Online Appendix
Search, Screening and Sorting

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Appendix A Proofs and Additional Results

A.1 Proof of Lemma 1

First, consider the application stage. Given queue length λ, a firm's number of applicants n_A in our benchmark model follows a geometric distribution with support $\mathbb{N}_0$ and mean λ, i.e. $\mathbb{P}\left[n_A = n \,|\, \lambda\right] = \frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^n$ for $n = 0, 1, 2, \ldots$. If $\sigma = 1$ (firms can interview all candidates), then we have

$$\phi(\mu, \lambda) = 1 - \sum_{n=0}^{\infty} \mathbb{P}\left[n_A = n \,|\, \lambda\right]\left(1 - \frac{\mu}{\lambda}\right)^n = \frac{\mu}{1+\mu},$$

where the first equality uses the fact that the probability that an applicant is high-type is μ/λ and is independent across applicants.

Next, consider the screening stage. A firm's *potential* number of interviews, n_C, follows a geometric distribution with support $\mathbb{N}_1$ and mean $(1-\sigma)^{-1}$. That is, $\mathbb{P}[n_C \geq n \,|\, \sigma] = \sigma^{n-1}$ for $n = 1, 2, \ldots$. Since interviewing might be constrained by the number of applications, the firm's *actual* number of interviews is $n_I = \min\{n_A, n_C\} \in \mathbb{N}_0$, distributed according to $\mathbb{P}\left[n_I \geq n \,|\, \lambda, \sigma\right] = \mathbb{P}\left[n_A \geq n \,|\, \lambda\right]\sigma^{n-1} = \left(\frac{\lambda}{1+\lambda}\right)^n \sigma^{n-1}$. An interview reveals a high-type worker with probability μ/λ, independently across applicants. The firm therefore interviews at least one high-type

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worker with probability

$$\phi(\mu,\lambda) = 1 - \sum_{n=0}^{\infty} \mathbb{P}\left[n_I = n|\lambda,\sigma\right]\left(1-\frac{\mu}{\lambda}\right)^n = \sum_{n=1}^{\infty} \mathbb{P}\left[n_I \geq n|\lambda,\sigma\right]\frac{\mu}{\lambda}\left(1-\frac{\mu}{\lambda}\right)^{n-1},$$

where the second equality follows from summation by parts. Substituting $\mathbb{P}\left[n_I \geq n \mid \lambda,\sigma\right] = \left(\frac{\lambda}{1+\lambda}\right)^n \sigma^{n-1}$ yields equation (2). □

A.2 Marginal Contributions

Adding more low-type workers to a submarket only increases λ, while adding more high-type workers increases both μ and λ. Thus, the marginal contribution of low-type and high-type workers at a firm of type y with queues (μ,λ) are $S_\lambda(\mu,\lambda,y)$ and $S_\mu(\mu,\lambda,y)+S_\lambda(\mu,\lambda,y)$, respectively. Because of constant returns to scale, the firm's marginal contribution is the difference between total surplus and the sum of the marginal contributions of its applicants, i.e. $S(\mu,\lambda,y)-\mu S_\mu(\mu,\lambda,y)-\lambda S_\lambda(\mu,\lambda,y)$.[36] Using $S(\mu,\lambda,y)$ from (3), $f^1 \equiv f(x_1,y)$ and $\Delta f = f(x_2,y) - f(x_1,y)$, we get

$$T_1(\mu,\lambda,y) = m'(\lambda)f^1 + \phi_\lambda(\mu,\lambda)\Delta f, \tag{26}$$

$$T_2(\mu,\lambda,y) = m'(\lambda)f^1 + \left(\phi_\mu(\mu,\lambda)+\phi_\lambda(\mu,\lambda)\right)\Delta f, \tag{27}$$

$$R(\mu,\lambda,y) = \left(m(\lambda)-\lambda m'(\lambda)\right)f^1 + \left(\phi(\mu,\lambda)-\mu\phi_\mu(\mu,\lambda)-\lambda\phi_\lambda(\mu,\lambda)\right)\Delta f, \tag{28}$$

where T_1, T_2 and R are the marginal contribution to surplus of low-type workers, high-type workers, and firms, respectively.

A.3 Concavity of Surplus Function $S(\mu,\lambda,y)$

Let $\kappa(y)$ be a measure of output dispersion, defined as the relative gain in output for a firm of type y from hiring a high- rather than a low-type worker, i.e.

$$\kappa(y) \equiv \frac{f(x_2,y)-f(x_1,y)}{f(x_1,y)} > 0. \tag{29}$$

The following lemma then presents the planner's second-order condition (SOC).[37]

[36] Alternatively, increase the number of firms by a factor $1+\Delta s$. The additional surplus is then $(1+\Delta s)S(\mu/(1+\Delta s),\lambda/(1+\Delta s),y) - S(\mu,\lambda,y)$, which yields the same result when $\Delta s \to 0$.

[37] We omit the arguments of the derivatives of $\phi(\mu,\lambda)$ and $m(\lambda)$ for simplicity.

Lemma 4. *Surplus $S(\mu,\lambda,y)$ is strictly concave at a point (μ,λ) with $0<\mu<\lambda$ if*

$$\frac{1}{\kappa(y)} > \frac{\phi_{\lambda\lambda}-\phi_{\mu\lambda}^2/\phi_{\mu\mu}}{-m''}. \tag{30}$$

Proof. The Hessian $\mathcal{H}(\mu,\lambda,y)$ of $S(\mu,\lambda,y)$ equals

$$\mathcal{H}(\mu,\lambda,y) = \begin{pmatrix} \phi_{\mu\mu}\Delta f & \phi_{\mu\lambda}\Delta f \\ \phi_{\mu\lambda}\Delta f & m''f^1+\phi_{\lambda\lambda}\Delta f \end{pmatrix}.$$

When $\sigma>0$, we have $\phi_{\mu\mu}<0$. So, the Hessian is negative definite if and only if its determinant is positive, i.e. $\Delta f\left[m''\phi_{\mu\mu}f^1+\left(\phi_{\mu\mu}\phi_{\lambda\lambda}-\phi_{\mu\lambda}^2\right)\Delta f\right]>0$. Using $\Delta f>0$ and the definition of $\kappa(y)$, we obtain condition (30). $\square$

The right-hand side of (30) is a rescaled version of the determinant of the Hessian matrix of $\phi(\mu,\lambda)$. It is zero if $\sigma=1$, which means that the SOC always holds in that case.[38] It is positive for $0<\sigma<1$ and converges to infinity when $\sigma\to 0$. That is, the SOC never holds when meetings are bilateral, as is well-known from Eeckhout and Kircher (2010); in what follows, we will therefore focus on the case $\sigma>0$, but our results extend to the bilateral case by continuity.

If the planner creates a submarket with queues (μ,λ), then (30) must hold, otherwise splitting the submarket increases total surplus.

A.4 Proof of Lemma 2

We prove this result and discuss it extensively in Cai et al. (2022). Here, we state the single-crossing condition and briefly argue why it leads to Lemma 2. To do so, we define $H(\mu,\lambda)$ as the right-hand side of (30), i.e.

$$H(\mu,\lambda) \equiv \frac{\phi_{\lambda\lambda}-\phi_{\mu\lambda}^2/\phi_{\mu\mu}}{-m''}. \tag{31}$$

Cai et al. (2022) then show that Lemma 2 holds whenever a meeting technology satisfies Property A0, A1, A2 and the following A3.

[38] Cai et al. (2017) describe a broader class of meeting technologies for which $\phi(\mu,\lambda)$ is jointly concave in (μ,λ) such that (30) is always satisfied. However, as they show, such technologies feature (weakly) positive meeting externalities, making them unsuitable for our paper.

A3. (single-crossing condition) At any point (ζ, λ) where $H(\lambda\zeta, \lambda) > 0$, we have $\partial H(\lambda\zeta, \lambda)/\partial\lambda > 0$ and

$$-\frac{\partial\phi_\mu(\lambda\zeta, \lambda)/\partial\zeta}{\partial\phi_\mu(\lambda\zeta, \lambda)/\partial\lambda} < -\frac{\partial H(\lambda\zeta, \lambda)/\partial\zeta}{\partial H(\lambda\zeta, \lambda)/\partial\lambda}. \tag{32}$$

Note that Property A0 states that $\partial\phi_\mu(\lambda\zeta, \lambda)/\partial\zeta < 0$, while Property A2 states that $\partial\phi_\mu(\lambda\zeta, \lambda)/\partial\lambda < 0$, making the left-hand side of (32) strictly negative. When $\phi(\mu, \lambda)$ is given by (2), direct computation reveals that both $H(\lambda\zeta, \lambda)$ and the right-hand side of (32) are strictly positive. Thus, Property A3 is trivially satisfied in this case.

Let $R(\mu, \lambda, y)$ and $T_2(\mu, \lambda, y)$ denote the marginal contributions to surplus of firms and high-type workers, respectively, as derived in Appendix A.2. The idea of the proof of Cai et al. (2022) is then as follows. Suppose that the marginal contribution to surplus of firms equals R^*. Property A3 then implies that, in the λ-ζ plane, the level curve $R(\lambda\zeta, \lambda, y) = R^*$ crosses the level curve $H(\lambda\zeta, \lambda) = 1/\kappa(y)$ at most once and from the left, as illustrated in Figure 1 of Cai et al. (2022). If the intersection exists, denote it by (λ^*, ζ^*). Along the level curve $R(\lambda\zeta, \lambda, y) = R^*$, the SOC (30) is then satisfied for $\zeta > \zeta^*$ and violated for $\zeta < \zeta^*$. The only feasible submarket when $\zeta < \zeta^*$ is therefore the corner solution $\zeta = 0$. Furthermore, along the level curve $R(\lambda\zeta, \lambda, y) = R^*$, the marginal contribution to surplus by high-type workers, $T_2(\lambda\zeta, \lambda, y)$ is monotonically decreasing in ζ for $\zeta \geq \zeta^*$. Since the marginal contribution of high-type workers must be the same among all submarkets containing such workers, there can exist only one submarket with $\zeta \geq \zeta^*$. Hence, there exist at most two submarkets: one with $\zeta = 0$ and the other with $\zeta \geq \zeta^*$. Cai et al. (2022) then show that there exists only one pair of (γ, Δ) which satisfies the FOC for the maximization problem in (4). Hence, the planner's solution is unique. □

A.5 Proof of Proposition 2

Consider first the degenerate case $x = x_1 = x_2$. Surplus in a submarket equals $m(\lambda)f(x, y)$, so the marginal contribution of a worker is $m'(\lambda)f(x, y)$, which must be the same across different submarkets. That is, the optimal queue length $\lambda(y)$ satisfies $m'(\lambda(y))f(x, y) = W$, where W is a constant such that $\int_{\underline{y}}^{\overline{y}} \lambda(y) = L$.

When x_2 is sufficiently close to $x_1 = x$, then the marginal contributions of low-and high-type workers will be close to W, which implies that to solve the planner's problem, it is without loss of generality to limit the queue length of each firm to $\overline{\lambda} \equiv 2\lambda(\overline{y})$, where $\lambda(\overline{y})$ is the optimal queue length of the firm with the highest type in the degenerate case. That is, to solve the planner's problem in (5), we can restrict $(\mu(y), \lambda(y))$ to be in the convex set $\Delta \equiv \{(\mu, \lambda) \,|\, 0 \leq \mu \leq \lambda \leq \overline{\lambda}\}$.

In this set Δ, the right-hand side of the firm's SOC (30) is bounded due to continuity. Hence, (30) will hold for all (μ, λ) in Δ when $\kappa(y)$, or equivalently $x_2 - x_1$, is sufficiently small. That is, for each firm type y, the surplus function $S(\mu, \lambda, y)$ is strictly concave on the set Δ, which implies that in the planner's problem in (5), we can replace $\widetilde{S}(\mu, \lambda, y)$ with $S(\mu, \lambda, y)$ because two submarkets in Δ are strictly suboptimal. Thus the planner solves a standard (strictly) concave maximization problem; the optimal solution $(\mu(y), \lambda(y))$ is unique and continuous. Furthermore, when $\mu(y)$ and $\lambda(y)$ satisfy $0 < \mu(y) < \lambda(y)$ for some firm type y, they are jointly determined by the FOCs (16) and (17).

As $x_2 \to x_1 = x$, the FOCs (16) and (17) converge to $m'(\lambda^*(y))f(x, y)$ (which is constant across firms with $\lambda^*(y) > 0$) and $\phi_\mu(\lambda^*(y)\zeta^*(y), \lambda^*(y))f_x(x, y)$ (which is constant across firms with $\zeta^*(y) \in (0, 1)$). Without loss of generality, assume that all firms are active: $\lambda^*(y) > 0$ (otherwise, $\lambda^*(y) = 0$ for firms with small y, in which case we can exclude them from consideration). By the implicit function theorem, $\zeta^*(y)$ is differentiable whenever it is interior. Since $\zeta^*(y)$ is continuous, PAM holds if whenever $\zeta^*(y)$ is interior, i.e. $\frac{d}{dy}h(\zeta^*(y), \lambda^*(y)) \geq 0$. Assume that $\zeta^*(y)$ is interior for some firm type y. Differentiating the two FOCs with respect to y yields

$$
\begin{aligned}
0 &= m''(\lambda(y))\lambda'(y)f(x, y) + m'(\lambda(y))f_y(x, y) \\
0 &= \left(\frac{\partial \phi_\mu}{\partial \zeta(y)}\zeta'(y) + \frac{\partial \phi_\mu}{\partial \lambda(y)}\lambda'(y)\right) f_x(x, y) + \phi_\mu f_{xy}(x, y),
\end{aligned}
$$

where we suppress the arguments of $\phi_\mu(\lambda(y)\zeta(y), \lambda(y))$ and the superscripts of $(\zeta^*(y), \lambda^*(y))$. Combining these two equations yields $\zeta'(y)$ and $\lambda'(y)$, which then implies that $\zeta'(y) \geq 0$ if and only if $\rho(x, y) \geq a^c(\zeta(y), \lambda(y))$ and $\frac{d}{dy}h(\zeta(y), \lambda(y)) \geq 0$ if and only if $\rho(x, y) \geq a^m(\zeta(y), \lambda(y))$.

Next, we show the necessity of (23) and (24). We only consider the case of PAC; the other cases (PAM, NAC and NAM) follow the same logic. Suppose that (23) does not hold for $i = c$, so that there exist x_0, y_0, ζ_0, and λ_0 such that $\rho(x_0, y_0) < a^c(\zeta_0, \lambda_0)$. We can then construct a counterexample in which worker/firm heterogeneity is small and NAC holds at the planner's solution. In particular, by continuity, we can assume that $0 < \zeta_0 < 1$ (note the strict inequality), and that there exists a small ϵ_0 such that the above inequality holds for all $x \in [x_0, x_0 + \epsilon_0]$, $y \in [y_0 - \epsilon_0, y_0 + \epsilon_0]$, $\zeta \in [\zeta_0 - \epsilon_0, \zeta_0 + \epsilon_0]$, and $\lambda \in [(1 - \epsilon_0)\lambda_0, (1 + \epsilon_0)\lambda_0]$. Fix ϵ_0 from now on and set $x_1 = x_0$, $Lz = \lambda_0\zeta_0$, $L(1 - z) = \lambda_0(1 - \zeta_0)$, $\underline{y} = y_0 - \epsilon_1$, and $\overline{y} = y_0 + \epsilon_1$ for some $\epsilon_1 \leq \epsilon_0$. Next, we reduce firm heterogeneity by letting $\epsilon_1 \to 0$. When ϵ_1 is sufficiently small, $\lambda(y) \in [(1-\epsilon_0)\lambda_0, (1+\epsilon_0)\lambda_0]$ and $\zeta(y) \in [\zeta_0-\epsilon_0, \zeta_0+\epsilon_0]$ for all y. Thus, NAC holds at the planner's solution. □

A.6 Proof of Lemma 3

We first consider $a^c(\zeta, \lambda)$. Since $\phi(\mu, \lambda)$ is given by equation (2) and $a^c(\zeta, \lambda)$ is defined by equation (21), direct calculation yields

$$a^c(\zeta, \lambda) = \frac{1 + \lambda}{2\lambda}\left(1 + \frac{1}{1 + (1 - \sigma)\lambda} - \frac{2}{1 + \sigma\zeta\lambda + (1 - \sigma)\lambda}\right). \tag{33}$$

Note that $a^c(\zeta, \lambda)$ is strictly increasing in ζ. Thus, we have $\max_\zeta a^c(\zeta, \lambda) = a^c(1, \lambda)$ and $\min_\zeta a^c(\zeta, \lambda) = a^c(0, \lambda)$. Moreover, (33) reveals that $a^c(0, \lambda) + a^c(1, \lambda) = 1$ and $\frac{da^c(1,\lambda)}{\lambda} = -\frac{\sigma(1-\sigma)}{2(1+(1-\sigma)\lambda)^2} < 0$. Therefore, $a^c(1, \lambda)$ approaches its supremum when $\lambda \to 0$ and $a^c(0, \lambda)$ approaches its infimum when $\lambda \to 0$. Hence, we have $\sup_{\zeta,\lambda} a^c(\zeta, \lambda) = \lim_{\lambda\to0} a^c(1, \lambda) = (1+\sigma)/2$ and $\inf_{\zeta,\lambda} a^c(\zeta, \lambda) = 1-\sup_{\zeta,\lambda} a^c(\zeta, \lambda) = (1-\sigma)/2$, where neither the infimum nor the supremum can be reached because we require $\lambda > 0$. Furthermore,

$$\frac{\partial a^c(\zeta, \lambda)}{\partial\sigma} = \frac{1 + \lambda}{2}\left(\frac{1}{(1 + \lambda(1 - \sigma))^2} - \frac{2(1 - \zeta)}{(1 + \lambda(1 - \sigma) + \lambda\sigma\zeta)^2}\right).$$

Hence, $a^c(\zeta, \lambda)$ is strictly increasing in σ if and only if $\frac{\lambda\zeta\sigma}{1+\lambda(1-\sigma)} > \sqrt{2(1 - \zeta)} - 1$.

Next, we consider $a^m(\mu, \lambda)$. Analogous to above, direct computation yields

$$a^m(\zeta, \lambda) = \frac{1}{2}\left(1 + \frac{\sigma(2\zeta - 1)}{1 + (1 - \sigma)\lambda}\right). \tag{34}$$

Note that $a^m(\zeta, \lambda)$ is strictly increasing in ζ. For a given λ, $a^m(\zeta, \lambda)$ therefore reaches its minimum at $\zeta = 0$ and its maximum at $\zeta = 1$. Because $a^m(0, \lambda) = a^c(0, \lambda)$ and $a^m(1, \lambda) = a^m(1, \lambda)$, we have $\overline{a}^m = \overline{a}^c$ and $\underline{a}^m = \underline{a}^c$. The above equation implies that $a^m(\zeta, \lambda)$ is strictly increasing in σ if and only if $\zeta > 1/2$. When $\zeta = 1/2$, $a^m(1/2, \lambda) = 1/2$, independent of λ.

Finally, note that

$$a^c(\zeta, \lambda) - a^m(\zeta, \lambda) = \frac{\zeta(1-\zeta)\sigma^2\lambda}{(1 + (1-\sigma)\lambda)(1 + \sigma\zeta\lambda + (1-\sigma)\lambda)} \geq 0.$$

Thus, when $\sigma > 0$, $a^c(\zeta, \lambda) = a^m(\zeta, \lambda)$ if and only if $\zeta = 0$ or $\zeta = 1$. □

A.7 Proof of Proposition 3

To prove PAM/PAC, we establish two results (in Secion A.7.3 and A.7.4, respectively). First, we show that if there exists a firm type y_m that is present in two submarkets, then $\zeta(y)$ must jump up around type y_m under the assumption $\underline{\rho} \geq (1+\sigma)/2$ (note that $\underline{\rho} > 1/2$ is actually sufficient; see Lemma 7).

Second, we show that if firm types have a unique optimal queue within some interval, then both $\zeta(y)$ and $h(\zeta(y), \lambda(y))$ are increasing in y within this interval when $\underline{\rho} \geq (1+\sigma)/2$.

These two results jointly imply that PAC/PAM holds at the planner's solution. Note that if there exist no firm types with two submarkets, then the second result above implies that PAC/PAM holds. Suppose that there exists a single firm type y_m which has two submarkets where $\zeta(y_m)$ is 0 and $\zeta_1 > 0$ (in Figures 1a and 1b, $y_m = 0.6$ and $\zeta_1 = 0.2$). Then when $y < y_m$ or $y > y_m$, firms of type y have a unique optimal queue. The second result above implies that both $\zeta(y)$ and $h(\zeta(y), \lambda(y))$ are increasing when $y < y_m$ and when $y > y_m$. Recall that $\mathcal{Q}(y)$ is the set of queues that firms of type y face at the planner's solution. Since $\mathcal{Q}(y)$ solves the maximization problem in (8), it is an upper hemi-continuous correspondence by the Theorem of the Maximum. The first result above then implies $\lim_{y \uparrow y_m} \zeta(y) = 0$ and $\lim_{y \downarrow y_m} \zeta(y) = \zeta_1$. Therefore, the resulting optimal queues must look like the one in Figure 1a. Hence, PAC/PAM holds.

Finally, note that there exists at most one firm type that is present in two submarkets when $\underline{\rho} \geq (1+\sigma)/2$. As before, suppose that firms of type y_m have two submarkets. Then, $\zeta(y_m)$ is 0 and $\zeta_1 > 0$. Then the first result above implies that firms with type y slightly above y_m have a unique submarket whose $\zeta(y)$ is close to ζ_1 and firms with types slightly below y_m have a unique submarket whose $\zeta(y)$ is close to 0. Therefore, firm types that have two submarkets are isolated from each other so that we can list them as $y_m^1 < \cdots < y_m^K$. Assume that $K \geq 2$, and that $\zeta(y_m^i)$ is either 0 or ζ_1^i for $i = 1, \ldots, K$. Then firms of type $y \in (y_m^i, y_m^{i+1})$ have a unique optimal queue, and by the first result above, $\lim_{y \downarrow y_m^i} \zeta(y) = \zeta_1^i$ and $\lim_{y \uparrow y_m^{i+1}} \zeta(y) = 0$, which contradicts with the second result above. Hence there exists at most one firm type that is present in two submarkets.

After presenting two helpful lemmas in Section A.7.1 and A.7.2, we prove the two main results in Section A.7.3 and A.7.4. Finally, we show that the planner's solution is unique in Section A.7.5.

A.7.1 The Elasticity of Complementarity Revisited.

Note that $\rho(x, y)$ is the ratio of the percentage change in $f_y(x, y)$ (the marginal output by firms) and the percentage change in $f(x, y)$ caused by increasing the worker type to $x + \Delta x$. That is, for sufficiently small $\Delta x > 0$, we have

$$\frac{f_y(x + \Delta x, y)}{f_y(x, y)} \approx 1 + \rho(x, y) \frac{f_x(x, y)}{f(x, y)} \Delta x \approx \left(\frac{f(x + \Delta x, y)}{f(x, y)} \right)^{\rho(x,y)} .$$

In general, when x is discrete and $\rho(x, y)$ is not necessarily constant, the elasticity of f_y with respect to f is bounded by $\underline{\rho}$ and $\overline{\rho}$, as summarized by the following lemma.

Lemma 5. *For given y, $f_y(x, y)/f(x, y)^{\underline{\rho}}$ is increasing in x, and $f_y(x, y)/f(x, y)^{\overline{\rho}}$ is decreasing in x. That is,*

$$\left(\frac{f(x_2, y)}{f(x_1, y)} \right)^{\underline{\rho}} \leq \frac{f_y(x_2, y)}{f_y(x_1, y)} \leq \left(\frac{f(x_2, y)}{f(x_1, y)} \right)^{\overline{\rho}}, \tag{35}$$

where the first (resp. second) inequality holds as equality if and only if $\underline{\rho}$ (resp. $\overline{\rho}$) is equal to $\rho(x, y)$ for all $x \in [x_1, x_2]$.

Proof. Given ρ_0, the derivative of $\log f_y(x, y) - \rho_0 \log f(x, y)$ with respect to x equals

$$\frac{\partial}{\partial x} (\log f_y - \rho_0 \log f) = \frac{f_{xy}}{f_y} - \rho_0 \frac{f_x}{f} = \frac{f_{xy} f - \rho_0 f_x f_y}{f f_y},$$

where we suppress the arguments of $f(x, y)$ and its partial derivatives for simplicity.

The right-hand side is weakly positive (resp. negative) if $\rho_0 = \underline{\rho}$ (resp. $\rho_0 = \overline{\rho}$), which means that $\log f_y(x_2, y) - \underline{\rho} \log f(x_2, y) \geq \log f_y(x_1, y) - \underline{\rho} \log f(x_1, y)$, and $\log f_y(x_2, y) - \overline{\rho} \log f(x_2, y) \geq \log f_y(x_1, y) - \overline{\rho} \log f(x_1, y)$, which jointly imply (35). □

A.7.2 A Technical Lemma

The first two parts of the following lemma are trivial, whereas the third part is non-trivial and critical for our results.

Lemma 6. *(i) If $\rho > 1$, then $\frac{1}{\kappa}((1+\kappa)^\rho - 1)$ is strictly increasing for $\kappa > 0$; (ii) if $\rho \in (0, 1)$, then $\frac{1}{\kappa}((1+\kappa)^\rho - 1)$ is strictly decreasing for $\kappa > 0$; and (iii) if $\rho \in (0, 1)$, then $\left(\frac{1}{\kappa} + \frac{1-\rho}{2}\right)((1+\kappa)^\rho - 1)$ is strictly increasing for $\kappa > 0$.*

Proof. For (i) and (ii), define $g(\kappa) = (1+\kappa)^\rho$, which is strictly concave if $\rho \in (0, 1)$ and strictly convex if $\rho > 1$. Observe that $((1+\kappa)^\rho - 1)/\kappa = (g(\kappa) - g(0))/(\kappa - 0)$, which is strictly increasing in κ if $g(\kappa)$ is strictly convex, and strictly decreasing in κ if $g(\kappa)$ is strictly concave.

For (iii), direct computation gives

$$\frac{d}{d\kappa}\left[\left(\frac{1}{\kappa} + \frac{1-\rho}{2}\right)((1+\kappa)^\rho - 1)\right] = \frac{2(1+\kappa)^{1-\rho} - 2 - \kappa(1-\rho)(2-\kappa\rho)}{2\kappa^2(1+\kappa)^{1-\rho}}.$$

The numerator on the right-hand side equals zero for $\kappa = 0$. Moreover, its derivative is $\frac{d}{d\kappa}[2(1+\kappa)^{1-\rho} - 2 - \kappa(1-\rho)(2-\kappa\rho)] = 2(1-\rho)[(1+\kappa)^{-\rho} - (1-\kappa\rho)] > 0$, because convexity of $(1+\kappa)^{-\rho}$ implies $(1+\kappa)^{-\rho} - (1-\kappa\rho) > 0$. Hence, the numerator on the right-hand side is strictly positive for $\kappa > 0$, which proves (iii). □

A.7.3 Local Analysis: Around a Firm Type with Two Submarkets

We now present a lemma which guarantees that the planner's choice is well behaved around a multiplicity point y_m. Note that the sufficient condition for PAC/PAM ($\underline{\rho} \geq (1+\sigma)/2$) is more than we need here ($\underline{\rho} \geq 1/2$) for the first case.

Lemma 7. *Suppose that at the planner's solution, firms of type y_m have two submarkets with queues $(0, \lambda_0)$ and $(\lambda_1\zeta_1, \lambda_1)$ and $\zeta_1 > 0$. If $\underline{\rho} > 1/2$, then there exists a small interval of firm types containing y_m such that within this interval, if $y > y_m$ then firms of type y have a single submarket whose queue is close to $(\lambda_1\zeta_1, \lambda_1)$, and* *if $y < y_m$ then firms of type y have a single submarket whose queue is close to $(0, \lambda_0)$.*

When $\overline{\rho} \leq (1-\sigma)/2$, then the conclusion is reversed: Within the interval, if $y > y_m$ then firms of type y have a single submarket whose queue is close to $(0, \lambda_0)$, and if $y < y_m$ then firms of type y have a single submarket whose queue is close to $(\lambda_1 \zeta_1, \lambda_1)$.

Proof. Suppose that the queues in the two submarkets for firms of type y_m are (ζ_0, λ_0) and $(\lambda_1 \zeta_1, \lambda_1)$, where $0 = \zeta_0 < \zeta_1$. Since the marginal contribution to surplus by firms of type y_m must be the same for the two submarkets, by (28) we have

$$m(\lambda_0) - \lambda_0 m'(\lambda_0) = m(\lambda_1) - \lambda_1 m'(\lambda_1) + \left(\phi(\zeta_1 \lambda_1, \lambda_1) - \lambda_1 \frac{d\phi(\zeta_1 \lambda_1, \lambda_1)}{d\lambda} \right) \frac{\Delta f}{f^1}, \tag{36}$$

where $\Delta f = f(x_2, y_m) - f(x_1, y_m)$ and $f^1 = f(x_1, y_m)$. The left-hand side is the firm's marginal contribution to surplus with a queue $(0, \lambda_0)$, divided by $f(x_1, y_m)$, and the right-hand side is the corresponding value with a queue $(\lambda_1 \zeta_1, \lambda_1)$.

If $\zeta_1 \in (0, 1)$, then low-type workers are present in both queues and their marginal contribution to surplus must be the same. Equation (26) then yields

$$m'(\lambda_0) = m'(\lambda_1) + \phi_\lambda(\zeta_1 \lambda_1, \lambda_1) \frac{\Delta f}{f^1} \quad \text{if } \zeta_1 \in (0, 1). \tag{37}$$

Low-type workers are not present in the shorter queue if $\zeta_1 = 1$. In this special case, optimality requires that the left-hand side of (37) is larger than the right-hand side.

Recall that $\mathcal{Q}(y)$ is the set of queues that firms of type y face at the planner's solution. By the Theorem of the Maximum, $\mathcal{Q}(y)$ is an upper hemi-continuous correspondence. That is, for firm types y close to y_m, the element(s) in $\mathcal{Q}(y)$ must be close to either $(0, \lambda_0)$ or $(\lambda_1 \zeta_1, \lambda_1)$.

By the envelope theorem, if a firm with type y close to y_m is constrained to choose only (μ, λ) close to $(\lambda_1 \zeta_1, \lambda_1)$, then its return is approximately (first-order) $\Pi(\zeta_1, \lambda_1, y_m) + \Pi_y(\zeta_1, \lambda_1, y_m)\Delta y$ where $\Delta y = y - y_m$. Similarly, if the firm is constrained to choose $\zeta = 0$, then its maximum expected profit is approximately

$\Pi(0, \lambda_0, y_m) + \Pi_y(0, \lambda_0, y_m)\Delta y$. Recall that $\Pi(\zeta_1, \lambda_1, y_m) = \Pi(0, \lambda_0, y_m)$. When $\Pi_y(\zeta_1, \lambda_1, y_m) > \Pi_y(0, \lambda_0, y_m)$, then a firm type $y > y_m$ strictly prefers to choose ζ around ζ_1 instead of around zero, and a firm type $y < y_m$ strictly prefers to choose ζ around zero instead of around ζ_1. As mentioned before, by continuity, it is without loss of generality to constrain the firm to choose between zero and all ζ close to ζ_1.

Note that by the envelope theorem, the condition $\Pi_y(0, \lambda_0, y_m) < \Pi_y(\zeta_1, \lambda_1, y_m)$ can be written as

$$m(\lambda_0) < m(\lambda_1) + \phi(\zeta_1\lambda_1, \lambda_1)\frac{\Delta f_y}{f_y^1}, \tag{38}$$

where $\Delta f_y = f(x_2, y_m) - f(x_1, y_m)$ and $f_y^1 = f_y(x_1, y_m)$. Similarly, $\Pi_y(0, \lambda_0, y_m) > \Pi_y(\zeta_1, \lambda_1, y_m)$ when the reverse inequality holds in (38).

First consider the case in which $\zeta_1 < 1$, such that (37) holds with equality. From (36) and (37), we can solve for $\kappa(y_m)$ and λ_0 in terms of ζ_1 and λ_1. This yields

$$\kappa(y_m) = \frac{4\sigma(1 + \lambda_1 - \lambda_1\sigma(1 - \zeta_1))^2}{(1 + \lambda_1)(\lambda_1 - \sigma - \lambda_1\sigma(1 - \zeta_1) + 1)^2}, \tag{39}$$

$$\lambda_0 = \frac{\lambda_1(\lambda_1 + \sigma(-\lambda_1 + (\lambda_1 + 2)\zeta_1 - 1) + 1)}{1 - \sigma - \lambda_1(1 - \sigma - \sigma\zeta_1)}. \tag{40}$$

Assume $\underline{\rho} > 1/2$. Rewrite (38) as

$$1 + \frac{m(\lambda_0) - m(\lambda_1)}{\phi(\zeta_1\lambda_1, \lambda_1)} < \frac{f_y(x_2, y_m)}{f_y(x_1, y_m)}. \tag{41}$$

Since $\underline{\rho} > 1/2$, $f_y(x_2, y_m)/f_y(x_1, y_m) > (1 + \kappa(y_m))^{1/2}$ by (35). Note that

$$(1 + \kappa(y_m)) - \left(1 + \frac{m(\lambda_0) - m(\lambda_1)}{\phi(\zeta_1\lambda_1, \lambda_1)}\right)^2 = \frac{4\lambda_1\sigma^3(1 - \zeta_1)(1 + \lambda_1(1 - \sigma(1 - \zeta_1)))}{(1 + \lambda_1)^2(1 - \sigma + \lambda_1(1 - \sigma(1 - \zeta_1)))^2} > 0,$$

hence (41) holds.

On the other hand, if $\overline{\rho} \leq (1-\sigma)/2$, then we have

$$\frac{\Delta f_y}{f_y^1} - \frac{m(\lambda_0) - m(\lambda_1)}{\phi(\zeta_1\lambda_1, \lambda_1)} < (1+\kappa(y_m))^{\overline{\rho}} - 1 - \frac{m(\lambda_0) - m(\lambda_1)}{\phi(\zeta_1\lambda_1, \lambda_1)}$$
$$< \frac{1-\sigma}{2}\kappa(y_m) - \frac{m(\lambda_0) - m(\lambda_1)}{\phi(\zeta_1\lambda_1, \lambda_1)} = -\frac{2\sigma^2\lambda_1(1-\sigma(1-\zeta_1))(1+\lambda_1(1-\sigma(1-\zeta_1)))}{(1+\lambda_1)\left(1-\sigma+\lambda_1(1-\sigma(1-\zeta_1))\right)^2} \leq 0,$$

where the first inequality follows from $f_y(x_2, y_m)/f_y(x_1, y_m) < (1+\kappa(y_m))^{\overline{\rho}}$ (see (35)), the second inequality follows from $(1+\kappa)^{\overline{\rho}} < 1 + \overline{\rho}\kappa \leq 1 + \frac{1-\sigma}{2}\kappa$, and the equality follows from equations (39) and (40). Hence, (38) holds with $>$.

Next, consider the case $\zeta_1 = 1$, where (36) holds with equality and (37) holds with $>$. From (36) we can solve

$$\frac{f(x_2, y_m)}{f(x_1, y_m)} = \kappa(y_m) + 1 = \frac{(\lambda_0/(1+\lambda_0))^2}{(\lambda_1/(1+\lambda_1))^2} \tag{42}$$

The sorting condition (38) becomes $\frac{\lambda_0/(1+\lambda_0)}{\lambda_1/(1+\lambda_1)} < \frac{f_y(x_2,y_m)}{f_y(x_1,y_m)}$, which, by (42), is equivalent to $\sqrt{\frac{f(x_2,y_m)}{f(x_1,y_m)}} < \frac{f_y(x_2,y_m)}{f_y(x_1,y_m)}$. If $\underline{\rho} > 1/2$, then the above inequality holds by Lemma 5; if $\underline{\rho} < 1/2$, then similarly, the above inequality holds with $>$. $\square$

A.7.4 Local Analysis: An Interval of Firm Types That Have Unique Queues and Both Types of Workers

We now consider an interval of firm types that have unique queues ($\mathcal{Q}(y)$ contains a single element) and attract both types of workers ($\zeta(y) \in (0,1)$). The FOCs (16) and (17) jointly determine $\lambda(y)$ and $\zeta(y)$. Differentiating (17) with respect to y yields

$$-\frac{1}{\phi_\mu}\left(\frac{\partial\phi_\mu}{\partial\zeta}\zeta'(y) + \frac{\partial\phi_\mu}{\partial\lambda}\lambda'(y)\right) = \frac{\Delta f_y}{\Delta f}, \tag{43}$$

which states that the percentage decrease in ϕ_μ must equal the percentage increase in Δf.

Similarly, differentiating (16) with respect to y yields

$$\zeta'(y)(W_2 - W_1) = m' f_y^1 + m'' \lambda'(y) f^1 + (\zeta(y)\phi_\mu + \phi_\lambda)\Delta f_y$$
$$+ \left[\zeta'(y)\phi_\mu + \zeta \frac{\partial \phi_\mu}{\partial \zeta}\zeta'(y) + \zeta \frac{\partial \phi_\mu}{\partial \lambda}\lambda'(y) + \frac{\partial \phi_\lambda}{\partial \zeta}\zeta'(y) + \frac{\partial \phi_\lambda}{\partial \lambda}\lambda'(y)\right]\Delta f,$$

where we have suppressed the arguments $\mu(y)$ and $\lambda(y)$ from the functions m and ϕ. By (17), we can substitute $\phi_\mu \Delta f$ for $W_2 - W_1$ on the left-hand side. The resulting equation and equation (43) are two linear equations in $\zeta'(y)$ and $\lambda'(y)$. A simple but tedious calculation then yields the percentage change of $m'(\lambda)$ across firm types,

$$-\frac{m''(\lambda(y))}{m'(\lambda(y))}\lambda'(y) = \frac{f_y^1}{f^1} \frac{1 - \frac{1}{m'}\left(\phi_\mu \frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}} - \phi_\lambda\right)\frac{\Delta f_y}{f_y^1}}{1 - \frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}} - \phi_{\lambda\lambda}\right)\frac{\Delta f}{f^1}}. \tag{44}$$

When the meeting technology exhibits no congestion externalities (i.e. $\sigma = 1$), the second factor on the right-hand side reduces to 1. That is, when we move towards more productive jobs, the percentage decrease in $m'(\lambda)$ (as a result of a longer queue) is independent of ζ and simply equals the percentage increase in $f(x_1, y)$. When there are congestion externalities between heterogeneous workers, however, the optimal queue involves a trade-off between quantity and quality, and more of one affects the marginal contribution of the other. The second factor on the right-hand side of (44) represents this complex interplay between quality and quantity.

Dividing both sides of (43) by the corresponding side of (44) then gives the relative change in ϕ_μ and $m'(\lambda)$ across firm types,

$$\frac{\frac{1}{\phi_\mu}\left(\frac{\partial \phi_\mu}{\partial \zeta}\zeta'(y) + \frac{\partial \phi_\mu}{\partial \lambda}\lambda'(y)\right)}{\frac{m''}{m'}\lambda'(y)} = \frac{f^1 \Delta f_y}{f_y^1 \Delta f} \frac{1 - \frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}} - \phi_{\lambda\lambda}\right)\frac{\Delta f}{f^1}}{1 - \frac{1}{m'}\left(\phi_\mu \frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}} - \phi_\lambda\right)\frac{\Delta f_y}{f_y^1}}. \tag{45}$$

The left-hand side reflects the relative change in ϕ_μ and $m'(\lambda)$ across firm types. Recall that $a^c(\zeta, \lambda)$, as defined by equation (21), measures the relative change in ϕ_μ and $m'(\lambda)$, while fixing ζ. Thus if the right-hand side of (45) is larger than $a^c(\zeta(y), \lambda(y))$, then it must be the case that $\zeta'(y) \geq 0$. Similarly, if the right-hand side of (45) is larger than $a^m(\zeta(y), \lambda(y))$, as defined by equation (22), then it must be the case that $\frac{d}{dy}h(\zeta(y),\lambda(y)) \geq 0$. We can summarize this in the following Lemma.

Lemma 8. *Assume that at the planner's solution, there exists an interval of firm types that have unique queues and attract both types of workers ($\zeta(y) \in (0,1)$). If type y is in this interval, then $\zeta'(y) \geq 0$ (resp. $\frac{d}{dy}h(\zeta(y),\lambda(y)) \geq 0$) if and only if*

$$\frac{f^1 \Delta f_y}{f_y^1 \Delta f} \geq a^i \frac{1 - \frac{1}{m'}\left(\phi_\mu \frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}} - \phi_\lambda\right)\frac{\Delta f_y}{f_y^1}}{1 - \frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}} - \phi_{\lambda\lambda}\right)\frac{\Delta f}{f^1}}, \tag{46}$$

where $i = c$ (resp. $i = m$), and we suppress the arguments of $\phi(\zeta(y)\lambda(y), \lambda(y))$, $m(\lambda(y))$ and $a^i(\zeta(y),\lambda(y))$.

Proof. Rearranging equation (45) gives

$$-\frac{1}{\phi_\mu}\frac{\partial \phi_\mu}{\partial \zeta}\zeta'(y) = \frac{f_y^1}{f^1}\left(\frac{f^1 \Delta f_y}{f_y^1 \Delta f} - \frac{\frac{1}{\phi_\mu}\frac{\partial \phi_\mu}{\partial \lambda}}{\frac{m''}{m'}} \frac{1 - \frac{1}{m'}\left(\phi_\mu \frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}} - \phi_\lambda\right)\frac{\Delta f_y}{f_y^1}}{1 - \frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}} - \phi_{\lambda\lambda}\right)\frac{\Delta f}{f^1}}\right) \tag{47}$$

where we used equation (44) to substitute out $\lambda'(y)$. Since $\phi(\mu,\lambda)$ is strictly concave, $\frac{\partial \phi_\mu}{\partial \zeta} = \lambda \phi_{\mu\mu} < 0$, which implies that $\zeta'(y) \geq 0$ if and only if the term in the parenthesis on the right-hand side is positive, i.e. (46) holds with $i = c$.

By definition, PAM is equivalent to $\frac{\partial h}{\partial \zeta}\zeta'(y) + \frac{\partial h}{\partial \lambda}\lambda'(y) \geq 0$. Combining (44) and (47) then shows that PAM is obtained if and only if (46) holds with $i = m$. $\square$

We now show that the necessary condition (23) implies that PAC/PAM holds locally at all interior points, so it is also sufficient. The same conclusion also applies to the case of NAC/NAM.

Recall $\kappa(y) \equiv \Delta f / f^1$. Throughout we will then use the following inequalities which result from rewriting (35):

$$\frac{(1+\kappa(y))^{\underline{\rho}} - 1}{\kappa(y)} \leq \frac{f^1 \Delta f_y}{f_y^1 \Delta f} \leq \frac{(1+\kappa(y))^{\overline{\rho}} - 1}{\kappa(y)}. \tag{48}$$

First, consider PAC/PAM. Assume the necessary condition (23) holds, i.e. $\underline{\rho} \geq \overline{a}^i$. Since $\overline{a}^i \geq 0$, this implies that $\Delta f_y \geq 0$ (i.e. f is supermodular) such that the left-hand side of (46) is positive. We now prove a stronger version of (46), i.e.

$$\frac{f^1 \Delta f_y}{f_y^1 \Delta f} \geq \overline{a}^i \frac{1 - \frac{1}{m'}\left(\phi_\mu \frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}} - \phi_\lambda\right)\frac{\Delta f_y}{f_y^1}}{1 - \frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}} - \phi_{\lambda\lambda}\right)\frac{\Delta f}{f^1}},$$

where $a^i(\zeta, \lambda)$ is replaced by its supremum $\overline{a}^i$. This is justified because if the second factor on the right-hand side is negative then we have nothing to prove; if it is positive, then we have a stronger version of the original inequality. Firms' SOC implies that the denominator of this factor is positive. Rearranging terms therefore gives

$$\frac{f^1 \Delta f_y}{f_y^1 \Delta f} + \frac{\Delta f_y}{f_y^1}\left[\overline{a}^i \frac{1}{m'}\left(\phi_\mu \frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}} - \phi_\lambda\right) - \frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}} - \phi_{\lambda\lambda}\right)\right] \geq \overline{a}^i. \tag{49}$$

Since $\phi(\mu, \lambda)$ is given by (2) and $\overline{a}^i = (1+\sigma)/2$ by Lemma 3, the above condition can be rewritten as

$$\frac{f^1 \Delta f_y}{f_y^1 \Delta f} + \frac{\Delta f_y}{f_y^1} \frac{(1-\sigma)(2+(1-\sigma)\lambda)(1+\lambda)^2}{4(1+\lambda(1-\sigma))(1+\lambda(1-\sigma)+\lambda\sigma\zeta)} \geq \frac{1+\sigma}{2}.$$

Consider now two subcases, determined by the value of $\underline{\rho}$. If $\underline{\rho} \geq 1$, then the first term on the left-hand side is greater than 1 by (48); hence the above condition holds. Next, consider the case $\underline{\rho} \in (0, 1)$. Note that

$$\frac{(1-\sigma)(2+(1-\sigma)\lambda)(1+\lambda)^2}{4(1+\lambda(1-\sigma))(1+\lambda(1-\sigma)+\lambda\sigma\zeta)} \geq \frac{(1-\sigma)(2+(1-\sigma)\lambda)(1+\lambda)}{4(1+\lambda(1-\sigma))} \geq \frac{1-\sigma}{2}$$

where the first inequality is because the denominator reaches its maximum at $\zeta = 1$, and the second one is because $1 + \lambda \geq 1 + (1-\sigma)\lambda$. Thus a sufficient condition for (49) is

$$\frac{f^1 \Delta f_y}{f_y^1 \Delta f} + \frac{\Delta f_y}{f_y^1} \frac{1-\sigma}{2} \geq \frac{1+\sigma}{2}.$$

Note that

$$\frac{f^1\Delta f_y}{f_y^1\Delta f}+\frac{\Delta f_y}{f_y^1}\frac{1-\sigma}{2}\geq\frac{(1+\kappa(y))^{\underline{\rho}}-1}{\kappa(y)}+\left((1+\kappa(y))^{\underline{\rho}}-1\right)(1-\underline{\rho})\geq\underline{\rho}\geq\frac{1+\sigma}{2},$$

where the first inequality holds by (48) and the assumption $\underline{\rho}\geq(1+\sigma)/2$, the second inequality holds because the second term reaches its minimum value $\underline{\rho}$ at $\kappa(y)=0$, by part (iii) of Lemma 6. Therefore, (49) holds when $\underline{\rho}\geq\overline{a}^i$.

Next, consider NAC/NAM. If $\Delta f_y\leq 0$, then the left-hand side of (46) is negative. The denominator on the right-hand side is positive because of the SOC, and the numerator is positive because

$$\phi_\mu\frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}}-\phi_\lambda=\frac{1-\sigma}{2\sigma(1+\lambda(1-\sigma)+\lambda\sigma\zeta)}\geq 0.$$

Thus, it follows immediately that (46) holds with $\leq$.

In contrast, if $\Delta f_y\geq 0$, then we have

$$\begin{aligned}\frac{f^1\Delta f_y}{f_y^1\Delta f}\frac{1}{m'}\left(\phi_\mu\frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}}-\phi_\lambda\right)-\frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}}-\phi_{\lambda\lambda}\right)&\leq\frac{1-\sigma}{2}\frac{1}{m'}\left(\phi_\mu\frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}}-\phi_\lambda\right)-\frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}}-\phi_{\lambda\lambda}\right)\\&=-\frac{\lambda(1+\lambda)^2(1-\sigma)^2}{4(1+(1-\sigma)\lambda)(1+\sigma\mu+(1-\sigma)\lambda)}\leq 0.\end{aligned}$$

which then implies

$$1\leq\frac{1-\frac{1}{m'}\left(\phi_\mu\frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}}-\phi_\lambda\right)\frac{\Delta f_y}{f_y^1}}{1-\frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}}-\phi_{\lambda\lambda}\right)\frac{\Delta f}{f^1}}.$$

Therefore, we have

$$\frac{f^1\Delta f_y}{f_y^1\Delta f}\leq\overline{\rho}\leq\underline{a}^i\leq a^i\leq a^i\frac{1-\frac{1}{m'}\left(\phi_\mu\frac{\phi_{\mu\lambda}}{\phi_{\mu\mu}}-\phi_\lambda\right)\frac{\Delta f_y}{f_y^1}}{1-\frac{1}{m''}\left(\frac{\phi_{\mu\lambda}^2}{\phi_{\mu\mu}}-\phi_{\lambda\lambda}\right)\frac{\Delta f}{f^1}},$$

where the three inequalities follow from (48), part ii) of Lemma 6, and our assumption $\overline{\rho}\leq\underline{a}^i$, respectively, and the last inequality follows from the result above. Hence, we have proved the case of NAC/NAM.

A.7.5 Uniqueness of the Planner's Solution

Suppose that the solution to the planner's problem is not unique: there exist two allocations $(\overline{\mu}(y), \overline{\lambda}(y))$ and $(\widetilde{\mu}(y), \widetilde{\lambda}(y))$ that solve (5). Consider a new allocation which has queue schedule $(\gamma\overline{\mu}(y) + (1-\gamma)\widetilde{\mu}(y), \gamma\overline{\lambda}(y) + (1-\gamma)\widetilde{\lambda}(y))$ for some $\gamma \in (0,1)$, which must yield the same maximum surplus as the original two allocations. Hence for each firm type y, we have $\gamma\widehat{S}(\overline{\mu}(y), \overline{\lambda}(y), y) + (1-\gamma)\widehat{S}(\widetilde{\mu}(y), \widetilde{\lambda}(y), y) = \widehat{S}(\gamma\overline{\mu}(y) + (1-\gamma)\widetilde{\mu}(y), \gamma\overline{\lambda}(y) + (1-\gamma)\widetilde{\lambda}(y), y)$.

Since the two allocations $(\overline{\mu}(y), \overline{\lambda}(y))$ and $(\widetilde{\mu}(y), \widetilde{\lambda}(y))$ are different, there exist at least two firm types y_1 and y_2 such that $(\overline{\mu}(y), \overline{\lambda}(y)) \neq (\widetilde{\mu}(y), \widetilde{\lambda}(y))$. Consider firms of type y_1. Recall that $\widehat{S}(\mu, \lambda, y_1)$ is linear in (μ, λ) on the line segment between $(\overline{\mu}(y_1), \overline{\lambda}(y_1))$ and $(\widetilde{\mu}(y_1), \widetilde{\lambda}(y_1))$. Given the average queue lengths $(\overline{\mu}(y), \overline{\lambda}(y))$ and $(\widetilde{\mu}(y), \widetilde{\lambda}(y))$, the planner must create two submarkets $(0, \lambda_a(y_1))$ and $(\mu_b(y_1), \lambda_b(y_1))$ in either case. The same is true for firms of type y_2. Therefore, in each of the two allocations $(\overline{\mu}(y), \overline{\lambda}(y))$ and $(\widetilde{\mu}(y), \widetilde{\lambda}(y))$, there are two firm types each of which has two submarkets, which contradicts with PAC/PAM. We have thus proved that the planner's solution must be unique. □

A.8 Proof of Proposition 4

When $\sigma = 1$, $\phi(\mu, \lambda)$ is independent of λ: $\phi_\lambda(\mu, \lambda) = 0$; hence $\phi(\mu, \lambda) = m(\mu)$. Therefore, $S(\mu, \lambda, y)$ in (3) reduces to $m(\lambda) f(x_1, y) + m(\mu)[f(x_2, y) - f(x_1, y)]$, which is strictly concave in (μ, λ). Thus, $\widehat{S}(\mu, \lambda, y) = S(\mu, \lambda, y)$, and the planner's problem in (5) is strictly concave, which implies a unique optimal solution $(\mu(y), \lambda(y))$ that is continuous in y, and is determined by the FOCs (16) and (17) and the complementary slackness conditions. Given that the surplus function is separable in μ and λ (see (50) and (51)), below we derive the FOCs with respect to μ and λ, which are equivalent but simpler than the corresponding version with $\zeta = \mu/\lambda$ and λ given by (16) and (17).

Our proof below consists of three steps: 1) we assume that no firms attract x_2 workers only and show that this assumption is valid if and only if the fraction of x_2 workers z is smaller than some threshold $\widehat{z}$. Furthermore, we derive some characterizations of the planner's solution under this assumption. 2) We show that for PAC/PAM to occur, this assumption is necessary when $\rho \in (0,1)$. 3) We derive the conditions for PAC/PAM, and by utilizing the characterizations derived in step

1, show that they hold if and only if z is sufficiently small.

Step 1: Assume that at the planner's solution, there exist no firms that attract x_2 workers only: if $\lambda(y) > 0$, then $\mu(y) < \lambda(y)$). Then the FOC with respect to λ is given by,

$$m'(\lambda(y))f(x_1, y) = W_1, \tag{50}$$

where W_1 is determined by the budget constraint: $\int_{\underline{y}}^{\overline{y}} \lambda(y) = L$. As long as the above assumption holds, then $\lambda(y)$ and W_1 are independent of z, since the FOC (50) and the corresponding budget constraint do not depend on z.

If $\mu(y) > 0$, then the FOC with respect to μ is

$$m'(\mu(y))\,[f(x_2, y) - f(x_1, y)] = W_2 - W_1. \tag{51}$$

where $W_2 - W_1$ and hence W_2 are determined by the budget constraint: $\int_{\underline{y}}^{\overline{y}} \mu(y) = Lz$. Therefore, for a given y if $f(x_2, y) - f(x_1, y) > W_2 - W_1$, then $\mu(y) > 0$ and is strictly decreasing in $W_2 - W_1$. Thus, $W_2 - W_1$ is strictly decreasing in z for a given λ.

Given $\lambda(y)$ and W_1, as long as $W_2 - W_1 > \max_{y \in [\underline{y}, \overline{y}]} m'(\lambda(y))[f(x_2, y) - f(x_1, y)]$, where the right-hand side is (51) evaluated at $\mu(y) = \lambda(y)$ (the knife-edge case), then no firms will attract only x_2 workers. Since $W_2 - W_1$ is strictly decreasing in z, there exists a threshold $\widehat{z}$ such that the above assumption holds if and only if $z < \widehat{z}$.

Step 2: Suppose that the above assumption fails and there exists some firm type y_1 with $0 < \mu(y_1) = \lambda(y_1)$. The FOCs for firms of type y_1 are: $m'(\mu(y_1))f(x_2, y_1) = W_2$ and $m'(\mu(y_1))f(x_1, y_1) \leq W_1$, which implies that

$$\frac{f(x_2, y_1) - f(x_1, y_1)}{f(x_1, y_1)} \geq \frac{W_2 - W_1}{W_1}$$

But, since $(\mu(y), \lambda(y))$ is continuous in y, there must exist some firm type y_2 with $0 < \mu(y_2) < \lambda(y_2)$. For firms of type y_2, both (50) and (51) must hold, which implies that

$$\frac{W_2 - W_1}{W_1} = \frac{m'(\mu(y_2))}{m'(\lambda(y_2))} \frac{f(x_2, y_2) - f(x_1, y_2)}{f(x_1, y_2)} > \frac{f(x_2, y_2) - f(x_1, y_2)}{f(x_1, y_2)}$$

Combining the above two equations implies that $f(x_2, y_1)/f(x_1, y_1) > f(x_2, y_2)/f(x_1, y_2)$. Since we assume $\rho < 1$, $f(x, y)$ is strictly log-submodular: $f(x_2, y)/f(x_1, y)$ is strictly decreasing in y. Thus $y_1 < y_2$ and PAC/PAM fails at the planner's solution.

Step 3: Assume $z < \widehat{z}$ or equivalently that there exist no firms that attract x_2 workers only. By differentiating (50) and (51) with respect to y (or equivalently equation (46) in Lemma 8 in Appendix A.7), PAC/PAM holds at the planner's solution if and only if for each y,

$$\frac{(1 + \kappa(y))^\rho - 1}{\kappa(y)} \geq a^i(\zeta(y), \lambda(y)) \tag{52}$$

where, as before, $i = c$ for the case of PAC and $i = m$ for the case of PAM. Note that when $\sigma = 1$, $a^m(\zeta, \lambda) = \zeta$ and $a^c(\zeta, \lambda) = \zeta(1 + \lambda)/(1 + \zeta\lambda) > \zeta$. Since $\rho < 1$, the left-hand side above is strictly decreasing in $\kappa(y)$ and at $\kappa(y) = 0$, it equals ρ. Thus, (52) implies that when PAC/PAM holds, $\rho > \zeta(y)$ for all y.

Recall that when $z < \widehat{z}$, both $\lambda(y)$ and W_1 are independent of z. For $i = c$ and m, define $\overline{\zeta}^i(y)$ as the value of $\zeta(y)$ such that (52) holds with equality. Since both $a^c(\zeta, \lambda)$ and $a^m(\zeta, \lambda)$ are decreasing in ζ, PAC/PAM holds if and only if for each y, $\zeta(y) \leq \overline{\zeta}^i(y)$. As before, as long as $W_2 - W_1 \geq \max_{y \in [\underline{y}, \overline{y}]} m'(\lambda(y)\overline{\zeta}^i(y))[f(x_2, y) - f(x_1, y)]$ (the knife-edge case), then $\zeta(y) \leq \overline{\zeta}^i(y)$ for all y and PAC/PAM holds. Thus following the same logic as before, there exists a threshold $\overline{z}^i$ such that $\zeta(y) \leq \overline{\zeta}^i(y)$ for all y if and only if $z \leq \overline{z}^i$. Since $a^c(\zeta, \lambda) > a^m(\zeta, \lambda)$, $\overline{\zeta}^m(y) > \overline{\zeta}^c(y)$ and thus $\overline{z}^m > \overline{z}^c$. □

A.9 Proof of Proposition 5

First, we consider the unconditional probability that an applicant generates a positive signal $\widetilde{x}_2$. The probability of this event equals $\mathbb{P}(\widetilde{x}_2) = \frac{\mu}{\lambda} + \frac{\lambda - \mu}{\lambda}(1 - \tau)$, and the queue length of such applicants is $\widetilde{\lambda} = \lambda\mathbb{P}(\widetilde{x}_2) = \mu + (\lambda - \mu)(1 - \tau)$. Given a positive signal $(\widetilde{x}_2)$, the probability that an applicant is of high type (x_2) is $\mathbb{P}(x_2 \,|\, \widetilde{x}_2) = \mathbb{P}(x_2)\mathbb{P}(\widetilde{x}_2 \,|\, x_2)/\mathbb{P}(\widetilde{x}_2) = \mu/\widetilde{\lambda}$, where the first equality is simply Bayes' rule.

Next, we consider the probability that the firm interviews at least one high-type worker, $\phi(\mu, \lambda)$. For this, we can ignore the existence of applicants with negative signals; they are low-type workers for sure and do not affect the meeting process between firms and workers with positive signals. By equation (2), the probability that a firm interviews someone from the queue μ of high-type applicants, given a queue $\widetilde{\lambda}$ of applicants with positive signals, is $\phi(\mu, \lambda) = \mu/(1 + \sigma\mu + (1 - \sigma)\widetilde{\lambda})$, which yields the desired result after substitution of $\widetilde{\lambda}$. $\square$

A.10 Market Equilibrium

Here we formally establish that the market equilibrium where firms post wage menus implements the planner's solution.

Beliefs. A firm of type y posting a wage menu $\boldsymbol{w}$ has to form beliefs about its queues $(\mu(\boldsymbol{w}, y), \lambda(\boldsymbol{w}, y))$. Following the standard approach in the literature, we restrict these beliefs in the spirit of subgame perfection through what is known as the *market utility condition.* To state this condition, consider a worker of type x_i. Define $V_i(\boldsymbol{w}, \mu, \lambda, y)$ as his expected payoff in a submarket $(\boldsymbol{w}, y)$ with queues (μ, λ), and his *market utility* U_i as the maximum expected payoff that he can obtain in equilibrium, either by visiting one of the submarkets or by remaining inactive. Firms' beliefs $(\mu(\boldsymbol{w}, y), \lambda(\boldsymbol{w}, y))$ must then satisfy equation (12).

For common meeting technologies, including our benchmark as we will show in Lemma 9 below, (12) admits a unique solution (μ, λ), which is then the firm's belief. For other technologies, there can be multiple solutions to (12). The standard assumption is then that firms are optimistic and expect the solution that maximizes their expected payoff $\pi(\boldsymbol{w}, \mu, \lambda, y)$. Explicit expressions for π and V_i are provided in Section 3.3.

Strategies. Let $G(\boldsymbol{w} \,|\, y)$ denote the (conditional) probability that a firm of type y offers a wage menu $\widetilde{\boldsymbol{w}} \leq \boldsymbol{w}$, where $\widetilde{\boldsymbol{w}} = (\widetilde{w}_1, \widetilde{w}_2)$, $\boldsymbol{w} = (w_1, w_2)$, $\widetilde{w}_1 \leq w_1$ and $\widetilde{w}_2 \leq w_2$. Given market utilities (U_1, U_2), firm optimality means that every $\boldsymbol{w}$ in the support of $G(\boldsymbol{w} \,|\, y)$ must maximize $\pi(\boldsymbol{w}, \mu, \lambda, y)$ subject to (12).

Similarly, let $H_i(\boldsymbol{w}, y)$ denote the probability that workers of type x_i apply to a firm with $\widetilde{\boldsymbol{w}} \leq \boldsymbol{w}$ and $\widetilde{y} \leq y$. The following accounting identities then link workers' strategies $H_1(\boldsymbol{w}, y)$ and $H_2(\boldsymbol{w}, y)$ to the queues in different submarkets.

$$H_1(\boldsymbol{w}, y) = \frac{1}{L(1-z)} \int_{\widetilde{y} \leq y} \int_{\widetilde{\boldsymbol{w}} \leq \boldsymbol{w}} [\lambda(\widetilde{\boldsymbol{w}}, \widetilde{y}) - \mu(\widetilde{\boldsymbol{w}}, \widetilde{y})] \, dG(\widetilde{\boldsymbol{w}} \,|\, \widetilde{y}) \, dJ(\widetilde{y}). \tag{53}$$

$$H_2(\boldsymbol{w}, y) = \frac{1}{Lz} \int_{\widetilde{y} \leq y} \int_{\widetilde{\boldsymbol{w}} \leq \boldsymbol{w}} \mu(\widetilde{\boldsymbol{w}}, \widetilde{y}) \, dG(\widetilde{\boldsymbol{w}} \,|\, \widetilde{y}) \, dJ(\widetilde{y}), \tag{54}$$

Optimality requires that workers must obtain exactly U_i at any firm to which they apply with positive probability, and weakly less at other firms i.e. (12) must hold. Further, note that no firm will post a wage menu $\boldsymbol{w} \geq \overline{\boldsymbol{w}} \equiv (f(x_1, \overline{y}), f(x_2, \overline{y}))$. Thus, $H_i(\overline{\boldsymbol{w}}, \overline{y})$ is the probability that workers of type x_i apply, which must equal 1 if $U_i > 0$, as the payoff from not sending an application is zero. This condition can be interpreted as "market clearing": in equilibrium, demand for each type of applicant must equal supply, which determines the "market prices" U_1 and U_2.

Equilibrium Definition. We can now define an equilibrium as follows.

Definition 4. *A (directed search) equilibrium is a triple* $(G, \{H_1, H_2\}, \{U_1, U_2\})$ *satisfying*

(i) Firm Optimality. *Given* (U_1, U_2)*, every wage menu* $\boldsymbol{w}$ *in the support of* $G(\cdot \,|\, y)$ *maximizes* $\pi(\boldsymbol{w}, \mu(\boldsymbol{w}, y), \lambda(\boldsymbol{w}, y), y)$ *for each firm type* y*, where the queue lengths* $(\mu(\boldsymbol{w}, y), \lambda(\boldsymbol{w}, y))$ *are determined by* (12)*.*

(ii) Worker Optimality. *Given* (U_1, U_2)*, the application strategy of high-type and low-type workers satisfies* (54) *and* (53)*, respectively, where the queue lengths* $(\mu(\boldsymbol{w}, y), \lambda(\boldsymbol{w}, y))$ *are determined by* (12)*. Further,* $H_i(\overline{\boldsymbol{w}}, \overline{y}) = 1$ *if* $U_i > 0$*.*

Uniqueness of Queues. In a submarket $(\boldsymbol{w}, y)$, the queues (μ, λ) are determined by (12). Since this is a system of non-linear equations, it is not immediate that there is a unique solution. Lemma 9 guarantees uniqueness.

Lemma 9. *Suppose that* ϕ *is given by* (2)*. Given market utilities* U_1 *and* U_2*, there exists exactly one solution* (μ, λ) *to the market utility condition* (12) *for any wage menu* $\boldsymbol{w}$*.*

Proof. Given U_1/w_1 and U_2/w_2, consider then the level curves $\psi_2(\lambda\zeta, \lambda) = U_2/w_2$

and $\psi_1(\lambda\zeta, \lambda) = U_1/w_1$ in the λ-ζ space. Note that

$$\psi_1(\lambda\zeta, \lambda) = \frac{1 + (1-\sigma)\lambda}{(1+\lambda)(1 + (1-\sigma+\sigma\zeta)\lambda)} \text{ and } \psi_2(\lambda\zeta, \lambda) = \frac{1}{1 + (1-\sigma+\sigma\zeta)\lambda},$$

both of which are strictly decreasing in ζ. We now show that the two curves intersect at most once so that there exists exactly one solution (μ, λ). At any intersection point, the difference between the slopes of the two level curves is

$$-\frac{\partial\psi_1(\lambda\zeta, \lambda)/\partial\lambda}{\partial\psi_1(\lambda\zeta, \lambda)/\partial\zeta} + \frac{\partial\psi_2(\lambda\zeta, \lambda)/\partial\lambda}{\partial\psi_2(\lambda\zeta, \lambda)/\partial\zeta} = \frac{1 + (1-\sigma+\sigma\zeta)\lambda}{\lambda(\lambda+1)(1 + (1-\sigma)\lambda)} > 0.$$

Hence, by a standard single-crossing argument, the two level curves cross each other at most once. Note that we can also derive the solution (μ, λ) explicitly. However, with this approach we need to discuss the conditions under which we have a corner solution ($\mu = 0$ or $\mu = \lambda$) or an interior solution ($0 < \mu < \lambda$). □

Productivity versus Profitability. We now show that it is without loss of generality to only consider wage menus satisfying (9). To do so, Lemma 10 establishes two results.[39] First, the maximum profit in (13) can always be obtained with a wage menu that satisfies (9). Second, a wage menu that violates (9) always yields a strictly lower profit. To understand the latter result, suppose that a firm posts a wage menu where low-type workers yield a higher profit ex post, i.e. $f(x_2, y) - w_2 < f(x_1, y) - w_1$, and attracts a queue (μ, λ) with $0 < \mu < \lambda$. Workers must obtain their market utility, so the expected transfer from the firm to the workers equals $\mu U_2 + (\lambda - \mu)U_1$. However, giving priority to low- rather than high-type workers reduces surplus relative to $S(\mu, \lambda, y)$ in (3). Hence, the firm's expected profit is strictly smaller than the maximum profit in (13).

Lemma 10. *A solution (μ, λ) (interior or corner) to an individual firm's problem (13) can be implemented with the wage menu $(w_1, w_2) = (U_1/\psi_1(\mu, \lambda), U_2/\psi_2(\mu, \lambda))$, which satisfies (9). Further, any wage menu violating (9) yields a strictly lower payoff than (w_1, w_2).*

Proof. We first show that given a solution (μ, λ) (interior or corner) to the firm's problem (13), the corresponding wage menu $(w_1, w_2) = (U_1/\psi_1(\mu, \lambda), U_2/\psi_2(\mu, \lambda))$

[39] A similar result appears in Shimer (2005) for urn-ball meetings. Lemma 10 generalizes his result to arbitrary meeting technologies.

satisfies (9). This proof is based on Shimer (2005), but extends his result to arbitrary $\phi(\mu,\lambda)$. Because $\phi(\mu,\lambda)$ is concave in μ, we have

$$\psi_1(\mu,\lambda) \le \phi_\mu(\mu,\lambda) \le \psi_2(\mu,\lambda), \tag{55}$$

where ψ_1 and ψ_2 are defined by equation (11). Consequently, the wages must satisfy

$$w_1 = \frac{U_1}{\psi_1(\mu,\lambda)} \ge \frac{U_1}{\phi_\mu(\mu,\lambda)} \quad \text{and} \quad w_2 = \frac{U_2}{\psi_2(\mu,\lambda)} \le \frac{U_2}{\phi_\mu(\mu,\lambda)}. \tag{56}$$

Moreover, the FOC of (13) with respect to μ implies $\phi_\mu(\mu,\lambda)(f(x_2,y)-f(x_1,y)) = U_2 - U_1$. Combining this FOC with (56) implies $w_2 - w_1 \le \frac{U_2-U_1}{\phi_\mu(\mu,\lambda)} = f(x_2,y) - f(x_1,y)$. The strict inequality in $f(x_2,y)-w_2 > f(x_1,y)-w_1$ then follows because the two inequalities in (55) cannot hold simultaneously; that would imply that $\phi(\mu,\lambda)$ is linear for $\mu \in [0,\lambda]$, in which case the firm's problem never has an interior solution.

Next, we show that posting a wage menu that violates (9) is always strictly suboptimal. Suppose that low-type workers are strictly preferred. The firms' expected profit in this case is

$$\pi(\boldsymbol{w},\mu,\lambda,y) = \phi(\lambda-\mu,\lambda)\left[f(x_1,y)-w_1\right] + \left[m(\lambda)-\phi(\lambda-\mu,\lambda)\right]\left[f(x_2,y)-w_2\right],$$

where $\phi(\lambda-\mu,\lambda)$ is the probability that firms interview at least one low-type worker. The matching probabilities in (11) become $\psi_1(\mu,\lambda) = \frac{\phi(\lambda-\mu,\lambda)}{\lambda-\mu}$ and $\psi_2(\mu,\lambda) = (m(\lambda)-\phi(\lambda-\mu,\lambda))/\mu$. The firms' expected profit can then be rewritten as $m(\lambda)f^1 + (m(\lambda)-\phi(\lambda-\mu,\lambda))\Delta f - \lambda U_1 - \mu(U_2-U_1)$. Note that the expected costs are the same as the case where high-type workers are preferred; both equal $\lambda U_1 + \mu(U_2 - U_1)$. However, surplus is strictly smaller than that in (13). The case where firms randomize between low-type and high-type workers follows the same logic. $\square$

If the solution is interior ($0 < \mu < \lambda$), then the wage menu that firms need to post to attract the optimal queue is unique. In a corner solution ($\mu = 0$ or $\mu = \lambda$), the wage menu is not unique, but Lemma 10 describes the maximum wages satisfying (9).[40]

[40] For example, if $\mu = \lambda$, then the optimal w_2 is uniquely given by $U_2/\psi_2(\lambda,\lambda)$, but the optimal

Observability of Firm Productivity. By Lemma 10, all firms will post wage contracts such that high-type workers are more profitable. Given the wage contract, the market utility condition then determines the queue length and composition. Since workers only care about their hiring probability and the wage, this then means that all our results carry through if they do not observe firm types.

A.11 Invariant Technologies with N Worker Types

Invariant technologies, such as urn-ball or geometric, exhibit perfect screening in the sense that the presence of low types does not make it harder (or easier) for a firm to identify a high-type applicant. That is, $\phi_\lambda(\mu,\lambda) = 0$ for all μ and λ, or equivalently, $\phi(\mu,\lambda) = \phi(\mu,\mu) \equiv m(\mu)$, where $m(\mu)$ is always assumed to be strictly concave (see Cai et al., 2017). Furthermore, **?** show that for invariant meeting technologies $m(\cdot)$ has the following representation.

$$m(\lambda) = \int_{[0,\infty)} \left(1 - e^{-\lambda s}\right) d\widetilde{L}(s). \tag{57}$$

where $\widetilde{L}(s)$ is a probability measure on $[0,\infty)$ (the positive real half-line) with $\int_{[0,\infty)} s\, d\widetilde{L}(s) \le 1$. For the urn-ball meeting technology, $\widetilde{L}(s)$ is degenerate at $s = 1$; for the geometric meeting technology, $\widetilde{L}(s) = 1 - e^{-s}$ (the standard geometric distribution).

To analyze this case, we first introduce two elasticities:

$$\varepsilon_0(\mu) = \frac{\mu m'(\mu)}{m(\mu)} \quad \text{and} \quad \varepsilon_1(\mu) = \frac{\mu m''(\mu)}{m'(\mu)}. \tag{58}$$

The following lemma then presents $a^c(\zeta,\lambda)$ and $a^m(\zeta,\lambda)$ for invariant technologies in terms of $\varepsilon_0(\cdot)$ and $\varepsilon_1(\cdot)$.

Lemma 11. *When the meeting technology is invariant, we have*

$$a^c(\zeta,\lambda) = \frac{\varepsilon_1(\lambda\zeta)}{\varepsilon_1(\lambda)} \quad \text{and} \quad a^m(\zeta,\lambda) = \frac{\varepsilon_1(\lambda\zeta)}{\varepsilon_1(\lambda)} \frac{\varepsilon_0(\lambda)}{\varepsilon_0(\lambda\zeta)}, \tag{59}$$

with extrema $\underline{a}^c = \underline{a}^m = 0$ *and* $\overline{a}^c, \overline{a}^m \ge 1$.

w_1 can take any value between zero and $w_1 = U_1/\psi_1(\lambda,\lambda)$.

Proof. The desired expression for a^c follows readily from equations (21) and (58). To derive the expression for a^m, note that $\phi(\mu,\lambda)=m(\mu)$ implies that $\phi_\mu(\mu,\lambda)=m'(\mu)$ and $h(\zeta,\lambda)=m(\zeta\lambda)/m(\lambda)$. Therefore, the last factor in (22) can be rewritten as

$$1-\frac{\partial\phi_\mu/\partial\zeta}{\partial\phi_\mu/\partial\lambda}\frac{\partial h/\partial\lambda}{\partial h/\partial\zeta}=1-\frac{\lambda m''(\zeta\lambda)}{\zeta m''(\zeta\lambda)}\frac{\frac{\zeta m'(\zeta\lambda)m(\lambda)-m(\zeta\lambda)m'(\lambda)}{m(\lambda)^2}}{\lambda m'(\zeta\lambda)/m(\lambda)}=\frac{m(\zeta\lambda)m'(\lambda)}{\zeta m'(\zeta\lambda)m(\lambda)}=\frac{\varepsilon_0(\lambda)}{\varepsilon_0(\zeta\lambda)}.$$

Note that $a^c(1,\lambda)=a^m(1,\lambda)=1$ which implies that $\overline{a}^c,\overline{a}^m\geq 1$.

Next, consider $\underline{a}^c$ and $\underline{a}^m$. Since $m(\mu)$ is strictly concave and strictly increasing, $\varepsilon_0(\mu)$ is strictly positive, and $\varepsilon_1(\mu)$ is strictly negative when $\mu>0$. Hence, $a^c(\zeta,\lambda)$ and $a^m(\zeta,\lambda)$ are always nonnegative. Since $m(\cdot)$ is given by equation (57), $\mu m''(\mu)=-\int_{[0,\infty)}\mu s^2e^{-\mu s}\,d\widetilde{L}(s)$, which converges to zero as $\mu\to 0$. Since $m'(0)>0$, we have $\lim_{\mu\to 0}\varepsilon_1(\mu)=\lim_{\mu\to 0}\mu m''(\mu)/m'(\mu)=0$. Similarly, $\lim_{\mu\to 0}\varepsilon_0(\mu)=\lim_{\mu\to 0}\mu m'(\mu)/m(\mu)=\lim_{\mu\to 0}1+\mu m''(\mu)/m'(\mu)=1$. Thus, $\lim_{\zeta\to 0}a^c(\zeta,\lambda)=\lim_{\zeta\to 0}\varepsilon_1(\lambda\zeta)/\varepsilon_1(\lambda)=0$, and $\lim_{\zeta\to 0}a^m(\zeta,\lambda)=\lim_{\zeta\to 0}\varepsilon_1(\lambda\zeta)/\varepsilon_1(\lambda)\cdot\varepsilon_0(\lambda)/\varepsilon_0(\lambda\zeta)=0$. Hence, $\underline{a}^c=\underline{a}^m=0$. □

Recall that the contact quality-quantity elasticity $a^c(\zeta,\lambda)$ measures the relative percentage changes of $\phi_\mu(\lambda\zeta,\lambda)$ and $m'(\lambda)$ while holding ζ constant. For invariant technologies, $\phi_\mu(\lambda\zeta,\lambda)=m'(\lambda\zeta)$; thus, $a^c(\zeta,\lambda)$ is simply $\varepsilon_1(\lambda\zeta)/\varepsilon_1(\lambda)$. Next, note that $a^m(\zeta,\lambda)$ measures the same relative percentage changes while holding $m(\lambda\zeta)/m(\lambda)$ constant. The latter requires the percentage changes of $m(\lambda\zeta)$ and $m(\lambda)$ to be equal, that is, the percentage change of $\lambda\zeta$ equals $\varepsilon_0(\lambda)/\varepsilon_0(\lambda\zeta)$ times the percentage change of λ; thus, $a^m(\zeta,\lambda)=a^m(\zeta,\lambda)\varepsilon_0(\lambda)/\varepsilon_0(\lambda\zeta)$.

For invariant technologies we consider $N\geq 2$ worker types, i.e. $x_1<x_2<\cdots<x_N$. To do so, define μ_i as the queue length of workers with type x_i *or higher*, for $i=1,2,\ldots,N$. That is, the queue length of workers of type x_i is $\mu_i-\mu_{i+1}$, with the convention that $\mu_{N+1}=0$, and the total queue length is μ_1.

Consider then a firm of type y that faces a queue $(\mu_1,\mu_2,\ldots,\mu_N)$. With probability $m(\mu_1)$, the firm meets at least one worker, which generates a surplus of at least $f(x_1,y)$; with probability $m(\mu_2)$ the firm meets at least one worker with a type higher than or equal to x_2, which generates an additional surplus of at least $f(x_2,y)-f(x_1,y)$, and so on. Using the convention $f(x_0,y)=0$, expected surplus therefore equals

$$S(\mu_1, \ldots, \mu_N, y) = \sum_{i=1}^{N} m(\mu_i)[f(x_i, y) - f(x_{i-1}, y)], \tag{60}$$

which generalizes equation (3) and is strictly concave in $(\mu_1, \ldots, \mu_N)$.

Similar to equation (13), the firms' problem can be written as

$$\max_{\mu_1,\ldots,\mu_N} S(\mu_1, \ldots, \mu_N, y) - (\mu_1 U_1 + \mu_2(U_2 - U_1) + \cdots + \mu_N(U_N - U_1))$$

where U_i is the market utility of workers of type x_i. The above problem is strictly concave, which implies that the optimal queue is unique and is denoted by $(\mu_1(y), \ldots, \mu_N(y))$.

Assuming an interior solution to simplify exposition, $\mu_i(y)$ is determined by the FOC $m'(\mu_i(y))[f(x_i, y) - f(x_{i-1}, y)] = U_i - U_{i-1}$, where U_i is the market utility of workers of type x_i, with the convention $U_0 = 0$. Defining $\zeta_i(y) = \mu_i(y)/\mu_1(y)$ and $\lambda(y) = \mu_1(y)$, differentiation of this FOC along the equilibrium path yields

$$-\frac{m''(\zeta_i(y)\lambda(y))}{m'(\zeta_i(y)\lambda(y))}\left(\zeta_i'(y)\lambda(y) + \zeta_i(y)\lambda'(y)\right) = \frac{f_y(x_i, y) - f_y(x_{i-1}, y)}{f(x_i, y) - f(x_{i-1}, y)},$$

which replicates (43) for the case of invariant technologies. Setting $i = 1$ and using the fact that $\zeta_1(y) = 1$, the above equation implies

$$-\frac{m''(\lambda(y))}{m'(\lambda(y))}\lambda'(y) = \frac{f_y(x_1, y)}{f(x_1, y)},$$

which replicates (44). Combing the above two equations yields

$$\frac{\frac{m''(\zeta_i(y)\lambda(y))}{m'(\zeta_i(y)\lambda(y))}\left(\zeta_i'(y)\lambda(y) + \zeta_i(y)\lambda'(y)\right)}{\frac{m''(\lambda(y))}{m'(\lambda(y))}\lambda'(y)} = \frac{f_y(x_i, y) - f_y(x_{i-1}, y)}{f(x_i, y) - f(x_{i-1}, y)}\frac{f(x_1, y)}{f_y(x_1, y)},$$

which replicates (45). Similar to Lemma 8, combining the above equations yields the condition for PAC/PAM,

$$\frac{f^1}{f^1_y}\frac{f_y(x_i, y) - f_y(x_{i-1}, y)}{f(x_i, y) - f(x_{i-1}, y)} \geq a^i\left(\zeta_i(y), \lambda(y)\right). \tag{61}$$

where $i = c$ for PAC and $i = m$ for PAM. The case of NAC/NAM corresponds to the reverse inequality ($\leq$ instead of $\geq$).

Proposition 6. *Suppose the meeting technology is invariant. The equilibrium then exhibits PAC (resp. PAM) for any distribution of agents' types if and only if $\underline{\rho} \geq \overline{a}^c$ (resp. $\underline{\rho} \geq \overline{a}^m$). In contrast, the equilibrium exhibits NAC/NAM for any distribution of agents' types if and only if $f(x, y)$ is submodular.*

Proof. The necessary conditions directly follows from Proposition 2, as its proof is valid for any non-bilateral technology (i.e. ϕ is strictly concave in μ). Equivalently, we can let $x_i \to x_1$ in equation (61). Hence, we only need to prove sufficiency.

Consider first the case of NAC/NAM. If $f(x, y)$ is submodular, then $f_y(x_i, y) \leq f_y(x_{i-1}, y)$, which implies that NAC/NAM holds.

Consider next the case of PAC/PAM. Note that $\underline{\rho} \geq \overline{a}^i$ by assumption. Further, $\overline{a}^i \geq 1$, by Lemma 11. Therefore,

$$\frac{f^1}{f_y^1}\frac{f_y(x_i, y) - f_y(x_{i-1}, y)}{f(x_i, y) - f(x_{i-1}, y)} \geq \frac{f(x_{i-1}, y)}{f_y(x_{i-1}, y)}\frac{f_y(x_i, y) - f_y(x_{i-1}, y)}{f(x_i, y) - f(x_{i-1}, y)} \geq \frac{(1 + \kappa_i(y))^{\underline{\rho}} - 1}{\kappa_i(y)} \geq \underline{\rho} \geq \overline{a}^i,$$

where the first inequality holds due to the log-supermodularity of $f(x, y)$, which implies that $f(x_1, y)/f_y(x_1, y) \geq f(x_{i-1}, y)/f_y(x_{i-1}, y)$, the second inequality is because of (35), and the third inequality follows from part i) of Lemma 6. $\square$

A.12 Endogenous Screening

In our baseline model, the screening intensity σ is exogenous. However, firms can generally influence the number of applicants that they interview. We therefore analyze an extension in which firms can choose (and post) their recruiting intensity $\sigma \in [0, 1]$ at a linear cost $c\sigma$ where $c \geq 0$.[41] That is, they solve

$$\max_{\sigma, \mu, \lambda} \quad \frac{\lambda}{1 + \lambda} f^1 + \frac{\mu}{1 + \sigma\mu + (1 - \sigma)\lambda} \Delta f - \lambda U_1 - \mu \Delta U - c\sigma. \tag{62}$$

[41]Posting contracts that include σ in addition to wages is necessary for constrained efficiency in this environment. More restrictive contract spaces and more general cost functions are left for future research. Wolthoff (2018) endogenizes σ in a similar way as us, but with a cost function that is sufficiently convex (in an otherwise quite different model). In the random search model of Birinci et al. (2023), firms have the option to learn all their applicants' types after paying a fixed cost.

Since the second term above is convex in σ and $c\sigma$ is linear, this profit function is *convex* in σ. The maximum is therefore reached at a corner, i.e. when $\sigma = 0$ or 1. To determine firms' choice, we compare the profits from the two options.

Profits with No Screening. Consider a firm of type y choosing $\sigma = 0$. This firm's optimal queue then consists of either low-type workers or high-type workers, but not both. Suppose the firm attracts workers of type x_i. Equation (62) then reduces to $\max_{\lambda_i} m(\lambda_i) f(x_i, y) - \lambda_i U_i$. Because $m(\lambda)$ is strictly concave, the FOC of this problem is both necessary and sufficient. Assuming that $f(x_i, y) > U_i$, the optimal queue length is $\lambda_i = \sqrt{f(x_i, y)/U_i} - 1$, which yields an expected payoff of

$$\pi_i(y) = \left(\sqrt{f(x_i, y)} - \sqrt{U_i}\right)^2. \tag{63}$$

Naturally, the firm chooses the type of workers it wishes to attract based on whether $\pi_1(y)$ or $\pi_2(y)$ is higher, which requires comparing $\sqrt{f(x_2, y)} - \sqrt{f(x_1, y)}$ with $\sqrt{U_2} - \sqrt{U_1}$. If the former is strictly increasing in y, i.e. f is strictly square-root supermodular, then there exists a unique y^{EK} such that $\pi_2(y) > \pi_1(y)$ if $y > y^{EK}$ and vice versa. This result is a special case of Eeckhout and Kircher (2010).

Profits with Perfect Screening. When the firm chooses $\sigma = 1$, (62) reduces to

$$\overline{\pi}(y) \equiv \max_{0 \le \mu \le \lambda} \frac{\lambda}{1+\lambda} f^1 + \frac{\mu}{1+\mu} \Delta f - \lambda U_1 - \mu \Delta U. \tag{64}$$

This problem is strictly concave in (μ, λ), so that the FOCs are both necessary and sufficient. The only complexity lies in the constraint $0 \le \mu \le \lambda$, which, as we illustrate in Figure 5, implies that there are four possibilities with respect to the optimal applicant pool:

(i) *No applicants.* If $f(x_1, y) \le U_1$ and $f(x_2, y) \le U_2$, then the firm will not attract any applicants, such that $\overline{\pi}(y) = 0$.

(ii) *Only low-type applicants.* If $f(x_1, y) > U_1$ and $f(x_2, y) - f(x_1, y) \le U_2 - U_1$, the firm will attract low-type workers, but not high-type workers as their marginal product is less than their marginal cost; in this case, $\overline{\pi}(y) = \pi_1(y)$.

(iii) *Only high-type applicants.* If $f(x_2, y) > U_2$ and $f(x_2, y)/f(x_1, y) \ge U_2/U_1$, the firm will attract only high-type workers since their relative productivity

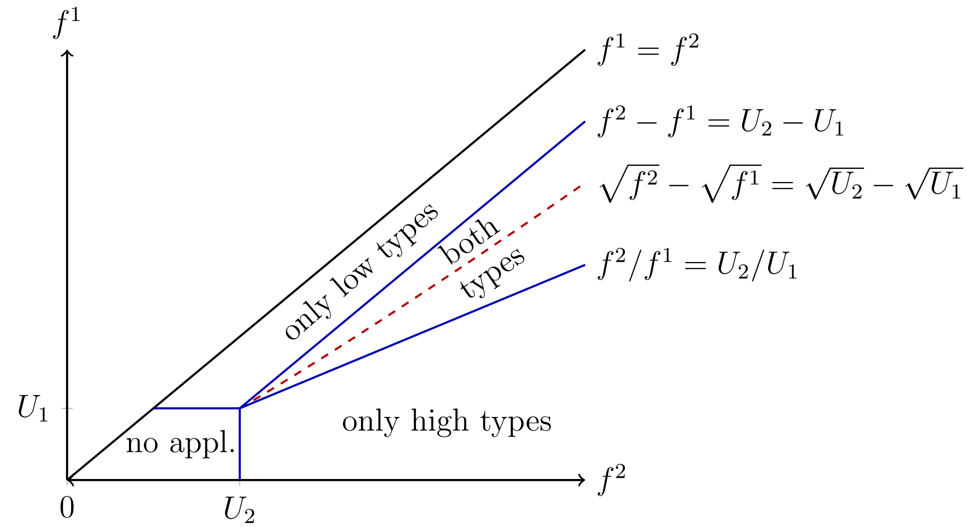

Figure 5: Optimal applicant pool for a firm, conditional on $\sigma = 1$.

is higher than their relative cost; in this case, $\overline{\pi}(y) = \pi_2(y)$.

(iv) *Both types of applicants.* If $f(x_2, y) - f(x_1, y) > U_2 - U_1$ and $f(x_2, y)/f(x_1, y) < U_2/U_1$, then the firm strictly prefers a mix of both types of workers in their application pool. By the FOCs, the optimal queue is given by $\mu = \sqrt{\Delta f/\Delta U} - 1$ and $\lambda = \sqrt{f^1/U_1} - 1$. In this case, $\overline{\pi}(y)$ is given by

$$\overline{\pi}(y) = \left(\sqrt{f^1} - \sqrt{U_1}\right)^2 + \left(\sqrt{\Delta f} - \sqrt{\Delta U}\right)^2. \tag{65}$$

Clearly, a necessary condition for $\sigma = 1$ to yield higher profits than $\sigma = 0$ is that the firm attracts both types of applicants. In what follows, we will therefore focus on this case, which occurs when

$$\Delta f > \Delta U \quad \text{and} \quad \frac{f(x_2, y)}{f(x_1, y)} < \frac{U_2}{U_1}. \tag{66}$$

As the red dashed line in Figure 5 shows, the region described by (66) is divided into two parts by the curve $\pi_1(y) = \pi_2(y)$, or equivalently

$$\sqrt{f^2} - \sqrt{f^1} = \sqrt{U_2} - \sqrt{U_1}. \tag{67}$$

We therefore have to distinguish between two cases when calculating the difference in profits between $\sigma = 0$ and $\sigma = 1$ in this region, i.e. $\Delta\pi(y) \equiv \overline{\pi}(y) - \max\{\pi_1(y), \pi_2(y)\}$. The following lemma formalizes this.

Lemma 12. *If a firm is indifferent between attracting low- and high-type workers conditional on $\sigma = 0$, i.e. $\pi_1(y) = \pi_2(y)$ or equivalently (67) holds, then this firm attracts both types of workers conditional on $\sigma = 1$, i.e. (66) also holds. In the region characterized by (66), the difference in profits between $\sigma = 1$ and $\sigma = 0$ equals*

$$\Delta\pi(y) = \begin{cases} \left(\sqrt{\Delta f} - \sqrt{\Delta U}\right)^2 & \text{if } \pi_1(y) \geq \pi_2(y), \quad \text{(68a)} \\ 2\left(\sqrt{f^2 U_2} - \sqrt{f^1 U_1} - \sqrt{\Delta f \Delta U}\right) & \text{if } \pi_1(y) \leq \pi_2(y). \quad \text{(68b)} \end{cases}$$

Proof. Equation (67) can be rewritten as $\sqrt{f^2/f^1} - 1 = \sqrt{U_1/f^1}(\sqrt{U_2/U_1} - 1)$. Since $U_1/f^1 < 1$, it follows that $\sqrt{U_2/U_1} - 1 > \sqrt{f^2/f^1} - 1$, and thus $U_2/U_1 > f^2/f^1$. Similarly, (67) can also be rewritten as $(f^2 - f^1)/(\sqrt{f^2} + \sqrt{f^1}) = (U_2 - U_1)/(\sqrt{U_2} + \sqrt{U_1})$. Because $f^1 > U_1$ and $f^2 > U_2$, we have $\Delta f > \Delta U$. Hence, (66) holds. Equation (68) then follows from substituting the relevant version of (63) into $\Delta\pi(y) = \overline{\pi}(y) - \max\{\pi_1(y), \pi_2(y)\}$. $\square$

Choice of Screening Intensity. The characterization of $\Delta\pi(y)$ completes the analysis of the firm's choice problem given by (62): the firm's optimal σ is 1 if $\Delta\pi(y) > c$, 0 if $\Delta\pi(y) < c$, and indeterminate in the knife-edge case $\Delta\pi(y) = c$. If the optimal σ is 1, then the optimal (μ, λ) must be interior, and given by $\mu = \sqrt{\Delta f/\Delta U} - 1$ and $\lambda = \sqrt{f^1/U_1} - 1$. When the optimal σ is 0, then the firm will attract either only low-type or only high-type workers, depending on whether $\sqrt{f^2} - \sqrt{f^1}$ is larger than $\sqrt{U_2} - \sqrt{U_1}$, as discussed after (63).

Sorting. In the special case $c = 0$, where all firms choose $\sigma = 1$, the necessary and sufficient condition for PAC/PAM (resp. NAC/NAM) is that $f(x, y)$ needs to be log-supermodular (resp. submodular). Proposition 7 below addresses the question whether the same conditions are sufficient for any screening cost c. For NAC/NAM, the answer is (almost) true: *strict* submodularity is sufficient for NAC/NAM for any distribution of agents' types and any screening cost c.

However, a sufficient condition for PAC/PAM for any distribution of agents' types and any screening cost c does not exist: For any log-supermodular $f(x, y)$, we can find counterexamples where PAC/PAM fails in equilibrium. The sufficient condition (69) in Proposition 7 for PAC/PAM is for a given distribution of agent types so that $\kappa(\underline{y})$, the lower bound of the output dispersion parameter, is fixed.[42] It requires that either production complementarity measured by $\underline{\rho}$, the lower bound of the production complementarities, or output dispersion measured by $\kappa(\underline{y})$ is sufficiently large. Note that condition (69) is quite sharp: in the proof of Proposition 7, we show that with CES production we can construct counterexamples where PAC/PAM fails in equilibrium whenever $\rho < \Omega(\kappa(\underline{y}))$.

Proposition 7. *In our environment with endogenous screening, the following holds:*

(i) Equilibrium exhibits NAC/NAM for any distribution of agents' types and any cost c if (resp. only if) $f(x, y)$ is strictly (resp. weakly) submodular.

(ii) Given any log-supermodular function f, we can find a distribution of agents' types and a screening cost c such that PAC/PAM fails in equilibrium. However, given a distribution of agents' types, PAC/PAM holds in equilibrium (for any screening cost c) if

$$\underline{\rho} \geq \Omega\left(\kappa(\underline{y})\right), \tag{69}$$

where $\kappa(\cdot)$ is defined by (29), $\underline{y}$ is the lowest firm type, and $\Omega(\kappa) \equiv 1/2 + \ln(\sqrt{\kappa}+\sqrt{1+\kappa})/\ln(1+\kappa)$, which is strictly decreasing with $\lim_{\kappa\to 0}\Omega(\kappa) = \infty$ and $\lim_{\kappa\to\infty}\Omega(\kappa) = 1$.

Proof. We first consider the case of NAC/NAM and then prove the case of PAC/PAM.

The Analysis of NAC/NAM. As mentioned in Appendix A.12, necessity of submodularity of $f(x, y)$ for NAC/NAM follows from the special case $c = 0$ (see Proposition 3). Next, we show that strict submodularity of $f(x, y)$ is sufficient for NAC/NAM. From the discussion after equation (63), it follows that when $f(x, y)$ is strictly submodular, and thus strictly square-root submodular, there exists a unique y^{EK} which solves (67). Furthermore, $\pi_2(y) > \pi_1(y)$ for firms with $y < y^{EK}$, and vice versa.

Since f is strictly submodular, both $f^2 - f^1$ and f^2/f^1 are strictly decreasing in y. The first part of Lemma 12 states that y^{EK} must belong to the region characterized by (66). There exists at most one $y' < y^{EK}$ such that $f(x_2, y')/f(x_1, y') =$

[42] Since f is assumed to be log-supermodular, $\kappa(y)$ is smallest at $y = \underline{y}$.

U_2/U_1 (otherwise set $y' = \underline{y}$), and at most one $y'' > y^{EK}$ such that $f(x_2, y'') - f(x_1, y'') = U_2 - U_1$ (otherwise set $y'' = \overline{y}$). The region characterized by (66) is thus $y \in (y', y'')$. The following Lemma establishes that $\Delta\pi(y)$ is single-peaked at $y = y^{EK}$.

Lemma 13. *Suppose that $f(x, y)$ is strictly submodular. In the region characterized by (66), $\Delta\pi(y)$ is strictly increasing in y for $y \leq y^{EK}$ and strictly decreasing in y for $y \geq y^{EK}$.*

Proof. For submodular f, $\pi_2(y) > \pi_1(y)$ if $y < y^{EK}$, and vice versa. As we remarked before, the region characterized by (66) is (y', y''), which contains y^{EK}. Hence,

$$\Delta\pi'(y) = \begin{cases} \left(1 - \frac{\sqrt{\Delta U}}{\sqrt{\Delta f}}\right) \Delta f_y & \text{if } y > y^{EK}, \quad \text{(70a)} \\ -\left(\sqrt{\frac{\Delta U}{\Delta f}} - \sqrt{\frac{U_2}{f^2}}\right) f_y^2 + \left(\sqrt{\frac{\Delta U}{\Delta f}} - \sqrt{\frac{U_1}{f^1}}\right) f_y^1 & \text{if } y < y^{EK}. \quad \text{(70b)} \end{cases}$$

To establish the sign of (70a), note that $\Delta f_y = f_y^2 - f_y^1 < 0$ when f is strictly submodular; hence, $\Delta\pi'(y) < 0$ for $y > y^{EK}$. To establish the sign of (70b), note that $f^2/f^1 < U_2/U_1$ is equivalent to $\Delta U/\Delta f > U_1/f^1$ or $\Delta U/\Delta f > U_2/f^2$. The coefficient of f_y^2 in (70b) is therefore negative. Since f is submodular, $f_y^2 \leq f_y^1$, and we have

$$\Delta\pi'(y) \geq -f_y^1 \left(\sqrt{\frac{\Delta U}{\Delta f}} - \sqrt{\frac{U_2}{f^2}}\right) + f_y^1 \left(\sqrt{\frac{\Delta U}{\Delta f}} - \sqrt{\frac{U_1}{f^1}}\right) = f_y^1 (\sqrt{U_2/f^2} - \sqrt{U_1/f^1}),$$

where the right-hand side is strictly positive because $U_2/U_1 > f^2/f^1$. Hence, $\Delta\pi'(y) > 0$ for $y < y^{EK}$, i.e. $\Delta\pi(y)$ is strictly increasing in y for $y \leq y^{EK}$. $\square$

This result implies that firms with type y^{EK} have the strongest incentive to screen. If all firms choose $\sigma = 1$ in equilibrium, then sufficiency follows from Proposition 3; if all firms choose $\sigma = 0$ in equilibrium, then sufficiency follows from Proposition 3 or Eeckhout and Kircher (2010). In the remaining case, where the equilibrium features both firms choosing $\sigma = 1$ and firms choosing $\sigma = 0$, we must have $\Delta\pi(y^{EK}) > c$ (otherwise all firms will choose $\sigma = 0$). There exist then two firm types $\underline{y}^s$ and $\overline{y}^s$ with $y' \leq \underline{y}^s < y^{EK} < \overline{y}^s \leq y''$, where firms of type $\underline{y}^s$ and $\overline{y}^s$ are indifferent between choosing $\sigma = 0$ and 1, i.e. $\Delta\pi(\underline{y}^s) = \Delta\pi(\overline{y}^s) = c$.

Firms with $y < \underline{y}^s$ will choose $\sigma = 0$ and attract only high-type workers; firms with $y \in (\underline{y}^s, \overline{y}^s)$ will choose $\sigma = 1$ and attract both types of workers; finally, firms with $y > \overline{y}^s$ will choose $\sigma = 0$ and attract only low-type workers. Since all firm types y between $\underline{y}^s$ and $\overline{y}^s$ choose $\sigma = 1$, submodularity implies that NAC/NAM holds within this interval. Combining the above results implies that NAC/NAM holds globally.

Note that we can not weaken the requirement of strict submodularity to mere submodularity for the sufficient condition. To see this, set $f(x, y) = x + y$ and initially set c large enough so that all firms choose $\sigma = 0$. Then for $y \geq y^{EK}$, $\Delta\pi(y)$ is a constant by equation (68a). If we set $c = \Delta\pi(y^{EK})$, all firms with $y \geq y^{EK}$ are indifferent between choosing $\sigma = 0$ with low-type applicants and $\sigma = 1$ with both types of applicants. This indeterminacy violates NAC/NAM.

The Analysis of PAC/PAM. First, with a slight abuse of notation, given x_1 and x_2, we define $\rho(x_1, x_2, y)$ as the solution to

$$\frac{f_y(x_2, y)}{f_y(x_1, y)} = \left(\frac{f(x_2, y)}{f(x_1, y)}\right)^{\rho(x_1, x_2, y)}. \tag{71}$$

By Lemma 5, $\rho(x_1, x_2, y) \in [\underline{\rho}, \overline{\rho}]$. Note that $\rho(x_1, x_2, y)$ is the discrete version of $\rho(x, y)$ defined in (1). We have $\rho(x_1, x_2, y) \to \rho(x, y)$ when $x_1, x_2 \to x$.

Second, to simplify exposition, we introduce a transformation $\Omega(\cdot)$ of $\kappa\,(y)$, the output dispersion parameter defined by equation (29). Define

$$\Omega(\kappa) \equiv \frac{1}{2} + \frac{\ln(\sqrt{\kappa} + \sqrt{1+\kappa})}{\ln(1+\kappa)}.$$

Lemma 14. *$\Omega(\kappa)$ is strictly decreasing with $\lim_{\kappa\to 0} \Omega(\kappa) = \infty$ and $\lim_{\kappa\to\infty} \Omega(\kappa) = 1$.*

Proof. By L'Hospital's Rule, $\lim_{\kappa\to 0} \Omega(\kappa) = \lim_{\kappa\to 0} \frac{1}{2} + \frac{1}{\sqrt{\kappa}+\sqrt{1+\kappa}}\left(\frac{1}{2\sqrt{\kappa}} + \frac{1}{2\sqrt{1+\kappa}}\right)(1+\kappa) = \infty$. In contrast, when $\kappa \to \infty$, we have $\kappa \approx 1 + \kappa$ and $\lim_{\kappa\to\infty} \Omega(\kappa) = \lim_{\kappa\to\infty} \frac{1}{2} + \frac{\ln(\sqrt{\kappa}+\sqrt{\kappa})}{\ln(\kappa)} = 1$.

Next, we prove that $\Omega(\kappa)$ is strictly decreasing. By direct computation,

$$\Omega'(\kappa) = \frac{\ln(1+\kappa) - 2\sqrt{\frac{\kappa}{1+\kappa}}\ln(\sqrt{\kappa} + \sqrt{1+\kappa})}{4\sqrt{\kappa(1+\kappa)}\ln(1+\kappa)}.$$

The derivative of the numerator above is $-\ln(\sqrt{\kappa}+\sqrt{1+\kappa})\sqrt{\frac{1+\kappa}{\kappa}}(1+\kappa)^{-2}<0$. At $\kappa=0$, the numerator is zero, which implies that it is strictly negative and hence $\Omega'(\kappa)<0$ when $\kappa>0$. □

We now provide a claim which is stronger than the statements in Proposition 7.

Claim. *Consider a log-supermodular function f. Given a distribution of agents' types, PAC/PAM holds in equilibrium as long as, for each y,*

$$\rho(x_1,x_2,y)\geq\Omega(\kappa(y)). \tag{72}$$

In contrast, given x_1,x_2 and $J(y)$, if for some $y^*\in(\underline{y},\overline{y})$, we have

$$\rho(x_1,x_2,y^*)<\Omega(\kappa(y^*)), \tag{73}$$

then we can find (L,z) and c such that PAC/PAM fails in equilibrium.

Since $\Omega(\cdot)$ is strictly decreasing and with log-supermodular f, $\kappa(y)$ is increasing in y), the right-hand side of (72) reaches its maximum at $y=\underline{y}$. Also since $\rho(x_1,x_2,y)\geq\underline{\rho}$, the sufficient condition (69) in Proposition 7 then implies (72). On the other hand, given any log-supermodular function, whenever $x_1,x_2\to x$, then $\kappa(y)\to 0$ and $\Omega(\kappa(y))\to\infty$, and (73) holds for all $y^*\in[\underline{y},\overline{y}]$, which, by the above claim, implies that we can find (L,z) and c such that PAC/PAM fails in equilibrium.

Note that for a CES production function, (72) reduces to $\rho\geq\Omega(\kappa(\underline{y}))$ and and (73) reduces to $\rho<\Omega(\kappa(\underline{y}))$. Thus, although the sufficient condition (69) is slightly weaker than (72), it is still sharp in the special case of CES production functions.

Similar to the analysis of NAC/NAM, since $f(x,y)$ is log-supermodular, and therefore strictly square-root supermodular, there exists a unique y^{EK} which solves (67). The first part of Lemma 12 states that y^{EK} must belong to the region characterized by (66). Furthermore, f^2-f^1 is strictly increasing so that there exists at most one $y'<y^{EK}$ such that $f^2-f^1=U_2-U_1$ (otherwise set $y'=\underline{y}$). Since we only assume weak log-supermodularity, f^2/f^1 is weakly increasing. Set $y''=\min\{y\mid f^2/f^1\geq U_2/U_1\}$ (if this set is empty, then set $y''=\overline{y}$). The region characterized by (66) is then $y\in(y',y'')$. The following Lemma establishes

that under the sufficient condition (72), $\Delta\pi(y)$ is single-peaked at $y = y^{EK}$, so PAC/PAM follows from the same logic that was used for the case of NAC/NAM.

Lemma 15. *Suppose that $f(x, y)$ is log-supermodular. In the region characterized by (66), $\Delta\pi(y)$ is strictly increasing in y for $y \leq y^{EK}$, and if condition (72) holds for each $y \in (\underline{y}, \overline{y})$, then it is strictly decreasing in y for $y \geq y^{EK}$.*

Proof. If $y \in (y', y^{EK}]$, then $\Delta\pi(y)$ is given by (68a) and its derivative is given by (70a), so it is strictly increasing in y since $\Delta f_y > 0$. If $y \in [y^{EK}, y'')$, then $\Delta\pi(y)$ is given by (68b) and its derivative is now given by (70b) and can be rewritten as

$$\Delta\pi'(y) = f_y^1 \sqrt{\frac{\Delta U}{\kappa(y) f^1}} \left[-(1+\kappa(y))^{\rho(y)} \left(1 - \sqrt{\frac{\kappa(y)}{1+\kappa(y)}} \sqrt{\frac{U_2}{\Delta U}} \right) + 1 - \sqrt{\frac{\kappa(y)}{\Delta U / U_1}} \right],$$

where, to simplify notation, we shorten $\rho(x_1, x_2, y)$ as $\rho(y)$, and we used the identities $f^2/f^1 = 1 + \kappa(y)$ and $f_y^2/f_y^1 = (1+\kappa(y))^{\rho(y)}$.

Furthermore, define

$$\delta(y) \equiv \sqrt{\frac{\kappa(y)}{\Delta U / U_1}}, \tag{74}$$

which implies $\sqrt{U_2/\Delta U} = \sqrt{(\kappa(y) + \delta(y)^2)/\kappa(y)}$, and $\Delta\pi'(y)$ can be rewritten as

$$\begin{aligned}
\Delta\pi'(y) &= f_y^1 \sqrt{\frac{\Delta U}{\kappa(y) f^1}} \left[(1+\kappa(y))^{\rho(y)} \left(\sqrt{\frac{\kappa(y) + \delta(y)^2}{1+\kappa(y)}} - 1 \right) + 1 - \delta(y) \right] \\
&= f_y^1 \sqrt{\frac{\Delta U}{\kappa(y) f^1}} \left[(1+\kappa(y))^{\rho(y) - \frac{1}{2}} \sqrt{\kappa(y) + \delta(y)^2} - \left((1+\kappa(y))^{\rho(y)} - 1 + \delta(y) \right) \right] \\
&= f_y^1 \sqrt{\frac{\Delta U}{\kappa(y) f^1}} \frac{(1+\kappa(y))^{2\rho(y)-1} \left(\kappa(y) + \delta(y)^2 \right) - \left((1+\kappa(y))^{\rho(y)} - 1 + \delta(y) \right)^2}{(1+\kappa(y))^{\rho(y) - \frac{1}{2}} \sqrt{\kappa(y) + \delta(y)^2} + \left((1+\kappa(y))^{\rho(y)} - 1 + \delta(y) \right)}.
\end{aligned}$$

Thus, $\Delta\pi'(y)$ has the same sign as the numerator of the last factor in the last line. Single out the numerator and define

$$\mathcal{S}(\delta, \kappa, \rho) = (1+\kappa)^{2\rho - 1} \left(\kappa + \delta^2 \right) - \left((1+\kappa)^{\rho} - 1 + \delta \right)^2, \tag{75}$$

which is a quadratic function of δ with a strictly positive second-order coefficient since we assume $\rho \geq 1$ (log-supermodularity). Note that $\mathcal{S}(1,\kappa,\rho) = 0$ and $\frac{\partial \mathcal{S}(\delta,\kappa,\rho)}{\partial \delta}\Big|_{\delta=1} = 2(1+\kappa)^{\rho}((1+\kappa)^{\rho-1}-1) \geq 0$. Therefore, if $\mathcal{S}(0,\kappa,\rho) \leq 0$, then $\mathcal{S}(\delta,\kappa,\rho) < 0$ for all $\delta \in (0,1)$. Note that $\mathcal{S}(0,\kappa,\rho) = \kappa(1+\kappa)^{2\rho-1} - ((1+\kappa)^{\rho}-1)^2$, Thus $\mathcal{S}(0,\kappa,\rho) \leq 0$ if and only if $\sqrt{\frac{\kappa}{1+\kappa}}(1+\kappa)^{\rho} \leq (1+\kappa)^{\rho}-1$, or equivalently $\rho \geq \Omega(\kappa)$.

If for each $y \in (\underline{y},\overline{y})$, we have $\rho(y) \geq \Omega(\kappa(y))$, then by the above argument, $\mathcal{S}(\delta(y),\kappa(y),\rho(y)) < 0$ and hence $\Delta\pi'(y) < 0$ for $y \in [y^{EK}, y'')$. $\square$

Similar to the case of NAC/NAM, we only need to consider the case where the equilibrium features both firms choosing $\sigma = 1$ and firms choosing $\sigma = 0$. Then there exist two firm types $\underline{y}^s$ and $\overline{y}^s$ that are indifferent between choosing $\sigma = 0$ and 1, where $y' \leq \underline{y}^s < y^{EK} < \overline{y}^s \leq y''$. Firms with $y < \underline{y}^s$ will choose $\sigma = 0$ and attract only low-type workers; firms with $y \in (\underline{y}^s, \overline{y}^s)$ will choose $\sigma = 1$ and attract both types of workers; finally, firms with $y > \overline{y}^s$ will choose $\sigma = 0$ and attract only high-type workers. Since all firms of y between $\underline{y}^s$ and $\overline{y}^s$ choose $\sigma = 1$, log-supermodularity implies that PAC/PAM holds within this interval. Combining the above results then implies that PAC/PAM holds globally.

Now consider the second part of the claim. Before we move to the detailed proof, we first give a brief sketch. If (73) holds, then we can find (L,z) and a large c such that all firms choose $\sigma = 0$ in equilibrium, and $\Delta\pi(y)$ reaches its maximum at some point $\widetilde{y} > y^{EK}$ (note that the maximum is between 0 and c here). Now decrease c gradually till firms near $\widetilde{y}$ find it optimal to choose $\sigma = 1$ and screen ex-post while firms with types slightly above y^{EK} will continue choosing $\sigma = 0$ and accordingly attract high-type applicants only. PAC/PAM then fails in this case. Below, we prove this claim formally.

We first prove the following. Given a log-supermodular function $f(x,y)$ and a distribution of agents' types, a necessary condition for PAC/PAM to hold for all c is that $\Delta\pi'_{+}(y^{EK}) \leq 0$ when c is sufficiently large (for example, $c \geq f(x_2,\overline{y})$) so that all firms choose $\sigma = 0$, where $\Delta\pi'_{+}(y^{EK})$ is the right derivative of $\Delta\pi(y)$ at point y^{EK}.

Suppose otherwise that $\Delta\pi'_{+}(y^{EK})$ is strictly positive; the maximum value of $\Delta\pi(y)$ must then be reached at some point $\widetilde{y} > y^{EK}$, since $\Delta\pi(y)$ is always strictly increasing when $y \in (y', y^{EK})$ (see Lemma 15). Now define $\widetilde{c} = \Delta\pi(\widetilde{y})$ and gradually decrease it from $f(x_2,\overline{y})$ to values around $\widetilde{c}$. What is the impact of this change on the sorting pattern? As long as $c \geq \widetilde{c}$, no firm is willing to invest in screening, so the equilibrium allocation remains the same. When c is slightly below $\widetilde{c}$, then firms with types sufficiently close to $\widetilde{y}$ will choose $\sigma = 1$. Note that the equilibrium market utilities U_1 and U_2 will change slightly, so that y^{EK} also changes only slightly. As before, firms with types slightly above y^{EK} will therefore choose $\sigma = 0$ and hire high-type workers only, while firms with types sufficiently close to $\widetilde{y}$ will attract both types of workers. Hence, PAC/PAM fails to hold when c is slightly below $\widetilde{c}$.

Below, we complete the proof by showing that for any log-supermodular function $f(x, y)$ and $(x_1, x_2, J(y))$, if (73) holds for some $y^* \in (\underline{y}, \overline{y})$, then we can choose (L, z) such that $\Delta\pi'_+(y^{EK}) > 0$ when c is sufficiently large that all firms choose $\sigma = 0$.

Step 1: Since $\rho(y^*) < \Omega(\kappa(y^*))$, we have $\mathcal{S}\left(0, \kappa(y^*), \rho(y^*)\right) > 0$, where $\mathcal{S}$ is defined in equation (75). Thus, by continuity, we can find a δ^* small enough such that $\mathcal{S}(\delta^*, \kappa(y^*), \rho(y^*)) > 0$. Next, we construct (U_1^*, U_2^*) from the following two equations,

$$\sqrt{f(x_2, y^*)} - \sqrt{f(x_1, y^*)} = \sqrt{U_2^*} - \sqrt{U_1^*}$$

$$\delta^* = \sqrt{\frac{(f(x_2, y^*) - f(x_1, y^*))/f(x_1, y^*)}{(U_2^* - U_1^*)/U_1^*}}.$$

These equations are reminiscent of (67) and (74), respectively. The main difference is that there we considered the market utilities as known and solved for y^{EK} and $\delta(y)$; here we treat y^* and δ^* as known and solve for market utilities instead.

Step 2: Given (U_1^*, U_2^*), y^* is then the firm type that corresponds to y^{EK} defined before. Since f is log-supermodular and hence strictly square-root supermodular, firms with types $y > y^*$ will attract only high-type applicants, and firms with types $y < y^*$ will attract only low-type applicants. The firms' problem is $\max_\lambda m(\lambda) f(x_1, y) - \lambda U_1^*$ for $y \leq y^*$, and $\max_\lambda m(\lambda) f(x_2, y) - \lambda U_2^*$ for $y \geq y^*$. Denote the solution by $\lambda(y)$ for all y.

Step 3: Set $L(1 - z) = \int_{\underline{y}}^{y^*} \lambda(y) dJ(y)$ and $Lz = \int_{y^*}^{\overline{y}} \lambda(y) dJ(y)$. Then, by construction, (U_1^*, U_2^*) are indeed the market utilities, $y^* = y^{EK}$ for the equilibrium where all firms choose $\sigma = 0$, and $\Delta\pi'_+(y^{EK}) > 0$ because $\mathcal{S}(\delta^*, \kappa(y^*), \rho(y^*)) > 0$ and $y^* = y^{EK}$. $\square$