Online Appendix for Financial Intermediation, Leverage, and Macroeconomic Instability

By Gregory Phelan

I. More on Leverage Constraints

A. Exogenous Leverage Constraints

Figure 1 plots the drifts and volatilities and prices for the given leverage restrictions. Tighter leverage constraints lead to lower asset price volatility (σ^q) and lower systemic volatility (σ^η). The drift in prices (μ^q) increases when leverage constraints are very tight because banks no longer drive down returns through competition. The effect on the drift in bank equity (μ^η), however, is not monotonic, reflecting that the effect on retained earnings can be complicated. A key reason that the system is more stable is that the price of good 1 is higher when leverage is constrained. In other words, intermediated investments yield higher returns when banks use less leverage.

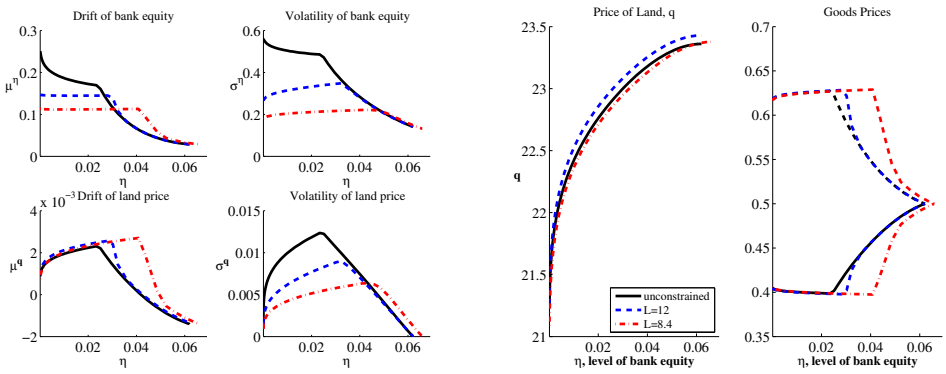


Figure 1. : Equilibrium Evolutions and Prices with Leverage Limits.

Figure 2 plots the Sharpe ratios and recovery rates given leverage constraints. Though Sharpe ratios at a given η are higher with tighter leverage constraints, the average Sharpe ratio is lower because the stationary distribution shifts toward η^* (the averages are 12.6% and 12.35% respectively for $L = 12$ and $L = 8.4$). I plot $g(\eta)$ against the distance from η^* rather than against η because η^* changes with borrowing constraints. The recovery function $g(\eta)$, defined in equation 9,

is the value of a bond that pays 1 when the economy returns to η^*. It is the present value of the next dividend payment to shareholders, and is a measure of the expected time to recovery. Recovery is faster with tighter leverage constraints.

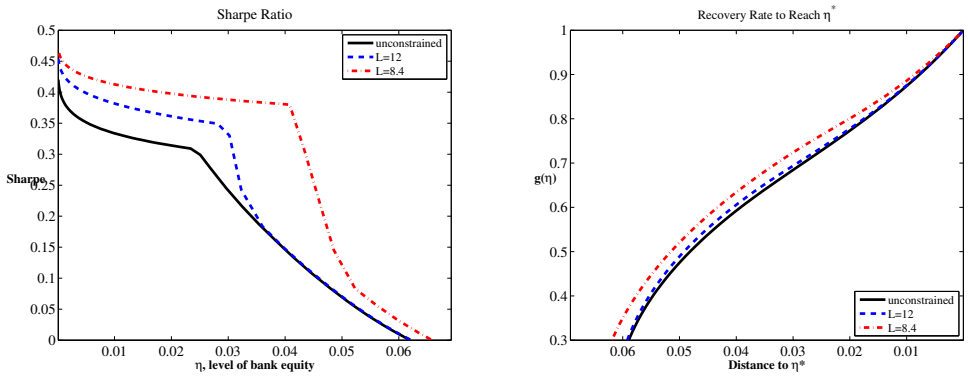


Figure 2. : Sharpe Ratios and Recovery Rates with Leverage Limits.

Figure 3 plots banks' value given leverage limits. Banks' values are hardly affected by leverage constraints, which is consistent with the results in Brunnermeier and Sannikov (2014). Notice that the change in bank value for $L = 8.4$ is primarily driven by the increase in η^*, i.e., the value function is shifted right.

Figure 4 plots how welfare varies with the maximum allowable leverage. The figure plots the maximum welfare the economy attains ($J(\eta*)$) against the leverage constraint L. There are two things to notice. First, there is a hump-shaped relationship between welfare and tighter leverage constraints. Second, the relationship is asymmetric. The welfare gains to tighter leverage constraints are generally not as large as the welfare losses from overly tight leverage regulation. Relative to the optimum, the costs of choosing too tight of leverage regulation is higher than the costs from choosing too loose of leverage regulation.

B. Illustrating the Results with a Simulation

I illustrate the main results by simulating economies with and without equity and leverage constraints. Figure 5 plots the simulations for flow utility and the asset price. These simulations show how banks affect flow utility (output, to a first-approximation) and asset prices and how limiting leverage can reduce volatility and improve the average level of flow utility.[1] The red line presents an economy in which banks are subject to no constraints: resources are efficiently allocated

[1] I've used $\sigma = 1\%$, $g = 3\%$, and $L = 15$ for these simulations in order to more clearly demonstrate the results, which are amplified for these parameters. The simulation is qualitatively the same, though muted, for the baseline parameters.

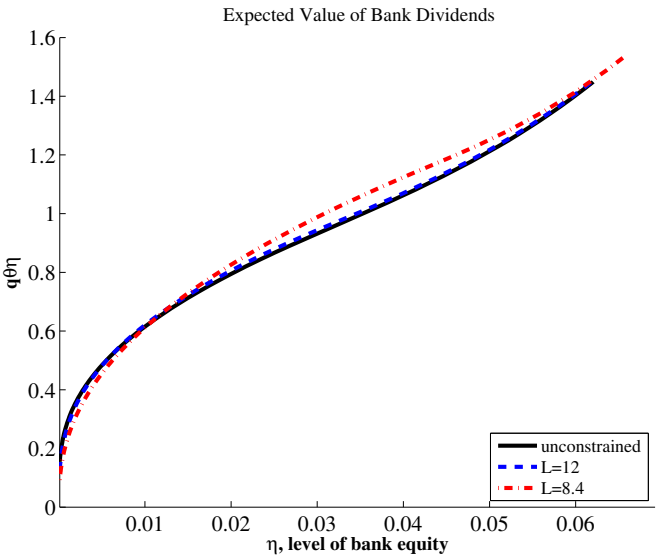


Figure 3. : Banks' Value with Leverage Limits.

and the economy is stable with no endogenous amplification or propagation. In such an economy, flow utility is stable and asset prices (the straight line on top) are constant even with fundamental shocks (because allocations do not change, prices are stable).

The black line shows an economy in which banks are limited in their ability to issue equity: there is significant endogenous amplification and persistence. Fundamental shocks affect banks' equity levels, and banks respond by decreasing their asset holdings, worsening allocations and decreasing asset prices. The decrease in asset prices feeds back into the value of banks' equity, further depressing its value, leading to more asset sales and misallocation. Importantly, the economy is much more volatile and volatility is almost entirely "downside" risk, because misallocation occurs when banks have low equity levels.

The blue line shows an economy in which banks' leverage and ability to issue equity are both limited. There are times when, because of leverage limits, banks hold fewer assets compared to the black economy and so flow allocations and flow utility suffers (this point is illustrated in the early part of the diagram). However, the economy is much more stable: fundamental shocks have a muted effect on banks' equity because banks use less leverage, and thus resources are on average allocated better and asset prices are less volatile. Limiting leverage decreases volatility, which has the dynamic effect of increasing the overall level of flow utility.

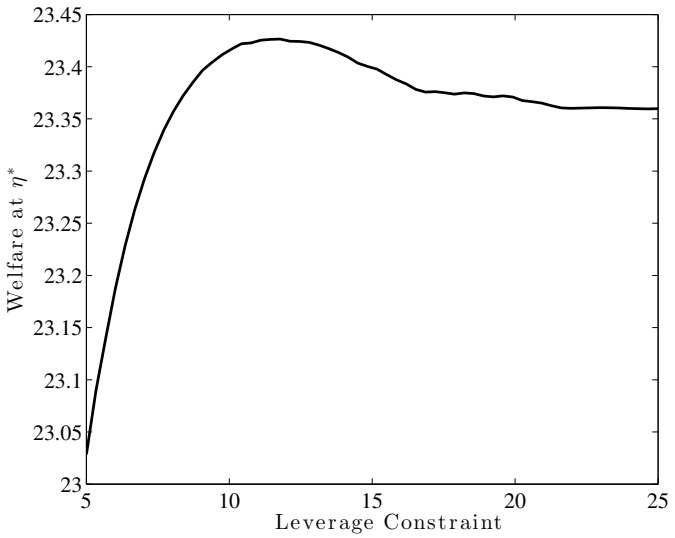


Figure 4. : Welfare at η^* Given Leverage Constraint.

C. Endogenous Borrowing Constraints: Value-at-Risk

Endogenous borrowing constraints may already limit leverage, and maybe too much. Empirical evidence suggests that bank leverage away from the steady state does not behave as in the model thus far.

Adrian and Shin (2010, 2011) show that leverage ratios for commercial and investment banks are typically acyclical and pro-cylical respectively; when volatility spikes, their leverage falls. Adrian and Shin present very strong evidence that these institutions maintain a constant value-at-risk ("VaR"). The value-at-risk constraint can be interpreted in at least 3 ways: The VaR constraint could capture (i) explicit capital constraints set by regulators, as in Basel II; (ii) collateral constraints with countercyclical margins; (iii) implicit agency problems resulting because borrowers can divert fund.[2]

Endogenous Borrowing Constraints, Value-at-Risk:

The endogenous borrowing constraint is a Value-at-Risk ("VaR") constraint that depends on equity and market volatility. Following Danielsson, Shin and

[2]The interpretation affects a regulator's ability to increase leverage, though only slightly. This is trivial in the first case. In the second case, because central banks can offer collateralized borrowing without the need to immediately liquidate seized collateral, they can offer collateralized borrowing at lower margins, and in fact they often do. In the third case, if regulators can, for a time, monitor the firm more closely than usual, then perhaps they can allow firm managers to take on more risk. Recent events suggest that it is possible for governments to increase the leverage that banks can take.

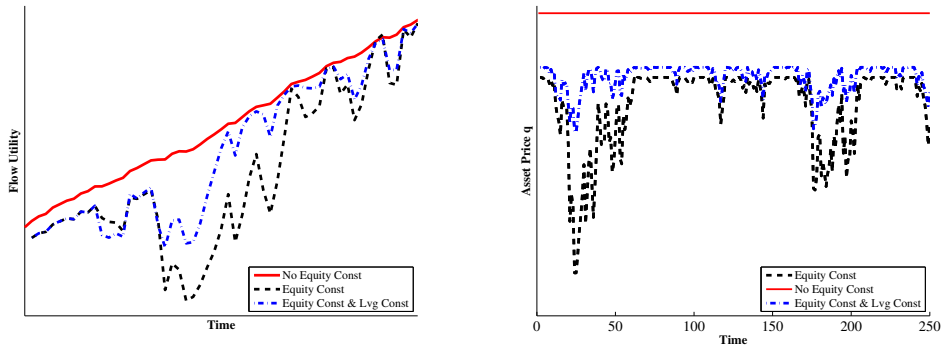


Figure 5. : Endogenous Instability and the Effect of Leverage Constraints.

Zigrand (2011), the value at risk for a bank with assets A_t with volatility $\sigma + \sigma_t^q$ with confidence level z^* is $A_t(\sigma + \sigma_t^q)z^*$. Thus a bank with equity n_t^b can hold assets of value A_t given by

$$A_t \leq \frac{n_t^b}{(\sigma + \sigma_t^q)z^*}, \tag{1}$$

where z^* can be interpreted as the z-score for the probability that the value of assets drop below the value of equity. This restricts leverage to $\left(\frac{1}{(\sigma+\sigma_t^q)z^*} - 1\right)$. With Gaussian returns, the expression $(\sigma+\sigma_t^q)z^*$ is the unit VaR with confidence level z^*. The 1-day 95% unit VaR is $1.645(\sigma + \sigma_t^q)$ and the 1-day 99% unit VaR is $2.326(\sigma + \sigma_t^q)$.

I solve the model for $z^* = 2.326\sqrt{3}$, $2.326\sqrt{5}$, and $2.326\sqrt{6}$, corresponding to 99% VaR at 3, 5, and 6 day horizons. Figure 6 plots equilibrium leverage and welfare. Since volatility is hump-shaped, VaR constraints cause leverage to be U-shaped *when constraints bind*. Notice that leverage rises local to $\eta = 0$; He and Krishnamurthy (2012, 2013) argue this is empirically correct.

The effects on equilibrium drifts and volatilities, the stationary distribution, recovery rate, and bank value are similar to those in the previous section. It is worth noting what happens to the distribution when borrowing constraints are very tight. Figure 7 plots the stationary distribution with VaR constraints. The shape of the distribution hardly changes as z^* increases, but the distribution overall shifts right. As z^* increases, the economy is more stable in the sense that higher η are more likely, but outcomes at each η are much lower because borrowing constraints lead to misallocation. Aggregate *outcomes* are not actually more stable: the distribution of outcomes is actually worse. Because banks face very tight borrowing constraints, they build up much more equity in order to hold more assets, but this response does not improve welfare, in part because

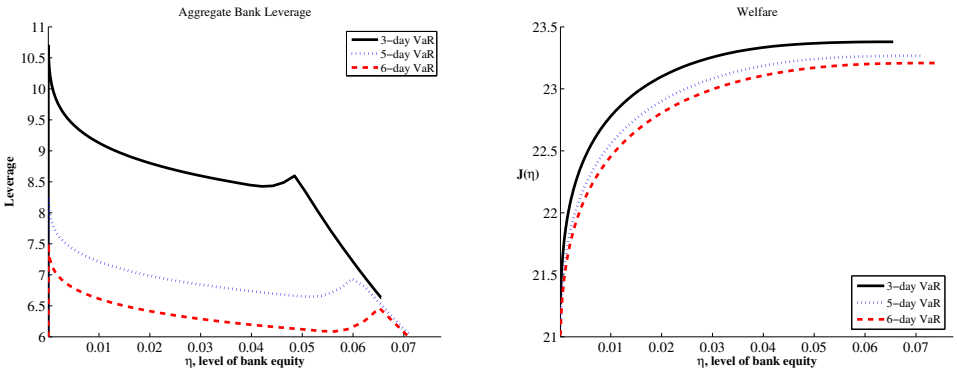


Figure 6. : Equilibrium Leverage and Welfare with VaR constraints.

liquidity services suffer and in part because misallocation is exacerbated by tight constraints.

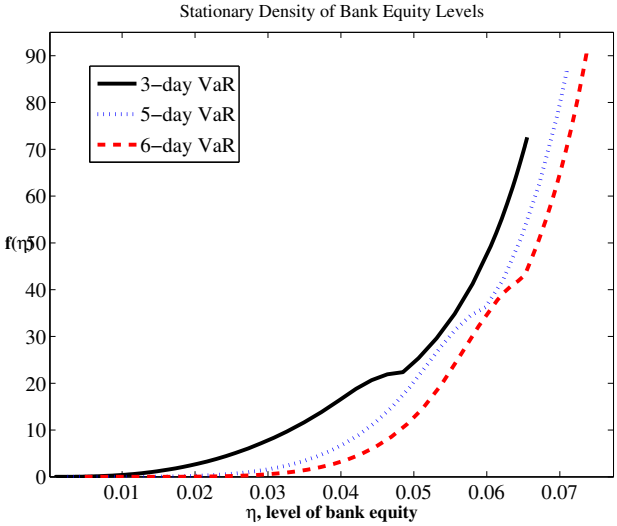


Figure 7. : Stationary Distribution with VaR Leverage Constraints.

Note that pro-cyclical leverage does not exacerbate crises by increasing volatility after bad shocks. One might worry that endogenous constraints might exacerbate crises since constraints bind precisely as volatility rises, forcing banks to dump assets and depress prices, which could further increase volatility. This does not

happen in this model because adjustments occur smoothly and instantaneously.[3]

In this example, tighter VaR constraints decrease welfare. (Proposition II.C still holds, so leverage at η^* is too high.) For low η, increasing leverage could improve welfare—this is the time when resource misallocation is the worst. Increasing leverage during bad times would risk more instability, but it could allow banks to rebuild equity quickly, and it would also improve land allocation.

To demonstrate this, I consider a policy of removing borrowing constraints during "bad times": for $\eta < .03$ there are no borrowing constraints, but for $\eta > .03$ banks are subject to a 5-day 99% VaR constraint. Figure 8 shows how equilibrium leverage changes and it shows the effect on welfare. Welfare is higher almost everywhere as a result of this intervention, and the effect is greatest for low values of η. For very low η, misallocation is the worst and the economy takes a long time to recover—misallocation costs persist. Thus, improving allocation during these times has the greatest effect on welfare.

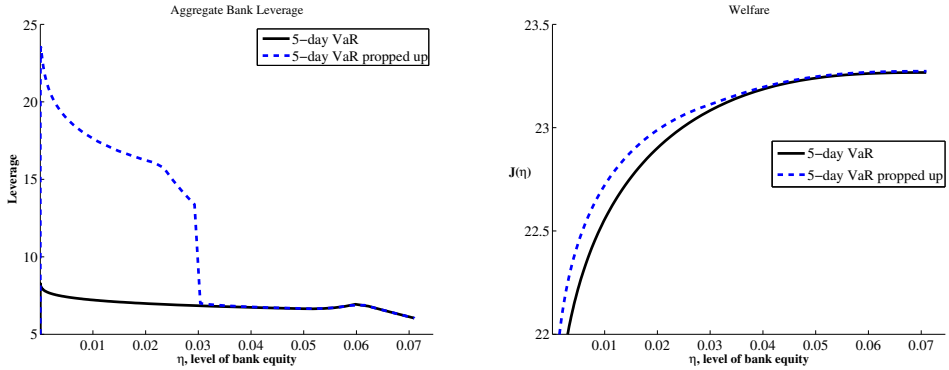


Figure 8. : Welfare Gains from Propping-up Leverage in Bad Times.

This policy does not significantly affect the economy's recovery time or the stationary distribution. Recovery is slightly worse for intermediate values of η, but recovery is faster for low η. Thus higher leverage has actually made the economy (slightly) more stable at times when stability is most valued. Increasing leverage in this way improves welfare only when borrowing constraints are tight. For example, this same policy hurts welfare significantly if $z^* = 2$.

II. Equilibrium With Costly Equity Issuance

Throughout the paper I assumed that banks cannot issue any new shares. In this section I relax that assumption slightly: banks can issue new shares but at a

[3]It would be worth investigating the effects of VaR constraints when markets are "illiquid" or when adjustments are lumpy rather than smooth.

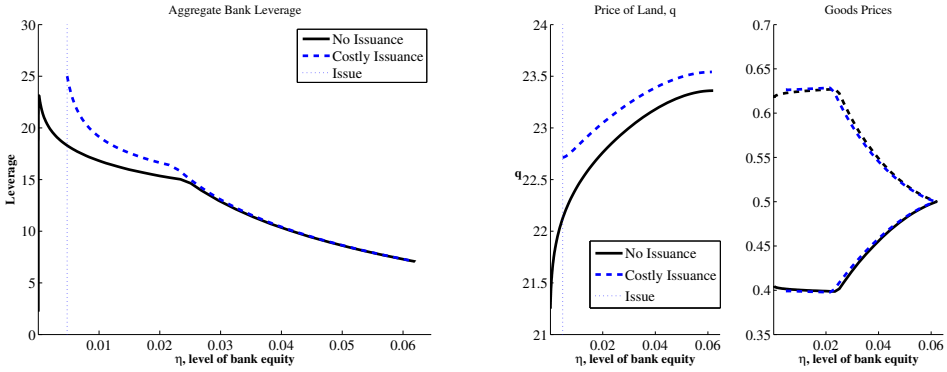
marginal cost $\kappa < 1$ per share. Thus, if $d\zeta_t^b < 0$ banks pay a cost $-\kappa d\zeta_t^b$. So long as banks do not issue shares too frequently, the results with no equity issuance are almost entirely unchanged.[4]

Modifying the budget constraint accordingly, the Bellman equation (with $y_{2t}^b = 0$) becomes:

$$r\theta_t n_t^b = \max_{y_{1t}^b \geq 0} \left\{ \theta_t r^L n_t^b + \theta_t \left((y_{1t}^b q_t)(\mathbb{E}[dr_t^{1b}] - r^L) \right) + \theta_t \mu_t^\theta n_t^b + \sigma_t^\theta \theta_t (y_{1t}^b q_t)(\sigma + \sigma_t^q) \right\}$$
$$+ \max_{d\zeta_t^b} \left\{ d\zeta_t^b (1 - \theta_t) \mathbb{1}_{d\zeta_t^b \geq 0} + d\zeta_t^b (1 - \theta_t + \theta_t \kappa) \mathbb{1}_{d\zeta_t^b < 0} \right\}.$$

Thus, banks will pay dividends when $\theta_t \leq 1$ and banks will issue new shares (i.e., set $d\zeta_t^b < 0$) when $\theta_t \geq \frac{1}{1-\kappa} = \underline{\theta}$.

The equilibrium changes as follows. The economy will fluctuate between $[\underline{\eta}, \eta^*]$ and banks will issue new shares (or new banks will enter) at $\underline{\eta}$. The ODEs are the same, but the boundary conditions are changed to reflect that $\underline{\eta}$ is a reflecting barrier with optimally chosen issuance. The conditions are now (i) $\theta(\eta^*) = 1$; (ii) $q'(\eta^*) = 0$; (iii) $\theta'(\eta^*) = 0$; (iv) $\theta(\underline{\eta}) = \underline{\theta}$; (v) $q'(\underline{\eta}) = 0$; and (vi) $\theta'(\underline{\eta})\underline{\eta} + \theta(\underline{\eta}) = 0$. The modifications are: $\theta(\underline{\eta}) = \underline{\theta}$ because banks issue shares at $\underline{\eta}$, and $q'(\underline{\eta}) = 0$ and $\theta'(\underline{\eta})\underline{\eta} + \theta(\underline{\eta}) = 0$ for smooth pasting at $\underline{\eta}$.

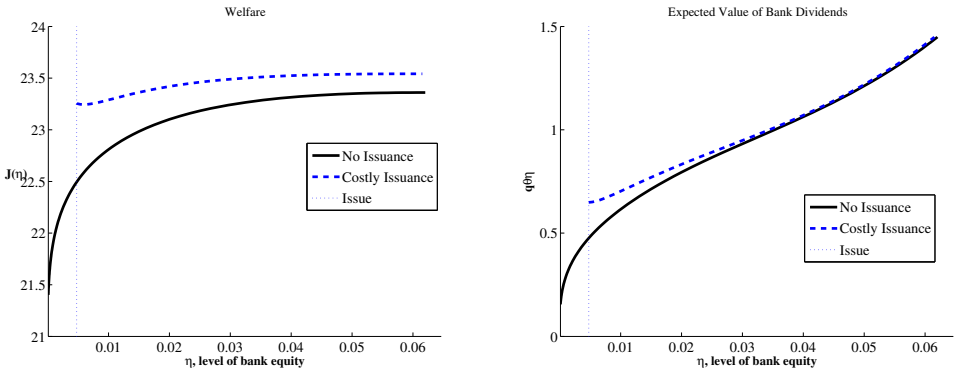


Bank Leverage q_t and p_{1t}, p_{2t}

Figure 9. : Equilibrium Leverage and Prices with Costly Equity Issuance.

[4] If banks issue shares frequently, the overall picture is similar, but some differences emerge because banks may not use the same level of precaution as their equity decreases.

Numerically solving with $\underline{\theta} = 6$, banks still issue new shares infrequently—but frequently enough that the economy is much more stable. Figures 9-11 show that compared to when banks cannot ever issue new shares, the economy is similar, except that at $\underline{\eta}$ endogenous volatility drops to zero and bank leverage spikes.



Welfare Bank Value

Figure 10. : Welfare and Bank Value with Costly Equity Issuance.

Crucially, when banks can issue equity: welfare is much higher, the economy is more stable (there is not a mass in the distribution near $\eta = 0$), the asset price is high, and endogenous volatility is lower even though banks use higher leverage.

III. Efficient Leverage at Steady State

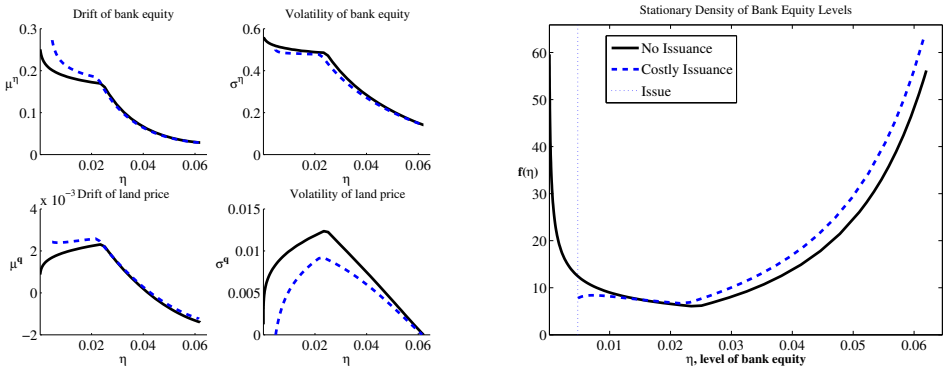
PROPOSITION III.1: *Let households have a value function that is separable in wealth and let banks maximize general preferences. Consider any economy with a stochastic steady state where the state-variable is an increasing function of bank equity, borrowing constraints do not bind at the stochastic steady state, volatility is positive, and at the stochastic steady state banks are not instantaneously more risk-averse than households. Then the marginal social value of bank leverage is negative at* η^*.

PROOF:

Denote the preferences of households by $v(c, b)$ and denote the preferences of banks by $u(\zeta, \eta)$, where ζ are dividends, and η is the state-variable. Let the final-stage production function be $F(Y_1, Y_2)$.

Denote the marginal utilities of households and banks as θ_l for $l = H, I$ respectively. The marginal utilities will follow an equilibrium path given by

$$\frac{d\theta_{lt}}{\theta_{lt}} = \mu_t^{\theta l} dt + \sigma_t^{\theta l} dW_t,$$



Drifts and Volatilities Stationary Distribution

Figure 11. : Evolutions and Stationary Distribution with Costly Equity Issuance.

where $\sigma_t^{\theta l} \leq 0$ reflects risk-aversion. If asset holdings are not constrained, then the following equations, reflecting martingale pricing, hold:

$$\begin{aligned}
\mathbb{E}[dr_t^{1b}] - r_t &\leq -\sigma_t^{\theta I}(\sigma + \sigma^q), \\
\mathbb{E}[dr_t^{2b}] - r_t &\leq -\sigma_t^{\theta I}(\sigma + \sigma^q), \\
\mathbb{E}[dr_t^{1h}] - r_t &\leq -\sigma_t^{\theta H}(\sigma + \sigma^q), \\
\mathbb{E}[dr_t^{2h}] - r_t &\leq -\sigma_t^{\theta H}(\sigma + \sigma^q),
\end{aligned}$$

where r_t is the return on deposits. Because banks have an advantage at cultivating good 1 but not good 2, the first and last equations will hold with equality always (that is, banks will always use land to cultivate good 1, and households for good 2). Define $\phi_I = g - g_B$.

Let $f(\eta_t) = F(y_{1t}, y_{2t})/Y_t$ be aggregate productivity and $g_Y = g - (\lambda - \psi)\, m - \psi\phi_I$ be the aggregate growth rate of land. Since the household value function is separable in wealth, we can write the first-order conditions for the optimal allocations as:

$$\begin{aligned}
\mathcal{L}(\psi_t) = v_c(c_t, d_t)\frac{\partial f(\psi)}{\partial \psi} + v_d(c_t, d_t)q \\
+ J'(\eta)\eta\frac{\partial \mu^\eta}{\partial \psi} + \left(J''(\eta)\sigma^\eta(\psi, \lambda, \eta) + J'\sigma\right)\eta\frac{\partial \sigma^\eta}{\partial \psi} + J\frac{\partial g_Y}{\partial \psi}.
\end{aligned}$$

Consider the first order condition for ψ at $\eta = \eta^*$. Smooth-pasting implies that

$J'(\eta^*) = 0$ so that we have

$$\begin{aligned}\mathcal{L}(\psi_t) &= v_c(c_t, d_t)\frac{\partial f(\psi, x)}{\partial \psi} + v_d(c_t, d_t)q + J\frac{\partial g_Y}{\partial \psi} + J''(\eta)\sigma^\eta(\psi, \lambda, \eta)\eta\frac{\partial \sigma^\eta}{\partial \psi} \\ &= v_c(c_t, d_t)\left(p_1 - p_2\right) + v_d(c_t, d_t)q + J\frac{\partial g_Y}{\partial \psi} + J''(\eta)\sigma^\eta(\psi, \lambda, \eta)\eta\frac{\partial \sigma^\eta}{\partial \psi}\end{aligned}$$

The interest rate on deposits is $r_t^b = r_t - v_d(c_t, d_t)$. Combining the equations for asset demands we can write

$$\frac{p_1}{q} - \frac{p_2}{q} + v_d(c_t, d_t) - \frac{\partial g_Y}{\partial \psi} + \sigma_t^{\theta I}(\sigma + \sigma^q) - \sigma_t^{\theta H}(\sigma + \sigma^q) = 0.$$

Note that $J'' < 0$ because $J(\eta^*) = \frac{z(\psi,\lambda)}{r - g_Y - \sigma^w/2} + \frac{1}{2(r-g_Y)}J''(\eta)(\sigma^\eta(\psi, \eta))^2$. Since the first term is the present discounted value if the system did not move from η^*, the total value is strictly less. Additionally, $\sigma^\eta > 0$ and $\frac{\partial \sigma^\eta}{\partial \psi} > 0$ and $J(\eta) \geq q(\eta)$ because of the no-equity-issuance constraint. From these inequalities, together with the hypothesis on risk-aversion, the first-order-condition is negative, $\mathcal{L}(\psi_t) < 0$, implying that the equilibrium ψ is too high.

IV. Solving for Equilibrium

DEFINITION IV.1 (Competitive Equilibrium): *Given an initial stock of land Y_0 and initial wealth and equity levels $\{n_0^h, n_0^b\}$, a competitive equilibrium is described by a group of stochastic processes on the filtered probability space defined by the Brownian motion $\{W_t, t \geq 0\}$: the price processes for land, goods, banks shares, and debt $\{q_t, p_{1t}, p_{2t}, p_{st}, r_t\}$, bank equity $\{n_t^b \geq 0\}$, portfolio holdings for banks and households $\{y_{1t}^b, y_{2t}^b, s_t^b, \delta_t^b, y_{1t}^h, y_{2t}^h, \delta_t^h\}$, goods demands and production $\{Y_{1t}, Y_{2t}, C_t\}$, and consumption and dividend choices $\{c_t^h, d\zeta_t^b\}$; such that*

1) Banks and Households optimize given prices.

2) Markets for goods clear

$$\begin{aligned}Y_{1t} &= \int y_{1t}^b db + \int y_{1t}^h dh, \\ Y_{2t} &= \int y_{2t}^b db + \int y_{2t}^h dh.\end{aligned}$$

3) Market for consumption goods clears

$$C_t = \int c_t^h dh.$$

4) Market for debt clears

$$\int \delta_t^b db = \int \delta_t^h dh.$$

5) Market for land clears

$$Y_t = \int (y_{1t}^h + y_{2t}^h) dh + \int (y_{1t}^b + y_{2t}^b) db.$$

A. Deriving Equilibrium Conditions

Deriving $d\eta_t$. — To derive the law of motion for η, use Ito's Lemma as follows:

$$\begin{aligned}
\frac{dY_t}{Y_t} =& g_Y dt + \sigma dW_t, \\
\frac{d(q_t Y_t)}{q_t Y_t} =& \left(g_Y + \mu^q + \sigma\sigma^q\right) dt + (\sigma + \sigma^q) dW_t, \\
d\left(\frac{1}{q_t Y_t}\right) q_t Y_t =& \left((\sigma + \sigma^q)^2 - g_Y - \mu^q - \sigma\sigma^q\right) dt - (\sigma + \sigma^q) dW_t, \\
d\eta_t =& d\left(N_t \frac{1}{q_t Y_t}\right) = dN_t \frac{1}{q_t Y_t} + d\left(\frac{1}{q_t Y_t}\right) N_t + Cov\left(N_t, \frac{1}{q_t Y_t}\right) dt.
\end{aligned}$$

Together with equations for returns and laws of motions, we have

$$\begin{aligned}
d\eta_t = & \left[r^L N_t dt + (\psi_t Y_t q_t) \left[\left(\mathbb{E}[dr_t^{1b}] - r^L\right) dt + (\sigma + \sigma^q) dW_t\right] - d\zeta_t^b\right] \frac{1}{q_t Y_t} \\
& + \left[\left((\sigma + \sigma^q)^2 - g(1 - \psi_t) + m(\lambda_t - \psi_t) - \psi_t g_B - \mu^q - \sigma\sigma^q\right) dt - (\sigma + \sigma^q) dW_t\right] \frac{N_t}{q_t Y_t} \\
& + \psi_t Y_t q_t (\sigma + \sigma^q) \left(-\frac{\sigma + \sigma^q}{q_t Y_t}\right) dt.
\end{aligned}$$

When borrowing constraints do not bind, $\mathbb{E}[dr_t^{1b}] - r^L = -\sigma^\theta(\sigma + \sigma^q)$, and so $-g - \mu^q - \sigma\sigma^q = \sigma^\theta(\sigma + \sigma^q) - r + \frac{p_{1t}}{q_t}$. Plugging in we have

$$\begin{aligned}
d\eta_t =& r^L \eta_t dt - \psi_t \sigma^\theta(\sigma + \sigma^q)dt \\
&+ \eta_t \left((\sigma + \sigma^q)^2 + m(\lambda_t - \psi_t) + \psi_t \phi_L + \sigma^\theta(\sigma + \sigma^q) + \frac{p_{1t}}{q_t} - r \right) dt \\
&+ (\psi_t - \eta_t)(\sigma + \sigma^q)dW_t - \psi_t(\sigma + \sigma^q)^2 dt - \frac{d\zeta_t}{q_t Y_t} \\
=& (\psi_t - \eta_t)(\sigma + \sigma^q) \left(dW_t - (\sigma^\theta + \sigma + \sigma^q)dt \right) \\
&+ \eta_t \left(\frac{p_{1t}}{q_t} + (\lambda_t - \psi_t)m - (1 - \psi_t)\phi_L \right) dt - \frac{d\zeta_t}{q_t Y_t}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mu_t^\eta &= -\frac{(\psi_t - \eta_t)}{\eta_t}(\sigma + \sigma^q)(\sigma^\theta + \sigma + \sigma^q) + \left(\frac{p_{1t}}{q_t} + (\lambda_t - \psi_t)m - (1 - \psi_t)\phi_L \right), \\
\sigma_t^\eta &= \frac{(\psi_t - \eta_t)}{\eta_t}(\sigma + \sigma^q), \\
d\Xi &= \frac{d\zeta_t}{N_t}.
\end{aligned}$$

When borrowing constraints bind, σ_t^η and $d\Xi_t$ are as before. Borrowing constraints affect only μ_t^η. Define $\mu_t^A = \frac{p_{1t}}{q_t} + g + \mu^q + \sigma\sigma^q$ to be the expected return banks get when owning land to cultivate good 1. Then we can write μ^η as

$$\mu_t^\eta = \frac{(\psi_t - \eta_t)}{\eta_t} \left(\mu_t^A - r - (\sigma + \sigma^q)^2 \right) + \left(\frac{p_{1t}}{q_t} + m(\lambda_t - \psi_t) - (1 - \psi_t)\phi_L \right),$$

since μ^η is given by

$$\begin{aligned}
\mu_t^\eta =& (r^L - g + m(\lambda_t - \psi_t) + \frac{\psi_t}{\eta_t}\phi_L - \mu^q - \sigma\sigma^q) \\
&+ \psi_t \left(\mathbb{E}[dr_t^{1b}] - r^L \right) - \frac{(\psi_t - \eta_t)}{\eta_t}(\sigma + \sigma^q)^2.
\end{aligned}$$

When borrowing constraints bind so that banks can only get leverage L, the Bellman equation (10) is

$$r\theta_t n_t^b = \theta_t \left(r^L n_t^b + (L+1)n_t^b(\mathbb{E}[dr_t^{1b}] - r^L) \right) + \theta_t \mu^\theta n_t^b + \sigma^\theta \theta_t (L+1) n_t^b (\sigma + \sigma^q).$$

Hence,

$$\begin{aligned}\mu^\theta =&\phi_L - (L+1)\left(\mathbb{E}[dr_t^{1b}] - r + \sigma_t^\theta(\sigma+\sigma^q)\right)\\ =&\phi_L - (L+1)\left(\mu_t^A - r + \sigma_t^\theta(\sigma+\sigma^q)\right).\end{aligned} \tag{2}$$

Notice that when leverage constraints do not bind, $\mu_t^A - r = -\sigma_t^\theta(\sigma+\sigma^q)$.

Deriving the System of Differential Equations. — To derive the ODE, proceed as follows. From equation (16)

$$\begin{aligned}\mu^q q(\eta) =&q'(\eta)\mu^\eta + \frac{1}{2}q''(\eta)(\sigma^\eta)^2 \quad \text{and} \quad \sigma^q q(\eta) = q'(\eta)\sigma^\eta,\\ \mu^\theta \theta(\eta) =&\theta'(\eta)\mu^\eta + \frac{1}{2}\theta''(\eta)(\sigma^\eta)^2 \quad \text{and} \quad \sigma^\theta \theta(\eta) = \theta'(\eta)\sigma^\eta,\end{aligned}$$

and hence,

$$q''(\eta) = 2\frac{\mu^q q(\eta) - \mu^\eta q'(\eta)}{(\sigma^\eta)^2} \quad \theta''(\eta) = 2\frac{\mu^\theta \theta(\eta) - \mu^\eta \theta'(\eta)}{(\sigma^\eta)^2}. \tag{3}$$

Thus, we need to solve for $\mu^q, \mu^\theta, \mu^\eta, \sigma^\eta$ to solve the differential equation. Note that using $d\eta_t$

$$\sigma^q q(\eta) = q'(\eta)(\psi-\eta)(\sigma+\sigma^q) \to \quad \sigma^q = \frac{q'(\eta)}{q(\eta)}\frac{(\psi-\eta)\sigma}{1-\frac{q'(\eta)}{q(\eta)}(\psi-\eta)}. \tag{4}$$

$$\sigma^\theta \theta(\eta) = \theta'(\eta)(\psi-\eta)(\sigma+\sigma^q) \to \quad \sigma^\theta = \frac{\theta'(\eta)}{\theta(\eta)}\frac{(\psi-\eta)\sigma}{1-\frac{q'(\eta)}{q(\eta)}(\psi-\eta)}. \tag{5}$$

Note that higher leverage increases asset price volatility.

Additionally, since households earn return r from cultivating good 2

$$\mu^q = r - g - \sigma\sigma^q - \frac{p_{2t}}{q_t}. \tag{6}$$

We solved for μ^θ in terms of returns and ψ in (2). Hence, we need to solve for λ, ψ to get our second derivatives using (3).

Solving for Allocations. — We solve for ψ, λ using household and bank demands for good 1 and good 2 investments. There are two possibilities: either $\lambda > \psi$ or $\lambda = \psi$.

When $\lambda > \psi$, households cultivate good 1 and therefore,

$$p_{1t} - p_{2t} = mq_t \implies \alpha \left(\frac{1-\lambda_t}{\lambda_t}\right)^{1-\alpha} - (1-\alpha)\left(\frac{\lambda_t}{1-\lambda_t}\right)^{\alpha} = mq_t.$$

This pins down λ. To get ψ when VaR constraints bind we use $\sigma + \sigma_q = \frac{\sigma}{1-\frac{q'(\eta)}{q(\eta)}(\psi-\eta)}$ in (1) to get

$$\psi \leq \frac{\eta}{z^*}\left(\frac{1+\eta\frac{q'(\eta)}{q(\eta)}}{\sigma + \frac{\eta}{z^*}\frac{q'(\eta)}{q(\eta)}}\right). \tag{7}$$

To get ψ when borrowing constraints do not bind, notice that that $-\sigma^{\theta}(\sigma+\sigma^q) = m$. Thus

$$\begin{aligned} m &= -\sigma^{\theta}(\sigma+\sigma^q) \\ m &= -\frac{\theta'(\eta)}{\theta(\eta)}\frac{(\psi-\eta)\sigma}{1-\frac{q'(\eta)}{q(\eta)}(\psi-\eta)}\left(\frac{\sigma}{1-\frac{q'(\eta)}{q(\eta)}(\psi-\eta)}\right). \end{aligned}$$

Let $x = \psi - \eta$. Then

$$\begin{aligned} m\left(1-\frac{q'(\eta)}{q(\eta)}x\right)^2 &= -\frac{\theta'(\eta)}{\theta(\eta)}\sigma^2 x \\ x^2\left(\frac{q'(\eta)}{q(\eta)}\right)^2 - x\left(2\frac{q'(\eta)}{q(\eta)}\right) + 1 &= -\frac{\theta'(\eta)}{\theta(\eta)}\frac{\sigma^2}{m}x \\ x^2\left(\frac{q'(\eta)}{q(\eta)}\right)^2 + x\left(\frac{\theta'(\eta)}{\theta(\eta)}\frac{\sigma^2}{m} - 2\frac{q'(\eta)}{q(\eta)}\right) + 1 &= 0. \end{aligned}$$

Define A, B, C so that $Ax^2 + Bx + C = 0$. Noting that $\psi \geq \eta$ and therefore $x \geq 0$ we can solve for x using the quadratic formula, and the relevant solution is $x = \frac{-B-\sqrt{B^2-4AC}}{2A}$.

Then $\psi_1 = x + \eta$. We can compare this to the borrowing constrained level and confirm that $\lambda > \psi$.

When, $\psi = \lambda$,

$$p_{1t} - p_{2t} = -\sigma^{\theta}(\sigma+\sigma^q)q_t,$$

hence, plugging in (3), (4), and (5)

$$\alpha\left(\frac{1-\psi}{\psi}\right)^{1-\alpha} - (1-\alpha)\left(\frac{\psi}{1-\psi}\right)^{\alpha} = -\frac{\theta'(\eta)}{\theta(\eta)}\frac{(\psi-\eta)\sigma}{1-\frac{q'(\eta)}{q(\eta)}(\psi-\eta)}\left(\frac{\sigma}{1-\frac{q'(\eta)}{q(\eta)}(\psi-\eta)}\right),$$

which pins down ψ. This is the solution if $-\sigma^\theta(\sigma+\sigma^q) \geq m$.

Sectoral allocation is not a state-variable. — To argue that λ is not a state-variable, let q_t have a general law of motion as above. It depends on something, but we do not have to decide what yet. In equilibrium, households earn expected return of r in good 2 investments. Thus

$$\frac{p_{2t}}{q_t} + g + \mu_t^q + \sigma\sigma_t^q = r.$$

For a given q_t and its evolution, there is a unique p_{2t} constituting equilibrium, and thus there is a unique λ_t. Hence, the allocation of land across sectors depends on whatever q_t depends on; the allocation of land is not a state-variable.

If borrowing constraints bind, then ψ is determined by N_t and σ_t^q. Thus, ψ_t depends on whatever q_t depends on and on N_t, but it is not a state-variable itself. Similarly, if borrowing constraints do not bind, then banks earn expected return

$$\frac{p_{1t}}{q_t} + g + \mu_t^q + \sigma\sigma_t^q = r - \sigma_t^\theta(\sigma+\sigma^q).$$

Thus, σ^θ depends on the same things as q. When borrowing constraints bind, μ^θ depends on ψ_t and market returns, and thus depends on the same things as q. When constraints do not bind, $\mu^\theta = \phi_L$. Hence, we can argue that θ depends on the same things as q, and we know that N_t matters for equilibrium.

Stationary Distribution and Recovery Rate. — The stationary distribution $f(\eta)$ given the equilibrium law of motion solves

$$\frac{\partial\left[(\sigma^\eta(\eta))^2 f(\eta)\right]}{\partial\eta} = 2\mu^\eta f(\eta), \tag{8}$$

which is derived from the Fokker-Planck equation when η^* is a regulated barrier. The stationary distribution exists so long as η does not get absorbed at 0. The conditions for this are similar to those found in Brunnermeier and Sannikov (2014) and require that μ^η be large enough compared to σ^η. Letting $D(\eta) = (\sigma^\eta(\eta))^2 f(\eta)$, then D solves $D'(\eta) = 2\frac{\mu^\eta}{\sigma^\eta(\eta))^2} D(\eta)$ (which is easier to solve) and then we back out f from D.

I use the following measure for the expected time for the economy to return to η^*. Let T be the time until $\eta_t = \eta^*$ (i.e., T is a "stopping-time"). Define $g(\eta) = \mathbb{E}\left[e^{-rT}\right]$, which varies from 0 to 1 and is higher when the expected recovery time is low, i.e., when the economy recovers quickly. The recovery function $g(\eta)$ is the value of a bond that pays 1 when the economy returns to η^*. It is the present value of the next dividend payment to shareholders, and is a measure of the

expected time to recovery. The recovery rate $g(\eta) = \mathbb{E}\left[e^{-rT}\right]$ is the discounted value of the time for the economy to go from η to η^*. It satisfies

$$rg(\eta) = \mu^\eta g'(\eta) + \frac{(\sigma^\eta)^2}{2} g''(\eta), \tag{9}$$

with boundary conditions $g(\eta^*) = 1$ and $g(0) = 0$. This is because at η^* the expected time to reach η^* is zero, and at $\eta = 0$ the economy never returns to η^* and so the recovery time is infinite.

THE WELFARE FUNCTION $J(\eta)$. — Welfare is given by

$$\begin{aligned} V(\eta_\tau, Y_\tau) = \mathbb{E}_\tau &\left[\int_\tau^\infty e^{-r(t-\tau)} \left(c_t^h + \phi_L \delta_t^h \right) dt \right], \\ &\text{with } c_t^h + \phi_L \delta_t^h = z(\eta_t) Y_t, \\ &\frac{d\eta_t}{\eta_t} = \mu^\eta dt + \sigma^\eta dW_t + d\Xi_t, \\ &\frac{dY_t}{Y_t} = g_Y(\eta_t) dt + \sigma dW_t. \end{aligned}$$

The welfare function $V(\eta, Y)$ solves the differential equation:

$$rV(\eta, Y) = z(\eta)Y + V_\eta \eta \mu^\eta + V_Y g_Y(\eta) Y + \frac{1}{2} V_{\eta\eta} (\eta\sigma^\eta)^2 + \frac{1}{2} V_{YY} (\sigma Y)^2 + V_{\eta Y} \eta \sigma^\eta \sigma Y.$$

Substituting $V(\eta, Y) = J(\eta)Y$ and collecting terms

$$\begin{aligned} rJ(\eta)Y &= z(\eta)Y + J'(\eta)Y\eta\mu^\eta + J(\eta)g_Y(\eta)Y + \frac{1}{2}J''(\eta)Y(\eta\sigma^\eta)^2 + \frac{1}{2}0(\sigma Y)^2 + J'(\eta)\eta\sigma^\eta\sigma Y \\ rJ(\eta) &= z(\eta) + J'(\eta)\eta\mu^\eta + J(\eta)g_Y(\eta) + \frac{1}{2}J''(\eta)(\eta\sigma^\eta)^2 + J'(\eta)\eta\sigma^\eta\sigma. \end{aligned}$$

Rearranging terms we get

$$(r - g_Y(\eta))J(\eta) = z(\eta) + J'(\eta)\eta(\mu^\eta + \sigma^\eta\sigma) + \frac{1}{2}J''(\eta)(\eta\sigma^\eta)^2.$$

We solve this second-order differential equation with the following two boundary conditions: (i) $J(0) = \frac{z(0)}{r - g_Y(0)} = \underline{q}$, because $\eta = 0$ is an absorbing state; (ii) $J'(\eta^*) = 0$, because η^* is a regulated barrier.

V. The Effect of Volatility on the Stationary Distribution

To demonstrate the effect of volatility on the stationary distribution of the state-variable, consider the following simple process. An investor has equity n, can borrow risk-free at rate r, and buys an asset with value Y, which evolves according to a geometric Brownian motion:

$$\frac{dY_t}{Y_t} = \mu dt + \sigma dW_t.$$

Define $\eta = n/Y$. Using Ito's Lemma, the evolution of η_t is given by:

$$d\eta_t = \underbrace{(1-\eta_t)(\mu - r - \sigma^2)}_{\mu_t^\eta}\, dt + \underbrace{(1-\eta)\sigma}_{\sigma_t^\eta}\, dW_t.$$

It is clear that higher volatility implies a lower drift term and a higher volatility term for this process. Now consider the following policy for the investor: if η reaches a level η^*, then consume. In this way η^* is a regulated upper barrier. The stationary distribution $f(\eta)$ given this process and this policy solves

$$\frac{\partial f(\eta)}{\partial \eta} = 2\frac{\mu^\eta}{(\sigma^\eta)^2} f(\eta),$$

which is derived from the Fokker-Planck equation.

Figure 12 plots numerical solutions for a range of σ with $\mu - r = .01$ and $\eta^* = .03$. Notice that as exogenous volatility rises, the mass of the distribution moves left. More generally, we can think of the economy as having a reflecting barrier at really good states and, potentially, another at really bad states. Higher volatility shifts the probability of states away from the barriers. The model has a reflecting barrier only for good states, and so higher volatility shifts the stationary distribution toward bad states. With reflecting barriers on both sides, if the good states are ex ante more likely than bad states, then the stationary distribution will move "toward the center" which is overall toward bad states.

We can apply this same exercise to the baseline model to see (i) how mass shifts to/away from the stochastic steady state, and (ii) to illustrate that finding a bimodal distribution depends on parameters. I calculate the stationary distribution in the baseline model for $\sigma = 4\%$, 2%, 1%, and $.5\%$. Figure 13 plots the stationary distribution, with the x-axis normalized to η/η^*, since η^* varies with volatility. Notice that for low volatility mass shifts toward η^* and the mode at zero virtually disappears. For high volatility mass shifts toward zero and the mode at η^* virtually disappears (and it would do so for high enough volatility). However, because of the "volatility paradox," decreasing exogenous volatility actually increases endogenous volatility, which is why the mode continues to persist in small degree.

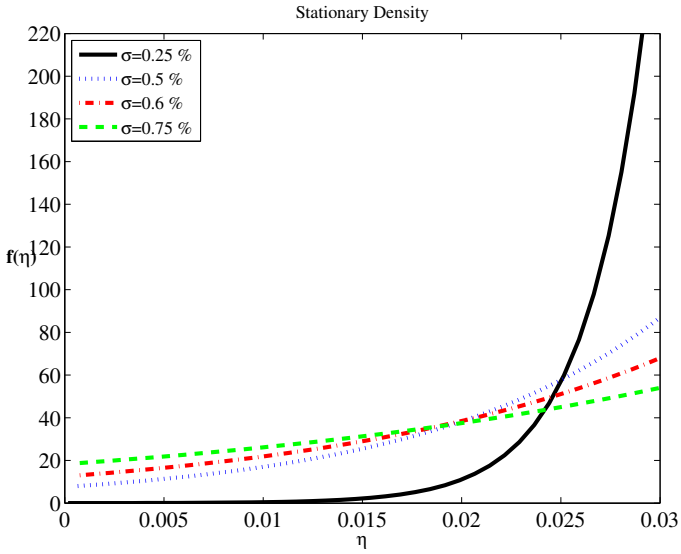


Figure 12. : The Effect of Volatility on the Capitalization Distribution.

In contrast, limiting leverage directly decreases endogenous volatility, which has the effect of killing the mode completely.

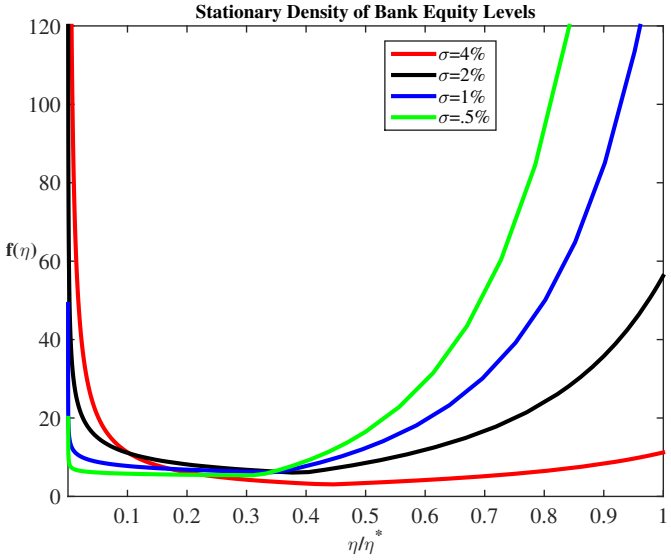


Figure 13. : The effect of volatility on the stationary distribution of the baseline model.

REFERENCES

Adrian, T., and H.S. Shin. 2010. "Liquidity and Leverage." *Journal of Financial Intermediation*, 19(3): 418–437.

Adrian, T., and H.S. Shin. 2011. "Procyclical Leverage and Value at Risk." Federal Reserve Bank of New York Staff Reports, N°338.

Brunnermeier, Markus K., and Yuliy Sannikov. 2014. "A Macroeconomic Model with a Financial Sector." *American Economic Review*, 104(2): 379–421.

Danielsson, Jon, Hyun Song Shin, and Jean-Pierre Zigrand. 2011. "Balance Sheet Capacity and Endogenous Risk." mimeo.

He, Zhiguo, and Arvind Krishnamurthy. 2012. "A Model of Capital and Crises." *The Review of Economic Studies*, 79(2): 735–777.

He, Zhiguo, and Arvind Krishnamurthy. 2013. "Intermediary Asset Pricing." *American Economic Review*, 103(2): 732–70.