Educational self-selection, tasks assignment and rising wage inequality*

Arnaud Dupuy†

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Abstract

This paper presents a general equilibrium assignment model of workers to tasks with endogenous human capital formation and multidimensionality of skills. The model has 2 key features. First, skills are endogenous and multidimensional. Second, two types of assignment occur, workers self-select their education and firms assign workers to tasks/machines. This assignment model yields two functions mapping skills of each type to tasks. Equilibrium is characterized by different wage functions for each type of skills, so that the wage distributions generally overlap. This model offers a unique framework to analyze changes in the wage structure within and between skills groups of workers and distinguishes between technological change that is related to machines (the technical

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†Corresponding address: ROA, Maastricht University PO Box 616, NL-6200 MD, The Netherlands. Email: a.dupuy@roa.unimaas.nl.

factor) or related to workers (the human factor). I show both theoretically and through simulations that the model can reproduce simultaneously i) the overlap in the wage distributions of college and high-school graduates, ii) the rise in the college-premium, iii) the rise in within wage inequality iv) the differential behavior of the between and within wage inequality in the 60s and 70s and, v) the decline of the wage at the 1^{st} decile of the overall wage distribution. A family of closed form solutions for the wage functions is proposed. In this family, the output of worker-task pairs is Cobb-Douglas, tasks are distributed according to a Beta distribution and the mapping functions have a logistic form.

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1 Introduction

This paper is about the assignment of heterogenous workers to heterogenous tasks and the impact of technical change on the distribution of wages. In assignment models, wage differentials between workers arise from productivity differentials between workers. These productivity differentials are initiated by skills differentials and magnified through the assignment of more able workers to more productive tasks in equilibrium. The equilibrium wage structure therefore depends on the distribution of skills, the distribution of tasks and the productivity of worker-task pairs. This means that changes in the wage structure could arise through the human factor,[1] i.e. changes in the distribution of skills, or through the technical factor, i.e. changes in the distribution of tasks,[2] but also through changes in the productivity of worker-task pairs over time. In turn, these productivity changes could be initiated either by the technical factor, i.e. improvements in the productivity of tasks,[3] or the human factor, i.e. improvements in the productivity of workers. While the technical factor is directly linked to technological change, the human factor could arise from either improvements of the way workers perform their tasks −a new technique− or improvements in variables affecting skills formation −and hence the distribution of skills− like the quality of school, family background, childhood environment or even genetical inheritance (see among others Cunha and Heckman (2007) and Heckman (2007)). Assignment models therefore offer a unique framework to analyze (changes in) the wage structure as they allow us to distinguish the contribution of technical factors from the contribution of human factors.

In this paper, I develop a general equilibrium assignment model of workers to tasks with endogenous human capital formation and multidimensional skills. The first key feature that distinguishes this model from the existing literature on assignment models (see Sattinger (1975, 1979 and 1993), Teulings (1995a, 1995b and 2005), Costrell and Loury (2004) and Terviö

[1] I owe the terminology technical factor versus human factor to Frank Levy.

[2] Think for instance of the job polarization observed in the US (see Autor et al. (2006)) and UK (see Goos and Manning (2007)) in recent years.

[3] This is the classical approach in the literature on skilled-biased technical change. See among others, Acemoglu (2002), Autor et al. (2003), Beaudry and Green (2003), Berman et al. (1994), Berman et al. (1998), Bound and Johnson (1992), Breshnahan (1999), Bresnahan et al. (2002), Brynjolfsson (1995), Card and Lemieux (2001), Katz and Murphy (1992), Krueger (1993) and Krusell et al. (2000).

(2007)) is that skills are endogenous and multidimensional. In existing assignment models the distribution of skills is exogenous. Human capital formation occurs outside of the model and a distinction between the contribution of either school quality, family background, childhood environment or genetical inheritance in changes in the distribution of skills over time and hence wage inequality is not possible. Moreover, skills are assumed to be unidimensional or, which boils down to the same thing, the importance of the respective types of skills is assumed constant over time so that the vector of skills can be aggregated into a single component. This assumption, however, seems to be at odds with recent empirical literature led by Heckman and Rubinstein (2001) that emphasizes the importance of noncognitive skills, such as personality traits, in explaining earnings. Moreover, the unidimensionality assumption fails to explain why the wage distributions of workers with different education overlap. This is of importance since the distribution of earnings of high-school graduates overlaps, in average over the last 4 decades, 1/3 of the distribution of earnings of college graduates as illustrated in Figure 1.[4] This overlap is extremely persistent as it remained around 25% in 2002 after 25 years of tremendous increase in the demand for skills.

Theoretically, the most plausible alternative to explain the overlap of the wage distributions of college and high-school graduates is imperfect capital markets. Suppose individuals are endowed with a single ability, for instance cognitive ability, but face short term liquidity constraints at the time they choose whether or not to go to college. In that case, smart but poor individuals will not have access to college education. Since these individuals have high ability however, they might end up with a higher wage than some college graduates. This short-term credit constraint generates an overlap of the earnings distribution of college and high-school graduates. At first sight, this explanation seems to be plausible as empirical results reveal a strong correlation between family income and post-secondary schooling. However, the empirical content of the imperfect capital markets explanation is weak. Carneiro and Heckman's (2002) careful analyzes show that "at most 8% of American youth are subject to short term liquidity constraints that affect their post-secondary schooling." Hence, the capital market imperfection explanation must be ruled out as it only explains a marginal share of the

[4] I consider herewith only white males working full-time full year, between 30 and 35 years old. The overlap is measured by Mann-Whitney statistic. This statistic is derived by matching N randomly selected high-school graduates with N randomly selected college graduates and reporting the percentage of pairs in which the high-school graduate earns more than the college graduates. Murphy and Welch (1992) documented an average overlap of 24% in the period 1985-1990, as measured by Mann-Withney statistic, for white males with equal experience (20-39 years) using wage data from the wage-earner questionnaire for outgoing rotations.

observed overlap.

To explain the apparent strong correlation between family income and educational choice, Carneiro and Heckman (2002) propose an alternative explanation to short term liquidity constraints. They suggest that family income at the time children make their educational decision is strongly correlated with family income over the child's life cycle so that the correlation between family income and educational choice does not reflect short term liquidity constraints but rather indicates that families with higher resources in a child's formative years help their children to develop their abilities relatively more compared to families with limited resources. At the time children make their post-secondary educational choice, "rich" children have higher abilities which lead them to choose for college degrees whereas "poor" children have limited abilities and quit school after high-school.

The model developed in this paper implicitly takes into account the fact that more favorable family backgrounds and other environment factors help fostering a child's cognitive and noncognitive abilities. In the model, ability endowments capture abilities *at the time individuals make their educational choice*, and hence, measure abilities at the end of compulsory education. The exact definition of abilities therefore encompasses pure ability endowment individuals were born with and the contribution of factors affecting a child's abilities up until the end of compulsory education, i.e. family background and environment.[5]

The second key feature that distinguishes the model from the existing literature on assignment models is that two types of assignment occur.[6] The first type of assignment is workers's educational self-selection. Workers are initially endowed with abilities of two types and can choose out of two types of education that each transforms abilities into marketable skills in

[5] Heckman et al. (2006) define their cognitive and noncognitive factors in a similar fashion.

[6] To my knowledge, the assignment model in this paper is the only model that yields two mapping functions and hence two wage functions in equilibrium. However, among others, Lucas (1978),(Lucas 1978) Rosen (1982) and more recently Garicano and Rossi-Hansberg (2006) developed one-sided assignment models in which two types of assignment occur. First, depending on their abilities, agents are assigned to occupations, either worker or manager. Above a certain threshold of ability agents become managers while below that threshold, agents become workers. After this initial assignment, groups of workers are assigned to managers. However, these models give rise to a single mapping function that maps ability to occupations. Hence, equilibrium is characterized by a single monotonic wage function so that the most able worker earns less than the least able manager, there is no overlap between the earnings of workers and managers. Interestingly enough, Epple et al. (2006) developed a general equilibrium model of the market for higher education in which students are heterogenous in terms of (household) income and endowed ability and colleges are heterogenous in terms of (mean students) quality and tuition. The problem for students is to choose the college that maximizes their (household) utility and the problem for colleges is to maximize (peer-)quality by setting the appropriate selection rule and tuition level. In equilibrium, each college has a distinct admision rule. These decision rules slant the income/ability plan into regions (colleges) in a way that is analogous to the assignment of agents to occupations in one-sided assignment models.

different proportions. Educational self-selection endogenizes human capital formation. Workers specialize and supply their skills of the type that maximizes their earnings. The second type of assignment is the assignment of workers to tasks. Each task refers to a different type of machine. To produce output, this machine needs to be operated by one and only one worker. Although the various machines can be operated by workers with different types and levels of skills, workers of different types and levels of skills differ in their productivity. For instance, if productivity is so that i) workers of each skills type have a comparative advantage on a different side of the support of tasks, ii) within types of skills, more skilled workers have an absolute advantage and iii) workers' skills complement the characteristics of machines in production then, following Ricardo's principles of comparative advantage[7] and differential rents, equilibrium in this model is characterized by two mapping functions, one for each type of skills supplied. The first mapping function is decreasing and maps skills of the first type to tasks on the left hand side of the support. The second mapping function is increasing and maps skills of the other type to tasks on the right hand side of the support. These two mapping functions generate two wage functions, one for each type of skills, that will in general overlap.

The remaining structure of the paper is as follows. Section 2 summarizes the related literature. Section 3 discusses the model of tasks assignment with endogenous human capital formation and multidimensional skills. Section 4 presents types of technical changes so that the model reproduces simultaneously i) a rise in the college supply, ii) a stable college-premium and, iii) a rise in within wage inequality as observed in the 70s in the US. In section 5, a family of closed form solutions for the wage functions is proposed. In this family, the output of worker-task pairs is Cobb-Douglas, tasks are distributed according to a Beta distribution and the mapping functions have a logistic form. For some parameter values, one can derive analytically the shape of the wage functions in equilibrium and the distribution of wages within and between educational groups. Numerical simulations show that the model can reproduce accurately i) the rise in the college supply, ii) the rise in the college-premium, iii) the rise in within wage inequality, iv) the differential behavior of the between and within wage inequality in the 70s and, v) the decline of the wage at the 1^{st} decile of the overall wage distribution. Section 6 concludes.

[7] The theory of comparative advantage in labor markets was formally developed by Sattinger (1975) (see also Sattinger (1993) for a survey of assignment models and comparative advantage) and the presence of comparative advantage was later demonstrated empirically in Sattinger (1978 and 1980).

2 Related literature

Related to the model developed in this paper is the model of self-selection proposed by Roy (1950 and 1951), and extended by Heckman and Sedlacek (1985 and 1990) and Heckman and Honore (1990) and estimated in the context of educational choice by Rosen and Willis (1979), and more recently in the context of occupational choice by Gould (2002). The self-selection model puts the emphasis on the *supply side* of the labor market by focusing on the heterogeneity of individuals. Individuals are endowed with different abilities and choose a sector among a small number of sectors. The demand for workers and wage rates by sectors are exogenous to the model, as opposed to the general equilibrium model developed in this paper. An interesting feature of Roy's model is that it can be used to model endogenous human capital formation by accounting for individuals' educational choice.[8] Individuals choose their educational profile based on their initial abilities and the exogenous market wages associated with each educational background.[9] However, used in the context of educational choice, the model does not spell out tasks or machines and therefore does not allow the distinction between human factors and technical factors in rising wage inequality.

Also related to the model is Rosen's (1978) tasks assignment model. Rosen's (1978) tasks assignment model puts the emphasis on the *demand side* of the labor market by focusing on the heterogeneity of tasks. Tasks differ from one another by the levels of the various types of skills they require. Workers are grouped in a small number of homogeneous skill groups (educational categories) and the supply of workers and wage rates by skill groups are assumed to be exogenous. The main advantage of this model is that it offers a very convenient framework to analyze substitution between skill groups of workers. Rosen (1978) considers the demand for labor by modeling firms' *indirect* production function resulting from the assignment of tasks to workers that maximizes output, given exogenous wages. However, human capital formation and workers heterogeneity within educational groups are not accounted for in Rosen's (1978) model. Moreover, labor is the only input and capital plays no role in production.[10] This makes the model less suited for studying the effect of technical change on the distribution of wages.

[8]See Willis and Rosen (1979) .

[9]Note that Heckman et al. (1998) have recently extended Roy's model to allow for investment and endogenous skills prices determination. However, jobs remain absent in their model.

[10]Dupuy (2006) extends Rosen's model to the case where 1), within skill groups, workers are heterogenous and 2) workers rent machines of different productive characteristics to produce output. The assignment problem is solved using Ricardo's differential rents principle.

Most closely related to the approach of this paper are the general hedonic models developed by Rosen (1974) and Lucas (1977) and in particular Sattinger (1975, 1979 and 1993), Teulings (1995a, 1995b and 2005), Costrell and Loury (2004) and Terviö (2007). In these models, wage differentials across individuals arise because of skills differentials.[11] These skills differentials are further amplified through assignment as abler workers are mapped to more productive tasks in equilibrium. These models are potentially valuable to explain the source of the recent rise in wage inequality. The general equilibrium structure of these models enables us to separate the contribution of changes in the distribution of skills in the rise of wage inequality from the contribution of technological change either in terms of increased productivity or changes in the distribution of jobs.[12] Moreover, by their very nature, they allow us to make the distinction between technological changes due to human factors or technical factors. However, these models are silent with respect to the overlap of the wage distribution of college and high-school graduates and the differential timing of the between and within rise in wage inequality since skills are unidimensional and exogenous. The model proposed in this paper contributes to the literature along those two points. First, the model acknowledges the multidimensionality of skills which yields two wage functions in equilibrium, one per type of skills, and therefore enables us to reproduce the overlapping wage distribution of college and high-school graduates. Second, human capital formation is endogenized by considering workers' educational self-selection.

3 A tasks assignment model with endogenous human capital formation and multidimensional skills

3.1 Supply of skills

Workers' endowed abilities and skills formation

Suppose workers are endowed with initial abilities vector $ab = \langle ab_1, ab_2 \rangle \in R_+^2$ where ab_1

[11] Tinbergen's (1956) allocation model also considers the mapping of heterogenous workers to heterogenous jobs but wage differentials arise as compensating differentials.

[12] Costrell and Loury (2004) normalize the tasks distribution to be uniform. In a single economy, this normalization is without loss of generality. However, when one is interested in comparing economies (over time or across countries) this normalization does not allow to identify separetely the effects of changes in productivity on assignment and wages from changes in the distribution of tasks.

and ab_2 represent an individual's ability of type 1 and 2, manual and intellectual ability for the sake of the argument. Define $\xi(ab_1, ab_2)$ the density[13] of workers whose abilities are ab_1 and ab_2. After compulsory education, workers make an educational choice: for instance, whether to go to college or quit after high-school or whether to study economics or mathematics. The educational system transforms the vector of initial abilities into a vector of skills through an educational production function[14] E_k, $k = 1, 2$.

Note that ability endowments vector ab captures abilities *at the time individuals make their educational choice.* Hence, ab measures abilities at the end of compulsory education. The exact definition of abilities therefore encompasses pure ability endowment individuals were born with and the contribution of factors affecting a child's ability vector up until the end of compulsory education. As noted earlier in the literature, families with higher resources, better educated parents and other favorable environment variables affect a child's ability (cognitive and noncognitive). Regarding cognitive abilities, Heckman (1995) and Heckman et al. (2006) show evidence that these abilities (IQ tests score) are set early in life and seems to be fairly set by the age 8. In contrast, noncognitive abilities are more malleable and seem to be so until the late adolescent years as argued by Heckman (2000) and Carneiro and Heckman (2003). This means that although ab_1 and ab_2 are fairly influenced by family background and environment variables, they are fairly set at the end of compulsory education.

The production of skills depends on educational choice. A worker with initial abilities $\langle ab_1, ab_2 \rangle$ will have skills $\langle t_1, t_2 \rangle = E_1(ab) = \langle e_{11}(ab_1); e_{12}(ab_2) \rangle$ with $e_{11}(ab_1) > ab_1$ and $e_{12}(ab_2) \geq ab_2$, if she selects education 1 and skills $\langle t'_1, t'_2 \rangle = E_2(ab) = \langle e_{21}(ab_1); e_{22}(ab_2) \rangle$ with $e_{21}(ab_1) \geq ab_1$ and $e_{22}(ab_2) > ab_2$, if she selects education 2. In general, this educational system has an heterogenous treatment effect in the sense that it enhances abilities by factors specific to each individual's abilities, i.e. $e''_{kj} > 0$ for $\langle k, j \rangle = 1, 2$.[15] Moreover, the educational system transforms initial abilities into skills in a non proportional way. A worker with initial abilities $\langle ab_1, ab_2 \rangle$ will have skills $\langle t_1, t_2 \rangle$ with $t_1/t_2 = e_{11}(ab_1)/e_{12}(ab_2) \geq ab_1/ab_2$ if she selects education 1 and skills with $t'_1/t'_2 = e_{21}(ab_1)/e_{22}(ab_2) \leq ab_1/ab_2$ if she selects education 2.

[13] The economy considered is sufficiently large so that the distributions of individuals and tasks can be described by continuous density functions.

[14] Hartog (2001) uses a similar approach to define the role of education in producing skills. However, Hartog uses linear transformation of abilities into skills, assumption I do not impose in the model.

[15] In the special case of a linear transformation of abilities into skills as assumed by Hartog (2001), the educational system has a common treatment effect: $t_k = e_{kk}(ab_k) = c_{kk}ab_k$ if education k is chosen or $t_k = c_{jk}ab_k$ if education j is chosen with $e''_{jk} = 0 \; \forall j, k$.

In order for educational choice to be non trivial, I further assume that education k develops abilities of type k relatively more than education j.

Assumption A1:

i) $e'_{jj} > 0$ for all j and $e'_{jk} \geq 0$ for $j \neq k, k = 1, 2$,

ii) $e_{jj}(ab_j)/e_{jk}(ab_k) \geq ab_j/ab_k$ for all $j, k = 1, 2$,

iii) $e_{kk}(ab_k) > e_{jk}(ab_k)$ for all j, k,

iv) $e_{kj}(0) = 0$ for all j, k.

Without assumption *A1 iii)*, for instance with $e_{jj}(ab_j) > e_{jk}(ab_k) > e_{kk}(ab_k) > e_{kj}(ab_j) \geq 1$, educational choice would still be endogenous but education j would be a strictly dominant educational strategy so that all workers would select education j. Workers with relatively low ability of type k still seek to enhance their ability of type j through education j only now workers with relatively high ability of type k choose education j to enhance their ability of type k. Clearly, although human capital is still endogenous this case is less interesting.

It is important to bear in mind that changes in the educational system that lead to relative improvements in the production of skills of one type will play a non neutral role in the educational choice of workers. Indeed, given the distribution of initial abilities and relative wages, i.e. $\frac{w_1(t_1)}{w_2(t_2)}$ where $w_k(t_k)$ is the wage of workers supplying t_k units of skills of type k and $w'_k > 0$, an increase in e_{kk} will increase the potential skills of type k compared to the potential skill of type j of every workers and lead more workers to select education k. The model therefore allows to investigate the effects of exogenous changes in the educational production of skills on educational choices and hence wage distribution. For instance, the model could be used to evaluate the general equilibrium impact of the recent introduction of the "Literacy Hour" in primary English schools (see Machin and McNally (2005)) or Singapore's mathematics teaching method in the USA −launched in order to offset the relative poor numeracy score of American pupils−[16] on wage inequality.

Foresight

[16] In the distribution of students' achievements in mathematics at the age of 13 years old across countries the USA rank 19^{th} out of 38 countries whereas Singapore ranks first. See TIMSS (2000).

Individuals can choose between various education (no education being one of them) in order to transform their endowed abilities into productive skills so as to maximize their expected future earnings. Prior to this educational choice, individuals evaluate the streams of expected earnings associated to each educational possibility. Since future wages are unobserved this complicates the assignment problem considerably. Most authors, see Friedman and Kutznets (1945), Mincer (1974 and 1993), Willis and Rosen (1979) and Heckman et al. (1998) for instance, have assumed that workers have perfect foresight. With perfect foresight, workers anticipate every shocks taking place in the future and therefore never make forecasting errors. This assumption simplifies considerably the model as it leads to strict specialization of workers. Workers who selected education k supply skills of type k as I will show below.

In contrast, imperfect information in the form of limited foresight[17] opens up new assignment possibilities as labor market shocks might be unexpected. For instance, think of a worker whose manual ability is as large as her intellectual ability, i.e. $ab = \langle ab_1; ab_2 \rangle = \langle 10; 10 \rangle$. Suppose this worker can select a manual degree and end up with $t = E_1(ab) = \langle 11; 10 \rangle$ or select a cognitive degree and end up with $t = E_2(ab) = \langle 10; 11 \rangle$. Furthermore, this worker *expects* to earn \$10 per unit of manual skills supplied and *expects* to earn \$12 per unit of intellectual skills supplied. Following her comparative advantage, this worker will select a cognitive degree to maximize her *expected* earnings. Suppose that at the time this worker graduates and enters the labor market, an unexpected shock occurs so that the pay-rate is \$12 per units of manual skills supplied and \$10 per unit of intellectual skills supplied. This worker's earnings will be maximized by supplying her manual skills full-time *rather* than her intellectual skills for which she studied (relative manual wage is $1.2 > 1.1$ relative manual skills).

Hence, with limited foresight, some workers will not supply the skills they developed most during education, but rather supply their alternative skills. Obviously, this result holds as long as the difference between expected wages and actual wages is large enough to compensate relative skills differential *ex post* $t_k/t_j = e_{kk}(ab_k)/e_{kj}(ab_j)$. Hence, if education increases one type of skills much more than the other type, few workers will be in a situation where supplying their alternative skills generates more earnings than supplying the skills they developed most during education. This means that the more unequally education enhances skills of the various types, the less restrictive the assumption of perfect foresight.

[17] See Siow (1984) for a comprehensive treatment of occupational choice under uncertainty.

The limited foresight world deserves particular attention.[18] In this paper however, we follow the mainstream of the literature, e.g. Willis and Rosen (1979) and assume that individuals have perfect foresight so that expected earnings are by definition equal to actual earnings.

Educational choice

Workers select their education to maximize earnings. Assume that workers can only use one type of skills at a time so that workers with skills $\langle t_1, t_2 \rangle$ can supply t_1 units of skills of type 1 for a share τ of their working time and supply t_2 units of skills of type 2 the rest of their working time.[19] Workers' earnings are therefore equal to $\tau w_1(t_1) + (1-\tau) w_2(t_2)$ and depend on educational choice since skills $t_j = e_{kj}(ab_j)$ if education k is chosen and $t_j = e_{jj}(ab_j)$ if education j is chosen.[20]

Earnings maximization resumes to:

$$\max \begin{bmatrix} \max_\tau \left(\tau w_1(e_{11}(ab_1)) + (1-\tau) w_2(e_{12}(ab_2))\right); \\ \max_\tau \left(\tau w_1(e_{21}(ab_1)) + (1-\tau) w_2(e_{22}(ab_2))\right) \end{bmatrix}$$

Conditional on their educational choice workers will maximize earnings by specializing and supplying full time their skills of type 1 or their skills of type 2 according to $w_1(t_1) \gtrless w_2(t_2)$. This means that earnings maximization is equivalent to max $[w_1(e_{11}(ab_1)); w_2(e_{12}(ab_2)); w_1(e_{21}(ab_1)); w_2($

Result R1:[21]

Wage functions are monotonic, $w_k'(t_k) > 0$ for all k.

[18]Note that some form of imperfect information in assignment models is treated in MacDonald (1982). MacDonald extends the comparative advantage model to labor markets with incomplete information about the types of workers.

[19]In Lazear's (2005) terminology, this means that my workers are assumed to be specialists, Lazear's entrepreneurs are excluded from the model. Note that in empirical work on wage inequality, the self-employed are usually excluded from the sample.

[20]Study costs could be explicitly accounted for in this model by assuming that during her study, a worker would receive a loan from the firm to a total of $Costs_k$. The worker will pay back this loan once working by seeing the firm punctions her wage as follows $w_k(t_k) = w_k^*(t_k) - Costs_k$, where $w_k^*(t_k)$ is the marginal productivity of this worker.

[21]It will be shown that an efficient assignment of workers to tasks, one that maximizes aggregate output, leads to monotonic increasing wage functions if within each educational group, the productivity of more skilled workers increases with the complexity of tasks (as stated below in assumption $A2$). The monotonicity of the wage functions is therefore a result of the efficient tasks assignment and not an assumption.

Given assumption $A1$ and result $R1$, we have $w_k(e_{kk}(ab_k)) > w_k(e_{jk}(ab_k))$ for all $j \neq k$, so that the maximization problem simplifies further to:

$$\max\left[w_1(e_{11}(ab_1)); w_2(e_{22}(ab_2))\right] \tag{1}$$

Using $W_j(.) \equiv w_j(e_{jj}(.))$ for notational simplicity, equation 1 implies that workers with abilities $ab = \langle ab_1, ab_2 \rangle$ so that $W_k(ab_k) > W_j(ab_j)$ follow their comparative advantage and select education k. Moreover, workers with education k will never supply their skills of type j under the assumption of perfect foresight.

Let $ab_j^*(ab_k) \equiv W_j^{-1}(W_k(ab_k))$.[22] Workers whose ability ab_j exceeds ab_j^* optimally select education j and workers with comparative advantage less than ab_j^* select education k. Note that ab_j^* is a strictly increasing function of ab_k under result $R1$ and assumption $A1$ i).[23] However, it could be concave, linear, convex or even locally concave and locally convex depending on the structural parameters, in contrast to Willis and Rosen's (1979) application of Roy's model in which t_j^* is a linear function of t_k.[24]

Figure 2 depicts the educational self-selection. The 95% contour of the bivariate distribution $\xi(ab_1, ab_2)$ is drawn assuming a positive correlation between abilities although the model does not require this assumption. The marginal distribution of each ability is drawn on the respective axis. Although educational self-selection is simply defined by the slant ab_1^*, understanding the assignment of tasks to workers later on requires to take a closer look at the graph. In particular, define groups of workers with the same level of ability of type 1 as a family. Formally, a worker is said to belong to family x if and only if her ability of type 1 is equal to ab_1^x. The size of each family is given by the marginal density distribution of ability 1. There is a infinite number of families and in Figure 2 I concentrate on families α, β and γ, with $ab_1^\alpha > ab_1^\beta > ab_1^\gamma$. Within families, workers have the same level type 1 ability

[22] Given result $R1$, $w_j(.)$ are invertible. Moreover, given assumption $A1$ i), $e_{jj}(.)$ are invertible. Therefore, $W_j(.)$ are invertible.

[23] Since $e_{jj}(.)$ and $w_j(.)$ are strictly increasing so are $W_j(.)$ and $W_j^{-1}(.)$. Hence the function $W_j^{-1}(W_k(.))$ is strictly increasing.

[24] In a single cross-section, the nonlinearity of relative wages does not matter since one could simply change the scales of both abilities so as to obtain a linear functional form. However, when one compares several economies, one must hold the scale of abilities constant across economies. Hence, by allowing the relative wage function to be concave, convex or linear, the model offers more flexibility than Willis and Rosen's linear specification.

but differ in their level of type 2 ability. Without loss of generality, I rank, within families, workers by increasing ability of type 2, $ab_2^{\alpha_1} < ..ab_2^{\alpha_{k_\alpha}}.. < ab_2^{\alpha_{N_\alpha}}$, $ab_2^{\beta_1} < ..ab_2^{\beta_{k_\beta}}.. < ab_2^{\beta_{N_\beta}}$ and $ab_2^{\gamma_1} < ..ab_2^{\gamma_{k_\gamma}}.. < ab_2^{\gamma_{N_\gamma}}$, so that worker x_{k_x} is indifferent between supplying either of her abilities, i.e. $ab_1^{x_{k_x}} = ab_1^*(ab_2^{x_{k_x}})$ and x_{N_x} is the size of each family. Hence, in equilibrium, only those workers with ability $ab_1^x > ab_1^*(ab_2^x)$ will supply their ability of type 1 so that the supply of workers with ability ab_1^x is a truncation of the marginal density at ab_1^x. Note that while those workers from family x that select education 1, supply the same level of ability ab_1^x, those that select education 2 supply different levels of ability of type 2, i.e. $ab_2^{x_{k_x}}, ..., ab_2^{x_{N_x}}$. Hence, while workers from family x selecting education 1 will all be assigned to the same task and earn equal wages, workers from family x that select education 2 will be assigned to different tasks corresponding to their respective levels of ability 2 and earn different wages.

Formally, the density of workers supplying ab_k is obtained by summing up all workers with $ab_j < ab_j^*(ab_k)$ for $j \neq k$. The density of workers supplying level ab_k of type k ability, say $s_k(ab_k)$ is therefore defined parametrically by:

$$s_1(ab_1) = \xi_1(ab_1) = \int_0^{ab_2^*(ab_1)} \xi(ab_1, ab_2) \cdot dab_2 \tag{2}$$

$$s_2(ab_2) = \xi_2(ab_2) = \int_0^{ab_1^*(ab_2)} \xi(ab_1, ab_2) \cdot dab_1 \tag{3}$$

where $ab_1^* = W_1^{-1}(W_2(ab_2))$ and $ab_2^* = W_2^{-1}(W_1(ab_1))$ with $W_1^{-1}(W_2(0)) = W_2^{-1}(W_1(0)) = 0$.

Note that the density of workers supplying ability ab_k depends on the wage functions $W_j(ab_j)$, $j = 1, 2$, through $ab_j^*(ab_k)$.

Finally, the supply of workers by education is given by:

$$S_1 = \int_0^{\infty} s_1(ab_1)\, dab_1 \tag{4}$$

$$S_2 = \int_0^{\infty} s_2(ab_2)\, dab_2 = 1 - S_1 \tag{5}$$

3.2 Demand for skills

Consider an economy producing a composite commodity by means of the input of different tasks. Each task is associated with a unit of capital, a machine for the sake of the argument, and the various tasks correspond to machines with different characteristics.[25] To produce output, each machine needs to be operated by a fixed proportion of workers, i.e. one and only one worker. The owner of a machine is loosely referred to as a firm. In this economy, output Y is obtained by summing up the production in each single task v from the continuum $v \in (0, 1)$. The distribution of tasks is exogenous and given by the density distribution of tasks $d(v)$ and cumulative distribution $F(v^*) = \int_0^{v^*} d(v)dv$. Assume further that there are as many machines as workers, i.e. the mass of workers and machines are the same.[26] Moreover, the economy is perfectly competitive so that no worker and no firm can affect the wage and rental rates.

Machine v can be operated by workers with different types and levels of skills. However, workers of different types and levels of skills differ in their productivity. Workers supplying t_k units of skills of type k can produce $p_k(v, t_k)$ units of output when assigned to machine v. Without loss of generality, I assume that workers supplying skills of type 1 have a comparative advantage in tasks v close to 0 and workers supplying skills of type 2 have a comparative advantage in tasks v close to 1 −to fix ideas, if ability of type 1 is manual ability and ability of type 2 is intellectual ability, then tasks close to 0 are for instance the tasks of a carpentrer and tasks close to 1 are the tasks of a rocket scientist. Similarily, machines close to 0 could be circular saws and machines close to 1 could be computers. Tasks in the middle of the support are the "anybody can do it as efficiently" tasks−. I assume further that productivity increases with the level of skills supplied. Moreover, I assume that type 1 skills and machines close to 0 and skills of type 2 and machines close to 1 are complementary. In other words, among workers supplying skills of type 1, those with higher t_1 skills are more productive in tasks v close to 0 and among workers supplying skills of type 2, those workers with higher t_2 skills are more productive in tasks v close to 1. Finally, workers supplying skill level 0 (the lowest possible skills of each type) are equally productive indifferently of the type of skills and the task to which they are assigned. These assumptions are summarized in Assumption $A2$.

[25] This part of the model is to a large extent similar to the differential rents models described in Sattinger (1979 and 1993). The terminology "task" and "machine" are interchangeable throughout the paper. In general I will use "task" for the sake of simplicity but when needed I will refer explicitly to machines.

[26] Without this assumption, the least productive machines or the least productive workers will inevitably remained unemployed in equilibrium.

Assumption A2:

i) Comparative advantage of skills types, i.e.$\frac{\partial p_1(v,t_1)}{\partial v} < 0$ and, $\frac{\partial p_2(v,t_2)}{\partial v} > 0 \; \forall v, t_k$,

ii) absolute advantage of skilled workers, $\frac{\partial p_k(v,t_k)}{\partial t_k} > 0 \; \forall v, t_k$,

iii) complementarity of skills types and machines, $\frac{\partial^2 p_1(v,t_1)}{\partial t_1 \partial v} \leq 0$ and, $\frac{\partial^2 p_2(v,t_2)}{\partial t_2 \partial v} \geq 0 \; \forall v, t_k$,

iv) $p_k(v,0) = \underline{p} \geq 0$ for all k.

3.3 Equilibrium

A competitive equilibrium consists of 1) two wage functions $W_k(ab_k)$ which indicate a worker's earnings associated to the level and type of ability this worker supplies and a rent function $r(v)$ which indicates the rents associated with a machine of type v, 2) an index function $ab_k^*(ab_j) = W_k^{-1}(W_j(ab_j))$ that assigns workers to education, 3) a marginal task ε that indicates the set of tasks assigned to workers of each type and 4) two mapping functions $V_k(ab_k)$ which indicate the type of machine assigned to workers of each type and level of ability such that i) workers maximize earnings and firms maximize rents and ii) both the labor and capital markets clear.

Following Sattinger (1979 and 1993), the general equilibrium of this model is derived in three steps once we assume that, in the period under consideration, the density distribution of tasks does not depend on wages. In the first step, we make a tentative assumption about the assignment of workers to tasks in equilibrium. The second step consists to derive the associated equilibrium wages for this assignment. Finally, in the third step, we check whether the second order conditions for equilibrium, i.e. profits and earnings are concave, are satisfied under the tentative assignment defined in step 1.

Step 1: *Tentative tasks assignment*

Given assumption *A*2 *i*), an efficient assignment of workers to tasks will maximize output by assigning workers supplying skills of type 1 to tasks $(0, \varepsilon)$ and workers supplying skills of type 2 to tasks $(\varepsilon, 1)$ where ε is the marginal task in equilibrium.[27] Given assumption *A*2 *ii)* and *iii)*, among workers supplying skills of type 1, those with the highest level of skill 1 will be

[27] Firms owing machine ε are indifferent between employing a worker with type 1 or type 2.

assigned to task 0 and so on until the marginal task ε is assigned to those workers supplying the lowest level of skill 1. By symmetry, the tasks $(\varepsilon, 1)$ are assigned to workers supplying skills of type 2. Workers supplying the lowest level of skills of type 2 are assigned to task ε and so on until those workers with the highest level of skills 2 are assigned to task 1.

This efficient tasks assignment results in a mapping function v_1 that associates a single value of ability ab_1 to each task $v \in (0, \varepsilon)$, i.e. $v = V_1(ab_1)$ with $V_1'(ab_1) < 0$, $\varepsilon = V_1(0)$ and $\lim_{ab_1 \to \infty} V_1(ab_1) = 0$, and a mapping function v_2 that associates a single value of ability ab_2 to each task $v \in (\varepsilon, 1)$, i.e. $v = V_2(ab_2)$ with $V_2'(ab_2) > 0$, $\varepsilon = V_2(0)$ and $\lim_{ab_2 \to \infty} V_2(ab_2) = 1$.[28] The marginal task is derived so that aggregate output in the economy is maximized. Since aggregate output is the sum of the product of each task,[29] i.e. $Y(\varepsilon) = \int_0^\varepsilon p_1(v, V_1^{-1}(v))d(v)dv + \int_\varepsilon^1 p_2(v, V_2^{-1}(v))d(v)dv$, the marginal task satisfies $\frac{\partial Y(\varepsilon)}{\partial \varepsilon} = 0$. Using Leibniz's formula, this reads as:

$$
\begin{aligned}
& p_1(\varepsilon, V_1^{-1}(\varepsilon))d(\varepsilon) + \int_0^\varepsilon \frac{\partial V_1^{-1}(v)}{\partial \varepsilon} \frac{\partial p_1(v, V_1^{-1}(v))}{\partial ab_1} d(v)dv \\
= \; & p_2(\varepsilon, V_2^{-1}(\varepsilon))d(\varepsilon) - \int_\varepsilon^1 \frac{\partial V_2^{-1}(v)}{\partial \varepsilon} \frac{\partial p_2(v, V_2^{-1}(v))}{\partial ab_2} d(v)dv
\end{aligned}
$$

This tasks assignment can be traced in Figure 2. For each family x, workers $x_1, ..., x_{k_x - 1}$, those with ability of type 2 lower than $ab_2^{x_{k_x}}$, are assigned to task $V_1(ab_1^x) \in (0, \varepsilon)$, whereas workers $x_{k_x}, ..., x_{N_x}$, those with ability of type 2 larger than $ab_2^{x_{k_x}}$ are assigned respectively to tasks $V_2(ab_2^{x_{k_x}}), ..., V_2(ab_2^{x_{N_x}})$ with $\varepsilon \leq V_2(ab_2^{x_{k_x}}) < ... < V_2(ab_2^{x_{N_x}}) \leq 1$.

Figure 3 shows this assignment of workers to tasks in the task-productivity space. The density distribution of tasks $d(v)$ is drawn below the horizontal axis. Assumption $A2$ $i)$ implies that the productivity of workers supplying ability of type 1 (respectively 2) decreases (increases) as we move to the right of the support of tasks. Assumption $A2$ $ii)$ implies that holding the task and the type of ability constant, productivity increases with the level of ability. Therefore, workers of family α supplying their ability of type 1 are more productive

[28]Note that $V_j(ab_j) \equiv v_j(e_{jj}(ab_j))$ where $v_1' < 0$ and $v_2' > 0$. Functions v_i, $i = 1, 2$ play the same role as the function $h(g)$ in Sattinger (1975) p. 356, where g is workers' ability (single scale) and $h(g)$ the difficulty (single scale) of the task performed by workers with ability g in equilibrium and, $c(u)$ in Teulings (1995a, 1995b and 2005) where u is the normalized level of skills and $c(u)$ the associated job complexity in equilibrium.

[29]Note that so far nothing guarentees that aggregate output will be bounded. Given assumption $A2$ $i)$ and $ii)$, a sufficient condition for aggregate output to be bounded is that $\lim_{v->0} p_1(v, v_1^{-1}(v))d(v) < \infty$ and $\lim_{v->1} p_2(v, v_2^{-1}(v))d(v) < \infty$.

in every tasks than workers of family β supplying their ability of type 1 whom are in turns more productive than workers γ supplying their type 1 ability.

Equilibrium is defined by the marginal task ε. This task is assigned to workers supplying level of ability 0 indifferently of the type of ability. Moving to the left, we reach first the task assigned to workers $\gamma_1, ..., \gamma_{k_\gamma - 1}$, then the task of workers $\beta_1, ..., \beta_{k_\beta - 1}$ and finally the task of workers $\alpha_1, ..., \alpha_{k_\alpha - 1}$. This defines the mapping function $V_1(ab_1)$. Going back to the marginal task and moving to the right, we reach successively the task assigned to workers γ_{k_γ}, β_{k_β}, γ_{N_γ}, α_{k_α}, β_{N_β} and finally α_{N_α}. This defines the mapping function $V_2(ab_2)$.

In this tentative equilibrium, the density of workers' skills is directly derived from the density of tasks by performing the transformation of variables $v = V_k(ab_k)$ and noting that $dv_k = V_k' \cdot dab_k$. This yields:

$$\int_{0=V_1(\infty)}^{\varepsilon=V_1(0)} d(v) \cdot V_1'(ab_1) \cdot dab_1 = \int_0^\infty s_1(ab_1) \cdot dab_1 \tag{6}$$

$$\int_{\varepsilon=V_2(0)}^{1=V_2(\infty)} d(v) \cdot V_2'(ab_2) \cdot dab_2 = \int_0^\infty s_2(ab_2) \cdot dab_2 \tag{7}$$

where $s_j(ab_j) = \xi_j(ab_j) = \int_0^{ab_k^*(ab_j)} \xi(ab_j, ab_k) \cdot dab_k$ and,
$ab_j^*(ab_k) = W_j^{-1}(W_k(ab_k))$.

Hence, the following result:

Result R2: In equilibrium the density of workers with ability ab_k is:

$$s_k(ab_k) = d(V_k(ab_k)) \cdot V_k'(ab_k) \tag{8}$$

Result $R2$ has important implications for the resolution of the assignment problem. At first sight, these equations look like independent first order nonlinear nonautonomous differential equations. In the one dimensional skill case, when both the distribution of skills and tasks are exogenous, as in Sattinger (1979 and 1993), equation 8 is indeed a first order nonlinear nonautonomous differential equation that admits closed form solutions when tasks and skills follow a Pareto or a Normal distribution. However, when the distribution of skills is endogenous, $s_j(ab_j)$ in equation 8 depends on equilibrium wages through the index function

$ab_k^*(ab_j) = W_k^{-1}(W_j(ab_j))$. As is shown below, equilibrium wages are derived from the first order condition to profit maximization. This condition stipulates that wage differentials are set equal to productivity differentials. The wage functions are therefore obtained by integrating the productivity differentials evaluated at the equilibrium task, i.e. replacing v by $V_k(ab_k)$, over abilities of the respective types. This means that each wage function depends not only on the shape of the production function of worker-task pairs for each type of workers but also on the associated mapping function. As a result, each $s_j(ab_j)$ is a function of both mapping functions. Hence, $V_1(ab_1)$ and $V_2(ab_2)$ are solutions of a system of first order nonlinear nonautonomous differential equations.

Closed form solutions are unlikely to exist. It should be noted, however, that the role of mapping functions in assignment models is to bring "flexibility" in an economy with fixed distribution of workers and tasks. To put it in Tinbergen's words,

> "The supply distribution of skills has to be deformed so as to coincide with the demand distribution otherwise there will not be equilibrium." Tinbergen (1956), p. 162.

The mapping functions deform (stretch) the density distribution of skills so as to make it fit the distribution of tasks. Allowing the distribution of skills to be endogenous, brings an additional source of flexibility in the economy. In the most extreme form of endogeneity, the distribution of skills would mimic perfectly the distribution of tasks and the mapping function would be identity. This suggests a legitimate solution to circumvent the problem of solving equation 8 for the mapping functions. Shifting the flexibility from the mapping function to the skills formation, one could impose the shape of the mapping functions V_k and solve equations 8 for the conditional density of workers with ability ab_k, $k = 1, 2$. Of course, since the type of endogeneity described in the model does certainly not refer to the case where the mapping functions are identity, the shape of the mapping functions must be flexible enough so that calibration of the parameters of these functions would reproduce accurately the conditional distribution of abilities observed in the economy or believed to be reasonably closed to the true distribution. In section 5, I will discuss a set of assumptions that provides closed form solutions for the wage and rent functions when the mapping functions have a logistic form.

Step 2: *First order conditions*

A firm owing machine v seeks to maximize the profits derived from its machine. The profits from assigning a worker with education k and with skills $t_k = e_{kk}(ab_k)$ are $p_k(v, e_{kk}(ab_k)) - W_k(ab_k)$. The firm will therefore compare the productivity increase to the wage increase associated to a worker with higher skills t_k or equivalently higher ability ab_k since $e'_{kk} > 0$ for all k. This yields the following first order condition:

$$\frac{\partial p_k(v, e_{kk}(ab_k))}{\partial ab_k} = W'_k(ab_k) \ \forall k = 1, 2 \tag{9}$$

Note that from assumptions $A1$ $i)$ and $A2$ $ii)$, we therefore have Result 1, $w'_k(t_k) = \frac{W'_k(ab_k)}{e'_{kk}(ab_k)} > 0 \ \forall k = 1, 2$.

The equilibrium rents are obtained in a similar fashion by noting that earnings are given by $W_k(ab_k) = p_k(v, e_{kk}(ab_k)) - r(v)$. Earnings maximization leads workers supplying ability ab_k to compare the productivity increase to the rent increase associated to a machine ranked to the left or the right of v. Hence, the first order conditions to earnings maximization are given by:

$$\frac{\partial p_k(v, e_{kk}(ab_k))}{\partial v} = r'(v) \ \forall k = 1, 2 \tag{10}$$

Moreover, firms owing machines ε are indifferent between employing the worker supplying the lowest level of skills of type 1, $t_{1,\varepsilon} = e_{11}(0)$, or the worker supplying the lowest level of skills of type 2, $t_{2,\varepsilon} = e_{22}(0)$. Stated otherwise, the rents of the owners of machines ε are equal whether worker $ab_1 = 0$ or $ab_2 = 0$ are assigned to machine ε: $p_1(\varepsilon, e_{11}(0)) - W_1(0) = p_2(\varepsilon, e_{22}(0)) - W_2(0)$.

Equation 9 gives the wage differential at task v and equation 10 gives the rent differential at ability ab_k. These wage and rent differentials do not hold for values of ab_k or v other than $ab_k = V_k^{-1}(v)$ and $v = V_k(ab_k)$ respectively and therefore depend on the equilibrium assignment.

Step 3: *Second order conditions*

The tentative assignment defined in step 1 is a valid one only when the firms' second order condition to profits maximization and workers' second order conditions to earnings maximization, that is profits are concave in ab_k and earnings are concave in v , are satisfied. Put in equation, the second order condition for profits maximization reads as:

$$\begin{aligned} \left[\frac{\partial^2 p_k(v, e_{kk}(ab_k))}{\partial ab_k^2}\right]_{v=V_k(ab_k)} - W_k^{"}(ab_k) &< 0\ \forall k = 1, 2 \\ &\Leftrightarrow \\ -\left[\frac{\partial^2 p_k(v, e_{kk}(ab_k))}{\partial ab_k \partial v} V_k'\right]_{v=V_k(ab_k)} &< 0 \end{aligned}$$

since $W_k''(ab_k) = \left[\frac{\partial^2 p_k(v, e_{kk}(ab_k))}{\partial ab_k^2}\right]_{v=V_k(ab_k)} + \left[\frac{\partial^2 p_k(v, e_{kk}(ab_k))}{\partial ab_k \partial v} V_k'(ab_k)\right]_{v=V_k(ab_k)}$.

The second order conditions for earnings maximization read as:

$$\begin{aligned} \left[\frac{\partial^2 p_k(v, e_{kk}(ab_k))}{\partial v^2}\right]_{ab_k=V_k^{-1}(v)} - r^{"}(v) &< 0\ \forall k = 1, 2 \\ &\Leftrightarrow \\ -\left[\frac{\partial^2 p_k(v, e_{kk}(ab_k))}{\partial ab_k \partial v} \frac{e_{kk}'(ab_k)}{V_k'}\right]_{ab_k=V_k^{-1}(v)} &< 0 \end{aligned}$$

since $r''(v) = \left[\frac{\partial^2 p_k(v, e_{kk}(ab_k))}{\partial v_k^2}\right]_{ab_k=V_k^{-1}(v)} + \left[\frac{\partial^2 p_k(v, e_{kk}(ab_k))}{\partial ab_k \partial v} \frac{e_{kk}'(ab_k)}{V_k'}\right]_{ab_k=V_k^{-1}(v)}$.

Since $e_{kk}' > 0$ from assumption $A1$ $i)$ and $V_1' < 0$ and $V_2' > 0$ these second order conditions therefore imply that:
$\left[\frac{\partial^2 p_1(v, e_{11}(ab_1))}{\partial ab_1 \partial v}\right]_{v=V_1(ab_1)} < 0$ and $\left[\frac{\partial^2 p_2(v, e_{22}(ab_2))}{\partial ab_2 \partial v}\right]_{v=V_2(ab_2)} > 0$. Hence, as long as $e_{kk}' > 0$ and the cross derivative $\frac{\partial^2 p_1(v, e_{11}(ab_1))}{\partial ab_1 \partial v}$ is negative and the cross derivative $\frac{\partial^2 p_2(v, e_{22}(ab_2))}{\partial ab_2 \partial v}$ is positive, that is as long as education increases skills (assumption $A1$ $i)$) and skills of type 1, respectively skills of type 2, complement machines close to 0, respectively 1 (assumption $A2$ $iii)$), an assignment where within educational groups more skilled workers get more productive machines, i.e. $V_1' < 0$ and $V_2' > 0$, is valid.

Equilibrium pricing functions

Evaluating the differential equation 9 at $v = V_k(ab_k)$ and integrating over ab_k yields the wage function for workers supplying ability of type k.

$$W_k(\overline{ab}_k) \equiv w_k(e_{kk}(\overline{ab}_k)) = w_{k0} + \int_0^{\overline{ab}_k} \left[\frac{\partial p_k(v, e_{kk}(ab_k))}{\partial ab_k}\right]_{v=V_k(ab_k)} dab_k \tag{11}$$

where w_{k0} is a constant of integration.

Similarly, evaluating the differential equation 10 at $ab_k = V_k^{-1}(v)$ and integrating over v yields the rent function as follows:

$$r(\overline{v}) = r_0 + \int_{\varepsilon}^{\overline{v}} \left[\frac{\partial p_k(v, e_{kk}(ab_k))}{\partial v}\right]_{ab_k=V_k^{-1}(v)} dv \tag{12}$$

where r_0 is a constant of integration.

The wage and rent functions are identified up to constants of integration. Following Sattinger (1979),[30] the model is closed by specifying exogenous reserve prices for the marginal workers and machine. For the least skilled workers in both groups to be indifferent between being assigned to machine ε or remaining unemployed we need $W_k(0) = w_{k0} = \widetilde{w} > 0$ where $\widetilde{w}$ is the reservation wage. Since firms owing machines ε are indifferent between employing the least skilled worker of each type, it follows from assumption *A2 iv)* that $r(\varepsilon) = r_0 = p_k(\varepsilon, e_{kk}(0)) - \widetilde{w} = \underline{p} - \widetilde{w} \geq 0 \ \forall\varepsilon$. Hence, for the firms owing machines ε to be indifferent between supplying the machine to the market or withholding the machine from the market we need $r_0 = \underline{p} - \widetilde{w} = \widetilde{r}$ where $\widetilde{r}$ is the reserve price for the owner of capital.

Wage distribution

We have established in Result 1 that, in equilibrium, wages are increasing in abilities. Moreover, the equilibrium in this economy is characterized by two functions mapping the ability of each type to tasks. Using this information, the wage density for both types of workers,

[30] Costrell and Loury (2004) consider a continuum of tasks in a single enterprise, not in the whole economy, and therefore close the model by a free entry condition that drives profits of each enterprise down to 0.

say $f_k(w)$, is directly derived from the density of tasks by performing the transformation of variables $v = V_k(W_k^{-1}(w))$ and noting that $dv = \frac{dV_k}{dab_k}\frac{dab_k}{dW}dW = \frac{V_k'}{W_k'}dW$. This yields:

$$\int_{\widetilde{w}}^{\infty} f_k(W)dW = \frac{1}{S_k}\int_{\varepsilon=W_k^{-1}(ab_k^{-}(\widetilde{w}))}^{0=W_k^{-1}(ab_k^{-1}(\infty))} d(V_k(W_k^{-1}(W)))\frac{V_k'}{W_k'}dW \tag{13}$$

where $S_k = \int_0^{\infty} d(V_k(W_k^{-1}(W))V_k'(W_k^{-1}(W)dW$ is the share of workers supplying ability of type k in equilibrium and noting that $W_k'(ab_k) = \frac{\partial p_k(v,e_{kk}(ab_k))}{\partial ab_k}$.

Hence, the following result:

Result R3: In equilibrium the wage density of workers of type k is:

$$f_k(W) = \frac{1}{S_k}d(V_k(W_k^{-1}(W)))\frac{V_k'}{W_k'} \tag{14}$$

Results $R3$ indicates that the wage density within education can be directly retrieved from the density of tasks, once we solve for the mapping functions V_k. These solutions are derived from the equilibrium employment condition.

4 Technological change, educational choice and the assignment of workers to tasks

The traditional approach to capture technological progress is to consider changes in the productivity of worker-task pairs, i.e. changes in $p_k(.,.)$. These changes in the shape of the production function of worker-task pairs could take an infinite number of different forms. It seems therefore judicious to focus on particular forms of changes and especially those yielding changes in educational self-selection and the distribution of wages that are close to changes observed in the data. The most puzzling finding in the rising wage inequality literature is the differential behavior of the between and within wage inequality in the US in the 60s and 70s. While within education wage inequality rose throughout the 60s and 70s, the between wage inequality remained fairly constant.

Using the model depicted above in this context, ab_1 could be thought of as manual ability and ab_2 as intellectual ability. Education 1 would then correspond to high-school education and education 2 to college education. The question arises as can the model depicted above reproduce a rise in within education wage inequality with constant between wage inequality and an increase in the supply of college graduates. Hence, can we change $p_k(v, e_{kk}(ab_k))$ so that i) S_2, the supply of college graduates, increases over time, ii) the ratio of the mean wage of college graduates to high-school graduates remains constant and iii) $f_k(w)$ flattens over time (increased within wage inequality).

4.1 Intuition

Let focus for the moment on workers educational self-selection and leave the assignment of tasks hidden in the back of our mind. Consider a simple example that provides a general intuition of the way formal conditions under which technical change will lead to i), ii) and iii) in the model can be derived. Instead of a continuum of manual and intellectual ability levels, consider a simplified version of the model with three levels of each type of ability, namely high, medium and low. The distribution of abilities is given by the matrix in Table 1. From result $R1$ of the model, wage rates are increasing in abilities and hence, the following inequalities must always hold $W_{jL} < W_{jM} < W_{jH}$ for $j = 1, 2$. Workers select either high-school education, supply their manual ability and earn the wage rate associated to their level of ability, i.e. W_{1J}, $J = L, M, H$ or select college education supply their intellectual ability and earn the associated wage rate W_{2J} $J = L, M, H$.

The initial equilibrium is depicted in Table 1. Given the wage rates reported in Table 1, the equilibrium assignment will be as follows. Starting from the bottom left corner of the matrix, since $W^0_{1L} = W^0_{2L} = 1$, 8 randomly chosen workers with low levels of both abilities will choose high-school, supply their manual ability and earn 1 while the remaining 8 workers will choose college education, supply their intellectual ability and earn 1. It follows from result $R1$ that the 9 workers with low intellectual ability but medium manual ability and the 8 workers with low intellectual ability but high manual ability select high-school education, supply their respective levels of manual ability and earn 2 and 3 respectively. Similarly, from result $R1$, the 9 workers with low manual ability but medium intellectual ability and the 8 workers with low manual ability and high intellectual ability select college education, supply their respective levels of intellectual ability and earn respectively 2 and 3. Since initial wage rates are so that $W^0_{1M} = W^0_{2M} = 2$, 8 randomly chosen workers with medium ability of both types select high-school education, supply their manual ability and earn 2 while the remaining 8 workers select college education, supply their intellectual ability and earn 2. From result $R1$, the 9 workers with medium manual ability but high intellectual ability select college education, supply their intellectual ability and earn 3. Similarly, from result $R1$, the 9 workers with medium intellectual ability but high manual ability select high-school education, supply their manual ability and earn 3. Finally, since $W^0_{1H} = W^0_{2H} = 3$, 8 randomly chosen workers with high abilities of both types select high-school education, supply their manual ability and earn 3 while the remaining 8 workers select college education, supply their intellectual ability and earn 3.

Denoting L_{jJ} the equilibrium employment of workers with education j and level J of ability of type j and s_{jJ} the share of workers with education j supplying level J of ability of type j in equilibrium (i.e. $s_{jJ} = \frac{L_{jJ}}{L_{j.}}$ where $L_{j.} = L_{jL} + L_{jM} + L_{jH}$ is the supply of workers with education j), between wage inequality is then given by $\frac{\sum_J s_{2J} W^0_{2J}}{\sum_J s_{1J} W^0_{1J}}$.

Note that this equilibrium assignment will hold as long as $W^t_{2L} = W^t_{1L} < W^t_{1M} = W^t_{2M} < W^t_{2H} = W^t_{1H}$ hold. Changing the values of the various wage rates but respecting these (in-)equalities would affect the wage distribution but not equilibrium assignment. Hence, technical change could lead to an increase in W^t_{jM} compared to W^t_{jL} and W^t_{jH} compared to W^t_{jM} and hence an increase in within wage inequality, without affecting the equilibrium assignment. For instance, consider a technical change that increases W^t_{jM} for $j = 1, 2$ from 2 to 4 and W^t_{jH} from 3 to 8, whereas W^t_{jL} remains constant over time. This technical change leaves equilibrium assignment constant, increases the mean wage in both groups from 2.3 to 5.5, leaving therefore between wage inequality unaffected, but increases the variance of wages within education from 0.54 to 7.2.

This type of technical change increases within wage inequality while maintaining between wage inequality constant but does not impact the equilibrium assignment and hence leaves the supply of college workers constant over time. To fully characterize the evolution of the labor market in the US in the 70s however, the supply of workers with college education should increase. One obvious way to increase the supply of college education while maintaining the between education wage inequality constant is to add workers with low intellectual ability to the group of college workers. To induce additional workers with low intellectual ability to select college education, the associated wage rate must increase compared to the wage rate for low manual ability. This new assignment will hold for wage rates so that $W^1_{1L} < W^1_{2L} < W^1_{1M} = W^1_{2M} < W^1_{2H} = W^1_{1H}$. Two forces of opposite sign impact the mean wage of both college and high-school graduates, i.e. the price effect and the composition effect. At constant supply, the relative increase in W_{2L} rises the mean wage of college graduates compared to high-school graduates and hence between wage inequality goes up. However, the price effect goes along with a change in the ability composition of both groups of workers. Indeed, since the additional college graduates have relative low intellectual ability, the share of low ability workers increases among college workers which decreases the mean wage of college graduates and compresses between wage inequality. In contrast, since the additional college workers also have relatively low manual ability, the density of low ability workers decreases among high-school graduates which increases the mean wage of high-school graduates and hence drives the between wage inequality upward. For the between wage inequality to remain constant, the relative increase in the wage rate for low intellectual ability compared to the wage rate for low manual ability must be large enough to compensate the relative composition effect in the two groups. Formally, this reads as:

$$W^1_{2L} = \frac{\frac{\sum_J s^0_{2J} W^0_{2J}}{\sum_J s^0_{1J} W^0_{1J}} \sum_J s^1_{1J} W^1_{1J} - \sum_{J \neq L} s^1_{2J} W^1_{2J}}{s^1_{2L}} \tag{15}$$

However, since this equilibrium assignment only holds as long as $W^t_{2L} < W^t_{1M} = W^t_{2M}$, the required increase in W^t_{2L} as derived from equation 15 must not be too large. For instance, for $W^1_{1L} = 1$, $W^1_{1M} = W^1_{2M} = 4$ and $W^1_{1H} = W^1_{2H} = 8$, W^1_{2L} as derived from equation 15 is equal to $2.6 > 2.1 = W^1_{1M} = W^1_{2M}$. In this case, the relative composition effect in the high-school group is too large. Stated otherwise, for feasible W^1_{2L}, the mean wage of college graduates will decrease compared to the mean wage of high-school graduates. One way to rebalance the relative mean wages in this example is to increase the wage rate for high intellectual ability compared to the wage rate for high manual ability. This will induce workers with high intellectual ability to select college education. Both the price and the composition effect will push the mean wage of college graduate up and the mean wage of high-school graduates down. Given the right magnitude, between wage inequality will return to its initial level. In other words, we have now two unknowns (W^t_{2L} and W^t_{2H}) so that $W^t_{1L} < W^t_{2L} < W^t_{1M} = W^t_{2M} < W^t_{1H} < W^t_{2H}$, and one equation, hence an infinity of solutions which satisfy the equilibrium conditions. For instance, suppose that $W^1_{1L} = 1$, $W^1_{1M} = W^1_{2M} = 4$, $W^1_{1H} = 8$ but W^1_{2H} is now equal to 9. From equation 15 we have $W^1_{2L} = 2$ which clearly lies between $W^1_{1L} = 1$ and $W^1_{1M} = 4$. This technical change has increased within wage inequality in both groups, from 0.54 to 4.3 and 9.3 respectively for high-school and college workers, maintained between wage inequality constant and increased the relative supply of college workers from 50% to 66%.

This simplification of the model provides an intuition about the type of technical change that leads simultaneously to an increase in the supply of college graduates and an increase of within wage inequality but maintains between wage inequality constant over time. This type of technical change increases wages associated with low and high intellectual ability compared to wages associated respectively to low and high manual ability.

4.2 Propositions

Denote $p_{k,t}(v, e_{kk}(ab_k))$ the productivity of a worker with ability ab_k at task v at time t. Since in assignment models wages are derived from the marginal productivity of workers, it is very convenient to define technical change in terms of the marginal productivity of workers rather than in terms of productivity. Let technical change be captured as $\Delta_t \frac{\partial p_{k,t}(v,e_{kk}(ab_k))}{\partial ab_k}$. This is a general definition that encompasses very different forms of technical change. Three families can be distinguished. The first family of technical change, characterized by $\Delta_t \frac{\partial p_{k,t}(v,e_{kk}(ab_k))}{\partial ab_k} = (\chi_k - 1) \frac{\partial p_{k,t-1}(v,e_{kk}(ab_k))}{\partial ab_k}$ with $\chi_k > 1$, increases the marginal productivity of workers proportionally in every tasks and for every workers. The second family of technical change relates to the technical factor as the marginal productivity of workers increases relatively more in tasks closed to 0 for workers supplying ability of type 1 and tasks closed to 1 for workers supplying ability of type 2. This family is characterized by $\Delta_t \frac{\partial p_{k,t}(v,e_{kk}(ab_k))}{\partial ab_k} = (\theta_k(v) - 1) \frac{\partial p_{k,t-1}(v,e_{kk}(ab_k))}{\partial ab_k}$ with $\theta_k(v) > 1$ and $\theta'_1 < 0$ and $\theta'_2(v) > 0$. Think for instance of improvement in the quality of red rubber tracks in athletics or the capacity of computers. The third family relates to the human factor as the marginal productivity of workers increases relatively more for more able workers. This family is characterized by $\Delta_t \frac{\partial p_{k,t}(v,e_{kk}(ab_k))}{\partial ab_k} = (\varphi_k(ab_k) - 1) \frac{\partial p_{k,t-1}(v,e_{kk}(ab_k))}{\partial ab_k}$ with $\varphi_k(ab_k) > 1$ and $\varphi'_k > 0$. Think for instance of the introduction of the Fosbury flop in high-jump or Bellman's equation in optimal dynamic programing.

Type 1: Neutral.

Formally, assume that technical change is captured by the parameters χ_k so that $\frac{\partial p_{k,1}(v,e_{kk}(ab_k))}{\partial ab_k} =$

$\chi_k \frac{\partial p_{k,0}(v,e_{kk}(ab_k))}{\partial ab_k}$ with $\chi_k > 1$. Hence, technical change increases the marginal productivity of workers with manual (intellectual) ability proportionally in every tasks. To derive conditions under which technical change will lead to i) ii) and iii), I propose to first derive conditions under which technical change will lead to ii) and iii) but leave equilibrium assignment unchanged.

Proposition 1 *If technical change, characterized by χ_k, is so that $\chi_1 = \chi_2$, then equilibrium assignment and between wage inequality remain constant while within wage inequality increases.*

Proof. First, note that a sufficient and necessary condition for the equilibrium assignment to remain constant over time at constant distribution of tasks is for the supply of abilities to remain constant over time. This condition reads as $ab^*_{1,1}(ab_2) = ab^*_{1,0}(ab_2)$. This is equivalent to imposing $ab^*_{1,1}(0) = ab^*_{1,0}(0)$ and $\frac{\partial ab^*_{1,1}(ab_2)}{\partial ab_2} = \frac{\partial ab^*_{1,0}(ab_2)}{\partial ab_2}$ for all $ab_2 > 0$. Since, in equilibrium, the least skilled workers get the reservation wage independently of their type of ability, as long as the reservation wage is constant over time we have $ab^*_{1,1}(0) = ab^*_{1,0}(0)$. Since $ab^*_{1,t}(ab_2) = W^{-1}_{1,t}(W_{2,t}(ab_2))$, we have:

$$\frac{\partial ab^*_{1,1}(ab_2)}{\partial ab_2} = \frac{\frac{\partial p_{2,1}(v,e_{22}(ab_2))}{\partial ab_2}}{\frac{\partial p_{1,1}(v,e_{11}(ab^*_{1,t}(ab_2)))}{\partial ab_1}} = \frac{\chi_2 \frac{\partial p_{2,0}(v,e_{22}(ab_2))}{\partial ab_2}}{\chi_1 \frac{\partial p_{1,0}(v,e_{11}(ab_1))}{\partial ab_1}}$$

It follows that:

$$\begin{aligned} \frac{\partial ab^*_{1,1}(ab_2)}{\partial ab_2} &= \frac{\partial ab^*_{1,0}(ab_2)}{\partial ab_2} \\ &\Leftrightarrow \\ \chi_1 &= \chi_2 \end{aligned}$$

Hence, as long as the technical change parameters χ_1 and χ_2 are equal, the equilibrium assignment of workers to education and workers to tasks will remain constant over time. This means that wage inequality between education remains constant. However, the slopes of the wage functions will increase since $\frac{\partial p_{k,1}(v,e_{kk}(ab_k))}{\partial ab_k} = \chi_k \frac{\partial p_{k,0}(v,e_{kk}(ab_k))}{\partial ab_k} > \frac{\partial p_{k,0}(v,e_{kk}(ab_k))}{\partial ab_k}$ for all k, v and ab_k, and so will wage inequality within education. ■

To fully characterize the evolution of the labor market in the US in the 70s, the supply of workers with college education should increase. Given the type of technical change defined by χ_j, the only way to do so is to increase χ_2 compared to χ_1. Though possible, it is very unlikely that changes in χ_2/χ_1 will leave the ratio of the mean wage of college graduates to high-school graduates constant.

Type 2: Technical factor.

Formally, assume that technical change is captured by the function $\theta_k(v)$ so that $\frac{\partial p_{k,1}(v,e_{kk}(ab_k))}{\partial ab_k} = \theta_k(v)\frac{\partial p_{k,0}(v,e_{kk}(ab_k))}{\partial ab_k}$ with $\theta_k(v) > 1$ and $\theta_1'(v) < 0$ and $\theta_2'(v) > 0$ for all v. Hence, technical change increases the marginal productivity of workers with manual (intellectual) ability relatively more in tasks close to 0 (respectively 1). To derive conditions under which technical change will lead to i) ii) and iii), I propose to follow the intuition provided in the previous section, that is, to first derive conditions under which technical change will lead to ii) and iii) but leave equilibrium assignment and hence the supply of college graduates unaffected over time and then depart from this condition by allowing technical change to affect the wage functions so that the additional workers selecting college education have either low or high intellectual ability compared to the initial college graduates.

Proposition 2 *If technical change, characterized by the functions $\theta_k(v)$, is so that $V_1^{-1}(\theta_1^{-1}(\theta_2(V_2(.)))) = W_{1,0}^{-1}(W_{2,0}(.))$ with $\theta_k(v) > 1$ and $\theta_1'(v) < 0$ and $\theta_2'(v) > 0$ for all v, then equilibrium assignment and between wage inequality remain constant while within wage inequality increases.*

Proof. Once again, for the equilibrium assignment to remain constant, we need supply to remain constant. Supply will remain constant if and only if:

$$\frac{\partial ab_{1,1}^*(ab_2)}{\partial ab_2} = \frac{\frac{\partial p_{2,1}(v,e_{22}(ab_2))}{\partial ab_2}}{\frac{\partial p_{1,1}(v,e_{11}(ab_{1,1}^*(ab_2)))}{\partial ab_1}} = \frac{\theta_2(v)\frac{\partial p_{2,0}(v,e_{22}(ab_2))}{\partial ab_2}}{\theta_1(v)\frac{\partial p_{1,0}(v,e_{11}(ab_1))}{\partial ab_1}}$$

It follows that:

$$
\begin{aligned}
\frac{\partial ab^*_{1,1}(ab_2)}{\partial ab_2} &= \frac{\partial ab^*_{1,0}(ab_2)}{\partial ab_2} \\
&\Leftrightarrow \\
\theta_1(v_1(ab^*_{1,0}(ab_2))) &= \theta_2(v_2(ab_2)) \\
&\Leftrightarrow \\
V_1^{-1}(\theta_1^{-1}(\theta_2(V_2(ab_2)))) &= ab^*_{1,0}(ab_2) \\
&= W_{1,0}^{-1}(W_{2,0}(ab_2))
\end{aligned}
$$

Hence, as long as the shapes of the technical change functions θ_k are so that $V_1^{-1}(\theta_1^{-1}(\theta_2(V_2(.)))) = W_{1,0}^{-1}(W_{2,0}(.))$ the equilibrium assignment of workers to education and workers to tasks will remain constant over time. This means that wage inequality between education remains constant. However, the slopes of the wage functions will increase since $\frac{\partial p_{k,1}(v,e_{kk}(ab_k))}{\partial ab_k} = \theta_k(v)\frac{\partial p_{k,0}(v,e_{kk}(ab_k))}{\partial ab_k} > \frac{\partial p_{k,0}(v,e_{kk}(ab_k))}{\partial ab_k}$ for all k, v and ab_k. This means that wage inequality will rise within education. ■

To fully characterize the evolution of the labor market in the US in the 70s, the supply of workers with college education should increase. Following the intuition provided in the previous section, suppose that technical change is so that:

$$
\begin{cases}
V_1^{-1}(\theta_1^{-1}(\theta_2(V_2(ab_2)))) < W_{1,0}^{-1}(W_{2,0}(ab_2)) \text{ for } 0 < ab_2 < ab_2^- \\
V_1^{-1}(\theta_1^{-1}(\theta_2(V_2(ab_2)))) = W_{1,0}^{-1}(W_{2,0}(ab_2)) \text{ for } ab_2^- \leq ab_2 \leq ab_2^+ \\
V_1^{-1}(\theta_1^{-1}(\theta_2(V_2(ab_2.)))) > W_{1,0}^{-1}(W_{2,0}(ab_2)) \text{ for } ab_2 > ab_2^+
\end{cases}
$$

Hence, this technical change leads workers with intellectual ability $ab_2 < ab_2^-$ or $ab_2 > ab_2^+$ and manual ability so that $ab^*_{1,0}(ab_2) < ab_1 < ab^*_{1,1}(ab_2)$ to switch from high-school to college education.

Type 3: Human factor.

Formally, assume that technical change is defined by the function $\varphi_k(ab_k)$ so that $\frac{\partial p_{k,1}(v,e_{kk}(ab_k))}{\partial ab_k} =$

$\varphi_k(ab_k)\frac{\partial p_{k,0}(v,e_{kk}(ab_k))}{\partial ab_k}$ with $\varphi_k(ab_k) > 1$ and $\varphi_k'(ab_k) > 0$ for all k and ab_k. Proposition 2 indicates a sufficient condition for this type of technical change to generate rising wage inequality within education and constant wage inequality between education leaving equilibrium assignment unaffected and hence the supply of college graduates constant over time.

Proposition 3 *If technical change, characterized by the functions $\varphi_k(ab_k)$, is so that $\varphi_1^{-1}(\varphi_2(.)) = W_{1,0}^{-1}(W_{2,0}(.))$ with $\varphi_k(ab_k) > 1$ and $\varphi_k'(ab_k) > 0$ for all k and ab_k, then equilibrium assignment and between wage inequality remain constant and within wage inequality increases.*

Proof. Once again, for the equilibrium assignment to remain constant, we need supply to remain constant. Supply will remain constant if and only if:

$$\begin{aligned} \varphi_2(ab_2) &= \varphi_1(ab_{1,0}^*(ab_2)) \\ &\Leftrightarrow \\ \varphi_1^{-1}(\varphi_2(ab_2)) &= ab_{1,0}^*(ab_2) \\ &= W_{1,0}^{-1}(W_{2,0}(ab_2)) \end{aligned}$$

Hence, as long as the shapes of the technical change functions φ_k are so that $\varphi_1^{-1}(\varphi_2(.)) = W_{1,0}^{-1}(W_{2,0}(.))$ the equilibrium assignment of workers to education and workers to tasks will remain constant over time. This means that wage inequality between education remains constant. However, the slopes of the wage functions will increase since $\frac{\partial p_{k,1}(v,e_{kk}(ab_k))}{\partial ab_k} = \varphi_k(ab_k)\frac{\partial p_{k,0}(v,e_{kk}(ab_k))}{\partial ab_k} > \frac{\partial p_{k,0}(v,e_{kk}(ab_k))}{\partial ab_k}$ for all k, v and ab_k, and so will wage inequality within education. ■

Similarly to the previous type of technical change, suppose that technical change is so that:

$$\begin{cases} \varphi_1^{-1}(\varphi_2(ab_2)) < W_{1,0}^{-1}(W_{2,0}(ab_2)) \text{ for } 0 < ab_2 < ab_2^- \\ \varphi_1^{-1}(\varphi_2(ab_2)) = W_{1,0}^{-1}(W_{2,0}(ab_2)) \text{ for } ab_2^- \leq ab_2 \leq ab_2^+ \\ \varphi_1^{-1}(\varphi_2(ab_2)) > W_{1,0}^{-1}(W_{2,0}(ab_2)) \text{ for } ab_2 > ab_2^+ \end{cases}$$

Hence, this technical change leads workers with intellectual ability $ab_2 < ab_2^-$ or $ab_2 > ab_2^+$

and manual ability so that $ab^*_{1,0}(ab_2) < ab_1 < ab^*_{1,1}(ab_2)$ to switch from high-school to college education.

It is important to note that the function φ_k could capture changes in the educational system that lead to relative improvements in the production of skills of one type compared to the other. Indeed, suppose for instance that $p_k(v, e_{kk}(ab_k))$ is multiplicatively separable, i.e. $p_k(v, e_{kk}(ab_k)) = f_k(v)g_k(e_{kk}(ab_k))$ with $f_1' < 0$, $f_2' > 0$ and $g_k' > 0$. The marginal productivity of workers at time t then reads as:

$$\frac{\partial p_{k,t}(v, e_{kk}(ab_k))}{\partial ab_k} = f_{k,t}(v)e'_{kk,t}(ab_k)g'_{k,t}(e_{kk,t}(ab_k))$$

Assuming that $f_{k,t}$ and $g_{k,t}$ are constant over time, we now have:

$$\varphi_k(ab_k) = \frac{e'_{kk,1}(ab_k)g'_{k,0}(e_{kk,1}(ab_k))}{e'_{kk,0}(ab_k)g'_{k,0}(e_{kk,0}(ab_k))}$$

Hence, the human factor could come about because workers find more efficient ways to use their machines over time or because, at given abilities, skills have increased through improvements of the human capital formation. Without proper empirical setting, there are no way one could distinguish between the two sources of human factor.

5 Closed form solutions

5.1 Tasks distribution

Assume that the density function of tasks is given by $d(v|d_1, d_2) = Av^{d_1}(1-v)^{d_2}$ with $d_j > 0$ and $A = 1/B(d_1+1, d_2+1)$ and $B(.)$ is the Beta function and cumulative distribution $F(v^*|d_1, d_2) = \Pr(v < v^*) = \int_0^{v^*} d(v|d_1, d_2)dv$. The mean task is given by $E[v] = \frac{d_1+1}{2+d_1+d_2}$ ($E[v] = \frac{1}{2}$ when $d_1 = d_2$) and the variance by $Var[v] = \frac{(d_1+1)(d_2+1)}{(2+d_1+d_2)(3+d_1+d_2)}$, with $\frac{\partial Var[v]}{\partial d_k} < 0$. Moreover, the distribution is skewed toward 0 when $d_1 > d_2$ and vice versa.

The Beta distribution is appealing because its support ranges from 0 to 1, it has only two parameters, and its shape is extremely flexible. If $d_1 > 0$ and $d_2 > 0$ the distribution is unimodal. If $d_1 = d_2 = d$ and $d = 0$ tasks are uniformly distributed. Moreover, for $d > 1$ the Beta distribution and the normal distribution with average $\frac{1}{2}$ and variance equal to $\frac{(d_1+1)(d_2+1)}{(2+d_1+d_2)(3+d_1+d_2)}$ look alike.

Given this specification we have in equilibrium:

$$s_k\left(ab_k\right) = Av_k(ab_k)^{d_1}(1 - v_k(ab_k))^{d_2} \cdot v_k'(ab_k)$$

5.2 The mapping functions

Assume that the mapping functions have a logistic form, with $\lim_{ab_k \to \infty} V_k'(ab_k) = 0$, $k = 1, 2$ since $\lim_{ab_k \to \infty} V_k(ab_k) = \begin{cases} 1 \text{ if } k = 2 \\ 0 \text{ if } k = 1 \end{cases}$. That is, let V_1 and V_2 be as follows:

$$\begin{aligned} V_1(ab_1) &= \frac{\varepsilon}{1 + ab_1^{r_1}} \\ V_2(ab_2) &= 1 - \frac{1 - \varepsilon}{1 + ab_2^{r_2}} \end{aligned} \tag{16}$$

with $ab_j^*(0) = W_j^{-1}(W_k(0))$ and $r_k > 0$. Note that $V_1' = -\frac{\varepsilon r_1 ab_1^{r_1 - 1}}{\left(1+ab_1^{r_1}\right)^2} < 0$ and $V_2' = \frac{(1-\varepsilon)r_2 ab_2^{r_2-1}}{(1+ab_2^{r_2})^2} > 0$.

The advantage of the logistic specification in equation 16 is that it allows for very flexible mapping functions that each depends on a single parameter r_k. The larger r_k, the more pronounce the $S - shape$ of the mapping function, that is the faster V_1 tends to 0 when ab_1 tends to infinity and the faster V_1 tends to ε when ab_1 tends 0, and the faster V_2 tends to 1 when ab_2 tends to infinity and the faster V_2 tends to ε when ab_2 tends 0.

Replacing $V_k(ab_k)$ by their expression in the employment equilibrium condition obtains:

$$\begin{aligned} s_1\left(ab_1\right) &= A\left(\frac{\varepsilon}{1 + ab_1^{r_1}}\right)^{d_1}\left(1 - \frac{\varepsilon}{1 + ab_1^{r_1}}\right)^{d_2}\frac{\varepsilon r_1 ab_1^{r_1-1}}{\left(1 + ab_1^{r_1}\right)^2} \\ s_2\left(ab_2\right) &= A\left(1 - \frac{1 - \varepsilon}{1 + ab_2^{r_2}}\right)^{d_1}\left(\frac{1 - \varepsilon}{1 + ab_2^{r_2}}\right)^{d_2}\frac{(1 - \varepsilon)r_2 ab_2^{r_2-1}}{(1 + ab_2^{r_2})^2} \end{aligned} \tag{17}$$

Equation 17 indicates that using data on workers containing information about their ability ab_1 and ab_2, conditional on the marginal task ε and the distribution of tasks, d_1 and d_2, we could calibrate r_k so that the right hand side is as close as possible to nonparametric estimation of the left hand side.

5.3 The production of skills and the product of worker-task pairs

Let skills production be multiplicative in abilities so that $e_{kk}(ab_k) = e_k ab_k$, with $e_k > 0$ for all k to satisfy assumption $A1$. Let the product of a worker-machine pair be Cobb-Douglas so that $p_1(v, e_{11}(ab_1)) = \underline{p} + p_1 \times (e_1 ab_1)^{m_1} \left(\frac{1}{v}\right)^{n_1}$ and $p_2(v, e_{22}(ab_2)) = \underline{p} + p_2 \times (e_2 ab_2)^{m_2} \left(\frac{1}{1-v}\right)^{n_2}$ with $\underline{p} = \widetilde{r} + \widetilde{w}$, $p_k > 0$, $m_k > 0$ and $n_k > 0$. Note that these shapes satisfy assumptions $A2$.

The first order conditions in equation 9 then simplify to:

$$
\begin{aligned}
p_1 m_1 e_1^{m_1} ab_1^{m_1-1} \left(\frac{1}{v}\right)^{n_1} &= e_1 w_1'(e_1 ab_1) \\
& \quad \textit{and} \\
p_2 m_2 e_2^{m_2} ab_2^{m_2-1} \left(\frac{1}{1-v}\right)^{n_2} &= e_2 w_2'(e_2 ab_2)
\end{aligned}
$$

Aggregate output is then given by:

$$
\begin{aligned}
Y(\varepsilon) &= \int_0^{\varepsilon} p_1(v, v_1^{-1}(v)) d(v) dv + \int_{\varepsilon}^{1} p_2(v, v_2^{-1}(v)) d(v) dv \\
&= \underline{p} + A \left(\begin{array}{c} \int_0^{\varepsilon} p_1 e_1^{m_1} (\varepsilon - v)^{m_1/r_1} v^{d_1 - n_1 - \frac{m_1}{r_1}} (1-v)^{d_2} dv \\ + \int_{\varepsilon}^{1} p_2 e_2^{m_2} (v - \varepsilon)^{m_2/r_2} v^{d_1} (1-v)^{d_2 - n_2 - \frac{m_2}{r_2}} dv \end{array} \right)
\end{aligned}
$$

Aggregate output will be bounded if and only if:

$$
\begin{aligned}
\lim_{v \to 0} p_1 e_1^{m_1} (\varepsilon - v)^{m_1/r_1} v^{d_1 - n_1 - \frac{m_1}{r_1}} (1-v)^{d_2} &< \infty \\
\lim_{v \to 1} p_2 e_2^{m_2} (v - \varepsilon)^{m_2/r_2} v^{d_1} (1-v)^{d_2 - n_2 - \frac{m_2}{r_2}} &< \infty
\end{aligned}
$$

Hence, for aggregate output to be bounded we need to impose $d_j - n_j - \frac{m_j}{r_j} \geq 0$. Since n_j, m_j and r_j are all positive, we have that $d_j \geq n_j + \frac{m_j}{r_j} > 0$ and therefore the model is limited to the case of unimodal distribution of tasks.

And, the marginal task is solution to the equality:

$$\begin{aligned} & \frac{p_1 m_1 e_1^{m_1}}{r_1} \int_0^{\varepsilon} (\varepsilon - v)^{\frac{m_1 - r_1}{r_1}} v^{d_1 - n_1 - \frac{m_1}{r_1}} (1-v)^{d_2} dv \\ = & \frac{p_2 m_2 e_2^{m_2}}{r_2} \int_{\varepsilon}^{1} (v - \varepsilon)^{\frac{m_2 - r_2}{r_2}} v^{d_1} (1-v)^{d_2 - n_2 - \frac{m_2}{r_2}} dv \end{aligned}$$

Unfortunately, general closed form solution for the marginal task do not exist. Close form solutions for the wage function will therefore be conditional on the (numerical) solution for the marginal task.

5.4 Equilibrium wages

Given these specifications, equilibrium wages have the following functional form:

$$\begin{aligned} w_1(e_1 ab_1) &= \widetilde{w} + p_1 e_1^{m_1} \left(\frac{1}{\varepsilon}\right)^{n_1} \int m_1 ab_1^{m_1-1} \left(1 + ab_1^{r_1}\right)^{n_1} dab_1 \\ w_2(e_2 ab_2) &= \widetilde{w} + p_2 e_2^{m_2} \left(\frac{1}{1-\varepsilon}\right)^{n_2} \int m_2 ab_2^{m_2-1} \left(1 + ab_2^{r_2}\right)^{n_2} dab_2 \end{aligned}$$

Although no general analytical solutions exist for real parameters values of n_k and numerical approximation techniques should be used, a family of analytical solutions exists for integer values. Indeed, for integer values of n_k all we need is to find a solution to integrals of the type $\int mx^{m-1} (1 + x^r)^n dx$. For $n = 1$, the solution is straightforward, $x^m \left(1 + \frac{m}{m+r} x^r\right)$. For $n \geq 2$, these integrals can be solved by successively integrating by parts. Integrating by part once we obtain:

$$\int mx^{m-1} (1 + x^r)^n dx = x^m (1 + x^r)^n - nr \int x^{m+r-1} (1 + x^r)^{n-1} dx$$

For $n = 2$, the integral in the right hand side of the equation has a simple solution, $x^{m+r}(\frac{1}{m+r} + \frac{1}{m+2r}x^r)$. If $n > 2$, this integral can be integrated by parts.

$$
\begin{aligned}
& \int x^{m+r-1} (1+x^r)^{n-1} \, dx \\
= & \frac{1}{m+1} \left(x^{m+r}(1+x^r)^{n-1} - (n-1)r \int x^{m+2r-1} (1+x^r)^{n-2} \, dx \right)
\end{aligned}
$$

Once again, if $n = 3$, the integral in the right hand side of the equation has a simple solution, $x^{m+2r}(\frac{1}{m+2r} + \frac{1}{m+3r}x^r)$. If $n > 3$, the integral can be once again integrated by parts. Since the parameter n_k are output elasticities with respect to the characteristics of the machines, this parameter is likely to be close to unity. Therefore I present herewith the closed form solution for the wage functions for $n_k = 1$ and $n_k = 2$.

For $n_k = 1$, equilibrium wages are given by:

$$
\begin{aligned}
w_1(e_1 ab_1) &= \widetilde{w} + p_1 e_1^{m_1} \frac{1}{\varepsilon} ab_1^{m_1} \left(1 + \frac{m_1}{m_1 + r_1} ab_1^{r_1}\right) \\
w_2(e_2 ab_2) &= \widetilde{w} + p_2 e_2^{m_2} \frac{1}{1-\varepsilon} ab_2^{m_2} \left(1 + \frac{m_2}{m_2 + r_2} ab_2^{r_2}\right)
\end{aligned}
$$

For $n_k = 2$, equilibrium wages are given by:

$$
\begin{aligned}
w_1(e_1 ab_1) &= \widetilde{w} + p_1 e_1^{m_1} \left(\frac{1}{\varepsilon}\right)^2 \\
& \times \left(ab_1^{m_1} (1 + ab_1^{r_1})^2 - 2r_1 ab_1^{m_1+r_1} (\frac{1}{m_1 + r_1} + \frac{1}{m_1 + 2r_1} ab_1^{r_1}) \right) \\
w_2(e_2 ab_2) &= \widetilde{w} + p_2 e_2^{m_2} \left(\frac{1}{1-\varepsilon}\right)^2 \\
& \times \left(ab_2^{m_2} (1 + ab_2^{r_2})^2 - 2r_2 ab_2^{m_2+r_2} (\frac{1}{m_2 + r_2} + \frac{1}{m_2 + 2r_2} ab_2^{r_2}) \right)
\end{aligned}
$$

5.5 Simulations

Can the model developed in this paper reproduce the changes in the wage structure observed in the last decades in the US? To answer this question, I use the parametric specification of sections 5.1 to 5.3 and look for parameters that best fit (changes in) the wage distribution for three key years, namely 1965, 1980 and 1995. These three years are selected because the evolution of the wage structure between 1965 and 1980 and between 1980 and 1995 was remarkable. While within wage inequality rose sharply and steadily over the whole period, the between wage inequality, as measured by the college premium, only started to rise from 1980 onwards. Moreover, while wages rose everywhere in the distribution between 1965 and 1980, the wage at the 1st decile decreased by 20% points between 1980 and 1995 to settle at 10% points below its 1965 level. The three years selected and the two periods they define provide therefore a good test for the model. Especially the differential timing of between and within wage inequality and the drop in the wage at the 1st decile are of interest.

The model has 13 parameters (p_j, m_j, n_j, r_j, d_j and e_j for $j = 1, 2$ and $\widetilde{w}$) which means that allowing each parameter to take x possible values on their respective domain, x^{13} simulations should be run. To reduce the computational burden, I first reduce the dimensionality of the problem by setting arbitrarily $\widetilde{w} = 1$, $e_1 = e_2 = 1$ and $d_1 = d_2 = 4$ so that the distribution of tasks is bell-shaped and constant over time. Preliminary simulations failed to provide numerical results for $m_j > 2$. I therefore restrict the domain of m_j to the interval $[0.5; 2]$. Also, the domain of n_j is restricted to the interval $[0.5; 3]$, the domain of p_j to $[0.5; 3]$ and the domain of r_j to $[3; 18]$.

For each of the 385,000 simulations ran,[31] values of the parameters r_j, p_j, m_j and n_j were drawn at random from their respective intervals. Each simulation is evaluated as a potential candidate to represent each year 1965, 1980 or 1995. In a first stage, I select for each year a set of simulations out of the 385,000 simulations. The selection is based on how well the generated wage structure fits the observed wage structures in that year. To evaluate the goodness of fit, I focused on 2 summary measures, namely the skill premium and the relative supply of college graduates. The skill-premium is measured by the (log) ratio of the median wage of college graduates to the median wage of high-school graduates and corresponding observations are taken from Figure 1 in Acemoglu (2002) and reported in the columns Data in Table 3.[32]

[31]Note that this is approximatively equivalent to a grid search where each of the 8 parameters takes on 5 different values, i.e. $5^8 = 390,625$.

[32]Note that Acemoglu (2002) uses the log of the *average* wage of college to high-school graduates (conditional on other explanatory variables). Unfortunately, the model does not produces closed form solution for the wage functions for all parameter values. Hence, while quantiles of the distribution of the wage distribution are readily accessible through numerical approximations, moments of the distribution (and hence the mean and variance) require numerical integrations of the wage distribution. I therefore herewith proxy the college premium as the log of the ratio of the median wage of college graduates to median wage of high-school graduates.

The relative supply of college graduates will be compared to observations provided in Figure 1 in Acemoglu (2002). Hence, for each year, I keep only those simulations for which both the generated relative supply and skill premium miss the target by less than 4%.[33] After this selection, there remained 89 simulations for 1965, 58 simulations for 1980 and 14 simulations for 1995.

The second stage consists of linking simulations from each year so as to generate a "time-series". Although there are $89 \times 58 \times 14 = 72,268$ possible time-series only those for which the evolution of the wage at the 1st decile of the overall distribution was closed enough to the observed behavior of the wage at the 1st decile were selected. The wage at the 1st decile of the overall wage distribution is proxied by the wage of the 1st decile of high-school graduates, and compared to observations shown in Figure 2 in Acemoglu (2002) and reported in Table 3. The goodness of fit was measured by the average of the errors for the change between 1965 and 1980 and between 1980 and 1995 and the selection criterion was set at the 5% level. Only those 2,067 time-series for which the average errors were less than 5% were selected for further investigation.

Finally, in the third stage, out of the 2,067 remaining time-series, I selected the two time-series for which the evolution of the wage gap between the 9th and 5th deciles and between the 9th and 1st deciles within education fitted the data best. Since the model generates two within wage distributions, one per education, I took the average of the two distributions to measure the within wage gaps. The generated within wage gaps are compared to the residual (log) wage gaps shown in Figure 3 in Acemoglu (2002) and reported in Table 3. Once again, the goodness of fit is measured by the mean errors for the 2 gaps in the 2 periods Both selected time-series had a goodness of fit approximately 97%.

The parameters associated to the two series are reported in Table 3. Remarkably enough, referring to a 50% change in a parameter as a large one, in both simulations, large changes in the *productivity* parameters occur between 1980 and 1995. In simulation I, changes occur for manual skills whereas in simulation II changes occur for intellectual skills. In simulation I, the slope of the productivity of manual workers p_1 drops while the technical factor contributes to an increase in productivity (the output elasticity with respect to tasks n_1 increases). This would

[33] When set at the 5% level, 134 simulations for 1965, 101 for 1980 and 19 for 1995 are selected. This means that 257,146 time-series must be generated in stage 2. Hence, setting the significance level at 4%, only 72,268 time-series need to be generated, which is still a significant amount but reduces the computational burden by a factor 3.6.

mean that the tasks (machines) to which manual workers have been assigned have become more productive over time especially those tasks extremely manual (close to 0). On the other hand, simulation II tends to indicate that the slope of the productivity of intellectual workers has increased between 1980 and 1995 (which could either reflect an increase in the productivity of their machines or an increase in the technique they use to operate their machines) while the human factor has contributed to a decrease in productivity especially in very intelectual tasks (close to 1) (the output elasticity with respect to intellectual skills decreases).

Although productivity parameters fluctuates between 1965 and 1980 in both simulations, the most remarkable changes between 1965 and 1980 are those observed for the mapping function parameters r_j. Since the distribution of tasks has been held constant, these changes in the mapping function over time can only be due to changes in the distribution of skills over time. This is of course in part the result of self-selection −the share of college graduates increases over the whole period− but could also be due to changes in the distribution of abilities over time. And since the exact definition of abilities in the model encompasses pure ability endowment individuals were born with and the contribution of factors affecting a child's ability vector up until the end of compulsory education, changes in $\xi(ab_1, ab_2)$ could either be due to genetics *(changes in pure ability endowments)* or changes in factors affecting a child's abilities up until the end of compulsory education, i.e. family background and environment. This is an interesting possibility that requires a proper experimental setting to be tested. It is important, however, to bear in mind that the simulations ran above are not such a setting and should not be taken as such. The simulations can not help us to isolate which sources *are* responsible for what share of the changes in the wage distribution but they at least spells out which sources *can be* responsible for the changes in the wage distribution over time.

6 Conclusion

This paper contributed to the literature on assignment models by presenting a general equilibrium assignment model that acknowledges the multidimensionality of skills and account for endogenous human capital formation through educational self-selection. The main characteristic of this model is that two types of assignment occur. The first type of assignment is workers's educational self-selection. Education is a mean for workers to transform their initially endowed abilities into marketable skills. Different education transform abilities of the two types in different proportions. Under mild conditions, spelled out in Assumption $A1$, workers specialize and supply their skills of the type that maximizes their earnings.

The second type of assignment is the assignment of workers to tasks. Each task is associated with a unit of capital, a machine for the sake of the argument, and the various tasks correspond to machines with different characteristics. To produce output, each machine needs to be operated by a fixed proportion of workers, i.e. one and only one worker to eliminate the intensive margin. The owner of a machine is loosely referred to as a firm. Although the various machines can be operated by workers with different types and levels of skills, workers of different types and levels of skills differ in their productivity. I show that if the productivity of worker-task pairs satisfies assumption $A2$ then, following Ricardo's principles of comparative advantage and differential rents, equilibrium in this model is characterized by two mapping functions, one for each type of skills supplied. The first mapping function is decreasing and maps skills of the first type to tasks on the left hand side of the support. The second mapping function is increasing and maps skills of the other type to tasks on the right hand side of the support. These two mapping functions generate two wage functions, one for each type of skills, that will in general overlap.

The multidimensionality of skills and self-selection is a unique asset of the model within the class of general equilibrium assignment models. This asset is fundamental to keep up with the recent empirical literature led by Heckman and Rubinstein (2001) that emphasizes the importance of noncognitive skills, such as personality traits, in explaining earnings. In addition to the multidimensionality of skills, self-selection offers to the model a natural explanation for the persistent overlap of the wage distributions of workers with different education (see Figure 1). In his seminal paper, Roy (1951) indeed spelled out the consequences of self-selection on the distribution of earnings. In his famous example of trout fishing and rabbit hunting economy, Roy showed that, unless abilities at fishing and hunting are perfectly and positively correlated, or stated otherwise, unless the rank of villagers is constant across abilities, the distribution of wages of hunters will overlap the distribution of wages of fishers.

This paper argues that general equilibrium assignment models offer a unique framework to study the impact of technical change on the wage structure. Assignment models enable us to distinguish between the contribution of human factors and the contribution of technical factors in rising wage inequality. While technical factors are directly linked to technical change, the human factors could take several forms. They could reflect changes in the way workers use their machines or perform their tasks −that is a change in the technique− or changes in the distribution of skills. In contrast to existing assignment models, educational self-selection in the model enables us to further distinguish between changes in the distribution of skills that come from changes in school (college) quality, i.e. changes in E_j, from changes that come from changes in the distribution of abilities, i.e. changes in $\xi(ab_1, ab_2)$. The exact definition of abilities in the model encompasses pure ability endowment individuals were born with and the contribution of factors affecting a child's ability vector up until the end of compulsory education. Hence changes in $\xi(ab_1, ab_2)$ could either be due to genetics *(changes in pure ability endowments)* or changes in factors affecting a child's abilities up until the end of compulsory education, i.e. family background and environment.

The numerical simulations ran in the paper have shown that the model can reproduce stylized facts hard to explain in a unified model, i.e. i) the overlap in the wage distributions of college and high-school graduates, ii) the differential behavior of the between and within wage inequality in the 70s and, iii) the decline of the wage at the 1^{st} decile of the overall wage distribution after 1980. Especially the decline of the wage at the 1^{st} decile of the wage distribution has been hard to interpret in models of technical change. Standard models of technical change would predict an increase in the wage at all deciles or at least a stagnant wage at the lowest deciles if new (more productive) technologies are not used by low wage workers.[34] That the assignment model with endogenous human capital formation is able to reproduce this stylized fact is a particularly important asset, but it raises the question: how *can* the model *explain* this decline?

The answer to this question is straightforward once we recognize that wages in the model reflect the productivity of worker-task pairs *in equilibrium*. The decline of the wage of workers at the 1^{st} decile indicates that the productivity of the workers at the 1^{st} decile has decreased over time. There are three possible reasons for that to have happened:

1. The machine or task to which workers at the 1^{st} decile are assigned have become less productive,

2. workers at the 1^{st} decile operate their machines in a less productive way and,

[34] Note that some other studies have proposed explanations for the fall of the wage at the 1^{st} decile. For instance, Galor and Moav (2000) argue that the drop follows from the "erosion effect" created by new tehcnologies, Acemoglu (1999) and Caselli (1999) argue that the capital-labor ratio for low wage workers falls as firms respond to technical change and Autor et al. (2003) suggests that computers substitute labor in manual routine tasks.

3. workers at the 1^{st} decile in 1995 have less skills than workers at the 1^{st} decile in 1980.

The first two reasons are very similar to the traditional technical change approach and hence suffer the same critics. In contrast, the third explanation is very plausible. As mentioned above, skills in the model are endogenous and depend on the educational production E_j and abilities which in turn depend on pure endowed abilities workers were born with and family background and environment during childhood. In this context, the model tells us that the decline in the skills supplied by workers at the 1^{st} decile could be either due to a decrease in the educational production E_j, a decline in the endowed abilities or a worsening of the family situation and environment of workers at the 1^{st} decile when they were children. Of the three explanations, the last one is certainly the most likely. Indeed, if anything, school quality has improved in the US between 1980 and 1995, see Card and Krueger (1992) for instance, and even if innate abilities are transmitted genetically, the process is most likely mean-reverting. However, it is plausible that the family background and social environment in general during the childhood of the worker at the 1^{st} decile in 1980 was more favorable in terms of ability development than that of the worker at the 1^{st} decile in 1995.

Finally, note that Autor et al. (2007) have argued that the wage polarization observed in the US in the 90s has been the result of a job polarization. Assignment models tells us that wage polarization could come about for two reasons: a S-shaparization of the wage function(s) $W_j(ab_j(v))$ due to changes in the productivity of worker-task pairs at constant tasks (and skill) distribution or job polarization (changes in $d(v)$). Hence job polarization is a sufficient condition for wage polarization yet not a necessary one. Just like standard assignment models, the assignment model proposed in this paper could easily capture this job polarization by allowing $d(v)$ to change over time. However, to highlight the fact that job polarization is not a necessary condition for the observed wage polarization, in this paper, simulations were ran holding $d(v)$ constant over time. These simulations have shown that it was possible to reproduce an increase in wage inequality simultaneously with a drop in the wage at the first decile which is an indication of wage polarization. Hence, the model put forward an alternative explanation for the wage polarization observed in the US in the 90s, namely a worsening of the factors affecting children's' abilities formation over time or at least between the childhood period of the worker at the 1^{st} decile in 1980 and the childhood period of the worker at the 1^{st} decile in 1995.

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Appendix

The data used to estimate the overlap of the wage distribution of high-school and college graduates are the CPS March supplements from 1964 to 2003. Since the model studies educational choice college drop outs are herewith classified with college graduates for they chose to go to college. Workers working less than 38 hours and less than 39 weeks were excluded as well as self-employed. Furthermore, each year sample includes only white males aged between 18 and 65 and all observations missing crucial information on wages, hours, weeks worked, education, industry and occupation are deleted.

The wage measure used is the hourly earnings of full-time full-weeks workers defined as the annual earnings divided by total weeks worked and total hours worked.[35] The hourly earnings are then deflated by the CPI-U (the Consumer Price Index for All Urban Consumers provided by the US department of Labor), to obtain a measure of real hourly earnings in 1996 US dollars.

As Katz and Murphy (1992), I excluded workers with real hourly earnings below one half of the real minimum wage of each year.[36] For the samples from 1964 to 1988, following Katz and Murphy (1992), I imputed to workers with topcoded earnings, annual earnings equal to 1.45 times the topcode amount. The factor 1.45 corresponds to the ratio of the estimated conditional average earnings of those with topcoded earnings by the topcode amount. From 1989 on, wage and salary incomes are collected into two separate variables, primary and secondary labor earnings, each with a different topcode amount. After adjusting for the topcodes, the primary and secondary earnings are added to form the annual earnings. For the primary earnings, workers with topcoded earnings were assigned the topcode until 1995. I multiply the values by 1.45 to obtain the primary earnings adjusted for topcodes. After 1996, topcoded workers were assigned the mean of all topcoded workers. I impute these workers the topcode times 1.45. For the secondary earnings, topcoded workers were assigned the topcoded value. I therefore impute these workers the topcode value times 1.45.

[35] Conform to the literature, for the 1964-1975 samples for which only interval of weeks worked are available, I imputed the mid-range of each interval.

[36] The series of nominal and real minimum wage rates are reported by the US department of Labor. The real series is obtained by deflating the nominal series by the CPI-U price index.

Figure 1: Overlapping wage distributions of College and High-school graduates. White males with equal experience (0-4 years) working full time full weeks. Overlap is the Mann-Whitney statistic defined as the percentage of randomly chosen high-school graduates earning above randomly chosen college graduates.

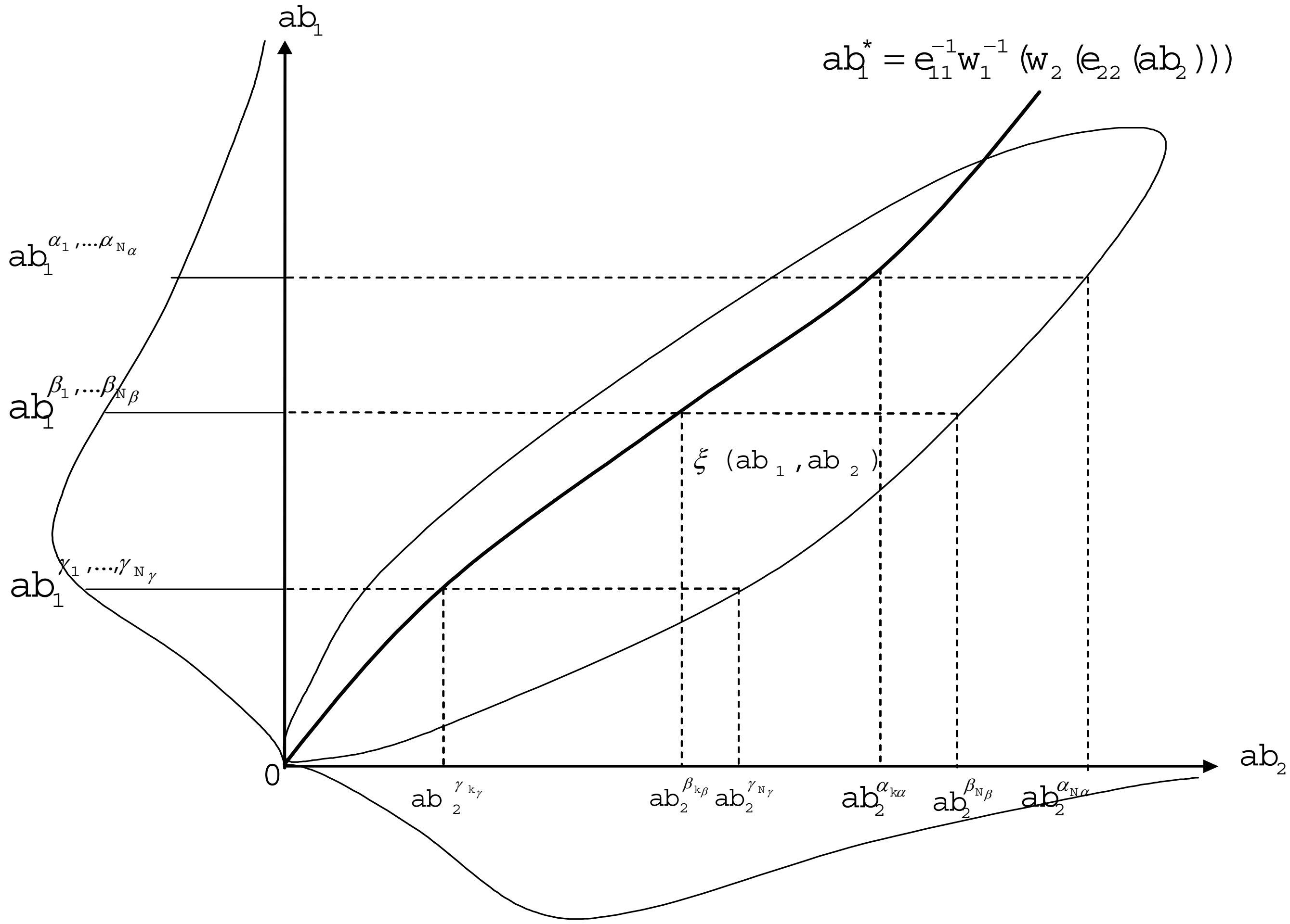

Figure 2: Educational self-selection

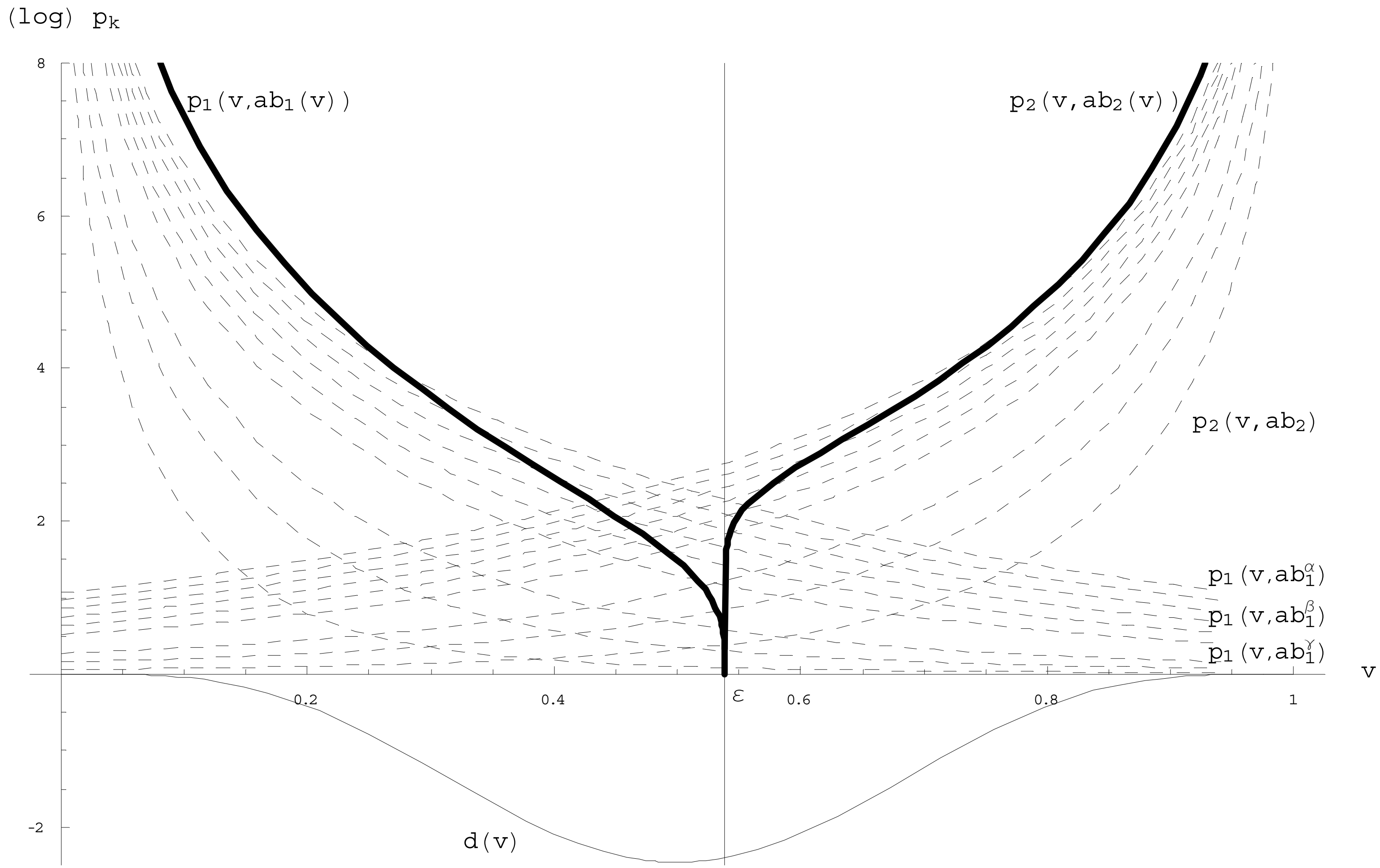

Figure 3: Tasks assignment. The figure is generated using the parametric specification 5.1-5.3 with: $p_1 = 1.7$, $p_2 = 1.9$, $m_1 = 2$, $m_2 = 1.5$, $n_1 = n_2 = 2.65$, $e_1 = e_2 = 1$, $r_1 = 4.5$, $r_2 = 7$, $d_1 = d_2 = 4$ and $\widetilde{w} = 1$.

Table 1: Example: Assignment and wage distribution at the initial conditions.

Ability types			2				
	Levels	L	M	H	N^a_{1J}	W_{1J}	L^a_{1J}
	H	8	9	16	33	3	25
1	M	9	16	9	34	2	17
	L	16	9	8	33	1	8
	N^a_{2J}	33	34	33	100		
	W_{2J}	1	2	3			$L_{1.} = 50$
	L^a_{2J}	8	17	25		$L_{2.} = 50$	

Summary Statistics:	*Mean*	*Within*[b]	*Overlap*[c]
High-school	2.3	0.54	
College	2.3	0.54	
Between	1		100%

[a]Note that N_{iJ} refers to the marginal density of workers with level J of ability of type i in the economy. In contrast, L_{iJ} is the conditional density in equilibrium (given the wage rates) or employment.

[b]Within wage inequality is measured herewith by the wage variance within education

[c]The overlap is measured herewith by the Mann-Whitney statistic. This statistic is equal to the probability that a randomly chosen worker with high-school education (ability_1) earns at least as much as a randomly chosen college graduate (with ability_2). In this example, this statistic is 1 since the wage distribution are identical.

Table 2: Example: The impact of technical change on assignment and wages.

Ability types			2				
	Levels	L	M	H	N^a_{1J}	W_{1J}	L^a_{1J}
	H	8	9	16	33	8	17
1	M	9	16	9	34	4	17
	L	16	9	8	33	1	0
	N^a_{2J}	33	34	33	100		
	W_{2J}	2	4	9			$L_{1.} = 34$
	L^a_{2J}	16	17	33		$L_{2.} = 66$	

Summary Statistics:	*Mean*	*Within*[b]	*Overlap*[c]
High-school	6	4.3	
College	6	9.3	
Between	1		$\frac{17\times16+17\times33}{34\times66} = 37\%$

[a]Note that N_{iJ} refers to the marginal density of workers with level J of ability of type i in the economy. In contrast, L_{iJ} is the conditional density in equilibrium (given the wage rates) or employment.

[b]Within wage inequality is measured herewith by the wage variance within education.

[c]The overlap is measured herewith by the Mann-Whitney statistic. This statistic is equal to the probability that a randomly chosen worker with high-school education (ability_1) earn at least as much as a randomly chosen college graduate (with ability_2).

In this example, this statistic is given by

$\Pr(\text{ability}_1$ is M and ability_2 is L$)$ +

$\Pr(\text{ability}_1$ is H and ability_2 is at most M$)$

Table 3: Simulations of the model: Wage distribution in the US 1965-1980-1995.

Parameters	1965			1980			1995		
	Model		Data*	Model		Data*	Model		Data*
	I	*II*		*I*	*II*		*I*	*II*	
p_1	2.55	2.94		2.69	2.89		0.94	2.18	
p_2	1.04	2.09		1.20	2.03		1.53	2.96	
m_1	1.78	1.09		1.49	1.58		1.44	1.60	
m_2	1.14	1.50		1.05	1.89		0.65	0.88	
n_1	1.47	2.09		1.64	2.20		2.75	2.51	
n_2	1.97	1.85		2.24	2.26		2.30	2.22	
r_1	6.77	5.66		4.84	6.66		4.66	4.67	
r_2	10.25	7.64		17.92	16.00		14.97	17.36	
S_2/S_1	0.26	0.25	0.25	0.43	0.43	0.42	0.68	0.68	0.70
$\ln \frac{w_2^{(5)}}{w_1^{(5)}}$	0.36	0.36	0.37	0.37	0.37	0.37	0.60	0.61	0.60
$w_1^{(1)}$	100	100	100	106	107	113	87	84	87
$\ln \frac{w^{(9)}}{w^{(5)}}$	100	100	100	107	110	107	125	120	117
$\ln \frac{w^{(9)}}{w^{(1)}}$	100	100	100	107	109	111	112	119	113

Note: for each year and each model we have $e_j = 1$, $d_j = 4$ and $\widetilde{w} = 1$.
∗Source: Acemoglu (2002), Figures 1, 2 and 3.