The Model Selection Curse

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Online Appendix: Proof of Proposition 1

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Fix the realization of sample noise ε. The coefficients b_0 and b_1 are given by the solution to the first-order conditions of

$$\min_{b_0,b_1} \qquad \sum_{x=0,1}(y_x - b_0 - b_1 x)^2 + c_0\mathbf{1}_{b_1\neq 0} + c_1|b_1|$$

where the dependence of the coefficients b_0 and b_1 on the noise realization ε is suppressed for notational ease.

The first-order condition with respect to b_0 is

$$(y_0 - b_0) + (y_1 - b_0 - b_1) = 0 \tag{1}$$

while the first-order condition with respect to b_1 when $b_1 \neq 0$ gives

$$2(y_1 - b_0 - b_1) = sign(b_1)c_1 \tag{2}$$

In particular, $2(y_1 - b_0 - b_1) = c_1$ when $b_1 > 0$, and $2(y_1 - b_0 - b_1) = -c_1$ when $b_1 < 0$.

From (1) we obtain

$$b_0 = \frac{1}{2}(y_0 + y_1 - b_1)$$

Plugging this into (2), we obtain the following characterization of b_1 conditional on it being non-zero:

$$b_1 = \begin{cases} \beta_1 + \varepsilon_1 - \varepsilon_0 - c_1 & if \quad b_1 > 0 \\ \beta_1 + \varepsilon_1 - \varepsilon_0 + c_1 & if \quad b_1 < 0 \end{cases}$$

This means in particular that when $\beta_1 + \varepsilon_1 - \varepsilon_0 \in (-c_1, c_1)$, $b_1 = 0$.

To complete the characterization of when $b_1 \neq 0$, we compute the difference between the Residual Sum of Squares (RSS) when the coefficient b_1 is admitted and when it is omitted. First,

$$RSS(b_1 \neq 0) = (b_0 - y_0)^2 + (b_0 + b_1 - y_1)^2$$

where b_0 and b_1 are given by (1) and (2). In contrast, when b_1 is omitted, $b_0 = \frac{1}{2}(y_0 + y_1)$, such that

$$RSS(b_1 = 0) = \left(\frac{1}{2}y_0 + \frac{1}{2}y_1 - y_0\right)^2 + \left(\frac{1}{2}y_0 + \frac{1}{2}y_1 - y_1\right)^2 = \frac{1}{2}(y_1 - y_0)^2$$

It follows that

$$\begin{aligned} RSS(b_1 &= 0) - RSS(b_1 \neq 0) = \frac{1}{2}(y_1 - y_0)^2 - (b_0 - y_0)^2 - (b_0 + b_1 - y_1)^2 \\ &= b_1[y_1 - y_0 - \frac{1}{2}b_1] \\ &= [y_1 - y_0 - sign(b_1)c_1][y_1 - y_0 - \frac{1}{2}(y_1 - y_0 - sign(b_1)c_1)] \\ &= \frac{1}{2}(y_1 - y_0)^2 - \frac{1}{2}(c_1)^2 \\ &= \frac{1}{2}(\beta_1 + \varepsilon_1 - \varepsilon_0)^2 - \frac{1}{2}(c_1)^2 \end{aligned}$$

The condition for $b_1 \neq 0$ is

$$RSS(b_1 = 0) - RSS(b_1 \neq 0) \geq c_0$$

i.e.

$$(\beta_1 + \varepsilon_1 - \varepsilon_0)^2 \geq (c_1)^2 + 2c_0$$

This concludes the proof.