(\tilde{x}_{it})V_{it}di}{\dot{N}_t}$$

by symmetry:

$$\chi w_t = V_{it} + \frac{\delta(\tilde{x}_{it})V_{it}}{\frac{\dot{N}_t}{N_t}}$$

$$\chi w_t = V_{it}(1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

B.5 Equilibrium when Firms Own Data

B.5.1 Solve for p_{bt}, Y_{it}, D_{sit}, D_{bit}, w_t as a function of $\tilde{x}_{it}$ and aggregates

Take the FOC w.r.t. to L_i in the firm problem, i.e. A.3:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}}+\alpha(\frac{p_{bt}}{1-\alpha})\right]\left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta}L_{it}^{\eta}=w_t$$

Now, from the date intermediary problem, i.e. A.6, $p_{bt}=\frac{p_{sit}}{N_t}=\lambda_{DI}N_t^{-\frac{1}{\epsilon}-1}\left(\frac{B_t}{\tilde{x}_{it}Y_{it}}\right)^{\frac{1}{\epsilon}}$. Substitute above:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})\tilde{x}_{it}p_{bt}N_t+\alpha(\frac{p_{bt}}{1-\alpha})\right]\left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta}L_{it}^{\eta}=w_t$$

Next, from A.2, $D_{it}=\frac{w_t}{p_{bt}}(1-\alpha)\eta L_{it}$, then

$$p_{bt}=\frac{w_t}{D_{it}}(1-\alpha)\eta L_{it}$$

Use (A.4) to get $\frac{Y_{it}}{D_{it}}=D_{it}^{\eta-1}L_{it}=\left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta-1}L_{it}^{\eta}$. Substitute:

$$\begin{aligned}p_{bt}&=\frac{w_t}{D_{it}}(1-\alpha)\eta L_{it}\\&=\left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta-1}L_{it}^{\eta}w_t(1-\alpha)\eta\left(\frac{Y_{it}}{L_{it}}\right)^{-1}\end{aligned}$$

Next, from equation (13), by symmetry, we know that $L_{it}=\frac{L_{pt}}{N_t}$. Moreover, by symmetry, $c_t=N_t^{\frac{\sigma}{\sigma-1}}c_{it}=N_t^{\frac{\sigma}{\sigma-1}}\frac{Y_{it}}{L_t}$ and therefore $Y_t=c_tL_t=N_t^{\frac{\sigma}{\sigma-1}}\frac{Y_{it}}{L_t}L_t=N_t^{\frac{\sigma}{\sigma-1}}Y_{it}$. Thus $\frac{Y_{it}}{L_{it}}=\left(\frac{Y_t}{N_t^{\frac{\sigma}{\sigma-1}}}\frac{N_t}{L_{pt}}\right)=N_t^{-\frac{1}{\sigma-1}}\frac{Y_t}{L_{pt}}$. Substituting above:

$$
\begin{aligned}
p_{bt} &= \left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta-1} L_{it}^{\eta} w_t(1-\alpha)\eta\left(\frac{Y_{it}}{L_{it}}\right)^{-1} \\
p_{bt} &= \left[\frac{w_t}{p_{bt}}(1-\alpha)\eta\right]^{\eta-1}\left(\frac{L_{pt}}{N_t}\right)^{\eta} w_t(1-\alpha)\eta N_t^{\frac{1}{\sigma-1}}\frac{L_{pt}}{Y_t} \\
p_{bt}^{\eta} &= \left[w_t(1-\alpha)\eta\right]^{\eta} L_{pt}^{\eta+1} N_t^{\frac{1}{\sigma-1}-\eta}\frac{1}{Y_t} \\
p_{bt} &= w_t(1-\alpha)\eta L_{pt} N_t^{\left(\frac{1}{\sigma-1}-\eta\right)\frac{1}{\eta}}\left(\frac{L_{pt}}{Y_t}\right)^{\frac{1}{\eta}} \\
p_{bt} &= (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\left(\frac{1}{\sigma-1}-\eta\right)\frac{1}{\eta}} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}}
\end{aligned}
\tag{A.7}
$$

Now, we need to solve Y_t. For that, from above $\frac{Y_{it}}{L_{it}} = N_t^{-\frac{1}{\sigma-1}}\frac{Y_t}{L_{pt}}$ and $L_{it} = \frac{L_{pt}}{N_t}$. Thus:

$$
Y_t = N_t^{\frac{\sigma}{\sigma-1}} Y_{it} \tag{A.8}
$$

But from the firm's problem we have

$$
Y_{it} = D_{it}^{\eta} L_{it} \tag{A.9}
$$

Therefore, solve for D_{it}, where:

$$
D_{it} = \alpha Y_{it} + (1-\alpha) D_{bit} \tag{A.10}
$$

But from the data intermediary problem $D_{bit} = B_t$ and from A.5, $D_{bit} = B_t = N_t D_{sit}$, and by definition $D_{sit} = \tilde{x}_{it} Y_{it}$. Thus:

$$
D_{bit} = N_t \tilde{x}_{it} Y_{it} \tag{A.11}
$$

Substitute A.11 in A.10and we get:

$$
\begin{aligned}
D_{it} &= \alpha Y_{it} + (1-\alpha) D_{bit} \\
&= \alpha Y_{it} + (1-\alpha) N_t \tilde{x}_{it} Y_{it} \\
&= Y_{it}(\alpha + (1-\alpha) N_t \tilde{x}_{it})
\end{aligned}
\tag{A.12}
$$

Substitute in A.9:

$$\begin{aligned} Y_{it} &= (Y_{it}(\alpha + (1-\alpha)N_t\tilde{x}_{it}))^{\eta}\, L_{it} \\ &= Y_{it}^{\eta}(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\eta} L_{it} \\ &= Y_{it}^{\eta}(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\eta}\left(\frac{L_{pt}}{N_t}\right) \\ \Rightarrow Y_{it} &= (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}} \end{aligned} \tag{A.13}$$

Substitute this expression for Y_{it} in A.8:

$$\begin{aligned} Y_t &= N_t^{\frac{\sigma}{\sigma-1}}(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}} \\ &= N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} \end{aligned} \tag{A.14}$$

Use this in A.7:

$$\begin{aligned} p_{bt} &= (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\left(\frac{1}{\sigma-1}-\eta\right)\frac{1}{\eta}} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}} \qquad \text{(A.15)} \\ &= (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\left(\frac{1}{\sigma-1}-\eta\right)\frac{1}{\eta}} L_{pt}^{\frac{1}{\eta}}\left(N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}\right)^{1-\frac{1}{\eta}} \\ &= (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\left(\frac{1}{\sigma-1}-\eta\right)\frac{1}{\eta}+\left(\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}\right)\left(1-\frac{1}{\eta}\right)} L_{pt}^{\frac{1}{\eta}+\frac{1}{1-\eta}\left(1-\frac{1}{\eta}\right)}(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\left(\frac{\eta}{1-\eta}\right)\left(1-\frac{1}{\eta}\right)} \\ &= (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\sigma-1}} L_{pt}^{0}\,(\alpha + (1-\alpha)\tilde{x_{it}}N_t)^{-1} \end{aligned}$$

Finally,

$$p_{bt} = (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\sigma-1}}\,(\alpha + (1-\alpha)\tilde{x_{it}}N_t)^{-1} \tag{A.16}$$

We will use this in the FOC with respect to L_{it}. Recall from A.3:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + (1-\frac{1}{\epsilon})Y_{it}^{-\frac{1}{\epsilon}}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}B_t^{\frac{1}{\epsilon}}\tilde{x}_{it}^{1-\frac{1}{\epsilon}} + \alpha(\frac{p_{bt}}{1-\alpha})\right]\left(\frac{Y_{it}}{L_{it}}\right) = w_t$$

But from A.6, $p_{bt} = \frac{p_{sit}}{N_t}$ and by the cost minimization problem of the data intermediary, $p_{bt} = \frac{p_{sit}}{N_t} = \frac{1}{N_t}\lambda_{DI}N_t^{-\frac{1}{\epsilon}}\left(\frac{B_t}{\tilde{x}_{it}Y_{it}}\right)^{\frac{1}{\epsilon}}$. Substituting above:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})p_{bt}\tilde{x}_{it}N_t+\alpha(\frac{p_{bt}}{1-\alpha})\right]\left(\frac{Y_{it}}{L_{it}}\right)=w_t$$

but from above $\frac{Y_{it}}{L_{it}}=N_t^{-\frac{1}{\sigma-1}}\frac{Y_t}{L_{pt}}$ and therefore:

$$\left[(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+(1-\frac{1}{\epsilon})p_{bt}\tilde{x}_{it}N_t+\alpha(\frac{p_{bt}}{1-\alpha})\right]N_t^{-\frac{1}{\sigma-1}}\frac{Y_t}{L_{pt}}=w_t$$

Also, using A.8:

$$\Rightarrow (1-\frac{1}{\sigma})N_t^{\frac{1}{\sigma-1}}+p_{bt}\left[(1-\frac{1}{\epsilon})\tilde{x}_{it}N_t+\frac{\alpha}{1-\alpha}\right]=N_t^{\frac{1}{\sigma-1}}\left(\frac{w_tL_{pt}}{Y_t}\right)$$

$$\Rightarrow (1-\frac{1}{\sigma})N_t^{\frac{1}{\sigma-1}}+(1-\alpha)\eta\left(\frac{w_tL_{pt}}{Y_t}\right)N_t^{\frac{1}{\sigma-1}}(\alpha+(1-\alpha)\tilde{x_{it}}N_t)^{-1}\left[(1-\frac{1}{\epsilon})\tilde{x}_{it}N_t+\frac{\alpha}{1-\alpha}\right]=N_t^{\frac{1}{\sigma-1}}\left(\frac{w_tL_{pt}}{Y_t}\right)$$

$$\Rightarrow (1-\frac{1}{\sigma})+(1-\alpha)\eta\left(\frac{w_tL_{pt}}{Y_t}\right)(\alpha+(1-\alpha)\tilde{x_{it}}N_t)^{-1}\left[(1-\frac{1}{\epsilon})\tilde{x}_{it}N_t+\frac{\alpha}{1-\alpha}\right]=\left(\frac{w_tL_{pt}}{Y_t}\right)$$

$$\Rightarrow (1-\frac{1}{\sigma})+(1-\alpha)\eta\left(\frac{w_tL_{pt}}{Y_t}\right)\frac{(1-\frac{1}{\epsilon})\tilde{x}_{it}N_t+\frac{\alpha}{1-\alpha}}{\alpha+(1-\alpha)\tilde{x_{it}}N_t}=\left(\frac{w_tL_{pt}}{Y_t}\right)$$

Next, define $f(N_t)=\frac{(1-\frac{1}{\epsilon})\tilde{x}_{it}N_t+\frac{\alpha}{1-\alpha}}{\alpha+(1-\alpha)\tilde{x_{it}}N_t}$ Thus:

$$\Rightarrow (1-\frac{1}{\sigma})+(1-\alpha)\eta\left(\frac{w_tL_{pt}}{Y_t}\right)f(N_t)=\left(\frac{w_tL_{pt}}{Y_t}\right)$$

$$\Rightarrow \frac{w_tL_{pt}}{Y_t}=\frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))} \tag{A.17}$$

When N_t is large, $\underset{N_t\to\infty}{lim}f(N_t)=\frac{(1-\frac{1}{\epsilon})}{(1-\alpha)}$. Therefore:

$$\Rightarrow \left(\frac{w_tL_{pt}}{Y_t}\right)=\frac{(\sigma-1)}{\sigma(1-\eta\frac{\epsilon}{\epsilon-1})} \tag{A.18}$$

Next, substitute A.14 in A.17 :

$$w_t = \frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}\frac{Y_t}{L_{pt}} \tag{A.19}$$
$$= \frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha+(1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1}$$

So far we have:

$$\begin{cases} \frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha+(1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1} = w_t \\ N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha+(1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} = Y_t \\ (1-\alpha)\eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{(\frac{1}{\sigma-1}-\eta)\frac{1}{\eta}} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}} = p_{bt} \\ p_{bt} = \frac{p_{sit}}{N_t} \\ 1 = x_{it} \\ Y_{it}(\alpha+(1-\alpha)N_t\tilde{x}_{it}) = D_{it} \\ N_t\tilde{x}_{it}Y_{it} = D_{bit} \\ Y_t N_t^{-\frac{\sigma}{\sigma-1}} = Y_{it} \\ N_t D_{sit} = B_t \\ \tilde{x}_{it}Y_{it} = D_{sit} \\ L_{it} = \frac{L_{pt}}{N_t} \end{cases} \tag{A.20}$$

B.5.2 The Value of the Firm and Profits

The firm's problem is:

$$r_t V_{it} = \max_{\{L_{it}, D_{bit}, x_{it}, \tilde{x}_{it}\}} \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} - p_{bt} D_{bit} + p_{sit}\tilde{x}_{it}Y_{it} - \delta(\tilde{x_{it}})V_{it} + \dot{V}_{it}$$

Thus:

$$V_{it} = \frac{\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} - p_{bt} D_{bit} + p_{sit}\tilde{x}_{it}Y_{it}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}}$$

Next, define $\pi_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - wL_{it}$. Then using the expressions above:

$$
\begin{aligned}
\pi_{it} &= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - wL_{it} \\
&= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w\left(\frac{L_{pt}}{N_t}\right) \\
&= \left(N_t^{\frac{1}{\sigma-1}}\right) N_t^{-\frac{\sigma}{\sigma-1}} Y_t - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1}\left(\frac{L_{pt}}{N_t}\right) \\
&= N_t^{-1} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1} \\
&= N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}-1} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}-1} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} \\
&= (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right)
\end{aligned} \tag{A.21}
$$

On the other hand:

$$
\begin{aligned}
-p_{bt}D_{bit} + p_{sit}\tilde{x}_{it}Y_{it} &= -\left(\frac{p_{sit}}{N_t}\right)(N_t\tilde{x}_{it}Y_{it}) + p_{sit}\tilde{x}_{it}Y_{it} \\
&= 0
\end{aligned} \tag{A.22}
$$

Hence, the value of the firm is given by:

$$
\begin{aligned}
V_{it} &= \frac{\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_tL_{it} - p_{bt}D_{bit} + p_{sit}\tilde{x}_{it}Y_{it}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \\
&= \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right) \\
&= \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(1 - \frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}\right) \\
&= \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(\frac{\sigma(1-(1-\alpha)\eta f(N_t)) - (\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}\right) \\
&= \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(\frac{1-\sigma(1-\alpha)\eta f(N_t)}{\sigma(1-(1-\alpha)\eta f(N_t))}\right)
\end{aligned} \tag{A.23}
$$

B.5.3 The Free Entry Condition

$$\chi w_t = V_{it}(1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

Using A.19 and A.23:

$$\chi \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)} N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}} (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}-1} = \ldots$$
$$\ldots \frac{(\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right)(1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

$$\Rightarrow \chi \frac{(1-\frac{1}{\sigma})N_t}{1-(1-\alpha)\eta f(N_t)} L_{pt}^{-1} = \frac{1}{r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right) \quad (1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

$$\left(\frac{N_t}{L_{pt}}\right) = \frac{(1-(1-\alpha)\eta f(N_t))\,\sigma}{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma-1)} \left(1 - \frac{\sigma-1}{\sigma(1-(1-\alpha)\eta f(N_t))}\right) \quad (1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

$$\Rightarrow \left(\frac{N_t}{L_{pt}}\right) = \frac{\sigma(1-(1-\alpha)\eta f(N_t)) - \sigma + 1}{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma-1)} \quad (1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

$$\Rightarrow \left(\frac{N_t}{L_{pt}}\right) = \frac{1 - \sigma(1-\alpha)\eta f(N_t)}{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma-1)} \quad (1 + \frac{\delta(\tilde{x}_{it})}{g_L})$$

$$\Rightarrow \left(\frac{L_{pt}}{N_t}\right) = \frac{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma-1)}{1 - \sigma(1-\alpha)\eta f(N_t)} \quad (\frac{g_L}{g_L + \delta(\tilde{x}_{it})}) \tag{A.24}$$

From the evolution of the number of varieties, we have $\dot{N}_t = \frac{1}{\chi}(L_t - L_{pt})$. Thus:

$$\frac{\dot{N}_t}{N_t} = \frac{1}{\chi}(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t})$$

In BGP: $\frac{\dot{N}_t}{N_t} = g_N$ and since $g_N = g_L$, then:

$$\Rightarrow g_L = \frac{1}{\chi}(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t})$$

$$\Rightarrow \frac{L_t}{N_t} = \frac{L_{pt}}{N_t} + \chi g_L$$

Next, using $\frac{\dot{V}_{it}}{V_{it}} = g_\pi$ and substituting A.24:

$$\Rightarrow \frac{L_t}{N_t} = \frac{\chi\left(r + \delta(\tilde{x}_{it}) - g_\pi\right)(\sigma - 1)}{1 - \sigma(1-\alpha)\eta f(N_t)} \left(\frac{g_L}{g_L + \delta(\tilde{x}_{it})}\right) + \chi g_L$$

$$\Rightarrow N_t = \frac{L_t}{\frac{\chi(r+\delta(\tilde{x}_{it})-g_\pi)(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}\left(\frac{g_L}{g_L+\delta(\tilde{x}_{it})}\right) + \chi g_L}$$

Define $\nu(N_t) = \frac{\chi(r+\delta(\tilde{x}_{it})-g_\pi)(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}\left(\frac{g_L}{g_L+\delta(\tilde{x}_{it})}\right)$, so that:

$$N_t = \frac{L_t}{\nu(N_t) + \chi g_L}$$

Moreover, define $\psi(N_t) = \frac{1}{\nu(N_t)+\chi g_L}$ so that:

$$N_t = L_t \psi(N_t) \tag{A.25}$$

B.6 Solution of the Competitive Equilibrium

B.6.1 Data shared with other firms

From above, the FOC for $\tilde{x}_{it}$ is:

$$p_{sit} Y_{it} \left(1 - \frac{1}{\epsilon}\right) - \delta'(\tilde{x_{it}}) V_{it} + \mu_{\tilde{x}0} - \mu_{\tilde{x}1} = 0$$

Assume an interior solution so $\mu_{\tilde{x}0} = \mu_{\tilde{x}1} = 0$.

$$p_{sit} Y_{it} \left(1 - \frac{1}{\epsilon}\right) - \delta'(\tilde{x_{it}}) V_{it} = 0$$

but from above $p_{bt} = \frac{p_{sit}}{N_t}$ and $p_{bt} = (1-\alpha)\eta \left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\sigma-1}} \left(\alpha + (1-\alpha)\tilde{x_{it}} N_t\right)^{-1}$ Therefore:

$$(1-\alpha)\eta \left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{\sigma}{\sigma-1}} \left(\alpha + (1-\alpha)\tilde{x_{it}} N_t\right)^{-1} Y_{it} \frac{(\epsilon - 1)}{\epsilon} - \delta'(\tilde{x_{it}}) V_{it} = 0$$

Substitute $\frac{w_t L_{pt}}{Y_t}$ using A.17:

$$(1-\alpha)\eta\frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}N_t^{\frac{\sigma}{\sigma-1}}\left(\alpha+(1-\alpha)\tilde{x_{it}}N_t\right)^{-1}Y_{it}\frac{(\epsilon-1)}{\epsilon}-\delta'(\tilde{x_{it}})V_{it}=0$$

Substitute Y_{it} using A.13:

$$(1-\alpha)\eta\frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}N_t^{\frac{\sigma}{\sigma-1}}\left(\alpha+(1-\alpha)\tilde{x_{it}}N_t\right)^{-1}\left(\alpha+(1-\alpha)N_t\tilde{x}_{it}\right)^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}\frac{(\epsilon-1)}{\epsilon}= \delta'(\tilde{x_{it}})V_{it}$$

Next, substitute V_{it} using

$$(1-\alpha)\eta\frac{(\sigma-1)}{\sigma(1-(1-\alpha)\eta f(N_t))}N_t^{\frac{\sigma}{\sigma-1}}\left(\alpha+(1-\alpha)\tilde{x_{it}}N_t\right)^{-1}\left(\alpha+(1-\alpha)N_t\tilde{x}_{it}\right)^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}\frac{(\epsilon-1)}{\epsilon}= \delta'(\tilde{x_{it}})\frac{\left(\alpha+(1-\alpha)N_t\tilde{x}_{it}\right)^{\frac{\eta}{1-\eta}}L_{pt}^{\frac{1}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}}{r+\delta(\tilde{x}_{it})-\frac{\dot{V}_{it}}{V_{it}}}\left(\frac{1-\sigma(1-\alpha)\eta f(N_t)}{\sigma(1-(1-\alpha)\eta f(N_t))}\right)$$

Simplify:

$$(1-\alpha)\eta(\sigma-1)N_t(\alpha+(1-\alpha)N_t\tilde{x}_{it})^{-1}\frac{(\epsilon-1)}{\epsilon}=\delta'(\tilde{x_{it}})\frac{1-\sigma(1-\alpha)\eta f(N_t)}{r+\delta(\tilde{x}_{it})-g_\pi}$$

$$\Rightarrow\frac{\delta'(\tilde{x_{it}})}{r+\delta(\tilde{x}_{it})-g_\pi}(\alpha+(1-\alpha)N_t\tilde{x}_{it})=\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta N_t\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

$$\Rightarrow\frac{\delta'(\tilde{x_{it}})}{r+\delta(\tilde{x}_{it})-g_\pi}(\frac{\alpha}{N_t}+(1-\alpha)\tilde{x}_{it})=\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

Next, use $\delta'(\tilde{x_{it}})=\delta_0\tilde{x}_{it}$ and $\delta(\tilde{x}_{it})=\frac{\delta_0}{2}\tilde{x}_{it}^2$ and N_t is large :

$$\Rightarrow\frac{(1-\alpha)\tilde{x}_{it}^2\delta_0}{r+\frac{\delta_0}{2}\tilde{x}_{it}^2-g_\pi}=\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

$$\Rightarrow (1-\alpha)\tilde{x}_{it}^2\delta_0 = \left(r + \frac{\delta_0}{2}\tilde{x}_{it}^2 - g_\pi\right)\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

$$\Rightarrow \tilde{x}_{it}^2\delta_0\left[(1-\alpha) - \frac{(\epsilon-1)}{2\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}\right] = (r-g_\pi)\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}$$

$$\Rightarrow \tilde{x}_{it} = \left(\frac{(r-g_\pi)\frac{(\epsilon-1)}{\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}}{\delta_0(1-\alpha) - \delta_0\frac{(\epsilon-1)}{2\epsilon}(1-\alpha)\eta\frac{(\sigma-1)}{1-\sigma(1-\alpha)\eta f(N_t)}}\right)^{\frac{1}{2}}$$

When N_t is large, $\underset{N_t\to\infty}{lim} f(N_t) = \frac{(1-\frac{1}{\epsilon})}{(1-\alpha)}$.

$$\Rightarrow \tilde{x}_{it} = \left(\frac{(r-g_\pi)\,2\,(\epsilon-1)\,(1-\alpha)\eta\frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}}{2\delta_0(1-\alpha) - \delta_0(\epsilon-1)(1-\alpha)\eta\frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}}\right)^{\frac{1}{2}}$$

$$\Rightarrow \tilde{x}_{it} = \left(\frac{(r-g_\pi)\,2\,(\epsilon-1)\,\eta\frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}}{2\delta_0 - (\epsilon-1)\delta_0\eta\frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}}\right)^{\frac{1}{2}}$$

$$\Rightarrow \tilde{x}_{it} = \left(\frac{2\,(r-g_\pi)\left[(\epsilon-1)\,\eta\frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}\right]}{\delta_0(2 - \left[(\epsilon-1)\eta\frac{(\sigma-1)}{\epsilon-\sigma\eta(\epsilon-1)}\right])}\right)^{\frac{1}{2}}$$

Define

$$\Gamma = (\epsilon-1)\,\eta\frac{(\sigma-1)}{\epsilon-\sigma\eta\,(\epsilon-1)} = \frac{(\sigma-1)}{\frac{\epsilon}{(\epsilon-1)\eta}-\sigma} = \frac{\eta(\sigma-1)}{\frac{\epsilon}{\epsilon-1}-\sigma\eta}. \tag{A.26}$$

Then:

$$\Rightarrow \tilde{x}_{it} = \left(\frac{2\,(r-g_\pi)\,\Gamma}{\delta_0(2-\Gamma)}\right)^{\frac{1}{2}}$$

Since $\rho = r - g_\pi$ then:

$$\Rightarrow \tilde{x}_{it} = \left(\frac{2\rho\Gamma}{\delta_0(2-\Gamma)}\right)^{\frac{1}{2}}. \tag{A.27}$$

B.6.2 Data used by own firm

From A.1 above:

$$x_f = 1$$

B.6.3 Firm Size

Use A.24:

$$L_{it} = \left(\frac{L_{pt}}{N_t}\right) = \frac{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma - 1)}{1 - \sigma(1-\alpha)\eta f(N_t)} \quad \left(\frac{g_L}{g_L + \delta(\tilde{x}_{it})}\right)$$

When N_t is large:

$$\begin{aligned} L_{it} &= \frac{\chi\left(r + \delta(\tilde{x}_{it}) - \frac{\dot{V}_{it}}{V_{it}}\right)(\sigma - 1)}{1 - \sigma\eta(1 - \frac{1}{\epsilon})} \quad \left(\frac{g_L}{g_L + \delta(\tilde{x}_{it})}\right) \\ &= \frac{\chi\left(\rho + \delta(\tilde{x}_{it})\right)(\sigma - 1)}{1 - \sigma\eta(1 - \frac{1}{\epsilon})} \quad \left(\frac{g_L}{g_L + \delta(\tilde{x}_{it})}\right) \\ &= \chi g_L \frac{\rho + \delta(\tilde{x}_{it})}{g_L + \delta(\tilde{x}_{it})} \cdot \frac{\sigma - 1}{1 - \sigma\eta(\frac{\epsilon - 1}{\epsilon})} \end{aligned}$$

And define

$$\nu_f = \chi g_L \frac{\rho + \delta(\tilde{x}_{it})}{g_L + \delta(\tilde{x}_{it})} \cdot \frac{\sigma - 1}{1 - \sigma\eta(\frac{\epsilon - 1}{\epsilon})}$$

These are equations (72) and (75).

B.6.4 Number of Varieties

From A.25, $N_t = L_t \psi(N_t)$ with $\psi(N_t) = \frac{1}{\nu(N_t) + \chi g_L}$ and $\nu(N_t) = \frac{\chi(r + \delta(\tilde{x}_{it}) - g_\pi)(\sigma - 1)}{1 - \sigma(1-\alpha)\eta f(N_t)}\left(\frac{g_L}{g_L + \delta(\tilde{x}_{it})}\right)$. Now, from above, we know that when N_t is large $\nu = \chi g_L \frac{\rho + \delta(\tilde{x}_{it})}{g_L + \delta(\tilde{x}_{it})} \cdot \frac{\sigma - 1}{1 - \sigma\eta(\frac{\epsilon - 1}{\epsilon})}$. Thus:

$$N_t = \frac{1}{\nu_f + \chi g_L} L_t = \psi_f L_t$$

B.6.5 Aggregate output

From A.14, $Y_t = N_t^{\frac{\sigma}{\sigma-1} - \frac{1}{1-\eta}} (\alpha + (1-\alpha) N_t \tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}$. But $(\frac{L_{pt}}{N_t})^f = \nu_f$ Therefore:

$$Y_t^f = (N_t)^{\frac{\sigma}{\sigma-1}} (N_t)^{\frac{\eta}{1-\eta}} \left(\frac{\alpha}{N_t} + (1-\alpha)\tilde{x}_{it}\right)^{\frac{\eta}{1-\eta}} (\nu_f)^{\frac{1}{1-\eta}}$$

When N_t is large:

$$\begin{aligned} Y_t^f &= (N_t)^{\frac{\sigma}{\sigma-1}} (N_t)^{\frac{\eta}{1-\eta}} ((1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} (\nu_f)^{\frac{1}{1-\eta}} \\ &= (\nu_f (1-\alpha)^{\eta} \tilde{x}_{it}^{\eta})^{\frac{1}{1-\eta}} (N_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}} \end{aligned}$$

and from equation (79), $N_t = \psi_f L_t$, thus:

$$Y_t^f = (\nu_f (1-\alpha)^{\eta} \tilde{x}_{it}^{\eta})^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}$$

B.6.6 Consumption per capita

$$c_t^f = \frac{Y_t^f}{L_t} \propto L_t^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}}$$

This is equation (83) in the paper.

B.6.7 Consumption per capita growth

Using equation (83):

$$g_c^f = (\frac{1}{\sigma-1} + \frac{\eta}{1-\eta}) g_L$$

This is equation (85) in the paper.

B.6.8 Firm production

Combining equation (80) $Y_t^f = (\nu_f (1-\alpha)^{\eta} \tilde{x}_{it}^{\eta})^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}$ and A.8:

$$\begin{aligned} Y_{it}^f &= (\nu_f (1-\alpha)^{\eta} \tilde{x}_{it}^{\eta})^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}} N_t^{-\frac{\sigma}{\sigma-1}} \\ &= (\nu_f (1-\alpha)^{\eta} \tilde{x}_{it}^{\eta})^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}} (\psi_f L_t)^{-\frac{\sigma}{\sigma-1}} \\ &= (\nu_f (1-\alpha)^{\eta} \tilde{x}_{it}^{\eta})^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\eta}{1-\eta}} \end{aligned}$$

and this is equation 91 in the paper.

B.6.9 Data Production

Combining equation 91 and A.12:

$$
\begin{aligned}
D^f_{it} &= Y_{it}(\alpha + (1-\alpha)N_t\tilde{x}_{it}) \\
&= (\nu_f(1-\alpha)^\eta \tilde{x}^\eta_{it})^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\eta}{1-\eta}} N_t(\frac{\alpha}{N_t} + (1-\alpha)\tilde{x}_{it}) \\
&= (\nu_f(1-\alpha)^\eta \tilde{x}^\eta_{it})^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\eta}{1-\eta}+1} (1-\alpha)\tilde{x}_{it} \\
&= (\nu_f(1-\alpha)\tilde{x}_{it}\psi_f L_t)^{\frac{1}{1-\eta}}
\end{aligned}
$$

and this is Equation 87 in the paper.

B.6.10 Aggregate data production

By definition $D_t = N_t D_{it}$, hence:

$$
\begin{aligned}
D^f_t &= \psi_f L_t(\nu_f(1-\alpha)\tilde{x}_{it}\psi_f L_t)^{\frac{1}{1-\eta}} \\
&= (\nu_f(1-\alpha)\tilde{x}_{it})^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{1}{1-\eta}+1}
\end{aligned}
$$

and this is exactly equation 89 in the paper.

B.6.11 Labor share

From A.18:

$$
\left(\frac{w_t L_{pt}}{Y_t}\right)^f = \frac{(1-\frac{1}{\sigma})}{1-\eta\frac{\epsilon}{\epsilon-1}} = \frac{\sigma}{\sigma\left(1-\eta\frac{\epsilon}{\epsilon-1}\right)}
$$

which is equation (93).

B.6.12 Profit share

From A.21, $\pi_t = (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \left(1 - \frac{(1-\frac{1}{\sigma})}{1-(1-\alpha)\eta f(N_t)}\right)$ and when N_t is large:

$$
\begin{aligned}
\pi_t &= (\alpha + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \left(\frac{(1-(1-\alpha)\eta f(N_t))\,\sigma - (\sigma-1)}{(1-(1-\alpha)\eta f(N_t))\,\sigma}\right) \\
&= ((1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+\frac{\eta}{1-\eta}} \left(\frac{1-(1-\alpha)\eta f(N_t)\sigma}{(1-(1-\alpha)\eta f(N_t))\,\sigma}\right) \\
&= ((1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+\frac{\eta}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{(1-\eta(\frac{\epsilon-1}{\epsilon}))\,\sigma}\right)
\end{aligned}
$$

Hence:

$$\begin{aligned}
\left(\frac{\pi_t N_t}{Y_t}\right)^f &= \frac{((1-\alpha)\tilde{x}_{it})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+1+\frac{\eta}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{(1-\eta(\frac{\epsilon-1}{\epsilon}))\sigma}\right)}{(\nu_f(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}} \\
&= \frac{L_{pt}^{\frac{1}{1-\eta}} (\psi_f L_t)^{\frac{1}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{(1-\eta(\frac{\epsilon-1}{\epsilon}))\sigma}\right)}{(\nu_f)^{\frac{1}{1-\eta}}} \\
&= \frac{(\nu_f N_t)^{\frac{1}{1-\eta}} (\psi_f L_t)^{-\frac{1}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{(1-\eta(\frac{\epsilon-1}{\epsilon}))\sigma}\right)}{(\nu_f)^{\frac{1}{1-\eta}}} \\
&= (N_t)^{\frac{1}{1-\eta}} (\psi_f L_t)^{-\frac{1}{1-\eta}} \left(\frac{1-\eta\sigma(\frac{\epsilon-1}{\epsilon})}{\left(1-\eta(\frac{\epsilon-1}{\epsilon})\right)\sigma}\right) \\
&= (\psi_f L_t)^{\frac{1}{1-\eta}} (\psi_f L_t)^{-\frac{1}{1-\eta}} \frac{1-\eta\sigma\left(\frac{\epsilon-1}{\epsilon}\right)}{\left(1-\eta(\frac{\epsilon-1}{\epsilon})\right)\sigma} \\
&= \frac{1-\eta\sigma\left(\frac{\epsilon-1}{\epsilon}\right)}{\left(1-\eta(\frac{\epsilon-1}{\epsilon})\right)\sigma}
\end{aligned}$$

which is equation (94).

B.6.13 Data Share

Use $p_{at} = \frac{\alpha}{1-\alpha} p_{bt}$ to value the data the firm owns, using the perfect substitutes argument. Then,

$$\begin{aligned}
p_{at} Y_{it} + p_{bt} D_{bt} &= p_{bt}\left(\frac{\alpha}{1-\alpha} Y_{it} + D_{bt}\right) \\
&= \frac{p_{bt}}{1-\alpha}(\alpha Y_{it} + (1-\alpha) D_{bt}) \\
&= \frac{p_{bt}}{1-\alpha} D_{it} \\
&= \frac{p_{bt}}{1-\alpha}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t] Y_{it} \\
&= \frac{p_{bt}}{1-\alpha}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t] Y_t N_t^{\frac{-\sigma}{\sigma-1}}
\end{aligned}$$

Now recall from (A.14) that $Y_t = N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t]^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}}$. Substituting this into the last equation above gives

$$\begin{aligned} p_{at}Y_{it} + p_{bt}D_{bt} &= \frac{p_{bt}}{1-\alpha}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t] N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t]^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{-\sigma}{\sigma-1}} \\ &= \frac{p_{bt}}{1-\alpha}[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t]^{\frac{1}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{-1}{1-\eta}} \end{aligned}$$

Now recall from (A.15) that $\frac{p_{bt}}{1-\alpha} = \eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\eta(\sigma-1)}-1} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}}$ and substitute this in to get

$$p_{at}Y_{it} + p_{bt}D_{bt} = \eta\left(\frac{w_t L_{pt}}{Y_t}\right) N_t^{\frac{1}{\eta(\sigma-1)}-1} L_{pt}^{\frac{1}{\eta}} Y_t^{1-\frac{1}{\eta}} [\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t]^{\frac{1}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{-1}{1-\eta}}$$

Again use equation (A.14) for Y_t to substitute for $Y_t^{-\frac{1}{\eta}}$ and simplify the exponents to get

$$p_{at}Y_{it} + p_{bt}D_{bt} = \eta\left(\frac{w_t L_{pt}}{Y_t}\right)\frac{Y_t}{N_t}$$

which gives the data share of GDP as

$$\frac{N_t(p_{at}Y_{it} + p_{bt}D_{bt})}{Y_t} = \eta\left(\frac{w_t L_{pt}}{Y_t}\right)$$

Finally, using the result for the labor share in equation (93), we have our result:

$$\frac{N_t(p_{at}Y_{it} + p_{bt}D_{bt})}{Y_t} = \frac{\eta}{1-\eta\frac{\epsilon-1}{\epsilon}} \cdot \frac{\sigma-1}{\sigma}$$

which is equation (101).

B.6.14 Price of a variety

From household problem, $p_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}$, then:

$$\begin{aligned} p_{it}^f &= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} \\ &= \left(\frac{Y_t}{Y_t N_t^{-\frac{\sigma}{\sigma-1}}}\right)^{\frac{1}{\sigma}} \\ &= N_t^{\frac{1}{\sigma-1}} \end{aligned}$$

and from equation (79), $N_t = \psi_f L_t$, thus:

$$p_{it}^f = (\psi_f L_t)^{\frac{1}{\sigma-1}}$$

which is equation (98).

C Competitive Equilibrium When Consumers Own Data

In this section we compute the Consumers Own Data equilibrium, including a production and entry subsidy. This allows us to compute the subsidies that align equilibrium allocations with those chosen by a social planner. The considered policies are a subsidy to firm revenue and a subsidy to entry, and thus do not appear directly in the Household Problem nor the Data Intermediary Problem. Of course, setting the subsidies to zero delivers the equilibrium allocations that arise without government intervention beyond setting property rights. It is straight forward to adapt the Firms Own Data equilibrium similarly.

C.1 Household Problem

The household problem is

$$U_0 = \max_{\{c_{it}, x_{it}, \tilde{x}_{it}\}} \int_0^\infty e^{-\tilde{\rho}} L_0 u(c_t, x_{it}, \tilde{x}_{it}) dt$$
$$s.t.\ c_t = \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}$$
$$\dot{a}_t = (r_t - g_L) a_t + w_t - \int_0^{N_t} p_{it} c_{it} di + \int_0^{N_t} x_{it} p_{st}^a c_{it} di + \int_0^{N_t} \tilde{x}_{it} p_{st}^b c_{it} di$$
$$= (r_t - g_L) a_t + w_t - \int_0^{N_t} q_{it} c_{it} di$$

The Hamiltonian of the problem is

$$H(c_{it}, x_{it}, \tilde{x}_{it}, a_t, \mu_t) = u\left(\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}, x_{it}, \tilde{x}_{it} \right) + \mu_t \left[(r_t - g_L) a_t + w_t - \int_0^{N_t} q_{it} c_{it} di \right]$$

The FOCs are:

$$\begin{cases} \dfrac{\partial H}{\partial c_{it}} = 0 \\ \dfrac{\partial H}{\partial x_{it}} = 0 \\ \dfrac{\partial H}{\partial \tilde{x}_{it}} = 0 \\ \dfrac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t \end{cases}$$

First, start with $\frac{\partial H}{\partial c_{it}} = 0$:

$$\frac{1}{\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}} \frac{\sigma}{\sigma - 1} \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{1}{\sigma-1}} \frac{\sigma - 1}{\sigma} c_{it}^{\frac{-1}{\sigma}} - \mu_t q_{it} = 0$$

$$\Rightarrow\ c_t^{\frac{1-\sigma}{\sigma}} c_{it}^{\frac{-1}{\sigma}} - \mu_t q_{it} = 0$$

$$\Rightarrow\ c_t^{\sigma-1} c_{it} = (\mu_t q_{it})^{-\sigma}$$

$$\Rightarrow\ \left(c_t^{\sigma-1} c_{it} \right)^{\frac{\sigma-1}{\sigma}} = (\mu_t q_{it})^{1-\sigma}$$

$$\Rightarrow\ c_t^{\frac{(1-\sigma)^2}{\sigma}} c_{it}^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} (q_{it})^{1-\sigma}$$

$$\Rightarrow\ c_t^{\frac{(1-\sigma)^2}{\sigma}} \int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di = (\mu_t)^{1-\sigma} \int_0^{N_t} q_{it}^{1-\sigma} di$$

$$\Rightarrow\ c_t^{\frac{(1-\sigma)^2}{\sigma}} c_t^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} q_{it}^{1-\sigma} di$$

$$\Rightarrow\ c_t^{\frac{\sigma-1}{\sigma}+\frac{(1-\sigma)^2}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} q_{it}^{1-\sigma} di$$

$$\Rightarrow\ \mu_t = \frac{1}{c_t \left(\int_0^{N_t} q_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}}$$

Next, define $P_t = \left(\int_0^{N_t} q_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}$. Thus:

$$\Rightarrow\ \mu_t = \frac{1}{c_t P_t} \tag{A.28}$$

Next, plug the expression for μ_t in the FOC:

$$c_t^{\frac{1-\sigma}{\sigma}} c_{it}^{\frac{-1}{\sigma}} - \frac{1}{c_t P_t} q_{it} = 0$$

$$\Rightarrow\ c_{it} = c_t \left(\frac{q_{it}}{P_t}\right)^{-\sigma}$$

Using the normalization $P_t = 1$ yields :

$$c_{it} = c_t (q_{it})^{-\sigma} \tag{A.29}$$

and therefore we get equation (52)

$$q_{it} = \left(\frac{c_t}{c_{it}}\right)^{\frac{1}{\sigma}} \tag{A.30}$$

Next, compute the FOC $\frac{\partial H}{\partial x_{it}} = 0:$

$$\frac{\partial H}{\partial x_{it}} = 0$$
$$\frac{\kappa}{N_t} x_{it} = \mu_t p^a_{st} c_{it} N_t$$
$$\frac{\kappa}{N_t^2} x_{it} = \mu_t p^a_{st} c_{it}$$

Next, compute the FOC $\frac{\partial H}{\partial \tilde{x}_{it}} = 0$:

$$\frac{\partial H}{\partial \tilde{x}_{it}} = 0$$
$$\frac{\tilde{\kappa}}{N_t} \tilde{x}_{it} = \mu_t p^b_{st} c_{it}$$

Finally, compute the FOC $\frac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t$:

$$\mu_t(r_t - g_L) = \tilde{\rho}\mu_t - \dot{\mu}_t$$
$$\Rightarrow \; (r_t - g_L) = \tilde{\rho} - \frac{\dot{\mu}_t}{\mu_t}$$

But $\mu_t = c_t^{-1}$ then $\frac{\dot{\mu}_t}{\mu_t} = -g_c$. Thus:

$$\Rightarrow \; (r_t - g_L) = \tilde{\rho} + g_c$$

Next, use A.28 and A.29 in the FOC of x_{it}:

$$\frac{\kappa}{N_t^2} x_{it} = p^a_{st} \left(q_{it}\right)^{-\sigma}$$

Thus:

$$x_{it} = \frac{N_t^2 p^a_{st}}{\kappa} \left(q_{it}\right)^{-\sigma} \tag{A.31}$$

Similarly, use A.28 and A.29 in the FOC of $\tilde{x}_{it}$:

$$\frac{\tilde{\kappa}}{N_t}\tilde{x}_{it} = p^b_{st}$$

$$\tilde{x}_{it} = \frac{N_t}{\tilde{\kappa}}\frac{1}{c_t}p^b_{st}c_t\left(q_{it}\right)^{-\sigma}$$

$$\Rightarrow \tilde{x}_{it} = \frac{N_t}{\tilde{\kappa}}p^b_{st}\left(q_{it}\right)^{-\sigma} \tag{A.32}$$

Now, we know that $q_{it} = \left(\frac{c_t}{c_{it}}\right)^{\frac{1}{\sigma}} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}$ and that $q_{it} = p_{it} - x_{it}p^a_{st} - \tilde{x}_{it}p^b_{st}$, which implies equation (53):

$$p_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st} \tag{A.33}$$

C.2 Firm Problem

The firm problem is given by:

$$r_tV_{it} = \max_{\{L_{it},D_{ait},D_{bit}\}}\left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right](1+s_y)Y_{it} - w_tL_{it} - p_{bt}D_{bit} - p_{at}D_{ait} - \delta(\tilde{x_{it}})V_{it} + \dot{V}_{it}$$

$$s.t.\ Y_{it} = D^{\eta}_{it}L_{it}$$

$$D_{it} = \alpha D_{ait} + (1-\alpha)D_{bit}$$

$$D_{ait} \geq 0$$

$$D_{bit} \geq 0$$

Start with the FOC w.r.t. L_{it}:

$$\frac{\partial \mathbb{L}}{\partial L_{it}} = 0$$

$$\left(\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right)(1+s_y)\frac{\partial Y_{it}}{\partial L_{it}} - (1+s_y)Y_{it}\left(\frac{1}{\sigma}\right)Y_t^{\frac{1}{\sigma}}Y_{it}^{-\frac{1}{\sigma}-1}\frac{\partial Y_{it}}{\partial L_{it}} - w_t = 0$$

$$(1+s_y)\frac{\partial Y_{it}}{\partial L_{it}}\left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st} - \left(\frac{1}{\sigma}\right)\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}\right] = w_t$$

But $\frac{\partial Y_{it}}{\partial L_{it}} = D^{\eta}_{it} = \frac{Y_{it}}{L_{it}}$ and therefore the FOC is:

$$(1+s_y)\frac{Y_{it}}{L_{it}}\left[\left(1-\frac{1}{\sigma}\right)\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = w_t$$

$$(1+s_y)\frac{Y_{it}}{L_{it}}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = w_t \tag{A.34}$$

Next, the FOC w.r.t. D_{ait}:

$$\frac{\partial \mathbb{L}}{\partial D_{ait}} = 0$$

$$\left(\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right)(1+s_y)\frac{\partial Y_{it}}{\partial D_{ait}} - \left(\frac{1}{\sigma}\right)Y_t^{\frac{1}{\sigma}}Y_{it}^{\frac{-1}{\sigma}-1}(1+s_y)Y_{it}\frac{\partial Y_{it}}{\partial D_{ait}} + -p_{at} = 0$$

$$(1+s_y)\frac{\partial Y_{it}}{\partial D_{ait}}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = p_{at} \tag{A.35}$$

Next, compute the derivative $\frac{\partial Y_{it}}{\partial D_{ait}}$:

$$\frac{\partial Y_{it}}{\partial D_{ait}} = \frac{\partial Y_{it}}{\partial D_{it}}\frac{\partial D_{it}}{\partial D_{ait}}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial D_{ait}} = \eta\frac{Y_{it}}{D_{it}}\alpha$$

Thus the FOC w.r.t. D_{ait} is:

$$\eta\alpha\frac{Y_{it}}{D_{it}}(1+s_y)\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = p_{at} \tag{A.36}$$

Similarly, the FOC w.r.t. D_{bit} is:

$$\eta(1-\alpha)\frac{Y_{it}}{D_{it}}(1+s_y)\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+x_{it}p^a_{st}+\tilde{x}_{it}p^b_{st}\right]=p_{bt} \tag{A.37}$$

We have computed the 3 FOCs. Now I solve the problem. First, divide A.36 by A.37 to get:

$$p_{at}=\frac{\alpha}{(1-\alpha)}p_{bt} \tag{A.38}$$

Second, divide A.34 by A.37:

$$\frac{Y_{it}}{L_{it}}\left[\eta(1-\alpha)\frac{Y_{it}}{D_{it}}\right]^{-1}=\frac{w_t}{p_{bt}}$$

$$\frac{D_{it}}{\eta(1-\alpha)L_{it}}=\frac{w_t}{p_{bt}}$$

$$\Rightarrow D_{it}=\frac{w_t}{p_{bt}}\eta(1-\alpha)L_{it}$$

but we know that $Y_{it}=D^{\eta}_{it}L_{it}$, therefore $L_{it}=\frac{Y_{it}}{D^{\eta}_{it}}$. Substituting L_{it}:

$$D^{\eta+1}_{it}=\frac{w_t}{p_{bt}}\eta(1-\alpha)Y_{it}$$

$$D_{it}=\left[\frac{w_t}{p_{bt}}\eta(1-\alpha)Y_{it}\right]^{\frac{1}{\eta+1}}$$

but $q_{it}=\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}$ therefore $Y_{it}=Y_t q^{-\sigma}_{it}$. Substituting:

$$D_{it}=\left[\frac{w_t}{p_{bt}}\eta(1-\alpha)Y_t q^{-\sigma}_{it}\right]^{\frac{1}{\eta+1}} \tag{A.39}$$

On the other hand, using A.34:

$$w_t=(1+s_y)D^{\eta}_{it}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}+x_{it}p^a_{st}+\tilde{x}_{it}p^b_{st}\right]$$

$$\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}=\frac{\sigma}{\sigma-1}\left(\frac{w_t}{(1+s_y)D^{\eta}_{it}}-x_{it}p^a_{st}-\tilde{x}_{it}p^b_{st}\right)$$

$$Y_{it} = Y_t \left[\frac{\sigma}{\sigma - 1} \left(\frac{w_t}{(1+s_y)D_{it}^{\eta}} - x_{it}p_{st}^a - \tilde{x}_{it}p_{st}^b \right) \right]^{-\sigma} \tag{A.40}$$

C.3 The 2 Data Intermediary Problems

The 2 data intermediary problems are given by:

$$\begin{aligned} &\max_{p_{ait}, D_{cit}^a} \int_0^{N_t} p_{ait} D_{ait} di - \int_0^{N_t} p_{st}^a D_{cit}^a di \\ &s.t. D_{ait} \leq D_{cit}^a \\ &\quad p_{ait} \leq p_{ait}^* \end{aligned}$$

and

$$\begin{aligned} &\max_{p_{bit}, D_{cit}^b} \int_0^{N_t} p_{bit} D_{bit} di - \int_0^{N_t} p_{st}^b D_{cit}^b di \\ &s.t. D_{bit} \leq B_t = \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} D_{cit}^{b\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \\ &\quad p_{bit} \leq p_{bit}^* \end{aligned}$$

As in the allocation when firms own the data, these problems can be decomposed in a cost minimization problem and a zero profit condition.

C.3.1 The Cost Minimization Problem

The cost minimization problems are:

$$\begin{aligned} &\min_{D_{cit}^a} \int_0^{N_t} p_{st}^a D_{cit}^a di \\ &s.t. D_{ait} \leq D_{cit}^a \end{aligned}$$

and

$$\begin{aligned} &\min_{D_{cit}^b} \int_0^{N_t} p_{st}^b D_{cit}^b di \\ &s.t. D_{bit} \leq B_t = \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} D_{cit}^{b\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}} \end{aligned}$$

The solutions are given by:

$$\begin{cases} D_{ait} = D^a_{cit} \\ D_{bit} = B_t = \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} D^{b\frac{\epsilon-1}{\epsilon}}_{cit} \, di \right)^{\frac{\epsilon}{\epsilon-1}} = N_t D^b_{cit} \end{cases} \tag{A.41}$$

where D_{ait} and D_{bit} are given by the firm problem.

C.3.2 The Zero Profit Condition for Each Data Intermediary Problem

By symmetry, the profit for each data intermediary is:

$$\begin{cases} \pi_{ait} = N_t \left[p_{ait} D_{ait} - p^a_{st} D^a_{cit} \right] = N_t D_{ait} \left[p_{ait} - p^a_{st} \right] \\ \pi_{bit} = N_t \left[p_{bit} D_{bit} - p^b_{st} D^b_{cit} \right] = N_t D_{bit} \left[p_{bit} - \frac{p^b_{st}}{N_t} \right] \end{cases}$$

The zero profit condition yields:

$$\begin{cases} p_{ait} = p^a_{st} \\ p_{bit} = \frac{p^b_{st}}{N_t} \end{cases} \tag{A.42}$$

and these are exactly equation (63).

C.4 Market Clearing Condition for Data

Imposing the market clearing condition in the data market, we get:

$$\begin{aligned} D^a_{cit} &= x_{it} c_{it} L_t \\ D^b_{cit} &= \tilde{x}_{it} c_{it} L_t \end{aligned} \tag{A.43}$$

but from the data intermediary problem $D_{bit} = B_t = \left(N_t^{-\frac{1}{\epsilon}} \int_0^{N_t} D^{b\frac{\epsilon-1}{\epsilon}}_{cit} \, di \right)^{\frac{\epsilon}{\epsilon-1}} = N_t D^b_{cit}$ and $D_{ait} = D^a_{cit}$ Thus the market clearing conditions yield

$$\begin{aligned} D_{ait} &= x_{it} c_{it} L_t \\ D_{bit} &= \tilde{x}_{it} c_{it} L_t N_t \end{aligned} \tag{A.44}$$

and therefore the supply of data is given by:

$$\begin{aligned} D_{it} &= \alpha D_{ait} + (1-\alpha) D_{bit} \\ &= \alpha x_{it} c_{it} L_t + (1-\alpha)\tilde{x}_{it} c_{it} L_t N_t \\ &= c_{it} L_t \left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t\right] \\ &= Y_{it} \left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t\right] \end{aligned} \tag{A.45}$$

Moreover, the zero profit condition means:

$$\pi_{ait} + \pi_{bit} = 0$$

$$\Rightarrow p_{at} D_{ait} + p_{bt} D_{bit} = p^a_{st} D^a_{cit} + D^b_{cit} p^b_{st}$$

Next, use A.41:

$$\Rightarrow p_{at} D^a_{cit} + p_{bt} N_t D^b_{cit} = p^a_{st} D^a_{cit} + D^b_{cit} p^b_{st}$$

Substitute using A.44:

$$\Rightarrow p_{at} x_{it} c_{it} L_t + p_{bt} N_t \tilde{x}_{it} c_{it} L_t = p^a_{st} x_{it} c_{it} L_t + \tilde{x}_{it} c_{it} L_t p^b_{st}$$

$$\Rightarrow p_{at} x_{it} + p_{bt} N_t \tilde{x}_{it} = p^a_{st} x_{it} + \tilde{x}_{it} p^b_{st}$$

Use A.38:

$$\Rightarrow \frac{\alpha}{(1-\alpha)} p_{bt} x_{it} + p_{bt} N_t \tilde{x}_{it} = p^a_{st} x_{it} + \tilde{x}_{it} p^b_{st}$$

And we finally get:

$$\Rightarrow \frac{p_{bt}}{(1-\alpha)} \left[\alpha x_{it} + (1-\alpha) N_t \tilde{x}_{it}\right] = p^a_{st} x_{it} + \tilde{x}_{it} p^b_{st} \tag{A.46}$$

C.5 Summary of Equations and Solution

C.5.1 Summary

Let's summarize all the equations we have so far, from the FOCs of the consumer and firm problems:

$$
\begin{aligned}
N_t^{-\frac{\sigma}{\sigma-1}} &= \frac{Y_{it}}{Y_t} = \frac{c_{it}}{c_t} = (q_{it})^{-\sigma} \\
x_{it} &= \frac{N_t^2 p_{st}^a}{\kappa}(q_{it})^{-\sigma} \\
\tilde{x}_{it} &= \frac{N_t}{\tilde{\kappa}} p_{st}^b (q_{it})^{-\sigma} \\
(1+s_y)\frac{Y_{it}}{L_{it}}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p_{st}^a + \tilde{x}_{it}p_{st}^b\right] &= w_t \\
\eta\alpha\frac{Y_{it}}{D_{it}}(1+s_y)\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p_{st}^a + \tilde{x}_{it}p_{st}^b\right] &= p_{at} \\
p_{at} &= \frac{\alpha}{(1-\alpha)}p_{bt} \\
D_{it} &= \left[\frac{w_t}{p_{bt}}\eta(1-\alpha)Y_t q_{it}^{-\sigma}\right]^{\frac{1}{\eta+1}} \\
Y_{it} = Y_t\left[\frac{\sigma}{\sigma-1}\left(\frac{w_t}{(1+s_y)D_{it}^{\eta}} - x_{it}p_{st}^a - \tilde{x}_{it}p_{st}^b\right)\right]^{-\sigma} & \\
D_{ait} &= D_{cit}^a \\
D_{bit} &= N_t D_{cit}^b \\
p_{at} &= p_{st}^a \\
p_{bt} &= \frac{p_{st}^b}{N_t} \\
D_{ait} &= x_{it}c_{it}L_t \\
D_{bit} &= \tilde{x}_{it}c_{it}L_t N_t \\
\frac{p_{bt}}{(1-\alpha)}\left[\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it}\right] &= p_{st}^a x_{it} + \tilde{x}_{it}p_{st}^b \\
D_{it} &= \left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right]Y_{it}
\end{aligned}
\tag{A.47}
$$

C.5.2 Solution for x_{it} and $\tilde{x}_{it}$

First, use $(q_{it})^{-\sigma} = N_t^{-\frac{\sigma}{\sigma-1}}$ in A.32 and A.31 to get:

$$x_{it} = \frac{p_{st}^a}{\kappa} N_t^{2-\frac{\sigma}{\sigma-1}} \tag{A.48}$$

$$\tilde{x}_{it} = \frac{p^b_{st}}{\tilde{\kappa}} N_t^{1-\frac{\sigma}{\sigma-1}} \tag{A.49}$$

Next, using $N_t^{-\frac{\sigma}{\sigma-1}} = \frac{Y_{it}}{Y_t}$ and A.45 in A.36:

$$(1+s_y)\eta\alpha\frac{Y_{it}}{D_{it}}\left[\frac{\sigma-1}{\sigma}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right] = p_{at}$$

$$(1+s_y)\eta\alpha\frac{\left[\frac{\sigma-1}{\sigma}N_t^{\frac{1}{\sigma-1}} + x_{it}p^a_{st} + \tilde{x}_{it}p^b_{st}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at} \tag{A.50}$$

Now, use A.48 and A.49 to substitute p^a_{st} and p^b_{st}:

$$(1+s_y)\eta\alpha\frac{\left[\frac{\sigma-1}{\sigma}N_t^{\frac{1}{\sigma-1}} + x^2_{it}\kappa N_t^{-2+\frac{\sigma}{\sigma-1}} + \tilde{x}^2_{it}\tilde{\kappa}N_t^{-1+\frac{\sigma}{\sigma-1}}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at}$$

$$(1+s_y)\eta\alpha\frac{\left[\frac{\sigma-1}{\sigma}N_t^{\frac{1}{\sigma-1}} + x^2_{it}\kappa N_t^{\frac{2-\sigma}{\sigma-1}} + \tilde{x}^2_{it}\tilde{\kappa}N_t^{\frac{1}{\sigma-1}}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at}$$

$$(1+s_y)\eta\alpha\frac{N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma} + x^2_{it}\kappa N_t^{-1} + \tilde{x}^2_{it}\tilde{\kappa}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at}$$

$$(1+s_y)\eta\alpha\frac{N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma} + x^2_{it}\kappa N_t^{-1} + \tilde{x}^2_{it}\tilde{\kappa}\right]}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t} = p_{at}$$

Use $p_{at} = \frac{\alpha}{(1-\alpha)}p_{bt}$ and get:

$$(1+s_y)\eta N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma} + x^2_{it}\kappa N_t^{-1} + \tilde{x}^2_{it}\tilde{\kappa}\right] = \frac{p_{bt}}{1-\alpha}\left(\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right) \tag{A.51}$$

Now, recall from A.46: $\frac{p_{bt}}{(1-\alpha)}\left[\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it}\right] = p^a_{st}x_{it} + \tilde{x}_{it}p^b_{st}$. With this, substitute the RHS in A.51:

$$(1+s_y)\eta N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma} + x^2_{it}\kappa N_t^{-1} + \tilde{x}^2_{it}\tilde{\kappa}\right] = p^a_{st}x_{it} + \tilde{x}_{it}p^b_{st}$$

Again, use A.48 and A.49 to substitute p^a_{st} and p^b_{st}:

$$
\begin{aligned}
(1+s_y)\eta N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma}+x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}\right] &= x_{it}^2\kappa N_t^{\frac{2-\sigma}{\sigma-1}}+\tilde{x}_{it}^2\tilde{\kappa}N_t^{\frac{1}{\sigma-1}} \\
(1+s_y)\eta N_t^{\frac{1}{\sigma-1}}\left[\frac{\sigma-1}{\sigma}+x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}\right] &= N_t^{\frac{1}{\sigma-1}}\left[x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}\right] \\
(1+s_y)\eta\frac{\sigma-1}{\sigma}+(1+s_y)\eta x_{it}^2\kappa N_t^{-1}+(1+s_y)\eta\tilde{x}_{it}^2\tilde{\kappa} &= x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa} \\
(1+s_y)\eta\frac{\sigma-1}{\sigma}+((1+s_y)\eta-1)x_{it}^2\kappa N_t^{-1}+((1+s_y)\eta-1)\tilde{x}_{it}^2\tilde{\kappa} &= 0
\end{aligned}
$$

Finally:

$$
\Rightarrow x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}=\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{\sigma-1}{\sigma} \tag{A.52}
$$

On other hand, divide A.48 by A.49:

$$
\frac{x_{it}}{\tilde{x}_{it}}=\frac{p^a_{st}N_t\tilde{\kappa}}{p^b_{st}\kappa}
$$

Use the zero profit condition for the data intermediary problems:

$$
\Rightarrow \frac{x_{it}}{\tilde{x}_{it}}=\frac{p_{at}N_t\tilde{\kappa}}{p_{bt}N_t\kappa}
$$

Use A.38:

$$
\frac{x_{it}}{\tilde{x}_{it}}=\frac{\alpha\tilde{\kappa}}{(1-\alpha)\kappa}. \tag{A.53}
$$

Hence, use A.53 in A.52:

$$
\tilde{x}_{it}^2\kappa N_t^{-1}\left(\frac{\alpha\tilde{\kappa}}{(1-\alpha)\kappa}\right)^2+\tilde{x}_{it}^2\tilde{\kappa}=\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{\sigma-1}{\sigma} \tag{A.54}
$$

When N_t is large, $\underset{N_t\to\infty}{lim} N_t^{-1}\left(\frac{\alpha\tilde{\kappa}}{(1-\alpha)\kappa}\right)^2 = 0$, therefore:

$$\tilde{x}^c = \left[\frac{1}{\tilde{\kappa}}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{\sigma-1}{\sigma}\right]^{\frac{1}{2}} \tag{A.55}$$

which is equation (65). Moreover, the expression for x^c is:

$$x^c = \frac{\alpha}{(1-\alpha)}\frac{\tilde{\kappa}}{\kappa}\tilde{x}^c \tag{A.56}$$

which is equation (69).

C.5.3 The prices p_{at} and p_{bt}

From A.51:

$$p_{bt} = (1+s_y)(1-\alpha)\eta N_t^{\frac{1}{\sigma-1}}\frac{\left[\frac{\sigma-1}{\sigma}+x_{it}^2\kappa N_t^{-1}+\tilde{x}_{it}^2\tilde{\kappa}\right]}{(\alpha x_{it}+(1-\alpha)\tilde{x}_{it}N_t)} \tag{A.57}$$

In the numerator, when N_t is large:

$$\begin{aligned}
p_{bt} &= (1+s_y)(1-\alpha)\eta N_t^{\frac{1}{\sigma-1}}\frac{\left[\frac{\sigma-1}{\sigma}+\tilde{x}_{it}^2\tilde{\kappa}\right]}{\alpha x_{it}+(1-\alpha)\tilde{x}_{it}N_t} \\
&= (1+s_y)(1-\alpha)\eta N_t^{\frac{1}{\sigma-1}}\frac{\left[\frac{\sigma-1}{\sigma}+\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{\sigma-1}{\sigma}\right]}{\alpha x_{it}+(1-\alpha)\tilde{x}_{it}N_t} \\
&= (1+s_y)(1-\alpha)\eta N_t^{\frac{1}{\sigma-1}}\frac{1}{1-\eta(1+s_y)}\frac{\sigma-1}{\sigma}\frac{1}{\alpha x_{it}+(1-\alpha)\tilde{x}_{it}N_t} \\
&= (1-\alpha)\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\cdot\frac{N_t^{\frac{1}{\sigma-1}}}{\alpha x_{it}+(1-\alpha)\tilde{x}_{it}N_t}
\end{aligned} \tag{A.58}$$

C.5.4 Wage w_t

Divide A.34 by A.36:

$$w_t = p_{at}\frac{D_{it}}{L_{it}\eta\alpha}$$

Substitute D_{it} and L_{it}:

$$w_t = p_{at}\frac{N_t Y_{it}\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t\right]}{L_{pt}\eta\alpha}$$

But $N_t^{-\frac{\sigma}{\sigma-1}} = \frac{Y_{it}}{Y_t}$, therefore:

$$w_t = p_{at}\frac{N_t^{1-\frac{\sigma}{\sigma-1}} Y_t\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t\right]}{L_{pt}\eta\alpha}.$$

Using N_t large, (A.58), and $p_{at} = \frac{\alpha}{(1-\alpha)} p_{bt}$ yields

$$w_t = \alpha\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\cdot\frac{N_t^{\frac{1}{\sigma-1}}}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t}\cdot\frac{N_t^{1-\frac{\sigma}{\sigma-1}} Y_t\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t\right]}{L_{pt}\eta\alpha}$$

Cleaning up yields

$$w_t = \frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{1}{\eta}\frac{Y_t}{L_{pt}}. \tag{A.59}$$

C.5.5 Value of the Firm and Profits

The value of a firm is:

$$V_t = \frac{\left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p_{st}^a + \tilde{x}_{it}p_{st}^b\right](1+s_y)Y_{it} - w_t L_{it} - p_{bt}D_{bit} - p_{at}D_{ait}}{r + \delta(\tilde{x_{it}}) - g_V}$$

We define components of firm profit to ease computation:

$$V_t = \frac{\overbrace{\left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p_{st}^a + \tilde{x}_{it}p_{st}^b\right](1+s_y)Y_{it}}^{C} - \overbrace{w_t L_{it}}^{B} \overbrace{- p_{bt}D_{bit} - p_{at}D_{ait}}^{A}}{r + \delta(\tilde{x_{it}}) - g_V}$$

First, we compute A:

$$A = p_{bt}D_{bit} + p_{at}D_{ait}$$

$$
\begin{aligned}
&= p_{bt}\left[D_{bit} + \frac{\alpha}{1-\alpha} D_{ait}\right] \\
&= \frac{p_{bt}}{(1-\alpha)}\left[(1-\alpha)D_{bit} + \alpha D_{ait}\right] \\
&= \frac{p_{bt}}{(1-\alpha)}\left[\alpha x_{it} c_{it} L_t + (1-\alpha)\tilde{x}_{it} c_{it} L_t N_t\right] \\
&= \frac{p_{bt}}{(1-\alpha)} Y_{it}\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t\right] \\
&= \frac{p_{bt}}{(1-\alpha)} N_t^{-\frac{\sigma}{\sigma-1}} Y_t\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t\right] \\
&= \frac{1}{(1-\alpha)}(1-\alpha)\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{N_t^{\frac{1}{\sigma-1}}}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t} \cdot N_t^{-\frac{\sigma}{\sigma-1}} Y_t\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t\right] \\
&= \frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{Y_t}{N_t}.
\end{aligned}
$$

Note, this term is equation (100), which gives the data share of the economy.

Next, let's compute B using (A.59):

$$
\begin{aligned}
B &= w_t L_{it} \\
&= \frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{1}{\eta}\frac{Y_t}{L_{pt}} L_{it} \\
&= \frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{1}{\eta}\frac{Y_t}{L_{pt}}\frac{L_{pt}}{N_t} \\
&= \frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{1}{\eta}\frac{Y_t}{N_t}
\end{aligned}
$$

Finally, compute C:

$$
\begin{aligned}
C &= \left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it} p^a_{st} + \tilde{x}_{it} p^b_{st}\right](1+s_y) Y_{it} \\
&= \left[N_t^{\frac{1}{\sigma-1}} + x_{it} p^a_{st} + \tilde{x}_{it} p^b_{st}\right](1+s_y) Y_{it} \\
&= \left[N_t^{\frac{1}{\sigma-1}} + x_{it}^2 \kappa N_t^{-2+\frac{\sigma}{\sigma-1}} + \tilde{x}_{it}^2 \tilde{\kappa}\right](1+s_y) Y_{it} \\
&= N_t^{\frac{1}{\sigma-1}}\left[1 + x_{it}^2 \kappa N_t^{-1} + \tilde{x}_{it}^2 \tilde{\kappa}\right](1+s_y) Y_{it}
\end{aligned}
$$

For N_t large:

$$
\begin{aligned}
&= N_t^{\frac{1}{\sigma-1}} \left[1+\tilde{x}_{it}^2\tilde{\kappa}\right](1+s_y)Y_{it} \\
&= N_t^{\frac{1}{\sigma-1}} \left[1+\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\right](1+s_y)Y_{it} \\
&= N_t^{\frac{1}{\sigma-1}} \left[1+\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\right](1+s_y)Y_t N_t^{-\frac{\sigma}{\sigma-1}} \\
&= (1+s_y)\left[1+\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\right]\frac{Y_t}{N_t}
\end{aligned}
$$

Using (A), (B),and (C) we can compute the value of the firm:

$$
\begin{aligned}
V_{it} &= \frac{\overbrace{\left[\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it}p_{st}^a + \tilde{x}_{it}p_{st}^b\right](1+s_y)Y_{it}}^{C} - \overbrace{w_tL_{it}}^{B} \overbrace{- p_{bt}D_{bit} - p_{at}D_{ait}}^{A}}{r+\delta(\tilde{x_{it}})-g_V} \\
&= \frac{1}{r+\delta(\tilde{x_{it}})-g_V}\left[(1+s_y)\left(1+\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\right) - \frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{1}{\eta} - \frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\right]\frac{Y_t}{N_t}
\end{aligned}
$$

Which, after simplifying, is

$$
= \frac{1}{(r+\delta(\tilde{x_{it}})-g_V)}\left[\frac{1-\sigma\eta(1+s_y)}{\sigma-\sigma\eta(1+s_y)}+\frac{s_y}{\sigma}\right]\frac{Y_t}{N_t}.
$$

C.6 Solution of the Competitive Equilibrium

Recall on a BGP $g_V = g_Y = g_N$ and $r - g_V = \rho$.

C.6.1 Firm Size

The free entry condition is

$$
(1-s_e)\chi w_t = V_{it}\left[1+\frac{\delta(\tilde{x}_{it})}{g_N}\right]
$$

Substitute V_{it} and w_t to get

$$
(1-s_e)\chi\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{1}{\eta}\frac{Y_t}{L_{pt}} = \left[1+\frac{\delta(\tilde{x}_{it})}{g_N}\right]\frac{1}{(r+\delta(\tilde{x}_{it})-g_V)}\left[\frac{1-\sigma\eta(1+s_y)}{\sigma-\sigma\eta(1+s_y)}+\frac{s_y}{\sigma}\right]
$$

$$\Rightarrow \frac{L_{pt}}{N_t} = \frac{(1-s_e)\chi\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)}\frac{1}{\eta}\left(r+\delta(\tilde{x}_{it})-g_V\right)}{\left[1+\frac{\delta(\tilde{x}_{it})}{g_N}\right]\left[\frac{1-\sigma\eta(1+s_y)}{\sigma-\sigma\eta(1+s_y)}+\frac{s_y}{\sigma}\right]}$$

Use $\rho = r - g_V$ and $g_N = g_L$ on BGP:

$$\begin{aligned}\frac{L_{pt}}{N_t} &= \frac{(1-s_e)\chi g_L\left(\rho+\delta(\tilde{x}_{it})\right)}{[g_L+\delta(\tilde{x}_{it})]}\cdot\frac{\sigma-1}{\sigma}\frac{(1+s_y)}{1-\eta(1+s_y)}\cdot\frac{1}{\left[\frac{1-\sigma\eta(1+s_y)}{\sigma-\sigma\eta(1+s_y)}+\frac{s_y}{\sigma}\right]}\\ &= (1-s_e)\chi g_L\frac{\rho+\delta(\tilde{x}_{it})}{g_L+\delta(\tilde{x}_{it})}\cdot\frac{\sigma-1}{1-\eta(\sigma+s_y)}\end{aligned}$$

Define $\nu_c = \frac{L_{pt}}{N_t}$. Finally:

$$\nu_c = (1-s_e)\chi g_L\frac{\rho+\delta(\tilde{x}_{it})}{g_L+\delta(\tilde{x}_{it})}\cdot\frac{\sigma-1}{1-\eta(\sigma+s_y)} \tag{A.60}$$

which is equation (74) in the paper (when the subsidies are zero).

C.6.2 Number of Varieties

From the evolution of the number of varieties, we have $\dot{N}_t = \frac{1}{\chi}(L_t - L_{pt})$. Thus:

$$\frac{\dot{N}_t}{N_t} = \frac{1}{\chi}\left(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t}\right).$$

In BGP: $\frac{\dot{N}_t}{N_t} = g_N$ and since $g_N = g_L$, then:

$$\begin{aligned} g_L &= \frac{1}{\chi}\left(\frac{L_t}{N_t}-\frac{L_{pt}}{N_t}\right)\\ \frac{L_t}{N_t} &= \frac{L_{pt}}{N_t}+\chi g_L\\ N_t &= \frac{L_t}{\nu_c+\chi g_L}\end{aligned}$$

Define $\psi_c = \frac{1}{\nu_c+\chi g_L}$. Then

$$N_t = \psi_c L_t, \tag{A.61}$$

which is equation (79) for the allocation when consumers own the data.

C.6.3 Aggregate Output

First, substitute $D_{it} = Y_{it}\left[\alpha x_{it} + (1-\alpha)\tilde{x}_{it}N_t\right]$ in $Y_{it} = D_{it}^{\eta}L_{it}$.

$$\begin{aligned} Y_{it} &= \left(Y_{it}(\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})\right)^{\eta} L_{it} \\ &= Y_{it}^{\eta}(\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\eta} L_{it} \\ &= Y_{it}^{\eta}(\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\eta}\left(\frac{L_{pt}}{N_t}\right) \end{aligned}$$

$$\Rightarrow Y_{it} = (\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}} \tag{A.62}$$

On the other hand, aggregate output is $Y_t = Y_{it}N_t^{\frac{\sigma}{\sigma-1}}$. Hence:

$$\begin{aligned} Y_t &= N_t^{\frac{\sigma}{\sigma-1}}(\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}} \\ &= N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it})^{\frac{\eta}{1-\eta}}L_{pt}^{\frac{1}{1-\eta}} \end{aligned}$$

When N_t is large:

$$\begin{aligned} Y_t^c &= (N_t)^{\frac{\sigma}{\sigma-1}}(N_t)^{\frac{\eta}{1-\eta}}\left((1-\alpha)\tilde{x}_{it}\right)^{\frac{\eta}{1-\eta}}(\nu_c)^{\frac{1}{1-\eta}} \\ &= \left(\nu_c(1-\alpha)^{\eta}\tilde{x}_{it}^{\eta}\right)^{\frac{1}{1-\eta}}(N_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}} \end{aligned}$$

and from equation (79), $N_t = \psi_c L_t$, thus:

$$Y_t^c = \left(\nu_c(1-\alpha)^{\eta}\tilde{x}_{it}^{\eta}\right)^{\frac{1}{1-\eta}}(\psi_c L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}$$

C.6.4 Firm production

$$\begin{aligned} Y_{it}^c &= Y_t N_t^{-\frac{\sigma}{\sigma-1}} \\ &\left(\nu_c(1-\alpha)^{\eta}\tilde{x}_{it}^{\eta}\right)^{\frac{1}{1-\eta}}(\psi_c L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}N_t^{-\frac{\sigma}{\sigma-1}} \\ &= \left(\nu_c(1-\alpha)^{\eta}\tilde{x}_{it}^{\eta}\right)^{\frac{1}{1-\eta}}(\psi_c L_t)^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}}(\psi_f L_t)^{-\frac{\sigma}{\sigma-1}} \\ &= \left(v_c(1-\alpha)^{\eta}\tilde{x}_{it}^{\eta}\right)^{\frac{1}{1-\eta}}(\psi_c L_t)^{\frac{\eta}{1-\eta}} \end{aligned}$$

which is equation (91) for allocation c.

C.6.5 Consumption per capita and growth

Consumption per capita is defined as $c_t^c = \frac{Y_t^c}{L_t}$, and using the expression for aggregate output, then:

$$c_t^c = \frac{Y_t^c}{L_t} \propto L_t^{\frac{1}{\sigma-1}+\frac{\eta}{1-\eta}}$$

which is equation (83) in the paper for the allocation c. Using this, the growth is:

$$g_c^c = \left(\frac{1}{\sigma-1} + \frac{\eta}{1-\eta}\right) g_L$$

which is equation (85) in the paper for the allocation c.

C.6.6 Data used by each firm and aggregate data

$$\begin{aligned} D_{it}^c &= Y_{it}(\alpha x_{it} + (1-\alpha)N_t\tilde{x}_{it}) \\ &= (\nu_c(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{\eta}{1-\eta}} N_t \left(\frac{\alpha}{N_t} x_{it} + (1-\alpha)\tilde{x}_{it}\right) \\ &= (\nu_c(1-\alpha)^\eta \tilde{x}_{it}^\eta)^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{\eta}{1-\eta}+1} (1-\alpha)\tilde{x}_{it} \\ &= (\nu_c(1-\alpha)\tilde{x}_{it}\psi_f L_t)^{\frac{1}{1-\eta}} \end{aligned}$$

By definition $D_t = N_t D_{it}$, hence:

$$\begin{aligned} D_t^c &= \psi_c L_t (\nu_c(1-\alpha)\tilde{x}_{it}\psi_c L_t)^{\frac{1}{1-\eta}} \\ &= (\nu_c(1-\alpha)\tilde{x}_{it})^{\frac{1}{1-\eta}} (\psi_c L_t)^{\frac{1}{1-\eta}+1} \end{aligned}$$

and this is equation (89).

C.6.7 Labor Share

From A.59:

$$\begin{aligned} \left(\frac{w_t L_{pt}}{Y_t}\right)^c &= \frac{\sigma-1}{\sigma} \frac{\eta(1+s_y)}{1-\eta(1+s_y)} \frac{1}{\eta} \frac{Y_t}{L_{pt}} \frac{L_{pt}}{Y_t} \\ &= \frac{\sigma-1}{\sigma} \frac{\eta(1+s_y)}{1-\eta(1+s_y)} \frac{1}{\eta} \end{aligned}$$

and this is equation (93) in the paper.

C.6.8 Profit share

Using the results from C.5.5, the profit is defined as $\pi_t = C - A - B$, where C, B and A are defined in C.5.5. Then:

$$\left(\frac{N_t \pi_t}{Y_t}\right) = \frac{1 - \sigma\eta(1+s_y)}{\sigma - \sigma\eta(1+s_y)} + \frac{s_y}{\sigma}$$

C.6.9 Prices

From A.47, $(q_{it})^{-\sigma} = N_t^{-\frac{\sigma}{\sigma-1}}$. Then:

$$\begin{aligned} q_{it} &= N_t^{\frac{1}{\sigma-1}} \\ &= (L_t \psi_c)^{\frac{1}{\sigma-1}} \end{aligned}$$

Next, from equation (53) in the paper, p_{it} is defined as $p_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} + x_{it} p^a_{st} + \tilde{x}_{it} p^b_{st}$. Now, from the computation of profit component C in C.5.5, we have

$$\begin{aligned} p_{it} &= N_t^{\frac{1}{\sigma-1}} \left[1 + \frac{\sigma-1}{\sigma} \frac{\eta(1+s_y)}{1-\eta(1+s_y)}\right] \\ &= (L_t \psi_c)^{\frac{1}{\sigma-1}} \left[1 + \frac{\sigma-1}{\sigma} \frac{\eta(1+s_y)}{1-\eta(1+s_y)}\right] \end{aligned}$$

C.6.10 Price p^b_{st}

Recall from A.58 that $p_{bt} = (1-\alpha)\frac{\sigma-1}{\sigma}\frac{\eta(1+s_y)}{1-\eta(1+s_y)} \cdot \frac{N_t^{\frac{1}{\sigma-1}}}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t}$ and we know that, by the zero profit condition in the data intermediary, $p_{bt} = \frac{p^b_{st}}{N_t}$. Then:

$$\begin{aligned} p^b_{st} &= N_t p_{bt} \\ &= N_t (1-\alpha) \frac{\sigma-1}{\sigma} \frac{\eta(1+s_y)}{1-\eta(1+s_y)} \cdot \frac{N_t^{\frac{1}{\sigma-1}}}{\alpha x_{it} + (1-\alpha)\tilde{x}_{it} N_t} \\ &= N_t (1-\alpha) \frac{\sigma-1}{\sigma} \frac{\eta(1+s_y)}{1-\eta(1+s_y)} \cdot \frac{N_t^{\frac{1}{\sigma-1}}}{N_t \left(\frac{\alpha x_{it}}{N_t} + (1-\alpha)\tilde{x}_{it}\right)} \end{aligned}$$

For N_t large:

$$
\begin{aligned}
&= \frac{\sigma - 1}{\sigma} \frac{\eta(1+s_y)}{1-\eta(1+s_y)} \cdot \frac{N_t^{\frac{1}{\sigma-1}}}{\tilde{x}_{it}} \\
&= \frac{\sigma - 1}{\sigma} \frac{\eta(1+s_y)}{1-\eta(1+s_y)} \frac{1}{\tilde{x}_c} \left(L_t \psi_c\right)^{\frac{1}{\sigma-1}}
\end{aligned}
$$

C.7 Optimal Subsidies

Recall that when Consumers Own Data,

$$
\begin{aligned}
\tilde{x}^c &= \left[\frac{1}{\tilde{\kappa}} \frac{\eta(1+s_y)}{1-\eta(1+s_y)} \frac{\sigma-1}{\sigma}\right]^{\frac{1}{2}} \\
x^c &= \frac{\alpha}{(1-\alpha)} \left(\frac{\tilde{\kappa}}{\kappa}\right) \tilde{x}^c,
\end{aligned}
$$

and that optimal data use chosen by the social planner is

$$
\begin{aligned}
\tilde{x}^{sp} &= \left(\frac{\eta}{1-\eta} \frac{1}{\tilde{\kappa}}\right)^{\frac{1}{2}} \\
x^{sp} &= \frac{\alpha}{(1-\alpha)} \left(\frac{\tilde{\kappa}}{\kappa}\right) \tilde{x}^{sp}.
\end{aligned}
$$

The revenue subsidy that aligns equilibrium and optimal data use is

$$
1 + s_y = \frac{1}{\eta + (1-\eta)\frac{\sigma-1}{\sigma}}.
$$

Notice that this is smaller than the typical subsidy in the classic monopolistic competition setup of $\frac{\sigma}{\sigma-1}$ exactly because of the importance of data, captured by η.

This revenue subsidy is not enough by itself to align the Consumers Own Data allocation with the Social Planner allocation, because, in addition to the distortion in the production of existing varieties, there is a distortion in the amount of firm entry. When Consumers Own Data,

$$
\nu_c = (1 - s_e)\chi g_L \frac{\rho + \delta(\tilde{x}_{it})}{g_L + \delta(\tilde{x}_{it})} \cdot \frac{\sigma - 1}{1 - \eta(\sigma + s_y)}
$$

but the optimal firm size chosen by the social planner is

$$\nu_{sp} = \left(\frac{L_{pt}}{N_t}\right)^{sp} = \frac{\rho\chi(\sigma-1)}{1-\eta}.$$

Thus, the entry subsidy that aligns equilibrium and optimal entry and firm size is

$$1 - s_e = \frac{\rho}{g_L}\frac{g_L + \delta(\tilde{x}_c)}{\rho + \delta(\tilde{x}_c)}\frac{1-\eta(\sigma+s_y)}{1-\eta},$$

where,

$$\sigma + s_y = \frac{\sigma - (1-\eta)\frac{\sigma-1}{\sigma}}{\eta + (1+\eta)\frac{\sigma-1}{\sigma}}.$$

Given the structure of the equilibrium and planner allocations when written as a function of $\tilde{x}$, ν, and parameters, setting subsidies to ensure $\tilde{x}_c = \tilde{x}_{sp}$ and $\nu_c = \nu_{sp}$ ensures that the subsidized Consumers Own Data equilibrium is optimal.

D Competitive Equilibrium With Outlaw Data Sharing

D.1 Household Problem

The problem is

$$\begin{aligned} U_0 = &\max_{\{c_{it}\}} \int_0^\infty e^{-(\tilde{\rho})} L_0 u(c_t, x_{it}) dt \\ s.t.\ c_t &= \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}} \\ \dot{a}_t &= (r_t - g_L)a_t + w_t - \int_0^{N_t} p_{it}c_{it}di \end{aligned}$$

Define the current value Hamiltonian with state variable a_t, control variable c_{it}, and co-state variable μ_t:

$$H(a_t, c_{it}, \mu_t) = u(\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}, x_{it}, \tilde{x}_{it}) + \mu_t\left[(r_t - g_L)a_t + w_t - \int_0^{N_t} p_{it}c_{it}di\right]$$

The FOCs are:

$$\begin{cases} \dfrac{\partial H}{\partial c_{it}} = 0 \\ \dfrac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t \end{cases}$$

First, start with $\frac{\partial H}{\partial c_{it}} = 0$:

$$\frac{1}{\left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{\sigma}{\sigma-1}}} \frac{\sigma}{\sigma-1} \left(\int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di\right)^{\frac{1}{\sigma-1}} \frac{\sigma-1}{\sigma} c_{it}^{\frac{-1}{\sigma}} - \mu_t p_{it} = 0$$

$$\Rightarrow \; c_t^{\frac{1-\sigma}{\sigma}} c_{it}^{\frac{-1}{\sigma}} - \mu_t p_{it} = 0$$

$$\Rightarrow \; c_t^{\sigma-1} c_{it} = (\mu_t p_{it})^{-\sigma}$$

$$\Rightarrow \; \left(c_t^{\sigma-1} c_{it}\right)^{\frac{\sigma-1}{\sigma}} = (\mu_t p_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_{it}^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} (p_{it})^{1-\sigma}$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} \int_0^{N_t} c_{it}^{\frac{\sigma-1}{\sigma}} di = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{(1-\sigma)^2}{\sigma}} c_t^{\frac{\sigma-1}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; c_t^{\frac{\sigma-1}{\sigma} + \frac{(1-\sigma)^2}{\sigma}} = (\mu_t)^{1-\sigma} \int_0^{N_t} p_{it}^{1-\sigma} di$$

$$\Rightarrow \; \mu_t = \frac{1}{c_t \left(\int_0^{N_t} p_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}}}$$

Next, use the fact that the price of c_t is normalized to 1, i.e., $P_t = \left(\int_0^{N_t} p_{it}^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} = 1$.

Next, plug the expression for μ_t in the FOC and this gives equation (39):

$$c_{it} = c_t p_{it}^{-\sigma}$$

Next, compute the FOC $\frac{\partial H}{\partial a_t} = \tilde{\rho}\mu_t - \dot{\mu}_t$:

$$\mu_t(r_t - g_L) = \tilde{\rho}\mu_t - \dot{\mu}_t$$

$$\Rightarrow \ (r_t - g_L) = \tilde{\rho} - \frac{\dot{\mu}_t}{\mu_t}$$

But $\mu_t = c_t^{-1}$ then $\frac{\dot{\mu}_t}{\mu_t} = -g_c$. Thus:

$$\Rightarrow \ (r_t - g_L) = \tilde{\rho} + g_c$$

D.2 Firm Problem

D.2.1 Define the Problem of the Firm

The firm problem is:

$$\begin{aligned} r_t V_{it} = & \max_{\{L_{it}, D_{bit}\}} \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it} + \dot{V}_{it} \\ & s.t.\ Y_{it} = D_{it}^{\eta} L_{it} \\ & \quad D_{it} = \alpha x_{it} Y_{it} \\ & \quad x_{it} \in [0; \bar{x}] \end{aligned}$$

D.2.2 Compute FOC

To solve this problem, write the Lagrangean:

$$\mathbb{L} = (Y_t)^{\frac{1}{\sigma}} Y_{it}^{1-\frac{1}{\sigma}} - w_t L_{it} + \dot{V}_{it}. + \mu_{x0}(x_{it}) + \mu_{x1}(\bar{x} - x_{it})$$

Now take the FOCs. Start with the FOC w.r.t. L_{it}:

$$\frac{\partial \mathbb{L}}{\partial L_{it}} = 0$$

$$\Leftrightarrow \ (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} \frac{\partial Y_{it}}{\partial L_{it}} = w_t$$

And using $Y_{it} = D_{it}^{\eta} L_{it}$ and assuming D_{it} depends on L_{it}, then by implicit derivation:

$$\frac{\partial Y_{it}}{\partial L_{it}} = \eta D_{it}^{\eta-1} L_{it} \frac{\partial D_{it}}{\partial L_{it}} + D_{it}^{\eta}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \eta \frac{Y_{it}}{D_{it}} \frac{\partial D_{it}}{\partial L_{it}} + \frac{Y_{it}}{L_{it}}$$

Next, compute $\frac{\partial D_{it}}{\partial L_{it}}$ using $D_{it} = \alpha x_{it} Y_{it}$:

$$\frac{\partial D_{it}}{\partial L_{it}} = \alpha x_{it} \frac{\partial Y_{it}}{\partial L_{it}}$$

Substituting above:

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \eta \frac{Y_{it}}{D_{it}} \alpha x_{it} \frac{\partial Y_{it}}{\partial L_{it}} + \frac{Y_{it}}{L_{it}}$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial L_{it}} = \frac{\frac{Y_{it}}{L_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}}$$

The FOC for L_{it} is then:

$$\frac{\frac{Y_{it}}{L_{it}}}{1 - \eta \frac{Y_{it}}{D_{it}} \alpha x_{it}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = w_t \tag{A.63}$$

Next, compute the FOC w.r.t. x_{it}:

$$\frac{\partial \mathbb{L}}{\partial x_{it}} = 0$$

$$\Leftrightarrow (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} \frac{\partial Y_{it}}{\partial x_{it}} + \mu_{x0} - \mu_{x1} = 0$$

$$\frac{\partial Y_{it}}{\partial x_{it}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = -\mu_{x0} + \mu_{x1}$$

Now to compute $\frac{\partial Y_{it}}{\partial x_{it}}$ use $Y_{it} = D_{it}^{\eta} L_{it}$:

$$\frac{\partial Y_{it}}{\partial x_{it}} = \eta \frac{Y_{it}}{D_{it}} \frac{\partial D_{it}}{\partial x_{it}}$$

and using $D_{it} = \alpha x_{it} Y_{it}$ and implicit derivation:

$$\frac{\partial D_{it}}{\partial x_{it}} = \alpha x_{it} \frac{\partial Y_{it}}{\partial x_{it}} + Y_{it}\alpha$$

Thus:

$$\Rightarrow \frac{\partial Y_{it}}{\partial x_{it}} = \eta \frac{Y_{it}}{D_{it}} \left[\alpha x_{it} \frac{\partial Y_{it}}{\partial x_{it}} + Y_{it}\alpha \right]$$

$$\Rightarrow \frac{\partial Y_{it}}{\partial x_{it}} = \frac{\eta \frac{Y_{it}}{D_{it}} Y_{it}\alpha}{1 - \alpha x_{it} \eta \frac{Y_{it}}{D_{it}}}$$

Then the FOC w.r.t. x_{it} is:

$$\frac{\eta \frac{Y_{it}}{D_{it}} Y_{it}\alpha}{1 - \alpha x_{it} \eta \frac{Y_{it}}{D_{it}}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = -\mu_{x0} + \mu_{x1}$$

Now, note that the LHS is > 0, then:

$$\mu_{x1} > \mu_{x0} \geq 0$$

$$\Rightarrow \mu_{x1} > 0$$

$$\Rightarrow x_{it} = \bar{x} \tag{A.64}$$

and the FOC w.r.t. x_{it} is:

$$\frac{\eta \frac{Y_{it}}{D_{it}} Y_{it}\alpha}{1 - \alpha x_{it} \eta \frac{Y_{it}}{D_{it}}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = \mu_{x1} \tag{A.65}$$

D.3 Free Entry and the Creation of New Varieties

The free entry condition is given by:

$$\chi w_t = V_{it}$$

D.4 Equilibrium with Outlaw Data Sharing

D.4.1 Expressions for Aggregate Output and Firm Output

Now, we need to solve Y_t. For that, from above $\frac{Y_{it}}{L_{it}} = N_t^{-\frac{1}{\sigma-1}} \frac{Y_t}{L_{pt}}$ and $L_{it} = \frac{L_{pt}}{N_t}$. Thus:

$$Y_t = N_t^{\frac{\sigma}{\sigma-1}} Y_{it} \tag{A.66}$$

But from the firm's problem we have

$$Y_{it} = D_{it}^{\eta} L_{it} \tag{A.67}$$

and $D_{it} = \alpha \bar{x} Y_{it}$. Hence:

$$\begin{aligned} Y_{it} &= (\alpha \bar{x} Y_{it})^{\eta} L_{it} \\ &= Y_{it}^{\eta} (\alpha \bar{x})^{\eta} L_{it} \\ &= Y_{it}^{\eta} (\alpha \bar{x})^{\eta} \left(\frac{L_{pt}}{N_t} \right) \end{aligned}$$

$$\Rightarrow Y_{it} = (\alpha \bar{x})^{\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t} \right)^{\frac{1}{1-\eta}} \tag{A.68}$$

Substitute this expression for Y_{it} in A.66:

$$\begin{aligned} Y_t &= N_t^{\frac{\sigma}{\sigma-1}} (\alpha \bar{x})^{\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t} \right)^{\frac{1}{1-\eta}} \\ &= N_t^{\frac{\sigma}{\sigma-1} - \frac{1}{1-\eta}} (\alpha \bar{x})^{\frac{\eta}{1-\eta}} L_{pt}^{\frac{1}{1-\eta}} \end{aligned} \tag{A.69}$$

D.4.2 Expression for w_t

From A.63, $\frac{\frac{Y_{it}}{L_{it}}}{1-\eta \frac{Y_{it}}{D_{it}} \alpha x_{it}} (1 - \frac{1}{\sigma}) \left(\frac{Y_t}{Y_{it}} \right)^{\frac{1}{\sigma}} = w_t$, hence:

$$\begin{aligned}
w_t &= \frac{\frac{Y_{it}}{L_{it}}}{1-\eta\frac{Y_{it}}{D_{it}}\alpha\bar{x}}(1-\frac{1}{\sigma})\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} \\
&= \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}L_{it}^{-1}}{1-\eta\frac{Y_{it}}{\alpha\bar{x}Y_{it}}\alpha\bar{x}}(1-\frac{1}{\sigma})N_t^{\frac{1}{\sigma-1}} \\
&= \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}L_{it}^{-1}}{1-\eta}(1-\frac{1}{\sigma})N_t^{\frac{1}{\sigma-1}} \\
&= \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left(\frac{L_{pt}}{N_t}\right)^{\frac{1}{1-\eta}}N_tL_{pt}^{-1}}{1-\eta}(1-\frac{1}{\sigma})N_t^{\frac{1}{\sigma-1}} \\
&= \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}}
\end{aligned}$$

Finally:

$$w_t = \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}} \tag{A.70}$$

D.4.3 Expressions for Profits and Value of the Firm

The firm's problem is:

$$r_tV_{it} = \max_{\{L_{it},D_{bit},x_{it},\tilde{x}_{it}\}}\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}Y_{it} - w_tL_{it} + \dot{V}_{it}$$

Thus:

$$V_{it} = \frac{\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}Y_{it} - w_tL_{it}}{r - \frac{\dot{V}_{it}}{V_{it}}}$$

Next, define $\pi_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}Y_{it} - wL_{it}$. Then using the expressions above:

$$
\begin{aligned}
\pi_{it} &= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - wL_{it} \\
&= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w\left(\frac{L_{pt}}{N_t}\right) \\
&= \left(N_t^{\frac{1}{\sigma-1}}\right) N_t^{-\frac{\sigma}{\sigma-1}} Y_t - \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}} \left(\frac{L_{pt}}{N_t}\right) \\
&= N_t^{-1}Y_t - \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \\
&= N_t^{-1}N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}L_{pt}^{\frac{1}{1-\eta}} - \frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} \\
&= L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} (\alpha\bar{x})^{\frac{\eta}{1-\eta}} \left[1 - \frac{\sigma-1}{\sigma(1-\eta)}\right]
\end{aligned} \tag{A.71}
$$

Then, the value of the firm is given by:

$$
\begin{aligned}
V_{it} &= \frac{\left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} Y_{it} - w_t L_{it}}{r - \frac{\dot{V}_{it}}{V_{it}}} \\
&= \frac{L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} (\alpha\bar{x})^{\frac{\eta}{1-\eta}} \left[1 - \frac{\sigma-1}{\sigma(1-\eta)}\right]}{r - g_V}
\end{aligned} \tag{A.72}
$$

D.5 Solution of the Competitive Equilibrium

D.5.1 Firm Size v_{os}

Use the free entry condition:

$$
\chi w_t = V_{it}
$$

Subsisting the expressions for w_t and V_{it} using A.70 and A.72:

$$
\chi\frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}} = \frac{L_{pt}^{\frac{1}{1-\eta}} N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}} (\alpha\bar{x})^{\frac{\eta}{1-\eta}} \left[1 - \frac{\sigma-1}{\sigma(1-\eta)}\right]}{r - g_V}
$$

$$
\chi\frac{1}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{-1}N_t = \frac{\left[1 - \frac{\sigma-1}{\sigma(1-\eta)}\right]}{r - g_V}
$$

$$\Rightarrow \frac{L_{pt}}{N_t} = \frac{\chi \frac{1}{1-\eta}\left(1-\frac{1}{\sigma}\right)(r-g_V)}{1-\frac{\sigma-1}{\sigma(1-\eta)}}$$

$$\Rightarrow \frac{L_{pt}}{N_t} = \frac{\chi \frac{1}{1-\eta}\left(\frac{\sigma-1}{\sigma}\right)(r-g_V)}{\frac{\sigma(1-\eta)-(\sigma-1)}{\sigma(1-\eta)}}$$

$$\Rightarrow \frac{L_{pt}}{N_t} = \frac{\chi(\sigma-1)(r-g_V)}{\sigma(1-\eta)-(\sigma-1)}$$

but $\rho = r - g_V$ then:

$$\Rightarrow \frac{L_{pt}}{N_t} = \frac{\chi\rho(\sigma-1)}{1-\sigma\eta}$$

Define $v_{os} = \frac{L_{pt}}{N_t}$. Then:

$$v_{os} = \frac{\chi\rho(\sigma-1)}{1-\sigma\eta} \tag{A.73}$$

which is equation (76).

D.5.2 Number of varieties

From the evolution of the number of varieties, we have $\dot{N}_t = \frac{1}{\chi}(L_t - L_{pt})$. Thus:

$$\frac{\dot{N}_t}{N_t} = \frac{1}{\chi}\left(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t}\right)$$

In BGP: $\frac{\dot{N}_t}{N_t} = g_N$ and since $g_N = g_L$, then:

$$\Rightarrow g_L = \frac{1}{\chi}\left(\frac{L_t}{N_t} - \frac{L_{pt}}{N_t}\right)$$

$$\Rightarrow \frac{L_t}{N_t} = \frac{L_{pt}}{N_t} + \chi g_L$$

Next, using $\frac{\dot{V}_{it}}{V_{it}} = g_\pi$ and substituting the firm size:

$$\Rightarrow \frac{L_t}{N_t} = \frac{\chi\rho(\sigma-1)}{1-\sigma\eta} + \chi g_L$$

$$\Rightarrow N_t = \frac{L_t}{\frac{\chi\rho(\sigma-1)}{1-\sigma\eta} + \chi g_L}$$

Define $\psi_{os} = \frac{1}{\frac{\chi\rho(\sigma-1)}{1-\sigma\eta}+\chi g_L} = \frac{1}{v_{os}+\chi g_L}$ so that:

$$N_t = L_t\psi_{os} \tag{A.74}$$

which is equation (79).

D.5.3 Solution for aggregate output Y_t^{os}.

From A.69, $Y_t = N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}L_{pt}^{\frac{1}{1-\eta}}$. Then: v_{os}

$$\begin{aligned} Y_t^{os} &= N_t^{\frac{\sigma}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}L_{pt}^{\frac{1}{1-\eta}} \\ &= N_t^{\frac{\sigma}{\sigma-1}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}v_{os}^{\frac{1}{1-\eta}} \\ &= (\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}v_{os} \\ &= (v_{os}\alpha^\eta\bar{x}^\eta)^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

and this is equation (82).

D.5.4 Consumption per capita and growth

$$c_t^{os} = \frac{Y_t^{os}}{L_t} \propto L_t^{\frac{1}{\sigma-1}}$$

which is equation (84). Thus, consumption per capita growth is:

$$g_c^{os} = (\frac{1}{\sigma-1})g_L$$

which is is equation (86).

D.5.5 Firm Production Y_{it}^{os}

Using equation (82) of the paper:

$$\begin{aligned}
Y_{it}^{os} &= Y_t N_t^{-\frac{\sigma}{\sigma-1}} \\
&= \left(v_{os}\alpha^{\eta}\bar{x}^{\eta}\right)^{\frac{1}{1-\eta}} \left(\psi_{os}L_t\right)^{\frac{\sigma}{\sigma-1}} N_t^{-\frac{\sigma}{\sigma-1}} \\
&= \left(v_{os}\alpha^{\eta}\bar{x}^{\eta}\right)^{\frac{1}{1-\eta}} \left(\psi_{os}L_t\right)^{\frac{\sigma}{\sigma-1}} \left(L_t\psi_{os}\right)^{-\frac{\sigma}{\sigma-1}} \\
&= \left(v_{os}\alpha^{\eta}\bar{x}^{\eta}\right)^{\frac{1}{1-\eta}}
\end{aligned}$$

which is equation (92).

D.5.6 Data used by the firm D_{it}^{os}

By definition, $D_{it} = \alpha x_{it} Y_{it}$. Then:

$$\begin{aligned}
D_{it}^{f} &= \alpha x_{it} Y_{it} \\
&= \left(v_{os}\alpha\bar{x}\right)^{\frac{1}{1-\eta}}
\end{aligned}$$

which is equation (88).

D.5.7 Aggregate Data used by the firm D_t^{os}

By definition $D_t = N_t D_{it}$, hence:

$$\begin{aligned}
D_t^{f} &= N_t \left(v_{os}\alpha\bar{x}\right)^{\frac{1}{1-\eta}} \\
&= \left(v_{os}\alpha\bar{x}\right)^{\frac{1}{1-\eta}} L_t \psi_{os}
\end{aligned}$$

which is equation (90).

D.5.8 Labor share $\left(\frac{w_t L_{pt}}{Y_t}\right)$

From A.70:

$$
\begin{aligned}
\left(\frac{w_t L_{pt}}{Y_t}\right)^{os} &= \frac{L_{pt}}{Y_t}\frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}} \\
&= ((v_{os}\alpha^\eta\bar{x}^\eta)^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}})^{-1}\frac{(\alpha\bar{x})^{\frac{\eta}{1-\eta}}}{1-\eta}(1-\frac{1}{\sigma})L_{pt}^{\frac{\eta}{1-\eta}+1}N_t^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}} \\
&= ((v_{os})^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}})^{-1}\frac{1}{1-\eta}(1-\frac{1}{\sigma})\,(v_{os}L_t\psi_{os})^{\frac{\eta}{1-\eta}+1}\,(L_t\psi_{os})^{\frac{1}{\sigma-1}-\frac{\eta}{1-\eta}} \\
&= \frac{1}{1-\eta}(1-\frac{1}{\sigma}) \\
&= \frac{1-\sigma}{\sigma(1-\eta)}
\end{aligned}
$$

which is equation (93).

D.5.9 Profit share $\left(\frac{N_t\pi_t}{Y_t}\right)$

From A.71, $\pi_t = L_{pt}^{\frac{1}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]$. Hence:

$$
\begin{aligned}
\left(\frac{\pi_t N_t}{Y_t}\right)^{os} &= \frac{N_t L_{pt}^{\frac{1}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]}{Y_t} \\
&= \frac{L_{pt}^{\frac{1}{1-\eta}}N_t^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+1}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]}{(v_{os}\alpha^\eta\bar{x}^\eta)^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}}} \\
&= \frac{(v_{os}L_t\psi_{os})^{\frac{1}{1-\eta}}(L_t\psi_{os})^{\frac{1}{\sigma-1}-\frac{1}{1-\eta}+1}(\alpha\bar{x})^{\frac{\eta}{1-\eta}}\left[1-\frac{\sigma-1}{\sigma(1-\eta)}\right]}{(v_{os}\alpha^\eta\bar{x}^\eta)^{\frac{1}{1-\eta}}(\psi_{os}L_t)^{\frac{\sigma}{\sigma-1}}} \\
&= 1-\frac{\sigma-1}{\sigma(1-\eta)} \\
&= \frac{1-\eta\sigma}{\sigma(1-\eta)}
\end{aligned}
$$

and this is equation (94).

D.5.10 Price of a variety p_{it}^{os}

From the household problem, $p_{it} = \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}}$, then:

$$\begin{aligned} p_{it}^{os} &= \left(\frac{Y_t}{Y_{it}}\right)^{\frac{1}{\sigma}} \\ &= \left(\frac{Y_t}{Y_t N_t^{-\frac{\sigma}{\sigma-1}}}\right)^{\frac{1}{\sigma}} \\ &= N_t^{\frac{1}{\sigma-1}} \end{aligned}$$

and from equation (79), $N_t = \psi_{os} L_t$, thus:

$$p_{it}^{f} = (\psi_{os} L_t)^{\frac{1}{\sigma-1}}$$

which is equation (99).

E Alternative Production Functions: Clarifying the Role of Data Nonrivalry

E.1 Cobb-Douglas Data Aggregation

In this section we solve an alternative model in which own-variety data is combined with the bundle of other-variety data by a Cobb-Douglas aggregator, as opposed to the perfect substitutes case in our baseline model.

First, some preliminaries of the economic environment that are the same as in the baseline model (imposing symmetry as usual):

$$\begin{aligned} Y_t &= c_t L_t = N_t^{\frac{\sigma}{\sigma-1}} Y_{it} \\ Y_{it} &= D_{it}^{\eta} L_{it} = D_{it}^{\eta} \frac{L_{pt}}{N_t} \\ \Rightarrow Y_t &= N_t^{\frac{1}{\sigma-1}} D_{it}^{\eta} L_{pt} \end{aligned}$$

The new data aggregation function is

$$D_{it} = (x_t Y_{it})^{\alpha} B_t^{1-\alpha}$$

$$D_{it} = (x_t Y_{it})^{\alpha} (N_t \tilde{x}_t Y_{it})^{1-\alpha}$$
$$D_{it} = (x_t)^{\alpha} (N_t \tilde{x}_t)^{1-\alpha} Y_{it}.$$

Combining the data aggregate with the aggregate production function yields

$$Y_t = N_t^{\frac{1}{\sigma-1}} \left(x_t^{\alpha} \tilde{x}_t^{1-\alpha}\right)^{\eta} N_t^{\eta(1-\alpha)} Y_{it}^{\eta} L_{pt}$$
$$Y_t = N_t^{\frac{1}{\sigma-1}} \left(x_t^{\alpha} \tilde{x}_t^{1-\alpha}\right)^{\eta} N_t^{\eta(1-\alpha)} Y_{it}^{\eta} L_{pt}$$
$$Y_t = \left[N_t^{\frac{1+\eta(1-\alpha)(\sigma-1)-\eta\sigma}{\sigma-1}} \left(x_t^{\alpha} \tilde{x}_t^{1-\alpha}\right)^{\eta} L_{pt}\right]^{\frac{1}{1-\eta}}$$

The social planner problem is the similar to that in the baseline model, with two changes. First, there is this new production function. Second, the privacy components of utility need to have different scaling in order to deliver interior solutions for data use.

$$\max_{\{L_{pt}, x_t, \tilde{x}_t\}} \int_0^{\infty} e^{-\tilde{\rho} t} L_0 \left(\log c_t - \frac{\kappa}{2} x_t^2 - \frac{\tilde{\kappa}}{2} \tilde{x}_t^2\right) dt, \quad \tilde{\rho} \equiv \rho - g_L$$

s.t.

$$c_t = Y_t / L_t$$
$$Y_t = \left[N_t^{\frac{1+\eta(1-\alpha)(\sigma-1)-\eta\sigma}{\sigma-1}} \left(x_t^{\alpha} \tilde{x}_t^{1-\alpha}\right)^{\eta} L_{pt}\right]^{\frac{1}{1-\eta}}$$
$$L_t = L_0 e^{g_L t}$$

Next, define Hamiltonian with state variable N_t, control variables $\{L_{pt}, x_t, \tilde{x}_t\}$ and co-state variable μ_t:

$$H(L_{pt}, x_t, \tilde{x}_t, N_t, \mu_t) = \log Y_t / L_t - \frac{\kappa}{2} x_t^2 - \frac{\tilde{\kappa}}{2} \tilde{x}_t^2 + \mu_t \frac{1}{\chi} (L_t - L_{pt})$$

The FOC are:

$$\begin{cases} \dfrac{\partial H}{\partial L_{pt}} = 0 \\ \dfrac{\partial H}{\partial x_t} = 0 \\ \dfrac{\partial H}{\partial \tilde{x}_t} = 0 \\ \tilde{\rho} = \dfrac{\partial H / \partial N_t}{\mu_t} + \dfrac{\dot{\mu}_t}{\mu_t} \end{cases}$$

Start with $\frac{\partial H}{\partial L_{pt}} = 0$, which implies

$$\frac{1}{Y_t}\frac{\partial Y_t}{\partial L_{pt}} = \frac{\mu_t}{\chi}$$

and therefore

$$L_{pt} = \frac{\chi}{1-\eta} \cdot \frac{1}{\mu_t}.$$

Next, consider $\frac{\partial H}{\partial x_t} = 0$:

$$\frac{1}{Y_t}\frac{\partial Y_t}{\partial x_t} = \kappa x_t.$$

Computing the marginal product and substituting gives

$$x_t = \left(\frac{\alpha}{\kappa}\frac{\eta}{1-\eta}\right)^{\frac{1}{2}}.$$

The next FOC is similar, $\frac{\partial H}{\partial \tilde{x}_t} = 0$:

$$\frac{1}{Y_t}\frac{\partial Y_t}{\partial \tilde{x}_t} = \tilde{\kappa}\tilde{x}_t.$$

Computing the marginal product and substituting gives

$$\tilde{x}_t = \left(\frac{1-\alpha}{\tilde{\kappa}}\frac{\eta}{1-\eta}\right)^{\frac{1}{2}}.$$

Finally, the last FOC is the arbitrage-like equation, $\tilde{\rho} = \frac{\partial H/\partial N_t}{\mu_t} + \frac{\dot{\mu}_t}{\mu_t}$. The derivative of the Hamiltonian $\frac{\partial H}{\partial N_t}$ is:

$$\frac{\partial H}{\partial N_t} = \frac{1}{Y_t}\frac{Y_t}{N_t}\left[\frac{1}{1-\eta} + \frac{\eta}{1-\eta}\left(1-\alpha-\frac{\sigma}{\sigma-1}\right)\right]$$

Substituting this into the FOC and rearranging to solve for μ_t gives

$$\begin{aligned}\mu_t &= \frac{\partial H/\partial N_t}{\tilde{\rho} - \dot{\mu}_t/\mu_t} \\ &= \frac{\frac{1}{N_t}\left[\frac{1}{1-\eta} + \frac{\eta}{1-\eta}\left(1-\alpha-\frac{\sigma}{\sigma-1}\right)\right]}{\tilde{\rho} - \dot{\mu}_t/\mu_t}.\end{aligned}$$

On a BGP

$$\mu_t = \frac{1}{N_t} \frac{\left[\frac{1}{1-\eta} + \frac{\eta}{1-\eta}\left(1 - \alpha - \frac{\sigma}{\sigma-1}\right)\right]}{\rho}$$

Plugging this into the FOC for L_{pt} yields

$$\begin{aligned} \nu_{sp}^{CD} := \frac{L_{pt}}{N_t} &= \frac{\rho \frac{\chi}{1-\eta}}{\frac{1}{1-\eta} + \frac{\eta}{1-\eta}\left(1 - \alpha - \frac{\sigma}{\sigma-1}\right)} \\ &= \frac{\rho\chi}{1 + \eta\left(1 - \alpha - \frac{\sigma}{\sigma-1}\right)} \end{aligned}$$

Finally, plugging this into the entry technology yields

$$N_t^{sp} = \frac{L_t}{\chi g_L + \nu_{sp}^{CD}} =: \psi_{sp}^{CD} L_t.$$

There are two main advantages to this alternative model. First, the algebra is simple for all N_t, not just as N_t gets large. Second, the importance of data nonrivalry shows up more explicitly here than in the baseline model, with $(1 - \alpha)$ now appearing in key equations that characterize equilibrium allocations.

E.2 CRS Goods Production: Rival Ideas and Nonrival Data

In this section, we try to highlight the distinction between the nonrivalry of ideas and the nonrivalry of data. Following Romer 1990, the nonrivalry of ideas is modeled by having the goods production function have constant returns to rival factors, but increasing returns to rival and nonrival factors. In this alternative model, we model ideas as if they were rival by modeling a goods production function with constant returns to scale in ideas and labor, but we still model data as nonrival. Data nonrivalry is modeled by allowing all firms to use all data, i.e., each firm can use the entire bundle of data.

First, some preliminaries of the economic environment that are different from the baseline model because the goods production function is different (imposing symmetry as usual):

$$\begin{aligned} Y_{it} &= D_{it}^{\eta} L_{it}^{1-\eta} = D_{it}^{\eta} \left(\frac{L_{pt}}{N_t}\right)^{1-\eta} \\ D_{it} &= \left(\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t\right) N_t Y_{it} \\ \Rightarrow Y_{it}^{1-\eta} &= N_t^{\eta} \left(\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t\right)^{\eta} \left(\frac{L_{pt}}{N_t}\right)^{1-\eta} \\ Y_{it} &= N_t^{\frac{\eta}{1-\eta}-1} \left(\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t\right)^{\frac{\eta}{1-\eta}} L_{pt} \end{aligned}$$

Since

$$Y_t = N_t^{\frac{\sigma}{\sigma-1}} Y_{it},$$

then

$$Y_t = N_t^{\frac{\sigma}{\sigma-1}+\frac{\eta}{1-\eta}-1} \left(\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t\right)^{\frac{\eta}{1-\eta}} L_{pt}.$$

The social planner problem is the same as in the baseline model, with this new production function:

$$\max_{\{L_{pt}, x_t, \tilde{x}_t\}} \int_0^{\infty} e^{-\tilde{\rho} t} L_0 \left(\log c_t - \frac{\kappa}{2}\frac{1}{N_t} x_t^2 - \frac{\tilde{\kappa}}{2}\tilde{x}_t^2\right) dt, \quad \tilde{\rho} \equiv \rho - g_L$$

s.t.

$$\begin{aligned} c_t &= Y_t / L_t \\ Y_t &= \left[N_t^{\frac{1+\eta(1-\alpha)(\sigma-1)-\eta\sigma}{\sigma-1}} \left(x_t^{\alpha} \tilde{x}_t^{1-\alpha}\right)^{\eta} L_{pt} \right]^{\frac{1}{1-\eta}} \\ \dot{N}_t &= \frac{1}{\chi}(L_t - L_{pt}) \\ L_t &= L_0 e^{g_L t} \end{aligned}$$

Next, define Hamiltonian with state variable N_t, control variables $\{L_{pt}, x_t, \tilde{x}_t\}$ and co-state variable μ_t:

$$H(L_{pt}, x_t, \tilde{x}_t, N_t, \mu_t) = \log Y_t / L_t - \frac{\kappa}{2}\frac{1}{N_t} x_t^2 - \frac{\tilde{\kappa}}{2}\tilde{x}_t^2 + \mu_t \frac{1}{\chi}(L_t - L_{pt})$$

The FOC are:

$$\begin{cases} \dfrac{\partial H}{\partial L_{pt}} = 0 \\ \dfrac{\partial H}{\partial x_t} = 0 \\ \dfrac{\partial H}{\partial \tilde{x}_t} = 0 \\ \tilde{\rho} = \dfrac{\partial H / \partial N_t}{\mu_t} + \dfrac{\dot{\mu}_t}{\mu_t} \end{cases}$$

Start with $\frac{\partial H}{\partial L_{pt}} = 0$, which implies

$$\frac{1}{Y_t}\frac{\partial Y_t}{\partial L_{pt}} = \frac{\mu_t}{\chi}$$

and therefore

$$L_{pt} = \frac{\chi}{\mu_t}.$$

Next, consider $\frac{\partial H}{\partial x_t} = 0$:

$$\frac{1}{Y_t}\frac{\partial Y_t}{\partial x_t} = \frac{\kappa}{N_t} x_t.$$

Computing the marginal product and substituting gives

$$x_t = \frac{\alpha}{\kappa}\frac{\eta}{1-\eta}\frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t}.$$

The next FOC is similar, $\frac{\partial H}{\partial \tilde{x}_t} = 0$:

$$\frac{1}{Y_t}\frac{\partial Y_t}{\partial \tilde{x}_t} = \tilde{\kappa}\tilde{x}_t.$$

Computing the marginal product and substituting gives

$$\tilde{x}_t = \frac{1-\alpha}{\tilde{\kappa}}\frac{\eta}{1-\eta}\frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t}.$$

Plug the solution for $\tilde{x}_t$ into FOC(x_t):

$$\begin{aligned} \tilde{x}_t &= \frac{1-\alpha}{\alpha}\frac{\kappa}{\tilde{\kappa}} x_t \\ \Rightarrow x_t &= \frac{\alpha}{\kappa}\frac{\eta}{1-\eta}\frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\frac{1-\alpha}{\alpha}\frac{\kappa}{\tilde{\kappa}} x_t} \end{aligned}$$

$$x_t = \left(\frac{\alpha}{\kappa} \frac{\eta}{1-\eta} \frac{1}{\frac{\alpha}{N_t} + (1-\alpha)\frac{1-\alpha}{\alpha}\frac{\kappa}{\tilde{\kappa}}} \right)^{\frac{1}{2}}$$

As N_t gets large:

$$x_t = \frac{\alpha}{1-\alpha} \frac{1}{\kappa} \left(\frac{\eta\tilde{\kappa}}{1-\eta} \right)^{\frac{1}{2}}$$
$$\tilde{x}_t = \frac{1}{\tilde{\kappa}} \left(\frac{\eta\tilde{\kappa}}{1-\eta} \right)^{\frac{1}{2}}.$$

Finally, the last FOC is the arbitrage-like equation, $\tilde{\rho} = \frac{\partial H/\partial N_t}{\mu_t} + \frac{\dot{\mu}_t}{\mu_t}$. The derivative of the Hamiltonian $\frac{\partial H}{\partial N_t}$ is:

$$\frac{\partial H}{\partial N_t} = \frac{1}{Y_t} \left[\frac{Y_t}{N_t} \left(\frac{\sigma}{\sigma-1} + \frac{\eta}{\eta-1} - 1 \right) - \frac{\eta}{\eta-1} \frac{Y_t}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t} \frac{\alpha x_t}{N_t^2} + \frac{\kappa}{2} \frac{x_t^2}{N_t^2} \right]$$
$$= \frac{1}{N_t} \left[\frac{1}{\sigma-1} + \frac{\eta}{\eta-1} - \frac{\eta}{\eta-1} \frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t} \frac{\alpha x_t}{N_t} + \frac{\kappa}{2} \frac{x_t^2}{N_t} \right]$$

Substituting this into the FOC and rearranging to solve for μ_t gives

$$\mu_t = \frac{\partial H/\partial N_t}{\tilde{\rho} - \dot{\mu}_t/\mu_t}$$
$$= \frac{\frac{1}{N_t} \left[\frac{1}{\sigma-1} + \frac{\eta}{\eta-1} - \frac{\eta}{\eta-1} \frac{1}{\frac{\alpha x_t}{N_t} + (1-\alpha)\tilde{x}_t} \frac{\alpha x_t}{N_t} + \frac{\kappa}{2} \frac{x_t^2}{N_t} \right]}{\tilde{\rho} - \dot{\mu}_t/\mu_t}.$$

On a BGP with N_t large

$$\mu_t = \frac{1}{N_t} \frac{\frac{1}{\sigma-1} + \frac{\eta}{1-\eta}}{\rho}$$

Plugging this into the FOC for L_{pt} yields

$$\nu_{sp}^{CRS} := \frac{L_{pt}}{N_t} = \frac{\rho\chi}{\frac{1}{\sigma-1} + \frac{\eta}{1-\eta}}$$

Finally, plugging this into the entry technology yields

$$N_t^{sp} = \frac{L_t}{\chi g_L + \nu_{sp}^{CRS}} =: \psi_{sp}^{CRS} L_t.$$

Even though there are no increasing returns to producing goods at the firm level, the growth rate of per capita income in this economy is the same as in the baseline economy. The $\frac{\eta}{1-\eta}$ term in the growth rate is due to the nonrivalry of data, not the increasing returns to goods production that captures the nonrivalry of ideas.