Affiliation and Positive Dependence in Auctions*

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This version: November 2007 ‡

Abstract

We consider private value auctions where bidders' types are dependent, a case usually treated by assuming affiliation. As any scientific assumption, affiliation has limitations and it is important to know them. We show that affiliation is a restrictive condition, that is, the set of affiliated distributions is small both in topological and measure-theoretical senses. The economic cases where affiliation holds do not correspond to the intuition usually given for affiliation. Affiliation's implications do not generalize to other definitions of positive dependence and may be frequently false. Nevertheless, some of these implications are true in a weaker sense and there are cases where affiliation can be well justified and used in theoretical models. Since these cases do not cover all economically relevant cases, it is desirable to seek a more general approach to dependence in auctions.

We propose a new approach that allow both theoretical and numerical characterization of pure strategy equilibrium of first-price auctions. We treat mainly symmetric auctions, but the approach can be extended to asymmetric auctions with dependence. New results about equilibrium existence and revenue ranking of auctions are provided.

JEL Classification Numbers: C62, C72, D44, D82.

Keywords: affiliation, positive dependence, dependence of types, auctions, pure strategy equilibrium, revenue ranking.

1 Introduction

Private information is a central theme in modern economics. It is often introduced in the economic models through (privately known) random variables. For mathematical

*This paper supersedes the first part of "Affiliation, Equilibrium Existence and the Revenue Ranking of Auctions". The second part of "Affiliation, Equilibrium Existence and the Revenue Ranking of Auctions" (still contained in this version as sections 4 and 5) will belong to a separate paper. I am grateful to seminar participants in 2006 Stony Brook Game Theory Festival, Washington University, University of Paris I, Universidad Carlos III, Universidad de Santiago de Compostela, University of Illinois, Universitat Pompeu Fabra, Penn State University and to Aloisio Araujo, Alain Chateauneuf, Maria Ángeles de Frutos, Ángel Hernando, Vijay Krishna, Humberto Moreira, Stephen Morris, Paulo K. Monteiro, Andreu Mas-Colell, Sergio Parreiras and Jeroen Swinkels for helpful suggestions. I am especially grateful to Flavio Menezes and Steven Williams for their comments. Find updated versions of this paper at www.impa.br/∼luciano.

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‡First version: January 2004.

convenience, it is common to assume that such random variables are independent, but this assumption is restrictive and unrealistic. It is restrictive because independence is a "knife-edged" assumption and it is unrealistic because there are many sources of correlation in the real world: education, culture, etc. Early recognizing these limitations, economists tried to surpass the mathematical difficulties and consider some sort of dependence in their models.

In auction theory, Wilson (1969 and 1977) was a pioneer in this task. After his important papers, a remarkable contribution was made by Milgrom and Weber (1982a), who introduced the concept of affiliation in auction theory.[1]

Affiliation is a generalization of independence — see the definition in section 3 — that was explained through the appealing *positive dependence intuition*: "Roughly, this [affiliation] means that a high value of one bidder's estimate makes high values of the others' estimates more likely." (Milgrom and Weber (1982a), p. 1096.) Among other results implied by affiliation, Milgrom and Weber (1982a) obtained the following: (i) affiliation ensures the existence of a symmetric monotonic (increasing) pure strategy equilibrium (SMPSE) for first price auctions;[2] (ii) under affiliation, the English and the second price auction[3] have higher expected revenue than the first price auction, which may explain the real world fact that English auctions are more common than first price auctions. The theoretical depth and elegance of the paper, the plausibility of the hypothesis of affiliation, as justified by a clear economic intuition, and the explanation of the predominance of English auctions may be considered reasons for the deep impact of this paper. Since then, affiliation became part of the foundations of auction theory and almost a synonymous of dependence in auctions.

After a quarter of century of intensive and successful use in auction theory, it seems opportune to make an assessment of affiliation as assumption. How strong is it? How robust are its implications? Do we really understand the consequences of dependence in auctions by studying affiliation? Is affiliation the definitive answer to the problem of dependence in auctions? The rethorical answer — this assumption is provisory and limited as any scientific construction — is not enough. It is important to know how severe such limitations are, because the directions of further developments may depend on this knowledge.

In section 3 we show that affiliation is restrictive in two senses. It is restrictive in a *topological* sense: the set of no affiliated probability density functions (p.d.f.'s) is open and dense in the set of continuous p.d.f.'s. It is also restrictive in a *measure-theoretic*

[1]In two previous papers, Paul Milgrom presented results that use a particular version of the same concept, under the name "monotone likelihood ratio property" (MLRP): Milgrom (1981a,b). Nevertheless, the concept is fully developed and the term affiliation first appears in Milgrom and Weber (1982a). See also Milgrom and Weber (1982b). When there is a density function, the property had been previously studied by statisticians under different names. Lehmann (1966) calls it Positive Likelihood Ratio Dependence (PLRD), Karlin (1968) calls it Total Positivity of order 2 (TP_2) for the case of two variables or Multivariate Total Positivity of Order 2 (MTP_2) for the multivariate case.

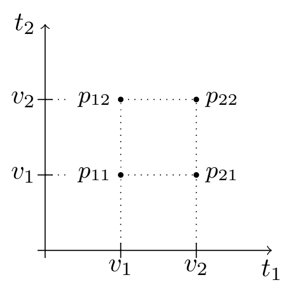
[2]They also proved the existence of equilibrium for second price auctions with interdependent values. In our set-up (private values), the second price auction always has an equilibrium in weakly dominant pure strategies, which simply consists of bidding the private value. Although equilibria in mixed strategies always exist (Jackson and Swinkels, 2005), first price auctions may fail to possess a pure strategy equilibrium when types are dependent.

[3]For private value auctions, which we mainly consider in this paper, English and second price auctions are equivalent — see Milgrom and Weber (1982a).

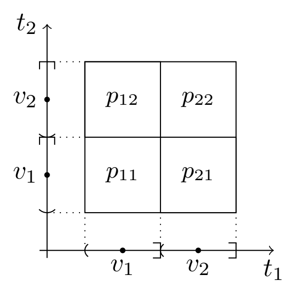
sense: if μ is a probability measure over the set of joint probabilistic density functions (p.d.f.'s), and if μ satisfies some weak conditions, then then μ puts zero measure in the set of affiliated p.d.f.'s. In a sense, affiliation is almost as "knife-edged" as independence. Although this observation is relevant, it is by no means definitive, since restrictive assumptions are widespread in economics. It is more important to know whether the assumption is reasonable in economically relevant situations and how robust its implications are.

From this, we reexamine the intuition used to introduce affiliation: the positive dependence intuition quoted above. We show that the intuition may be misleading, as there are many different (and weaker) definitions of positive dependence. We show that there are some economic models where affiliation is ensured, but they do not cover all the economically relevant cases. We also show that some of the main implications of affiliation — equilibrium existence and the revenue ranking of auctions — do not extend for other (still restrictive) definitions of positive dependence. From this, it seems important to find a new method to deal with dependence in auctions.

We offer a new method, based in a simple idea. See Figure 1.



(a) $p_{ij} = \Pr(t_1 = v_i, t_2 = v_j)$



(b) $p_{ij} = f(v_i, v_j)$

Figure 1: Discrete values, such as in (a), capture the relevant economic possibilities in a private value model, but preclude the use of calculus. We use continuous variables, but consider only simple density functions (constant in squares), such as in (b).

Let us expand the above explanation. Consider the setting of symmetric private value auctions with two risk neutral players, but general dependence of types. Since we are analyzing auctions of single objects, it would be sufficient to consider the case where bidders' types are distributed according to a finite number of values (the values can be specified only up to cents and are obviously bounded). Nevertheless, to work with discrete values precludes us from using the convenient tools of differential calculus, which allow, for instance, a complete characterization of equilibrium strategies. Maintaining the advantage of continuum variables, but without requiring unnecessary richness in the set of distributions, we focus on the set of densities which are constant in some squares around fixed values. This imposes no economic restriction on the cases considered, but allows a complete characterization of symmetric monotonic (increasing) pure strategy equilibrium (SMPSE) existence (see subsection 4.1).

It is easy to see that, as we take arbitrarily small squares, we can approximate any p.d.f. (including non-continuous ones). Thus, even if the reader insists on mathematical generality, that is, to include other distributions, our results are still meaningful because they cover a dense set and the equilibrium existence has some continuity properties.

For this set of simple p.d.f.'s, we are able to provide an algorithm, implementable by a computer, that completely characterizes whether or not a pure strategy equilibrium exists. The computer experiments (simulations) allow an exploration of facts about auctions in general. These experiments can illuminate directions of research: after exploring simulations and detecting a stable pattern, a theorist can try to prove that such pattern is mandatory, that is, prove a theorem about it. We illustrate how this can be done, by proving new results regarding pure strategy equilibrium existence.

The paper is organized as follows. Section 2 gives a brief exposition of the standard auction model. Section 3 makes an assessment of affiliation and shows that it is important to advance the study of dependence in auctions. Section 4 introduces the method proposed and presents the equilibrium existence results. Section 5 deals with the problem of revenue ranking in auctions. Section 6 considers related literature and presents concluding remarks. The more important and short proofs are given in an appendix, while lengthy constructions are presented in a separate supplement to this paper.

2 Basic model and definitions

Our model and notations are standard. There are n bidders, $i = 1, ..., n$. Bidder i receives private information $t_i \in [\underline{t}, \bar{t}]$ which is the value of the object for himself. The usual notation $t = (t_i, t_{-i}) = (t_1, ..., t_n) \in [\underline{t}, \bar{t}]^n$ is adopted. The values are distributed according to a p.d.f. $f : [\underline{t}, \bar{t}]^n \to \mathbb{R}_+$ which is symmetric, that is, if $\pi : \{1, ..., n\} \to \{1, ..., n\}$ is a permutation, $f(t_1, ..., t_n) = f(t_{\pi(1)}, ..., t_{\pi(n)})$.[4] Let $\overline{f}(x) = \int f(x, t_{-i})\, dt_{-i}$ be a marginal of f. Our main interest is the case where f *is not* the product of its marginals, that is, the case where the types are dependent. We denote by $f(t_{-i} \mid t_i)$ the conditional density $f(t_i, t_{-i}) / \overline{f}(t_i)$. After knowing his value, bidder i places a bid $b_i \in \mathbb{R}_+$. He receives the object if $b_i > \max_{j \neq i} b_j$. We consider both first and second price auctions. As Milgrom and Weber (1982a) argue, second price and English auctions are equivalent in the case of private values, as we assume here. In a first price auction, if $b_i > \max_{j \neq i} b_j$, bidder i's utility is $u(t_i - b_i)$ and is $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. In a second price auction, bidder i's utility is $u(t_i - \max_{j \neq i} b_j)$ if $b_i > \max_{j \neq i} b_j$ and $u(0) = 0$ if $b_i < \max_{j \neq i} b_j$. For both auctions, ties are randomly broken.

By reparametrization, we may assume, without loss of generality, $[\underline{t}, \bar{t}] = [0, 1]$. It is also useful to assume $n = 2$, but this is not needed for most of the results. For most of the paper, we assume risk neutrality, that is, $u(x) = x$. Thus, unless otherwise stated, the results will be presented under the following set-up:

BASIC SETUP: *There are $n = 2$ risk neutrals bidders, that is, $u(x) = x$, with private values distributed according to a symmetric density function $f : [0, 1]^2 \to \mathbb{R}_+$.*

[4] For the reader familiar with Mertens and Zamir (1986)'s construction of universal type spaces: we make the usual assumption in auction theory that the model is "closed" at the first level, that is, all higher level beliefs are consistently given by (and collapse to) f.

A pure strategy is a function $b : [0,1] \to \mathbb{R}_+$, which specifies the bid $b(t_i)$ for each type t_i. The interim payoff of bidder i, who bids β when his opponent $j \neq i$, follows $b : [0,1] \to \mathbb{R}_+$ is given by

$$\Pi_i (t_i, \beta, b(\cdot)) = u(t_i - \beta) F\left(b^{-1}(\beta) \mid t_i\right) = u(t_i - \beta) \int_{\underline{t}}^{b^{-1}(\beta)} f(t_j \mid t_i)\, dt_j,$$

if it is a first price auction and

$$\Pi_i (t_i, \beta, b(\cdot)) = \int_{\underline{t}}^{b^{-1}(\beta)} u(t_i - b(t_j)) f(t_j \mid t_i)\, dt_j,$$

if it is a second price auction.

We focus attention on symmetric monotonic pure strategy equilibrium (SMPSE), which is defined as $b(\cdot)$ such that $\Pi_i (t_i, b(t_i), b(\cdot)) \geq \Pi_i (t_i, \beta, b(\cdot))$ for all β and t_i. The usual definition requires this inequality to be true only for almost all t_i. This stronger definition creates no problem and makes some statements simpler, as those about the differentiability and continuity of the equilibrium bidding function (otherwise, such properties should be always qualified by the expression "almost everywhere"). Finally, under our assumptions, the second price auction always has a SMPSE in a weakly dominant strategy, which is $b(t_i) = t_i$.

3 Affiliation and Positive Dependence

A key aspect of most auctions is that bidders' values are private information. These private pieces of information are usually modeled as random variables. Although it is mathematically easier to assume that such random variables are independent, this is unrealistic.[5] Indeed, many real world institutions can act as correlation devices: culture, education, common sources of information, evolution, etc. Thus, a deeper understanding of auctions requires dealing with dependence of private information.

The introduction of affiliation in auction theory was a milestone in this enterprise. Milgrom and Weber (1982a) borrowed a statistical concept (multivariate total positivity of order 2, MTP_2) and applied it to a general model of symmetric auctions. In this fashion, they were able to prove many important results, including a revenue ranking that is different from the provisions of the revenue equivalence theorem (Vicrkey, 1961 and Myerson, 1981). The formal definition is as follows:

Definition 1 *The density function* $f : \left[\underline{t}, \bar{t}\right]^n \to \mathbb{R}_+$ *is affiliated if* $f(t) f(t') \leqslant f(t \wedge t') f(t \vee t')$*, where* $t \wedge t' = (\min\{t_1, t_1'\}, ..., \min\{t_n, t_n'\})$ *and* $t \vee t' = (\max\{t_1, t_1'\}, ..., \max\{t_n, t_n'\})$.

[5] For some problems, as those considered in mechanism design, there is an *economic* justification for assuming independence. As shown by Crémer and McLean (1987), dependence of types can allow full extraction of the bidders' surplus. This is not important for the problems that we are considering here, where the mechanisms are fixed (first and second price auctions).

For a quarter of a century, auction theorists used affiliation's properties to derive important conclusions. Affiliation's monotonicity properties (see Theorem 5 of Milgrom and Weber, 1982a) combine well with natural properties of auctions, simplifying the analysis and allowing useful predictions. In sum, affiliation provided foundation for a successful theory, as auction theory is considered (see e.g. Maskin, 2004).

However, as any scientific acchievement, affiliation has limitations. It is important to know what and how relevant such limitations are. The purpose of this section is to offer an assessment of these aspects. In subsection 3.1, we show that the set of affiliated distributions is a small subset or, in other words, affiliation is a restrictive assumption. Although this observation is relevant, it is by no means definitive, since restrictive assumptions are widespread in economics. It is more important to know whether the assumption is reasonable in economically relevant situations and how robust its implications are. For this, in subsection 3.2 we review the economic intuition used to describe affiliation and, in subsection 3.3, some cases where affiliation holds. We review some of the affiliation's implications in subsection 3.4. A summary of the findings is presented in subsection 3.5.

3.1 Affiliation is restrictive

In this subsection, we show that affiliation is a restrictive assumption, that is, the set of affiliated densities is small in the set of all densities. There are two ways to characterize a set as small: topological and measure-theoretic. We consider both in the sequel, beginning with the topological.

Let $\mathcal{C}$ denote the set of continuous density functions $f : [0,1]^2 \to \mathbb{R}_+$ and let $\mathcal{A}$ be the set of affiliated densities. For convenience and consistency with the notation in next sections, we are including in $\mathcal{A}$ all affiliated densities and not only the continuous one, which creates no problem. Endow $\mathcal{C}$ with the topology of the uniform convergence, that is, the topology defined by the norm of the sup:

$$\|f\| = \sup_{x \in [0,1]^2} |f(x)| .$$

The following theorem shows that the set of continuous affiliated densities is small in the topological sense.

Theorem 2 *The set of continuous affiliated density functions $\mathcal{C} \cap \mathcal{A}$ is meager.*[6] *More precisely, the set $\mathcal{C} \backslash \mathcal{A}$ is open and dense in $\mathcal{C}$.*
Proof. *See the appendix.* ■

The proof of this theorem is given in the appendix, but the main idea is simple. To prove that $\mathcal{C} \backslash \mathcal{A}$ is open, we take a p.d.f. $f \in \mathcal{C} \backslash \mathcal{A}$ which does not satisfy the affiliated inequality for some points $t, t' \in [0,1]^2$, that is, $f(t) f(t') > f(t \wedge t') f(t \vee t') + \eta$, for some $\eta > 0$. By using such η, we can show that for a function g sufficiently close to f, the above inequality is still valid, that is, $g(t) g(t') > g(t \wedge t') g(t \vee t')$ and, thus,

[6] A meager set (or set of first category) is the union of countably many nowhere dense sets. A set is nowhere dense if its closure has empty interior. Thus, the theorem says more than that $\mathcal{C} \cap \mathcal{A}$ is meager: $\mathcal{C} \cap \mathcal{A}$ is itself a nowhere dense set, by the second claim in the theorem.

is not affiliated. To prove that $\mathcal{C} \backslash \mathcal{A}$ is dense, we choose a small neighborhood V of a point $\hat{t} \in [0,1]^2$, such that for all $t \in V$, $f(t)$ is sufficiently close to $f(\hat{t})$ — this can be done because f is continuous. Then, we perturb the function in this neighborhood to get a failure of the affiliation inequality.

Maybe more instructive than the proof is to understand why the result is true: simply, affiliation requires an inequality to be satisfied everywhere (or almost everywhere). This is a strong requirement and it is the source of affiliation's restrictiveness.

Affiliation is also restrictive in the measure-theoretic sense, that is, in an informal way, it is of "zero measure". Obviously, we need to be careful with the formalization of this, since we are now dealing with an infinite-dimensional set (the set of distributions or densities). As is well known, there are no "natural" measures for infinite dimension sets, that is, measures with all of the properties of the Lebesgue measure — see Yasamaki (1985), Theorem 5.3, p. 139.

Thus, before we formalize our results, we informally explain what we mean by "measure-theoretic". Let $\mathcal{D}$ be the set of probabilistic density functions (p.d.f.'s) $f : [0,1]^2 \to \mathbb{R}_+$ and assume that there is a measure μ over it. We define below a sequence $\mathcal{D}^k$ of finite-dimensional subspaces of $\mathcal{D}$ and take the measures μ^k over $\mathcal{D}^k$ induced by the projection of $\mathcal{D}$ over $\mathcal{D}^k$. The result is as follows: if μ^k is absolutely continuous with respect to the Lebesgue measure λ^k over $\mathcal{D}^k$ — as seems reasonable — then μ puts zero measure on the set $\mathcal{A}$ of affiliated p.d.f.'s.

Remark 3 *There is an alternative method of characterizing smallness in the measure-theoretic sense: to show that the set is* shy*, as defined by Anderson and Zame (2001), generalizing a definition of Christensen (1974) and Hunt, Sauer and Yorke (1992). Stinchcombe (2000) points out some drawbacks in this approach. We discuss the characterization in this approach in the supplement to this paper.*

Now, we formalize our method. Endow $\mathcal{D}$ with the L^1-norm, that is, $\|f\|_1 = \int |f(t)|\, dt$. When there is no peril of confusion with the sup norm previously defined, we write $\|f\|$ for $\|f\|_1$.

For $k \geq 2$, define the transformation $T^k : \mathcal{D} \to \mathcal{D}$ by

$$T^k(f)(x,y) = k^2 \int_{\frac{p-1}{k}}^{\frac{p}{k}} \int_{\frac{m-1}{k}}^{\frac{m}{k}} f(\alpha,\beta)\, d\alpha d\beta,$$

whenever $(x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$, for $m, p \in \{1, 2, ..., k\}$. Observe that $T^k(f)$ is constant over each square $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Let $\mathcal{D}^k$ be the image of $\mathcal{D}$ by T^k, that is, $\mathcal{D}^k \equiv T^k(\mathcal{D})$. Thus, T^k is a projection.

Observe that $\mathcal{D}^k$ is a finite dimensional set. In fact, a density function $f \in \mathcal{D}^k$ can be described by a matrix $A = (a_{ij})_{k \times k}$, as follows:

$$f(x,y) = a_{mp} \text{ if } (x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right], \tag{1}$$

for $m, p \in \{1, 2, ..., k\}$. The definition of f at the zero measure set of points $\{(x,y) = \left(\frac{m}{k}, \frac{p}{k}\right) : m = 0 \text{ or } p = 0\}$ is arbitrary.

The following result is important to our method:

Proposition 4 *f is affiliated if and only if for all k, $T^k(f)$ also is. In mathematical notation: $f \in \mathcal{A} \Leftrightarrow T^k(f) \in \mathcal{A}$, $\forall k \in \mathbb{N}$, or yet: $\mathcal{A} = \cap_{k\in\mathbb{N}} T^{-k}\left(\mathcal{A} \cap \mathcal{D}^k\right)$.*
Proof. *See the supplement to this paper.* ■

The set of affiliated distributions $\mathcal{A}$ is the countable intersection of the sets $T^{-k}\left(\mathcal{A} \cap \mathcal{D}^k\right)$, and these sets themselves are small. $T^{-k}\left(\mathcal{A} \cap \mathcal{D}^k\right)$ is small in $\mathcal{D}$ because $\mathcal{A} \cap \mathcal{D}^k$ is small in $\mathcal{D}^k$ (by definition, T^k is surjective). In fact, we have the following:

Proposition 5 *If λ^k denotes the Lebesgue measure over $\mathcal{D}^k$, then $\lambda^k\left(\mathcal{A} \cap \mathcal{D}^k\right) \downarrow 0$ as $k \to \infty$.*
Proof. *See the supplement to this paper.* ■

The convergence is extremely fast, as shown in the following table, obtained by numerical simulations, with 10^7 cases (see the supplement to this paper for the description of the numerical simulation method and other results).

	$k=3$	$k=4$	$k=5$	$k=6$
$\lambda^k\left(\mathcal{A} \cap \mathcal{D}^k\right)$	1.1%	∼0.01%	$\sim 10^{-6}$	$<10^{-7}$

Table 1 - Proportion of affiliated distribution in the sets $\mathcal{D}^k$.

Now, define the measure μ^k over $\mathcal{D}^k$ as follows: if $E \subset \mathcal{D}^k$ is a measurable subset, put $\mu^k(E) = \mu\left(T^{-k}(E)\right)$. Now, it is easy to obtain the main result of this subsection:

Theorem 6 *If $\mu^k \leq M\lambda^k$ for some $M > 0$ then, $\mu(\mathcal{A}) = 0$.*[7]
Proof. *By Proposition 4, $\mathcal{A} \subset T^{-k}\left(\mathcal{A} \cap \mathcal{D}^k\right)$ for every k. Thus,*

$$\mu(\mathcal{A}) \leq \mu\left(T^{-k}\left(\mathcal{A} \cap \mathcal{D}^k\right)\right) = \mu^k\left(\mathcal{A} \cap \mathcal{D}^k\right) \leq M\lambda^k\left(\mathcal{A} \cap \mathcal{D}^k\right).$$

Since $\lambda^k\left(\mathcal{A} \cap \mathcal{D}^k\right) \downarrow 0$ as $k \to \infty$, by Proposition 5, we have the conclusion. ■

As the reader may note from the above proof, it is possible to change the condition $\mu^k \leq M\lambda^k$ for some $M > 0$ by $\mu^k \leq M^k\lambda^k$ for a sequence M^k, as long as M^k does not go to infinity as fast as $\lambda^k\left(\mathcal{A} \cap \mathcal{D}^k\right)$ goes to zero. Since the convergence $\lambda^k\left(\mathcal{A} \cap \mathcal{D}^k\right) \downarrow 0$ is extremely fast, as we noted above, this assumption seems mild.

[7] The reader may note that the assumption is slightly stronger than absolute continuity of μ^k with respect to λ^k. In fact, absolute continuity requires only that $\lambda^k(A) = 0$ implies $\mu^k(A) = 0$. Nevertheless, by the Radon-Nikodym Theorem, absolute continuity implies the existence of a measurable function m^k such that $\mu^k(A) = \int_A m^k d\lambda^k$. Thus, the above assumption is really requiring this function m^k to be bounded: $m^k \leq M$. As we discuss in the paragraph after the Theorem, this bound does not need to be uniform in k.

It is useful to observe that Theorem 6 is not empty, that is, there are many measures over $\mathcal{D}$ that satisfy it. A way to see this is to recall that a measure over $\mathcal{D}$ can be constructed from the measures over the finite-dimensional sets $\mathcal{D}^k$ by appealing to the Kolmogorov Extension Theorem (see Aliprantis and Border 1991, p. 491). The interested reader will find more comments about this in the supplement to this paper.

The results presented in this section up to now are new, but they may be non surprising for some auction specialists. It seems to be known that affiliation is restrictive, although we were unable to find references for this fact in auction theory. Nevertheless, the following finding may be surprising even for those auction specialists: the set of affiliated distributions is still small in the set distributions with pure strategy equilibrium. This fact can be seen as surprising because known cases with SMPSE are just those with affiliation. Thus, the fact that affiliation is a small set as a subset of the set of distributions with SMPSE is a complete novelty.

For observing this, we use Theorem 16 of section 4, which develops a method to determine equilibrium existence through numerical simulations. That is, for each trial $f \in \mathcal{D}^k$, we test whether the auction with bidders' types distributed according to f has a SMPSE and, when it has, we test whether it is affiliated or not. The results are shown in the Table 2 below.

Set of Distributions with SMPSE	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
Affiliated	7.7%	0.07%	$< 10^{-7}$	$-$	$-$
Non-affiliated	92.3%	99.9%	$\sim$100%	100%	100%

Table 2 - Proportion of affiliated $f \in \mathcal{D}^k$ among those f with symmetric monotonic pure strategy equilibrium (SMPSE).

Table 2 shows that affiliation is restrictive even in the set of p.d.f.'s with SMPSE. It is useful to record this fact separately. For this, let us introduce some useful notations. Let λ and λ^k denote the natural measures defined over $\mathcal{D}^\infty = \cup_{k=1}^{\infty} \mathcal{D}^k$ and $\mathcal{D}^k$, respectively, as constructed in the supplement to this paper and let $\mathcal{P}$ and $\mathcal{P}^k$ denote the set of p.d.f.'s in $\mathcal{D}^\infty$ and $\mathcal{D}^k$, respectively, which have a SMPSE. From the above table, we extract the following:

Observation 7 *Let $\lambda^k\left(\cdot|\mathcal{P}^k\right)$ denote the measure induced by the Lebesgue measure λ^k in the set $\mathcal{D}^k \cap \mathcal{P}^k$. Then, we have $\lambda^k\left(\mathcal{A} \cap \mathcal{D}^k|\mathcal{P}^k\right) \downarrow 0$ as $k \to \infty$.*

Another way to say this is: there are many more cases with pure strategy equilibrium than affiliation allows us to prove. This result strengthens this subsection's conclusion that affiliation is a restrictive condition. Nevertheless, this conclusion may be not so important, if affiliation is valid in the economically relevant situations. Because of this consideration, in the next subsection we discuss the intuition given for affiliation, which describes the economically relevant cases where affiliation is supposed to hold.

3.2 The intuition for affiliation may be misleading

Affiliation was introduced through the *positive dependence intuition*: "a high value of one bidder's estimate makes high values of the others' estimates more likely", Milgrom and Weber (1982a, p. 1096). This intuition is very appealing, since positive dependence is likely the most common kind of dependence in the real world. In fact, many authors introduce affiliation through this intuition or some of its variances.

It is easy to see that affiliation captures this intuition, as we illustrate in Figure 2, below. Affiliation requires that the product of the weights at points (x', y') and (x, y) (where both values are high or both are low) be greater than the product of weights at (x, y') and (x', y) (where they are high and low, alternatively). In other words, the distribution puts more weight in the points in the diagonal than outside it.

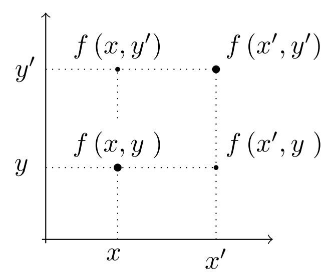


Figure 2: The p.d.f. f is affiliated if $x \leq x'$ and $y \leq y'$ imply $f(x, y') f(x', y) \leq f(x', y') f(x, y)$.

However, as long as we are interested in *positive dependence*, as the given intuition suggests, affiliation is not the only definition available. In the statistical literature many concepts were proposed to correspond to the notion of positive dependence. For simplicity, let us consider only the bivariate case, and assume that the two real random variables X and Y have joint distribution F and strictly positive density function f. The following concepts are formalizations of the notion of positive dependence:[8]

Property I - X and Y are positively correlated (PC) if $cov(X, Y) \geqslant 0$.

Property II - X and Y are said to be positively quadrant dependent (PQD) if $cov(g(X), h(Y)) \geqslant 0$, for all non-decreasing functions g and h.

Property III - The real random variables X and Y are said to be associated (As) if $cov(g(X, Y), h(X, Y)) \geqslant 0$, for all non-decreasing functions g and h.

Property IV - Y is said to be left-tail decreasing in X (denoted LTD($Y|X$)) if $\Pr[Y \leqslant y | X \leqslant x]$ is non-increasing in x for all y. X and Y satisfy property IV if LTD($Y|X$) and LTD($X|Y$).

[8]Most of the concepts can be properly generalized to multivariate distributions. See, e.g., Lehmann (1966) and Esary, Proschan and Walkup (1967). The hypothesis of strictly positive density function is made only for simplicity.

Property V - Y is said to be positively regression dependent on X (denoted PRD($Y|X$)) if $\Pr[Y \leqslant y|X = x] = F(y|x)$ is non-increasing in x for all y. X and Y satisfy property V if PRD($Y|X$) and PRD($X|Y$).

Property VI - Y is said to be Inverse Hazard Rate Decreasing in X (denoted IHRD($Y|X$)) if $\frac{F(y|x)}{f(y|x)}$ is non-increasing in x for all y, where $f(y|x)$ is the p.d.f. of Y conditional to X. X and Y satisfy property VI if IHRD($Y|X$) and IHRD($X|Y$).

We have the following:

Theorem 8 *Let affiliation be Property VII. Then, the above properties are successively stronger, that is,*

$$(VII) \Rightarrow (VI) \Rightarrow (V) \Rightarrow (IV) \Rightarrow (III) \Rightarrow (II) \Rightarrow (I),$$

and all implications are strict.
Proof. *See the appendix.*[9] ■

For this theorem, we used only seven concepts for simplicity. Yanagimoto (1972) defines more than thirty concepts of positive dependence and, again, affiliation is the most restrictive of all, but one.

One can say that the main contribution of this subsection is not the mathematical result presented as Theorem 8, but the observation that: 1) positive dependence was our primary target in the study of dependence in auctions; 2) affiliation *is not* positive dependence but just one among many possible definitions — and it is, in fact, one of the most restrictive.

This observation is important for an assessment of the assumption. If we believe that *positive dependence* corresponds to the set of economically relevant cases, then affiliation may not be the correct assumption or, in other words, the received intuition may be misleading. Accepting the intuition, we may believe that we are covering exactly the important cases, when we are not. The contribution here is to warn of this potential gap.

Of course, we may think that the positive dependence cases are *not*, in fact, the economically relevant ones. Instead, maybe the economically relevant cases are exactly those where affiliation holds. Thus, we need to consider more precisely these cases. This is the theme of the next subsection.

[9] Some implications of Theorem 8 are trivial and others were previously established. Our contribution regards Property VI, that we use later to prove convenient generalizations of equilibrium existence and revenue ranking results. We prove that Property VI is strictly weaker than affiliation and is sufficient for, but not equivalent to Property V.

3.3 In which economic models affiliation is well justified?

There are meaningful economic models where affiliation holds. The first (trivial) example is that of independence. Although independence is a restrictive and unrealistic assumption in general, there are economic situations where it can be considered reasonable. For instance, if the knowledge of one bidder's private value does not change the belief about the other bidders' values, then we have a justification for assuming independence. In fact, it is conceivable that some situations follow this intuitive condition.

Important cases with actual dependence can be found in the conditional independence models. These models assume that the signals of the bidders are conditionally independent, given a variable v (which can be the intrinsic value of the object, for instance). Some specialists seem to believe that conditional independence is a general property (which implies no loss of generality) in symmetric models, because symmetry (a synonymous of exchangeability) is the main assumption of De Finneti's Theorem:

Theorem 9 *(De Finneti's Theorem) Consider a sequence of random variables X_1, X_2,..., and assume that they are exchangeable, that is, assume that the distribution of $(X_1, ..., X_n)$ is equal to the distribution of $\left(X_{\pi(1)}, ..., X_{\pi(n)}\right)$, for any n and any permutation $\pi : \mathbb{N} \to \mathbb{N}$. Then, there is a random variable Q such that all X_1, X_2,..., are conditionally independent (and identically distributed) given Q.*[10]

Unfortunately, however, de Finneti's theorem *is not valid* in standard models of auction theory, even assuming symmetry. The reason is that standard models of auctions consider a finite number of players and, hence, a finite number of private values. De Finneti's theorem is valid only for a *sequence*, that is, an infinite number of random variables. The following example illustrates the problem:[11]

Example 10 *Consider two random variables, X_1 and X_2, taking values in $\{0, 1\}$, with joint distribution given by: $P(X_1 = 0, X_2 = 1) = P(X_1 = 1, X_2 = 0) = \frac{1}{2}$ and $P(X_1 = 0, X_2 = 0) = P(X_1 = 1, X_2 = 1) = 0$. It is easy to see that X_1 and X_2 are symmetric (exchangeable). Assume that there is a variable Q such that X_1 and X_2 are conditionally independent and identically distributed given Q. Let $p(Q) = \Pr[X_i = 1|Q]$, for $i = 1, 2$. Thus,*

$$0 = \int (p(Q))^2 \mu(dQ) = \int (1 - p(Q))^2 \mu(dQ).$$

This implies that $p(Q) = 0$ μ-almost surely and *$p(Q) = 1$ μ-almost surely. This is obviously impossible.*

[10] De Finneti proved this theorem for the case where the X_i are Bernoulli variables. Hewitt and Savage (1955) extended to the general setting. The above statement is somewhat vague. A precise statement is as follows: Let X_1, X_2,..., be an exchangeable sequence of random variables with values in $\mathcal{X}$. Then there exists a probability measure μ on the set of probability measures $\mathcal{P}(\mathcal{X})$ on $\mathcal{X}$ such that:

$$P(X_1 \in A_1, ..., X_n \in A_n) = \int Q(A_1) \cdots Q(A_n) \mu(dQ).$$

[11] This example is adapted from Diaconis and Freedman (1980), which presents a partial generalization of De Finneti's theorem for a finite set of random variables.

Thus, De Finneti's Theorem *does not* imply that conditional independence is a general condition in symmetric auctions. Of course, we may justify conditional independence by other means.[12] However, even if we are ready to assume conditional independence, some care is still necessary before getting affiliation. To see this, assume that the p.d.f. of the signals conditional to v, $f(t_1,...,t_n|v)$, is C^2 (twice continuously differentiable) and has full support. It can be proven that the signals are affiliated if

$$\frac{\partial^2 \log f(t_1,...,t_n|v)}{\partial t_i \partial t_j} \geqslant 0,$$

and

$$\frac{\partial^2 \log f(t_1,...,t_n|v)}{\partial t_i \partial v} \geqslant 0, \tag{2}$$

for all i, j.[13] It is important to note that conditional independence implies only that

$$\frac{\partial^2 \log f(t_1,...,t_n|v)}{\partial t_i \partial t_j} = 0.$$

Thus, conditional independence is not sufficient for affiliation. To obtain the latter, one needs to assume (2) or that t_i and v are affiliated. In other words, to obtain affiliation from conditional independence, one has to assume affiliation itself. Thus, conditional independence does not give economic justification to affiliation.

The fact that we are not able to find a justification in the general model of conditional independence does not imply that it does not exist, at least in special cases. In fact, there is a particular case of this model where affiliation can be justified. Assume that the signals t_i are a common value plus an individual error, that is, $t_i = v + \varepsilon_i$, where the ε_i are independent and identically distributed. Now, we *almost* have the result that the signals t_1, ..., t_n are affiliated: it is still necessary to assume an additional condition. Let g be the p.d.f. of the ε_i, $i = 1, ..., n$. Then, t_1, ..., t_n are affiliated if and only if g is a strongly unimodal function.[14,15]

Another instance of justification of affiliation occurs when we have some reason to believe or accept that the bidders' values are distributed according to some specific distribution. If such distribution has the affiliation property, then the use of affiliation is justified by the reasons for adopting the distribution.[16]

Finally, it may be the case that the focus of the model are some consequences, not the soundness of the assumptions. For instance, the economist may be interested in

[12] For instance, suppose that the knowledge of some variable is sufficient for an estimation of the bidders' values, that is, after knowing such variable, to know some bidder's value does not change the estimation of the other bidders' values. In this case, we have an intuitive economic justification for conditional independence.

[13] See Topkis (1978), p. 310.

[14] The term is borrowed from Lehmann (1959). A function is strongly unimodal if $\log g$ is concave. A proof of the affirmation can be found in Lehmann (1959), p. 509, or obtained directly from the previous discussion.

[15] Even if g is strongly unimodal, so that $t_1, ..., t_n$ are affiliated, it is not true in general that t_1, ..., t_n, ε_1, ..., ε_n, v are affiliated.

[16] This is the case, for instance, when we assume that the distributions are in some family of affiliated copulas as do Li, Paarsch and Hubbard (2007).

some monotone comparative statics, as Milgrom and Shannon (1994). Since affiliation implies nice monotonicity properties, as Milgrom and Weber (1982a) show, there is an advantaged of assuming it. In a sense, the assumption is justified by its methodological advantages, not because it is (approximately) true. This kind of justification is, in fact, very common. For instance, one of the best justifications for using independence is exactly its methodological simplicity.

Yet, the inability of providing a good general economical justification for affiliation does not necessarily implies that it is false in the real world. Only empirical tests can assert this. Such tests are necessary, but they are amazingly absent in the empirical literature.[17]

Nevertheless, even if affiliation is not valid in the real world, what is more important is its implications. As Friedman (1953) argues, the most important criterion for judging an assumption is whether the resulting theory "yields sufficiently accurate predictions"(p. 14). Because of this, we analyze affiliation's implications in the next subsection.[18]

3.4 Affiliation's implications are not robut

Many results were proved using affiliation. They can be classified in two groups: facts that are already true for the independent case (affiliation allows a generalization) and predictions qualitatively different from the case of independence. In this subsection, we will focus in one ele in each of these groups. The first one is the symmetric monotonic pure strategy equilibrium (SMPSE) existence for first price auctions, generalized from independence to affiliation. The second one is the revenue ranking of auctions: under affiliation, the English and the second price auction give expected revenue at least as high as the first price auction (a fact that we denote by $R_2 \geq R_1$). This last result is in contrast with the case of independence, where the revenue equivalence theorem implies the equality of the expected revenues ($R_2 = R_1$).[19,20] Both implications were obtained by Milgrom and Weber (1982a) and we choose them because of their importance. The purpose of this subsection is to verify whether these implications (SMPSE existence and $R_2 \geq R_1$) are true in a more general setting.

Is SMPSE existence true under other definitions of positive dependence (see subsection 3.2)? For private values auctions, it is not difficult to see that the same proof from Milgrom and Weber (1982a) can be used to prove equilibrium existence for Property VI. Indeed, as we show in Theorem 11 below, the following property is sufficient:[21]

Property VI$'$ - The joint (symmetric) distribution of X and Y satisfy property VI$'$

[17] See Laffont (1997) for a survey of empirical literature on auctions.

[18] The empirical literature has tested affiliation's implication that the English auction gives higher revenue than the first price auction, but there is no clear confirmation of this. See Laffont (1997).

[19] Since affiliation contains independence as a special case, the results can be *qualitatively* different, but must have an overlap.

[20] Both the revenue ranking under affiliation and the revenue equivalence theorem requires symmetry, risk neutrality and the same payoff by the lowest type of bidders.

[21] Recently, Monteiro and Moreira (2006) obtained further generalizations of equilibrium existence for non-affiliated variables. Their results are not directly related to positive dependence properties.

if for all x, x' and y in $[0,1]$, $x \geq y \geq x'$ imply

$$\frac{F(y|x')}{f(y|x')} \geq \frac{F(y|y)}{f(y|y)} \geq \frac{F(y|x)}{f(y|x)}.$$

It is easy to see that Property VI implies Property VI′ (under symmetry and full support). Thus, the question becomes whether the generalization of SMPSE existence is possible or not for Property V or even further.

If we define $\Pi(x,y) = (x - b(y)) F(y|x)$, where $b(\cdot)$ is a candidate for the symmetric equilibrium,[22] then equilibrium existence is equivalent to $\Pi(x,x) \geq \Pi(x,y)$. Since $b(\cdot)$ is monotonic, one may conjecture that the monotonicity of $F(y|x)$ — as Property V assumes — may be sufficient to equilibrium existence, through some single crossing arguments (see Athey, 2001). Since property V is still a strong property of positive dependence, this conjecture may be considered reasonable. It turns out that this conjecture is wrong: the following theorem clarifies that SMPSE existence does not generalize beyond Property VI.

Theorem 11 *If $f : [0,1]^2 \to \mathbb{R}$ satisfies property VI′, there is a symmetric pure strategy monotonic equilibrium. Nevertheless, property V is not sufficient for equilibrium existence.*
Proof. *See the appendix.* ■

Although there were reasons to expect SMPSE existence for Property V, as we discussed above, one can rationalize the above theorem by remembering that equilibrium existence requires the inequality $\Pi(x,x) \geq \Pi(x,y)$ to be satisfied in many places (for every pair of points$(x,y) \in [0,1]^2$). Since there are many opportunities to break this inequality, we may understand the previous theorem through the intuition that Property V is not sufficiently strong to control this inequality everywhere. Nevertheless, the next implication — $R_2 \geq R_1$ — does not have this problem, because it is a comparison over expected values, that is, over integrals. Even if the inequality could be wrong for some realizations, it should be true in average for the cases of positive dependence. Thus, one could have the intuition that the revenue ranking implication should be stable across the cases of positive dependence.

There is yet another way of reaching the same conclusion: it is the intuition for the revenue ranking $R_2 \geq R_1$, which is a contribution of Klemperer (2004, p. 48-9):

> [In a first-price auction,] A player with value $v + dv$ who makes the same bid as a player with a value of v will pay the same price as a player with a value of v when she wins, but because of affiliation she will expect to win a bit less often [than in the case of independence]. That is, her higher signal makes her think her competitors are also likely to have higher signals, which is bad for her expected profits.
>
> But things are even worse in a second-price afilliated private-values auction for the buyer. Not only does her probability of winning diminish,

[22] This candidate is increasing and unique, as we show in section 4.

as in the first-price auction, but her costs per victory are higher. This is because affiliation implies that contingent on her winning the auction, the higher her value the higher expected second-highest value which is the price she has to pay. Because the person with the highest value will win in either type of auction they are both equally efficient, and therefore the higher consumer surplus in first-price auction implies higher seller revenue in the second-price auction.

This intuition appeals mainly to the notion of positive dependence. Thus, the intuition should lead us to believe that the revenue ranking is still valid under other definitions of positive dependence. Despite these intuitive arguments, however, the following theorem shows that the implication $R_2 \geq R_1$ is not robust to weaker definitions of positive dependence. As before, Property V is again not sufficient for this result.

Theorem 12 *If f satisfies Property VI' (see definition above), then the second price auction gives greater revenue than the first price auction ($R_2 \geq R_1$). Specifically, the revenue difference is given by*

$$\int_0^1 \int_0^x b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx$$

where $b(\cdot)$ is the first price equilibrium bidding function, or by

$$\int_0^1 \int_0^x \left[\int_0^y L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx, \qquad (3)$$

where $L(\alpha|t) = \exp\left[-\int_\alpha^t \frac{f(s|s)}{F(s|s)} ds \right]$. Moreover, Property V is not sufficient for this revenue ranking.
Proof. *See the appendix.* ■

The results of this subsection are essentially negative: affiliation's implications are not robust. Nevertheless, this conclusion can change if we accept a weaker sense for the implications. This is to require the implications to be true not for each distribution but just in average. That is, if we consider a natural measure over the set of distributions, then the expected value (over distributions) of the expected revenue (for a given distribution) is bigger for second price auctions. In this *weaker sense*, the first implication — SMPSE existence — is still not true, but $R_2 \geq R_1$ is. These results are reported in sections 4 and 5.

Now, we need to summarize and discuss our findings.

3.5 Discussion about the use of affiliation in auction theory

In this section, we have shown that affiliation is restrictive, does not capture the positive dependence cases (which implies that the given intuition may be misleading), and that some of its implications are not robust. Nevertheless, there are some economic models where it can be safely assumed and some of the implications can be true in more general cases, but in a weaker sense. How these two sides of affiliation's assessment compare?

First, for economic situations that can be well described by the models in subsection 3.3, affiliation is justified and its implications are ensured. As long one can accept the intuitions described there, there is absolutely no problem in doing auction theory with affiliation, despite the observation about restrictiveness. As long as we are in a setting in which affiliation is justified, we are free of all perils.

It is useful to illustrate the importance of this point by examining the situations where affiliation (under different names) is used in other sciences. For instance, affiliation is used in statistics, as Positive Likelihood Ratio Dependence (PLRD), the name given by Lehmann (1966) when he introduced the concept. In reliability theory, affiliation is generally referred to as Total Positivity of order 2 of order two (TP_2) for the case of two variables or Multivariate Total Positivity of Order 2 (MTP_2) for n variables, after Karlin (1968). MTP_2 is used when the problem in study has some natural distributions and these distributions satisfy the MTP_2 condition. An example of this can be seen in the historical notes of Barlow and Proschan (1965) about reability theory. It is natural to assume that the failure rates of components or systems follow specific probabilistic distributions (exponentials, for instance) and such special distributions have the TP_2 property. Thus, the corresponding theory of total positive distributions can be advantegously used. Another example of this is the use of copulas. If we assume that the distribution is in a family of copulas that have the MTP property, then the use of affiliation's properties and implications is advantageous and are justified, by the choice of the set of distribution functions analyzed, as we discussed in subsection 3.3.[23]

However, we shall remember that the random variables (types) in auction theory represent information gathered by the bidders. There are some situations where we can assume special forms of the types' distributions (as the cases described in subsection 3.3), but in general there is no justification for assuming specific distributions. In fact, they are rarely assumed in the theory. Thus, there are meaningful and important economic situations that are not covered by affiliation.

From this, we conclude the following: (1) affiliation is useful as a theoretical tool, and can be safely assumed in *some* economic models; (2) there are economically relevant cases where affiliation may be not valid and its implications may be not true; and, (3) there is need for considering a more general approach to dependence in auctions. The next section proposes an instance of such more general approach.

4 A method for dealing with the complexity of auctions

The complexity of auction models requires new tools for dealing with dependence and asymmetry. For instance, even for single object auctions with independent types, but asymmetric bidders, it is not possible to obtain a complete characterization of the equilibrium strategies (see Lebrun 2006). Also under symmetry, but dependent types, there is not a developed theory beyond affiliation. If we want to treat asymmetry and

[23]Li, Paarsch and Hubbard (2007) use copulas to model dependence in auction theory. They are able to find evidence of correlation between the bids.

general dependence, the conclusions seem to be beyond the reach of purely theoretical results.

The problem is not restricted to the characterization of equilibria. Equilibria existence itself is yet a not completely solved problem. We know that mixed strategy equilibria always exists, at least for private value auctions (see Jackson and Swinkels, 2005). Nevertheless, in auction theory is more common to work with pure strategy equilibria. There is no general equilibrium existence results for auctions with dependence (beyond affiliation). It is not clear whether the pure strategy equilibria are common but difficult to prove or, instead, they are rare and because of that it is impossible to give a general existence result. We need methods to treat all of these problems.

In subsection 4.1 we argue that the auction phenomena related to dependence can be modeled and analyzed by considering a simpler but sufficiently rich class of distributions, which we introduce there. Working in this class, we are able to completely characterize the SMPSE existence question in subsection 4.2. In this subsection, we also show that the proportion of distributions with SMPSE is small in the set of all densities considered.

4.1 The class of distributions

Modeling types as continuous real variables is a widespread practice in auction theory. The reason for that is clear: continuous variables allow the use of the convenient tools of calculus, such as derivatives and integrals, to obtain precise characterizations and uniqueness results. This is a very important advantaged, that should not be underestimated. (See Remark 17 below for a consequence of this). On the other hand, working with a continuous of types requires to rely only on analytical arguments to establish equilibrium results. As we show below, the set of first-price auctions with dependence where the standard arguments are able to establish pure strategy equilibrium existence is small in the set of all cases. Thus, continuous variables bring a benefit of characterization at the cost of loosing generality. We offer a method that has both advantages: it gives precise characterizations and is as general as an economist needs.[24] The idea is as follows.

Observe that the value of the single object in the auction is expressed up to cents and is obviously bounded. Thus, the number of actual possible values is finite. Nevertheless, instead of sticking to the (actual) case of discrete values, we allow them to be continuous, but impose, on the other hand, that the density functions are simple (see Figure 1 in the Introduction). In fact, it is sufficient to consider the particular set of simple symmetric functions $\mathcal{D}^k$, as defined in subsection 3.1. A density f in $\mathcal{D}^k$ can be described by a matrix, as the figure 3 below illustrates.[25]

[24]The costs go to the complexity of the tools that are needed in the background.

[25]In this subsection, we will restrict our description to $n = 2$ players. In the case of generic n players, the density function can be described by an array $[f] \in \mathbb{R}^{k^n}$. The reader can find details of this in the supplement of the paper.

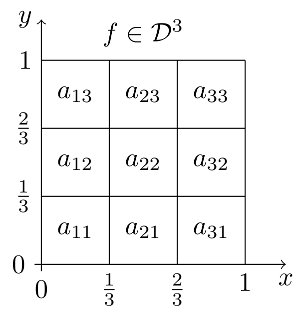


Figure 3: A density $f \in \mathcal{D}^k$ can be represented by a matrix $A = (a_{ij})$.

The restriction to $\mathcal{D}^\infty = \cup_{k\in\mathbb{N}}\mathcal{D}^k$ is a mathematical restriction that implies no economic restriction to the problem we are studying. Note also that the closure $\overline{\mathcal{D}^\infty}$ is the set of all densities $\mathcal{D}$. Now we describe how the equilibrium existence problem can be completely solved in the set $\mathcal{D}^\infty$.

First, recall the standard result of auction theory on SMPSE in private value auctions: if there is a differentiable symmetric increasing equilibrium, it satisfies the differential equation (see Krishna 2002 or Menezes and Monteiro 2005):

$$b^{'}(t) = \frac{t - b(t)}{F(t|t)} f(t|t).$$

If f is Lipschitz continuous, one can use Picard's theorem to show that this equation has a unique solution and, under some assumptions (basically, Property VI' of the previous subsection), it is possible to ensure that this solution is, in fact, equilibrium. Now, for $f \in \mathcal{D}^\infty$, the right hand side of the above equation is not continuous and one cannot directly apply Picard's theorem. We proceed as follows.

First, we show that if there is a symmetric increasing equilibrium b, under mild conditions (satisfied by $f \in \mathcal{D}^\infty$), b is continuous. We also prove that b is differentiable at the points where f is continuous. Thus, for $f \in \mathcal{D}^\infty$, b is continuous everywhere and differentiable everywhere but, possibly, at the points of the form $\frac{m}{k}$. See figure 4.

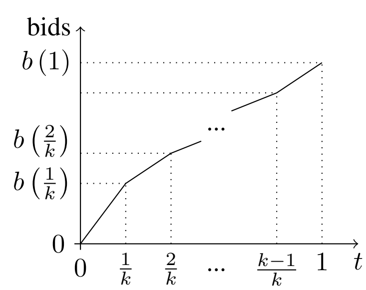


Figure 4: Bidding function for $f \in \mathcal{D}^k$.

With the initial condition $b(0) = 0$ and the above differential equation being valid for the first interval $\left(0, \frac{1}{k}\right)$, we have uniqueness of the solution on this interval and, thus, a unique value of $b\left(\frac{1}{k}\right)$. Since b is continuous, this value is the initial condition for the interval $\left(\frac{1}{k}, \frac{2}{k}\right)$, where we again obtain a unique solution and the uniqueness of the value $b\left(\frac{2}{k}\right)$. Proceeding in this way, we find that there is a unique b which can be a symmetric increasing equilibrium for an auction with $f \in \mathcal{D}^{\infty}$. In the supplement to this paper we prove the following:

Theorem 13 *Assume that u is twice continuously differentiable, $u' > 0$, $f \in \mathcal{D}^{k}$, f is symmetric and positive ($f > 0$). If $b : [0,1] \to \mathbb{R}$ is a symmetric increasing equilibrium, then b is continuous in $(0,1)$ and is differentiable almost everywhere in $(0,1)$ (it is may be non-differentiable only in the points $\frac{m}{k}$, for $m = 1, ..., k$). Moreover, b is the unique symmetric increasing equilibrium. If $u(x) = x^{1-c}$, for $c \in [0,1)$, b is given by*

$$b(x) = x - \int_{0}^{x} \exp\left[-\frac{1}{1-c}\int_{\alpha}^{x} \frac{f(s \mid s)}{F(s \mid s)} ds\right] d\alpha. \tag{4}$$

Proof. *See the supplement to this paper.* ■

Having established the uniqueness of the candidate for equilibrium, our task is reduced to verifying whether this candidate is, indeed, an equilibrium. We complete this task in the next subsection.

Remark 14 *The benefits of chosing $f \in \mathcal{D}^{\infty}$ are deeply related to our focus in pure strategy equilibrium. This kind of equilibrium is the most common in auction theory. It has an important advantage over mixed strategy equilibria: the later can be hardly characterized, even for bimatrix games, while the former is explictly and uniquely determined in general. Since we already know that mixed strategy equilibrium exist, and little can be said beyond its existence, it seems very natural to follow the standard practice in auction theory and restrict attention to pure strategy equilibrium, as we do.*

Even if the reader insists on considering the more general set of p.d.f.'s $\mathcal{D}$ — being aware that this is a matter of mathematical generality, but not of economic generality — our set $\mathcal{D}^{\infty}$ is still dense in $\mathcal{D}$ and, thus, may arbitrarily approximate any conceivable p.d.f. in $\mathcal{D}$. In fact, the following result shows that equilibrium existence in the set $\mathcal{D}^{\infty}$ is sufficient for equilibrium existence in $\mathcal{D}$. This provides an additional justification of the method.

Proposition 15 *Let $f \in \mathcal{D}$ be continuous and symmetric. If $T^{k}(f)$ has a differentiable symmetric pure strategy equilibrium for all $k \geq k_0$, then so does f, and it is the limit of the equilibria of $T^{k}(f)$ as k goes to infinity.*[26]
Proof. *See the supplement to this paper.* ■

[26]See the definition of T^{k} in subsection 3.1.

4.2 Equilibrium existence results for 2 bidders

In the previous subsection, we established the uniqueness of the candidate for symmetric increasing equilibrium for $f \in \mathcal{D}^{\infty}$. Let $b(\cdot)$, given by (4) with $c = 0$, denote such a candidate. Let $\Pi(y, b(x)) = (y - b(x)) F(x|y)$ be the interim payoff of a player with type y who bids as type x, when the opponent follows $b(\cdot)$. Let $\Delta(x, y)$ represent $\Pi(y, b(x)) - \Pi(y, b(y))$. It is easy to see that $b(\cdot)$ is equilibrium if and only if $\Delta(x, y) \leq 0$ for all x and $y \in [0, 1]^2$. Thus, the content of the next theorem is that it is possible to prove equilibrium existence by checking this condition only for a finite set of points:

Theorem 16 *Consider Symmetric Risk Neutral Private Value Auction with* 2 *players with* $f \in \mathcal{D}^{\infty} = \cup_{k \geq 1} \mathcal{D}^k$. *There exist an algorithm that decides in finite time if there is or not a symmetric monotonic pure strategy equilibrium for this auction. For* $f \in D^k$, *the algorithm requires less than* $3\left(k^2 + k\right)$ *comparisons. Errors occur only in elementary operations as sums, multiplications, divisions, comparisons and square and third degree roots.*

Proof. *See the supplement to this paper.* ∎

Remark 17 *It is important to compare this result with the best algorithms for solving simpler games as bimatrix games (see Savani and von Stengel 2006). While best known algorithms for bimatrix games requires operations that grow* exponentially *with the size of the matrix, our algorithm requires operations that increases proportionately to the size of the matrix (* k^2*). We do not state that the algorithm runs in polynomial time because our problem is in continuous variables, not in discrete ones. "Polynomial time" would be slightly vague here, since errors of approximations are possible. Nevertheless, as stated, the possible errors are elementary and require a small number of operations. This allows one to realize the important benefits of working with continuous variables but density functions in* $\mathcal{D}^k$, *as we propose. The characterization of the strategies obtained through differential equations allows one to drastically reduce the computational effort, by reducing the equilibrium candidates to one. The fact that we restrict our focus to densities in* $\mathcal{D}^k$ *— an economically motivated restriction, as we previously emphasized — allows to precisely characterize a small number of points to be tested for the equilibrium condition. This characterization makes possible to have a fast and precise method. The speed of the method allows auction theorists to run simulations for a big number of trials and get a good figure of what happens in general. From this, conjectures for theoretical results can also be derived.*

It is useful to say that the theorem is not trivial, since $\Delta(x, y)$ is not monotonic in the squares $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$. Indeed, the main part of the proof is the analysis of the non-monotonic function $\Delta(x, y)$ in the sets $\left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right]$ and the determination of its maxima for each of these sets. It turns out that we need to check a different number of points (between 1 and 5) for some of these squares.

Using Theorem 16, we can explore the set of distributions with SMPSE and derive observations from this. This may suggest results that can be proven. An example is the following: the set of p.d.f.'s with SMPSE is small. This result is proved formally in the following:

Theorem 18 *The measure of the set of densities $f \in \mathcal{D}^k$ which has SMPSE goes to zero as k increases, that is $\mu^k\left(\mathcal{P}^k\right) \downarrow 0$. Consequently, the measure of the set of densities $f \in \mathcal{D}^\infty$ with SMPSE is zero, that is, $\mu(\mathcal{P}) = 0$.*
Proof. *See the supplement to this paper.* ■

The proof of this theorem follows a simple idea: the equilibrium existence depends on a series of inequalities, the number of which increases with k. Although some care is needed for rigorously establishing the result, this simple observation is the heart of the argument. This gives us the intuition that the equilibrium constraints defining equilibrium increase faster than the degrees of freedom of the problem, when k increases.

The following table provides the numbers that come from numerical simulations and show that the convergence of $\mu^k\left(\mathcal{P}^k\right)$ to zero is also very fast.

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
With SMPSE	43.3%	22.2%	11.4%	5.6%	2.7%	1.3%	0.6%
Without SMPSE	56.7%	77.8%	88.7%	94.4%	97.3%	98.7%	99.4%

Table 3 - Proportion of $f \in \mathcal{D}^k$ with and without SMPSE.

The result summarized in Table 3 is negative in the sense that it suggests that the focus on symmetric monotonic equilibrium may be too narrow. Nevertheless, this is not yet sufficient to conclude that most of the equilibria are in mixed strategies. In fact, while we know that mixed strategy equilibria always exist (Jackson and Swinkels 2005), there is the possibility — not considered in our results — that there are equilibria in asymmetric or non-monotonic pure strategies.

The result presented in Theorem 18 — the fact that symmetric monotonic pure strategy equilibrium (SMPSE) existence is a restrictive property in the set of all distributions — is a novelty in auction theory. This is a negative result because auction theory usually relies on pure strategy equilibria. Nevertheless, maybe it is not too negative, because we already know that equilibria mixed strategies exist and, maybe, they are close and have similar properties. But this last claim is just a conjecture, with little justification beyond the continuity properties that equilibria seem to have (see Lebrun, 2002).

4.3 Equilibrium results for n players and asymmetric auctions

The results of the previous subsection are restricted to the narrow auction setup of symmetric risk neutral private values auctions with 2 players. In this subsection, we show that our approach can be extended well beyond this.

Let us begin with the symmetric risk neutral private values auctions with n players. The mathematics for treating this case is obviously more complex, but the ideas are essentially the same. For both cases, the equilibrium is given and it is sufficient to test whether $\Delta(x,y) = \Pi(x,b(y)) - \Pi(x,b(x))$ is non-positive. We can test the signal of $\Delta(x,y)$ for $(x,y) \in \left(\frac{m-1}{k},\frac{m}{k}\right] \times \left(\frac{p-1}{k},\frac{p}{k}\right]$, for $m,p \in \{1,...,k\}$. This is simplified to check non-positive of a polynomial over $[0,1]^2$, even for n players. The only difference is that for $n=2$, this polynomial is of degree 3 and we can analytically solve it. For $n>2$, the polynomial has degree $n+1$ and we have to rely in numerical methods for finding roots. Thus, we have the following:

Theorem 19 *Consider symmetric risk neutral private value auction with n players with $f \in \mathcal{D}^\infty$. There exist an algorithm that decides in finite time if there is or not a symmetric monotonic pure strategy equilibrium for this auction. Errors are commited in finding roots of polynomials of degree $n+1$ and in elementary operations.*
Proof. *See the supplement to this paper.* ■

Note that we did not make statements about the speed of the method. This is just because this speed depends on the numerical method used to find roots of polynomials. We were unable to find good characterizations of the running time of solutions to this problem.

The method can also be applied to the general setup, that is, asymmetric interdependent values auctions with n risk averse players. For this general case, there are two important differences with respect to the previous case. With symmetry, it is possible to know exactly what is the unique candidate for equilibrium. In the asymmetric case, there is no explicit solution to the system of differential equations. Thus, an algorithm for testing for equilibrium has to include the additional step of finding the solution of the system of differential equations. The second difference is that the interdependent values case allow different value functions $v_i(t)$, which implies that the values are themselves arbitrary functions. In this case, the idea of using only density functions $f \in \mathcal{D}^\infty$ has to be extended also to the value functions. The idea is basically the same, because the values are again expressed only in finite terms, but it is beyond the scope of this paper to describe the algorithms for this case. The important point is that the method is useful even in this more general auction setup.

5 The Revenue Ranking of Auctions

Now, we illustrate how the method described in the previous section can be used to address the problem of revenue ranking of the first price and second price auctions. The point of departure is that, for each p.d.f.'s $f \in \mathcal{D}^k$, it is easy to obtain the expression of the expected revenue difference $\Delta_R^f \equiv R_2^f - R_1^f$ between the two auctions. In order to make a relative comparison, we define $r \equiv \frac{R_2^f - R_1^f}{R_2^f}$, for each f. Generating a uniform sample of $f \in \mathcal{D}^k$, we can obtain the probabilistic distribution of Δ_R^f or of r. The procedure to generate $f \in \mathcal{D}^k$ uniformly is described in the supplement to this paper. The results are shown in subsection 5.1 below.

Moreover, we can also obtain theoretical results about what happens for $\mathcal{D}^k$ for a large k and even for $\mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$. Nevertheless, for the last case, one has to be careful with the meaning of the "uniform" distribution. In the supplement to this paper we show that a natural measure can be defined for $\mathcal{D}^\infty$, which is analogous to Lebesgue measure, although it cannot have all the properties of the finite dimensional Lebesgue measure.

In this fashion, we are able to obtain previsions based on simulations and also theoretical results. One possible objection to this approach is that it considers too equally the p.d.f.'s in the sets $\mathcal{D}^k$. But this is just because we are not assuming any specific information about the context where the auction runs — in some sense, this is a "context-free"approach. If one has information on the environment where the auction runs, so that one can restrict the set of suitable p.d.f.'s, then the uniform measure should be substituted by the empirical measure obtained from this environment. Obviously, the method can be easily adapted to this, once one has such "empirical measure" of the possible distributions.

Now, we present the results that one can obtain using this approach.

5.1 Results on Revenue Ranking

In the supplement to this paper, we develop the expression of the revenue differences from the second price auction to the first price auction for $f \in \mathcal{D}^k$. Let us denote by R_2^f the expected revenue (with respect to $f \in \mathcal{D}^k$) of the second price auction. Similarly, R_1^f denotes the expected revenue (with respect to $f \in \mathcal{D}^k$) of the first price auction. When there is no need to emphasize the p.d.f. $f \in \mathcal{D}^k$, we write R_1 and R_2 instead of R_1^f and R_2^f. Below, μ refers to the natural measure defined over $\mathcal{D}^\infty = \cup_{k=1}^\infty \mathcal{D}^k$, as further explained in the supplement to this paper. We observe the following fact in the simulations made:

Observation 20 *The expectation of the (expected) revenue differences, $R_2 - R_1$, is non-negative, that is, $E_\mu \left[R_2^f - R_1^f | f \in \mathcal{P}^k \right] \geq 0$, where $\mathcal{P}^k$ denotes the set of those $f \in \mathcal{D}^k$ for which there is a SMPSE in the first price auction.*[27]

The simulations were made as follows. We generated the distributions $f \in \mathcal{D}^k$ as described in the supplement to this paper. We evaluate the revenue difference percentage, given by:

$$r = \frac{R_2^f - R_1^f}{R_2^f} \cdot 100\%,$$

that is, we carried out the following:

Numerical experiments

In what follows, we will treat the numerical simulations as giving an "experimental distribution" of r. No confusion should arise between the "experimental distribution" of r and the distributions generated by each $f \in \mathcal{D}^k$. We generated 10^7 distributions $f \in \mathcal{D}^k$, for $k = 3, ..., 9$ and obtained r for each such f. The "experimental distribution" of r is characterized by the table below. It is worth saying that the results are already stable for 10^6 trials.

[27]This was verified for $k \leq 10$, but seems to be valid for larger k's.

Distribution:	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
Expectation	4.5%	8.0%	10.3%	12.1%	13.4%	14.6%	15.5%
Variance	5.3%	6.9%	7.3%	7.2%	7.0%	6.8%	6.6%
5% quantile	−4%	−3%	−2%	0%	1%	3%	4%
10% quantile	−2%	−1%	0%	2%	3%	4%	6%
25% quantile	0%	2%	4%	6%	6%	8%	8%
50% quantile	2%	6%	8%	10%	10%	12.5%	12.5%
75% quantile	6%	10%	12.5%	15%	15%	17.5%	17.5%
90% quantile	10%	15%	17.5%	17.5%	19%	19%	19%
96% quantile	12.5%	17.5%	20%	20%	20%	20%	20%
99% quantile	15%	20%	25%	25%	25%	25%	25%

Table 4 - Expectation of the relative revenue differences (r) for $f \in \mathcal{D}^k$ with SMPSE.

Figure 6 shows the "experimental density" (histogram) of r for $k = 4$.

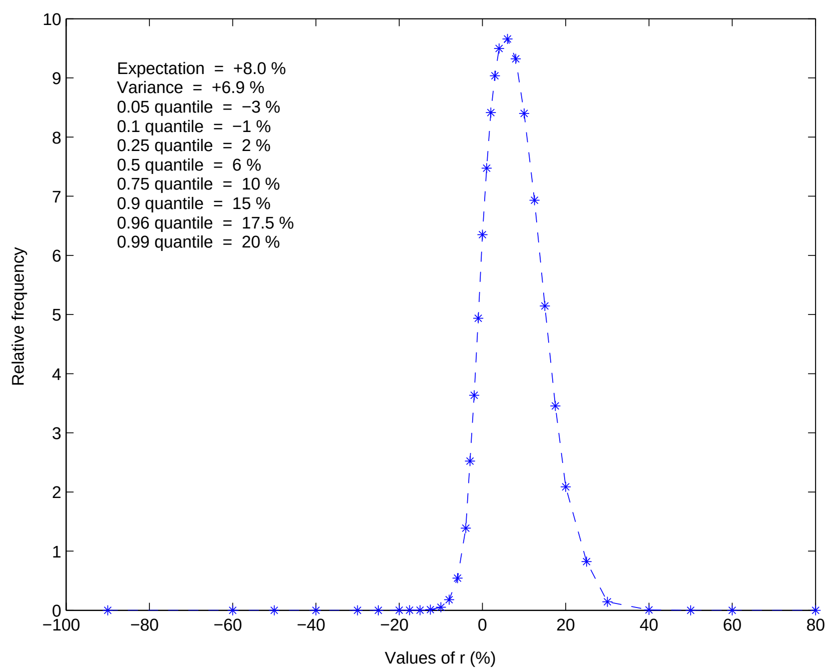


Figure 6: Histogram of r for $k = 4, c = 0$ — for those $f \in \mathcal{D}^k$ with SMPSE.

In Table 4, we displayed the results only for those f with SMPSE. If we consider all distributions, with and without SMPSE, we obtain the results in Table 5 below. This shows that the restriction of SMPSE existence matters for the distribution of r.

Distribution	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
Expectation	-0.08%	0.13%	0.28%	0.38%	0.46%	0.53%	0.57
Variance	10.5%	10.9%	10.5%	9.9%	9.4%	9.0%	8.5%
5% quantile	-25%	-20%	-20%	-17.5%	-17.5%	-15%	-15%
10% quantile	-15%	-15%	-15%	-15%	-12.5%	-12.5%	-12.5%
25% quantile	-8%	-8%	-8%	-8%	-8%	-8%	-8%
50% quantile	-1%	-1%	-1%	-1%	-1%	-1%	-1%
75% quantile	4%	6%	6%	6%	4%	4%	4%
90% quantile	10%	12.5%	12.5%	10%	10%	10%	10%
96% quantile	15%	15%	15%	15%	15%	15%	12.5%
99% quantile	25%	25%	25%	25%	25%	25%	25%

Table 5 - Expectation of the relative revenue differences (r) for all cases (with and without SMPSE).

From the last two tables, one may conjecture that the positivity of the expected revenue differences shown in Table 4 is explained, in fact, by the selection that the SMPSE existence makes. When we do not make this selection, the expected revenue differences are close to zero, as Table 5 shows. In other words, this suggests that SMPSE existence is more likely in the cases when first price auctions give less revenue than second price auctions. This observation means that maybe affiliation gives the right intuition, but its results about revenue ranking are valid in general only in this "weaker sense" (in average).

The following table allows one to compare the effects of dependence and risk aversion to the expected revenue differences. For this, we restrict ourselves to the case of CRRA bidders, that is, bidders with utility function $u(x) = x^{1-c}$, where $c \in [0,1)$.[28]

Expect.	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$
$c=0$	4.6%	8.0%	10.3%	12.1%	13.4%	14.6%	15.4%
$c=0.05$	3.7%	6.9%	8.8%	10.0%	10.8%	11.3%	11.6%
$c=0.1$	3.0%	5.7%	7.2%	8.0%	8.4%	8.6%	8.6%
$c=0.15$	2.2%	4.5%	5.7%	6.2%	6.3%	6.3%	6.3%
$c=0.2$	1.4%	3.4%	4.1%	4.4%	4.4%	4.4%	4.2%
$c=0.3$	0.0%	1.0%	1.2%	1.2%	1.1%	1.1%	1.0%
$c=0.4$	-1.4%	-1.5%	-1.8%	-1.8%	-1.8%	-1.7%	-1.5%
$c=0.52$	-2.9%	-4.7%	-5.3%	-5.2%	-4.9%	-4.6%	-4.1%
$c=0.65$	-3.7%	-8.8%	-9.6%	-9.3%	-8.6%	-7.9%	-7.1%
$c=0.8$	-6.3%	-14.7%	-15.9%	-15.2%	-14.1%	-13.1%	-12.1%

Table 6 - Expectation of the relative revenue differences (r) for bidders with CRRA function $u(x) = x^{1-c}$, where $c \in [0,1)$.

[28]In Table 6, we restrict our study to the cases where SMPSE exists for $c = 0$. We did not implemented the generalized algorithm for testing SMPSE existence with risk aversion. Thus, we were not able to test directly SMPSE existence for $c > 0$. The results in Table 6 should be considered with this in mind.

6 Related literature, the contribution and future work

A few papers have pointed out restrictions or limitations to the implications of affiliation. Perry and Reny (1999) presented an example of a multi-unit auction where the linkage principle fails and the revenue ranking is reversed, even under affiliation. Thus, their criticism seems to be restricted to the generalization of the consequences of affiliation to multi-unit auctions. In contrast, we considered single-unit auctions and non-affiliated distributions.

Klemperer (2003) argues that, in real auctions, affiliation is not as important as asymmetry and collusion. He illustrates his arguments with examples of the 3G auctions conducted in Europe in 2000-2001.

Nevertheless, much more was written in accordance with the conclusions of affiliation. McMillan (1994, p.152) says that the auction theorists working as consultants to the FCC in spectrum auctions, advocated the adoption of an open auction using the linkage principle (Milgrom and Weber 1982a) as an argument: "Theory says, then, that the government can increase its revenue by publicizing any available information that affects the licensee's assessed value". The disadvantages of the open format in the presence of risk aversion and collusion were judged "to be outweighed by the bidders' ability to learn from other bids in the auction" (p. 152). Milgrom (1989, p. 13) emphasizes affiliation as the explanation of the predominance of the English auction over the first price auction.

This paper presents evidence that affiliation is a restrictive assumption. After developing an approach to test the existence of symmetric monotonic pure strategy equilibrium (SMPSE) for simple density functions, we are able to verify that many cases with SMPSE do not satisfy affiliation. Also, the superiority of the English auction is not maintained even for distributions satisfying strong requirements of positive dependence. Nevertheless, we show that the original conclusion of Milgrom and Weber (1982a) (that positive dependence implies that English auctions gives higher revenue than first price auction) is true for a much larger set of cases , but in a weaker sense — "on average".

We can summarize the main contributions of this paper as the following:

- affiliation is a good theoretical assumption for some cases, but there are economically relevant cases not covered by it. In this cases, affiliation's implications may not hold.

- It is possible to approach the problem of dependence in auctions using a special, but sufficiently general set of density functions. Using this set, we can give a characterization of equilibrium existence and revenue ranking that allow numerical experiments and meaningful theoretical results.

Appendix

Proof of Theorem 2

First, we prove that $C\backslash A$ is open. If $f \in C\backslash A$, then $f(x) f(x') > f(x \wedge x') f(x \vee x')$, for some $x, x' \in [0,1]^n$. Fix such x and x' and define $K = f(x) + f(x') + f(x \wedge x') + f(x \vee x') > 0$. Choose $\varepsilon > 0$ such that $2\varepsilon K < f(x) f(x') - f(x \wedge x') f(x \vee x')$ and let $B_\varepsilon(f)$ be the open ball with radius ε and centre in f. Thus, if $g \in B_\varepsilon(f)$, $\|f - g\| < \varepsilon$, which implies $g(x) > f(x) - \varepsilon$, $g(x') > f(x') - \varepsilon$, $g(x \wedge x') < f(x \wedge x') + \varepsilon$, $g(x \vee x') < f(x \vee x') + \varepsilon$, so that

$$\begin{aligned} & g(x) g(x') - g(x \wedge x') g(x \vee x') \\ > \; & [f(x) - \varepsilon][f(x') - \varepsilon] - [f(x \wedge x') + \varepsilon][f(x \vee x') + \varepsilon] \\ = \; & f(x) f(x') - f(x \wedge x') f(x \vee x') - \varepsilon [f(x) + f(x') + f(x \wedge x') + f(x \vee x')] \\ = \; & f(x) f(x') - f(x \wedge x') f(x \vee x') - \varepsilon K \\ > \; & \varepsilon K > 0, \end{aligned}$$

which implies that $B_\varepsilon(f) \subset C\backslash A$, as we wanted to show.

Now, let us show that $C\backslash A$ is dense, that is, given $f \in C$ and $\varepsilon > 0$, there exists $g \in B_\varepsilon(f) \cap C\backslash A$. Since $f \in C$, it is uniformly continuous (because $[0,1]^n$ is compact), that is, given $\eta > 0$, there exists $\delta > 0$ such that $\|x - x'\|_{\mathbb{R}^n} < 2\delta$ implies $|f(x) - f(x')| < \eta$. Take $\eta = \varepsilon/4$ and the corresponding δ.

Choose $a \in (4\delta, 1 - 4\delta)$ and define x (x') by specifying that their first $\lfloor \frac{n}{2} \rfloor$ coordinates are equal to $a - \delta$ $(a + \delta)$ and the last ones to be equal to $a + \delta$ $(a - \delta)$. Thus, $x \wedge x' = (a - \delta, ..., a - \delta)$ and $x \vee x' = (a + \delta, ..., a + \delta)$. Let x_0 denote the vector $(a, ..., a)$. For $y = x$, x', $x \wedge x'$ or $x \vee x'$, we have: $|f(y) - f(x_0)| < \eta$. Let $\xi : (-1,1)^n \to \mathbb{R}$ be a smooth function that vanishes outside $\left(-\frac{\delta}{2}, \frac{\delta}{2}\right)^n$ and equals 1 in $\left(-\frac{\delta}{4}, \frac{\delta}{4}\right)^n$. Define the function g by

$$\begin{aligned} g(y) &= f(y) + 2\eta\xi(y - x) + 2\eta\xi(y - x') \\ &\quad -2\eta\xi(y - x \wedge x') - 2\eta\xi(y - x \vee x'). \end{aligned}$$

Observe that $\|g - f\| = 2\eta = \varepsilon/2$, that is, $g \in B_\varepsilon(f)$. In fact, $g \in B_\varepsilon(f) \cap C\backslash A$, because

$$\begin{aligned} g(x) &= f(x) + 2\eta > f(x_0) + \eta; \\ g(x') &= f(x) + 2\eta > f(x_0) + \eta; \\ g(x \wedge x') &= f(x \wedge x') - 2\eta < f(x_0) - \eta; \\ g(x \vee x') &= f(x \vee x') - 2\eta < f(x_0) - \eta, \end{aligned}$$

which implies

$$\begin{aligned} & g(x) g(x') - g(x \wedge x') g(x \vee x') \\ > \; & [f(x_0) + \eta]^2 - [f(x_0) - \eta]^2 \\ = \; & 4\eta > 0. \blacksquare \end{aligned}$$

Proof of Theorem 8.

It is obvious that $(III) \Rightarrow (II) \Rightarrow (I)$. The implication $(IV) \Rightarrow (III)$ is Theorem 4.3. of Esary, Proschan and Walkup (1967). The implication $(V) \Rightarrow (IV)$ is proved by Tong (1980), chap. 5, p. 80. Thus, we have only to prove that $(VI) \Rightarrow (V)$, since the implication $(VII) \Rightarrow (VI)$ is Lemma 1 of Milgrom and Weber (1982a). Assume that $H(y|x) \equiv \frac{f(y|x)}{F(y|x)}$ is non-decreasing in x for all y. Then, $H(y|x) = \partial_y [\ln F(y|x)]$ and we have

$$1 - \ln [F(y|x)] = \int_y^\infty H(s|x)\, ds \geqslant \int_y^\infty H(s|x')\, ds = 1 - \ln [F(y|x')],$$

if $x \geqslant x'$. Then, $\ln [F(y|x)] \leqslant \ln [F(y|x')]$, which implies that $F(y|x)$ is non-increasing in x for all y, as required by property V.

The counterexamples for each passage are given by Tong (1980), chap. 5, except those involving property (VI): $(V) \nRightarrow (VI)$, $(VI) \nRightarrow (VII)$. For the first counter example, consider the following symmetric and continuous p.d.f. defined on $[0,1]^2$:

$$f(x,y) = \frac{d}{1 + 4(y-x)^2}$$

where $d = [\arctan(2) - \ln(5)/4]^{-1}$ is the suitable constant for f to be a p.d.f. We have the marginal given by

$$f(y) = \frac{k}{2} [\arctan 2(1-y) + \arctan 2(y)]$$

so that we have, for $(x,y) \in [0,1]^2$:

$$f(x|y) = 2\left[1 + 4(y-x)^2\right]^{-1} [\arctan 2(1-y) + \arctan 2(y)]^{-1},$$

$$F(x|y) = \frac{[\arctan 2(x-y) + \arctan 2(y)]}{\arctan 2(1-y) + \arctan 2(y)}$$

and

$$\frac{F(x|y)}{f(x|y)} = 2\left[1 + 4(y-x)^2\right] [\arctan(2x-2y) + \arctan(2y)].$$

Observe that for $y' = 0.91 > y = 0.9$ and $x = 0.1$,

$$\frac{F(x|y')}{f(x|y')} = 0.366863 > 0.366686 = \frac{F(x|y)}{f(x|y)},$$

which violates property (VI). On the other hand,

$$\partial_y [F(x|y)] = \frac{\frac{2}{1+4y^2} - \frac{2}{1+4(x-y)^2}}{\arctan(2-2y) + \arctan(2y)} - \frac{[\arctan(2x-2y) + \arctan(2y)]\left[\frac{2}{1+4y^2} - \frac{2}{1+4(1-y)^2}\right]}{[\arctan(2-2y) + \arctan(2y)]^2}$$

In the considered range, the above expression is non-positive, so that property (V) is satisfied. Then, $(V) \nRightarrow (VI)$.

Now, fix an $\varepsilon < 1/2$ and consider the symmetric density function over $[0,1]^2$:

$$f(x,y) = \begin{cases} k_1, & \text{if } x + y \leqslant 2 - \varepsilon \\ k_2, & \text{otherwise} \end{cases}$$

where $k_1 > 1 > k_2 = 2\left[1 - k_1\left(1 - \varepsilon^2/2\right)\right]/\varepsilon^2 > 0$ and $\varepsilon \in (0, 1/2)$. For instance, we could choose $\varepsilon = 1/3$, $k_1 = 19/18$ and $k_2 = 1/18$. The conditional density function is given by

$$f(y|x) = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ \frac{k_1}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ \frac{k_2}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise} \end{cases}$$

and the conditional c.d.f. is given by:

$$F(y|x) = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ \frac{k_1 y}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ \frac{k_2(y+x+\varepsilon-2)+k_1(2-\varepsilon-x)}{k_2(x+\varepsilon-1)+k_1(2-\varepsilon-x)}, & \text{otherwise} \end{cases}$$

and

$$\frac{F(y|x)}{f(y|x)} = \begin{cases} 1, & \text{if } x \leqslant 1 - \varepsilon \\ y, & \text{if } x > 1 - \varepsilon \text{ and if } y \leqslant 2 - \varepsilon - x \\ y + x + \varepsilon - 2 + k_1/k_2(2 - \varepsilon - x), & \text{otherwise} \end{cases}$$

Since $1 - k_1/k_2 < 0$, the above expression is non-increasing in x for all y, so that property (VI) is satisfied. On the other hand, it is obvious that property (VII) does not hold:

$$f(0.5, 0.5)\, f\left(1 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right) = k_2 k_1 < k_1^2 = f\left(0.5, 1 - \frac{\varepsilon}{2}\right) f\left(0.5, 1 - \frac{\varepsilon}{2}\right).$$

This shows that $(VI) \nRightarrow (VII)$.■

Proof of Theorem 11

The equilibrium existence follows from Milgrom and Weber (1982a)'s proof. For the counterexample, consider the p.d.f. defined in the proof of Theorem 8:

$$f(x,y) = \frac{d}{1 + 4(y-x)^2},$$

where $d = [\arctan(2) - \ln(5)/4]^{-1}$. In the proof of Theorem 8, we established that this p.d.f. satisfies Property V but not Property VI and that:

$$F(x|y) = \frac{[\arctan 2(x-y) + \arctan 2(y)]}{\arctan 2(1-y) + \arctan 2(y)}.$$

From Theorem 13, it is sufficient to prove that

$$b(y) = y - \int_0^y \exp\left[-\frac{1}{2}\int_z^y \frac{1}{\arctan 2w} dw\right] dz$$

cannot be an equilibrium, that is, to verify the existence of x and y such that

$$(y - b(y)) F(y|y) < (y - b(x)) F(x|y).$$

This simplifies to the condition:

$$\frac{\int_0^y \exp\left[-\frac{1}{2}\int_z^y \frac{1}{\arctan 2w} dw\right] dz}{y - x + \int_0^x \exp\left[-\frac{1}{2}\int_z^x \frac{1}{\arctan 2w} dw\right] dz} < \frac{\arctan 2(x-y)}{\arctan 2y} + 1.$$

Let $y = 0.5$ and $x = 1$. Mathematica gives $\int_0^y \exp\left[-\frac{1}{2}\int_z^y \frac{1}{\arctan 2w} dw\right] dz = 0.391128$ and $\int_0^x \exp\left[-\frac{1}{2}\int_z^x \frac{1}{\arctan 2w} dw\right] dz = 0.745072$. Thus, we have:

$$\frac{0.391128}{-0.5 + 0.745072} = 1.59597 < 2 = \frac{\arctan 2(x-y)}{\arctan 2y} + 1,$$

which concludes the verification for the counterexample of SMPSE existence.

Proof of Theorem 12

The dominant strategy for each bidder in the second price auction is to bid his value: $b^2(t) = t$. Then, the expected payment by a bidder in the second price auction, P^2, is given by:

$$\begin{aligned} P^2 &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} y f(y|x)\, dy \cdot f(x)\, dx = \\ &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} [y - b(y)] f(y|x)\, dy \cdot f(x)\, dx + \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y) f(y|x)\, dy \cdot f(x)\, dx, \end{aligned}$$

where $b(\cdot)$ gives the equilibrium strategy for symmetric first price auctions. Thus, the first integral can be substituted by $\int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b'(y) \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx$, from the first order condition: $b'(y) = [y - b(y)] \frac{f(y|y)}{F(y|y)}$. The last integral can be integrated by parts, to:

$$
\begin{aligned}
& \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b(y) f(y|x)\, dy \cdot f(x)\, dx \\
= \; & \int_{[\underline{t},\bar{t}]} \left[b(x) F(x|x) - \int_{[\underline{t},x]} b^{'}(y) F(y|x)\, dy \right] \cdot f(x)\, dx \\
= \; & \int_{[\underline{t},\bar{t}]} b(x) F(x|x) \cdot f(x)\, dx - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) F(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

In the last line, the first integral is just the expected payment for the first price auction, P^1. Thus, we have

$$
\begin{aligned}
D &= P^2 - P^1 \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \frac{F(y|y)}{f(y|y)} f(y|x)\, dy \cdot f(x)\, dx \\
&\quad - \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) F(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} f(y|x) - F(y|x) \right] dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} b^{'}(y) \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

Remember that $b(t) = \int_{[\underline{t},t]} \alpha dL(\alpha|t) = t - \int_{[\underline{t},t]} L(\alpha|t)\, d\alpha$, where $L(\alpha|t) = \exp\left[-\int_{\alpha}^{t} \frac{f(s|s)}{F(s|s)} ds \right]$. So, we have

$$
\begin{aligned}
b^{'}(y) &= 1 - L(y|y) - \int_{[\underline{t},y]} \partial_y L(\alpha|y)\, d\alpha \\
&= \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
D &= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \frac{f(y|y)}{F(y|y)} \int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \left[\frac{F(y|y)}{f(y|y)} - \frac{F(y|x)}{f(y|x)} \right] f(y|x)\, dy \cdot f(x)\, dx \\
&= \int_{[\underline{t},\bar{t}]} \int_{[\underline{t},x]} \left[\int_{[\underline{t},y]} L(\alpha|y)\, d\alpha \right] \cdot \left[1 - \frac{F(y|x)}{f(y|x)} \cdot \frac{f(y|y)}{F(y|y)} \right] \cdot f(y|x)\, dy \cdot f(x)\, dx
\end{aligned}
$$

This is the desired expression. For the counterexample, consider the matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1.6938 & 0.3812 & 0.4140 \\ 0.3812 & 2.1318 & 0.5817 \\ 0.4140 & 0.5817 & 2.4206 \end{bmatrix},
$$

and define the p.d.f. as follows:

$$f(x,y) = a_{mp} \text{ if } (x,y) \in \left(\frac{m-1}{k}, \frac{m}{k}\right] \times \left(\frac{p-1}{k}, \frac{p}{k}\right],$$

for $m, p \in \{1,2,3\}$ and $k = 3$. This distribution satisfies property V (but not property VI) and has a pure strategy equilibrium. The expected revenue from a second price auction is 0.4295, while the expected revenue of a first price auction is 0.4608, which is nearly 7% above.

■

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